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Invariants of B–Type Links via an Analog of Turaev Theorem*

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Abstract.

In this paper we find a method for constructing invariants of **B**–type links within general Turaev framework for constructing link invariants. Such method is shown to be compatible with skein relations of **B**–type. The examples of link invariants of **B**–type relating to the quantum group $U_q(sl(2))$ is considered by means of an analog of the Turaev theorem for **B**–type links.

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Introduction

The study of new (or quantum) invariants of links in 3–manifolds is the field of the growing interest both for mathematicians and physicists during last time. The main ingredient for constructing such invariants, from the physical point of view, is Chern–Simons (CS) theory in three dimensions, where these invariants appear as the vacuum expectation values of Wilson loop operators in perturbative CS theory. In mathematical terms, the construction of new (or quantum) link invariants is given mainly for the case of the trivial topology of the underlying space (i.e., for the case of $\mathbf{R}^3$ or for three-dimensional sphere $\mathbf{S}^3$). There are (at least) three different constructions.

The first one is due to V. F. R. Jones [11] which uses the methods of operator algebras, and, in fact, closely related to thermodynamic limits of integrable models of statistical physics. (This construction can be interpreted via state models and skein categories, see [12]).

The second one is due to V. G. Turaev and N. Yu. Reshetikhin [23], [22] and is combinatorial. Their construction uses the methods of category theory and quasi–Hopf algebras (or: quantum groups, see also [27]). This construction also gives the method of constructing 3–manifold invariants (see also [24]).

The third construction is due to V. A. Vassiliev [25], M. Kontsevich [14], and D. Bar–Natan [3], and is based on the notion of the algebras of chord diagrams. Roughly speaking, such method can be considered as the mathematical formulation of the physical idea of pertubative expansion of CS invariants over the Feynman diagrams (*e.g.*[13]). In this way one can obtain so called finite type invariants of links and corresponding finite type invariants of 3–manifolds (see [18, 19, 20, 21]). Finite type invariants of link can be combine into one generating function, the Kontsevich integral for this link, taking its values in the chord algebra. The Kontsevich integral gives an analytic expression for link invariants. The relationship of such integral to combinatorial expression for link invariants (at least, for the case of knots) is given in [1].

The problem of evaluation of link invariants can be posed for the link in spaces with topology less trivial than $\mathbf{R}^3$, or $\mathbf{S}^3$ (*e.g.*[2], [26]). In this paper we deal with links in solid torus (handlebody of genus 1) and give the construction of the invariants of links in solid torus (or **B**–type links) via an analog of the Turaev theorem [23] for such links. [1] Such approach seems to be very natural from the point of view of the combinatorial way in constructing link invariants.

This paper is organized as follows.

In Section 1 the background on enhanced Yang–Baxter operators (after [23]) coupled to the Reflection Equation operators is given.

In Section 2 we prove an analog of the Turaev theorem in the case of **B**–type links and derive skein relations in this case. The first skein relation is quite similar to that in [23], and the second one reflects the *clasp rule* for **B**–type links.

In Section 3 the two examples of the application of such analog are considered.

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1) It should be noted that recently some algebraic and categorical–theoretic aspects in constructing invariants of solid torus links were considered in [6], [8], [9], [10].

The first example is so called R^2–matrix solution of the reflection equation. The Jones type invariants of **B**–type links constructing from this solution correspond to the well known embeddding the solid torus braid group $B_{1,n} \subset B_{n+1}$. The second example deals with a simple, but nontrivial, diagonal solution of the reflection equation. These invariants of **B**–type links can be interpreted from the point of view of the analog of Turaev theorem. We also derive the corresponding clasp rule which is reminiscent to that in [17].

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1. Yang–Baxter Operators, Reflection Equation Operators

In this Section we define the needed algebraic preliminaries relating the Yang–Baxter (YB) operator (or: R–matrix) and Reflection Equation (RE) operator. YB operator appears naturally in the integrable models of quantum field theory and statistical physics (see [4, 5]) and can be considered as the $\mathfrak{K}$–linear isomorphism [2] $R : V^{\otimes^2} \to V^{\otimes^2}$, for some finite–dimensional space V (over $\mathfrak{K}$), satisfying the Yang–Baxter (YB) Equation (see [4]). YB Equation (without spectral parameter) can be written as

$$R_1 \circ R_2 \circ R_1 = R_2 \circ R_1 \circ R_2, \tag{1.1}$$

where for any $\mathfrak{K}$–linear isomorphism $R : V^{\otimes^2} \to V^{\otimes^2}$ we adopt the following convention:

$$R_i(n) := Id_V^{\otimes^{i-1}} \otimes R \otimes Id_V^{\otimes^{n-i-1}},$$

for any $n \in \mathbf{N}$.

Remark. *YB Equation for YB operator implies that the operators $R_i \equiv R_i(n)$ form a finite–dimensional representation of Artin n–string braid group $B_n, n \geq 1$.*

RE operator appears in the integrable models of statistical physics [15], [16] and can be regarded as the $\mathfrak{K}$–linear isomorphism $K : V \to V$, satisfying the Reflection Equation (RE) without spectral parameter:

$$(K \otimes Id) \circ R \circ (K \otimes Id) \circ R = R \circ (K \otimes Id) \circ R \circ (K \otimes Id), \tag{1.2}$$

where R is YB operator (without spectral parameter).

Remark. *As noted in [15], the operators $K_0(n) := K \otimes Id^{\otimes^{n-1}}, R_i, i \geq 1$, form a finite–dimensional repersentation of solid torus n–string braid group $B_{1,n}$. The explicit presentation of $B_{1,n}$ in terms of generators and relations will be given in Section 2.*

In what follows we need in more general notion of RE operator which takes into account "the inner degrees of freedom" of boundary. Namely, by the RE operator

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[2]) Throughout the paper by $\mathfrak{K}$ is denoting the ground field.

in this paper we mean a $\mathfrak{K}$–linear isomorphism $K : V^{\otimes l+1} \to V^{\otimes l+1}$ satisfying the Reflection Equation (RE) written in more general form that (1.2):

$$(K\otimes Id)(Id_l\otimes R)(K\otimes Id)(Id_l\otimes R) = (Id_l\otimes R)(K\otimes Id)(Id_l\otimes R)(K\otimes Id), \qquad (1.3)$$

where $Id_l := Id\otimes...\otimes Id$ (l times). To proceed further we recall the notion of "operator trace" after [23]. Namely, for any homomorphism $f : V^{\otimes^n} \to V^{\otimes^n}$ its "operator trace" $Tr_n(f)$ is defined to be homomorphism $V^{\otimes^{n-1}} \to V^{\otimes^{n-1}}$, which can be written as

$$Tr_n(f)(v_{i_1}\otimes...\otimes v_{i_{n-1}}) = \sum_{1\le j_1,\ldots,j_{n-1},j\le m} f^{j_1,\ldots j_{n-1},j}_{i_1,\ldots,i_{n-1},j} v_{j_1}\otimes...\otimes v_{j_{n-1}},$$

for any $i_1,...i_{n-1} \in \{1,2,...,m\}$, and $m = dim_{\mathfrak{K}}V$. Here $v_1,...v_m$ is a basis in V, and this definition is independent of the choice of the basis. It is easy to see that $Tr(Tr_n(f)) = Tr(f) \in \mathfrak{K}$, where Tr is the usual trace of homomorphism. The next definition is modification of the corresponding definition in [23] adapted to the situation of solid torus links.

Definition. *An enhanced YB operator coupled to RE operator is said to be a quadriple $\mathfrak{S} = (R, K, \mu, (\alpha, \beta))$, satisfying to the following conditions:*

$$1.\ [R, \mu\otimes\mu] = 0$$

(The homomorphism $\mu\otimes\mu : V^{\otimes^2} \to V^{\otimes^2}$ commutes with R);

$$2.\ Tr_2(R\circ(\mu\otimes\mu)) = \alpha\beta\mu; \quad Tr_2(R^{-1}\circ(\mu\otimes\mu)) = \alpha^{-1}\beta\mu;$$

$$3.\ [K, \mu^{\otimes l+1}] = 0.$$

(The homomorphism $\mu^{\otimes l+1} : V^{\otimes l+1} \to V^{\otimes l+1}$ commutes with K)

Remark. *As remarked in [23], if μ is isomorphism, then the Condition 2 is equivalent to the following Condition $2^{'}$:*

$$Tr_2(R^{\pm 1}\circ(Id_V\otimes\mu)) = \alpha^{\pm 1}\beta\cdot Id_V$$

In what follows we shall assume that μ is isomorphism presented as diagonal matrix with respect to some basis as in [23].

2. An Analog of Turaev Theorem for B–Type Links

In this Section we will prove an analog of Turaev theorem for B-type links making use the technique of *Markov moves* for B-type links.

The oriented **B**–type links (or: oriented links in solid torus M) can be interpreted as the *mixed links* in three-dimensional sphere $\mathbf{S}^3$ containing the complementary solid torus $\hat{I}$, since the solid torus M can be considered as the complement in $\mathbf{S}^3$ of the another solid torus $\hat{I}$. The topology of **B**–type links was considered in [17] in details. For our combinatorial approach to constructing invariants of **B**–type links is essential that the analogs of Alexander's and Markov's theorems exist. (These theorems will be formulated below). This motivates a search of a construction of the finite–dimensional representations of braid group $B_{1,n}$ of n strings in solid torus via YB and RE operators. To every YB operator $R : V^{\otimes^2}\to V^{\otimes^2}$ (without spectral parameter) coupled to RE operator $K : V^{\otimes l+1}\to V^{\otimes l+1}$) corresponds a finite–dimensional representation $b_{R,K} : V^{\otimes l+n}\to V^{\otimes l+n}$ of the braid group $B_{1,n}$ of n strings in the solid handlebody of genus 1 (solid torus) which is defined via the generators an relations:

$$\Big\{\sigma_1,..,\sigma_{n-1},\tau: \quad \sigma_i\sigma_j=\sigma_j\sigma_i, \quad \text{for} \quad |i-j|\geq 2;$$

$$\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \text{for} \quad i=1,...,n-1; \quad \tau\sigma_1\tau\sigma_1=\sigma_1\tau\sigma_1\tau\Big\}.$$

Namely, we put $b_{R,K}(\sigma_i) := R_i^{(l)} = Id_V^{(l+i)-1}\otimes R\otimes Id_V^{n-i-1}$, and $b_{R,K} := K_0(n) = K\otimes Id_V^{\otimes n-1}$. From the properties of YB and RE operators follows that the relations defining the solid torus braid group are fullfilled. Thus we have a unique homomorphism $b_{R,K} : B_{1,n}\to Aut(V^{\otimes^{(l+n)}})$ which transforms σ_i into $R_i^{(l)}$ (for all i=1,...,n-1), and τ into K_0. Now we are in a position to define a mapping $T_{\mathfrak{S}} : \coprod_{n\geq 1}B_{1,n}\to\mathfrak{K}$ which is defined via the formula:

$$T_{\mathfrak{S}}(\xi)=\alpha^{-w(\xi)}\beta^{-n}Tr(b_{R,K}\circ\mu^{\otimes^{(l+n)}} : V^{\otimes^{(l+n)}}\to V^{\otimes^{(l+n)}}),$$

where $\xi\in B_{1,n}$, the homomorphism $\mu\otimes\mu\otimes...\otimes\mu$ ($l+n$ times) is denoted by $\mu^{\otimes^{(l+n)}}$ and homomorphism w from $B_{1,n}$ to the additive group of integers is defined by sending $\sigma_1,..,\sigma_{n-1}$ into 1.

The properties of this mapping are described by the following theorem which can be viewed as the analog of Turaev theorem in [23].

Theorem 1.

For any $\xi,\eta\in B_{1,n}$ we have

$$T_{\mathfrak{S}}(\eta^{-1}\xi\eta)=T_{\mathfrak{S}}(\xi)=T_{\mathfrak{S}}(\xi\sigma_n)=T_{\mathfrak{S}}(\xi\sigma_n^{-1}).$$

Proof. Repeat the corresponding proof from the paper [23] modifying it slightly. Recall the following lemma from [23] which deals with taking of the partial traces.

Lemma.

If f,g,h are endomorphisms respectively of $V^{\otimes^{(n+1)}},V^{\otimes^n},V^{\otimes^2}$, then

$$Tr_{n+1}(f\circ(g\otimes Id_V))=Tr_{n+1}(f)\circ g;$$

$$Tr_{n+1}((g\otimes Id_V)\circ f))=g\circ Tr_{n+1}(f);$$

$$Tr_{n+1}(Id_V^{\otimes^{n-1}}\otimes h)=Id_V^{\otimes^{n-1}}\otimes Tr_2(h).$$

Having this Lemma in mind we continuing the proof of Theorem 1.

It folllows from the definition of enhanced YB operator coupled to RE operator that $\mu^{\otimes^{(l+n)}}$ commutes with $b(\eta) := b_{R,K}(\eta)$, for any $\eta \in B_{1,n}$ (where $b_{R,K} \equiv b : B_{1,n} \longrightarrow Aut(V^{\otimes^{(l+n)}})$).

Therefore, we have

$$Tr((b(\eta^{-1}\xi\eta) \circ \mu^{\otimes(l+n)}) = Tr(b(\eta^{-1}) \circ (b(\xi) \circ \mu^{\otimes(l+n)}) \circ b(\eta)) = Tr(b(\xi) \circ \mu^{\otimes(l+n)}),$$

and $T_{\mathfrak{S}}(\eta^{-1}\xi\eta) = T_{\mathfrak{S}}(\xi)$, since $w(\eta^{-1}\xi\eta) = w(\xi)$.

The proof of the second part of the Theorem 1 is quite similar to that in [23]. Repeat these calculations.

$$b(\xi\sigma_n) = (b(\xi) \circ Id) \circ R_n^{(l)} : V^{\otimes^{l+n+1}} \rightarrow V^{\otimes^{l+n+1}}.$$

Therefore, using the properties of Tr_{l+n}, we have

$$Tr(b(\xi\sigma_n) \circ \mu^{\otimes^{l+n+1}}) = Tr[(b(\xi) \otimes Id_V) \circ R_n^{(l)} \circ (Id_V^{\otimes^{l+n-1}} \otimes \mu \otimes \mu) \circ (\mu^{\otimes^{l+n-1}} \otimes Id_V^{\otimes^2})] =$$

$$Tr\left\{ Tr_{n+1}[(b(\xi) \otimes Id) \circ (Id_V^{\otimes^{(l+n-1)}} \otimes [R \circ (\mu \otimes \mu) \circ (\mu^{\otimes^{(l+n-1)}} \otimes Id_V^2)] \right\}.$$

Then by Lemma it follows that the expression in figured brackets is equal to

$$b(\xi) \circ [Id_V \otimes Tr_2(R \otimes (\mu \otimes \mu))] \circ (\mu^{\otimes^{(l+n-1)}} \otimes Id_V)$$

By definition of enhanced YB operator coupled to RE operator this expression is equal to $\alpha\beta(b(\xi) \otimes \mu^{\otimes^{l+n}}$, and

$$Tr(b(\xi\sigma_n) \circ \mu^{\otimes^{(l+n+1)}}) = \alpha\beta Tr(b(\xi) \circ \mu^{\otimes^{l+n}})$$

with $w(\xi\sigma_n) = w_\xi + 1$. It follows that $T_{\mathfrak{S}}(\xi\sigma_n) = T_{\mathfrak{S}})(\xi)$, and in the similar way one can prove that $T_{\mathfrak{S}}(\xi\sigma_n^{-1}) = T_{\mathfrak{S}}(\xi)$, for any $\xi \in B_{1,n}$.

This completes the proof of Theorem 1. ♣

Theorem 1 which can be viewed as the analog of the corresponding Turaev theorem (namely, Theorem 3.1.1 in [23]). It gives an approach to constructing invariants of B-type links. Namely, recall Alexander's and Markov's theorems for B-type links in the Lampropoulou approach [17] (where "links of B–type" and "braids of B–type" are called as "mixed links" and "mixed braids", respectively).

Alexander's theorem for mixed links.

Any oriented mixed link is isotopic to the closure of some mixed braid.

Markov's theorem for solid torus links.

Let $M = S^3 \backslash \hat{I}$ be a solid torus with $\hat{I}$ also a solid torus; let L_1, L_2 be two oriented links in M and $I \bigcup B_1, I \bigcup B_2$ be mixed braids in $\coprod_{n=1}^{\infty} B_{1,n}$, corresponding to L_1, L_2.Then L_1 is isotopic to L_2 in M if and only if $I \bigcup B_1$ is equivalent to $I \bigcup B_2$ in $\coprod B_{1,n}$, under equivalence generated by the augmented braid relations together with the following two moves:

(i) Conjugation: If $\alpha, \beta \in B_{1,n}$, then $\alpha \simeq \beta^{-1} \alpha \beta$;

(ii) Markov moves: If $\alpha \in B_{1,n}$, then $\alpha \simeq \alpha \sigma_n^{\pm 1} \in B_{1,n+1}$.

Therefore, by Theorem 1, we have the following corollary.

Corollary.

For any enhanced YB operator coupled to RE operator $\mathfrak{S} = (R, K, \mu, (\alpha, \beta))$ the mapping $T_{\mathfrak{S}} : \coprod B_{1,n} \longrightarrow \mathfrak{K}$ which is given by Theorem 1 induces a mapping of the set of oriented isotopy classes of B–type links into $\mathfrak{K}$.

Both mappings will be denoted by $T_{\mathfrak{S}}$.

Now we can formulate such properties of the constructed link invariant as the skein relations for $T_{\mathfrak{S}}$. These skein relations reflect the corresponding behavoir of R– and K–matrices. First of all, recall that (usually) the R–matrix (as the automorphism of $V^{\otimes^2}$) satisfies the equation $\sum\limits_{i=p}^{q} k_i R^i = 0$ (with $k_p, .., k_q \in \mathfrak{K}$), and such property of R–matrix implies the skein relation for $T_{\mathfrak{S}}$ (see [23]):

$$\sum_{i=p}^{q} k_i \alpha^i T_{\mathfrak{S}}(L_i) = 0, \tag{2.4}$$

where oriented links $L_p, L_{p+1}, .., L_q$ have the same diagrams coinciding outside some disk, in which these diagrams looking as in Figure 1.

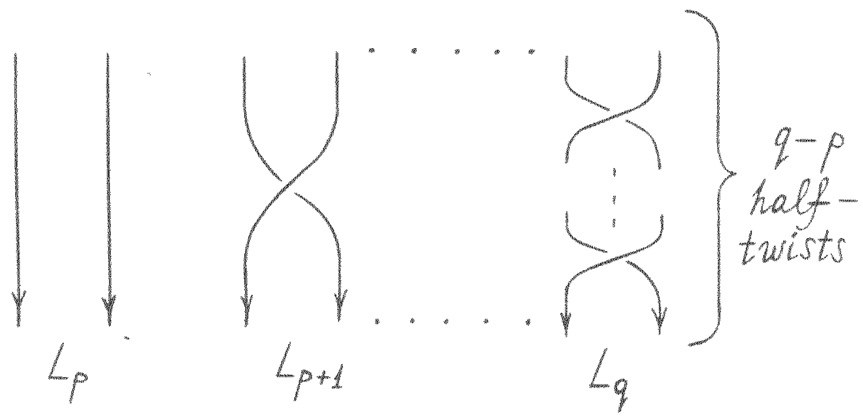

Figure 1.

Secondly, if the K–matrix satisfies the quadratic equation (we will concetrate on this case only):

$$l_1 K + l_0 + l_{-1} K^{-1} = 0, \tag{2.5}$$

(with $l_1, l_0, l_{-1} \in \mathfrak{K}$) this implies the following skein relation:

$$l_1 T_{\mathfrak{S}}(M_1) + l_0 T_{\mathfrak{S}}(M_0) + l_{-1} T_{\mathfrak{S}}(M_{-1}) = 0. \tag{2.6}$$

The links M_1, M_0, M_{-1} in (2.6) have diagrams coinciding outside some disk, in which these diagrams looking as in Figure 2.

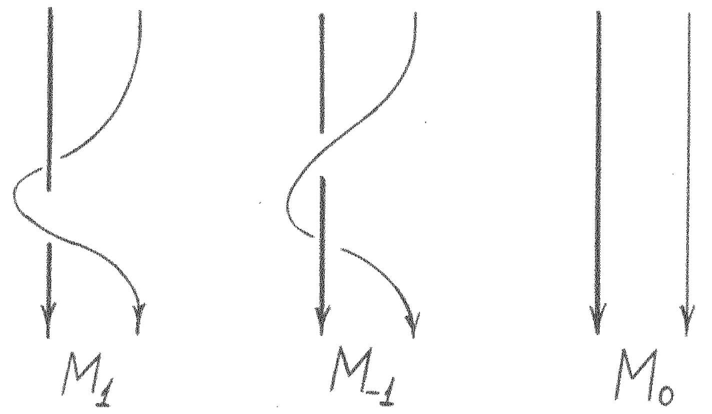

Figure 2.

We collect these observations in the following theorem.

Theorem 2.

Let $\mathfrak{S} = (R, K, \mu, (\alpha, \beta))$ be an enhanced YB operator (coupled to RE operator) such that:

1. the automorphism R of $V^{\otimes^2}$ satisfies the equation $\sum\limits_{i=p}^{q} k_i R^i = 0$, with $k_p, ., k_q \in \mathfrak{K}$,

2. the automorphism K of V satisfies the equation $l_1 K + l_0 + l_{-1} K^{-1} = 0$, with $l_1, l_0, l_{-1} \in \mathfrak{K}$,

then the following skein relations hold for link invariant $T_{\mathfrak{S}}$:

$$\sum_{i=p}^{q} k_i {\alpha}^i T_{\mathfrak{S}}(L_i) = 0, \tag{2.7}$$

$$l_1 T_{\mathfrak{S}}(M_1) + l_0 T_{\mathfrak{S}}(M_0) + l_{-1} T_{\mathfrak{S}}(M_{-1}) = 0, \tag{2.8}$$

where links $T_p, .., T_q$ (resp. M_1, M_0, M_{-1}) have the diagrams coinciding outside some disk, in which these diagrams looking as in Figure 1 (resp. Figure 2).

Proof. Repeat the corresponding proof in [23] modifying it slightly.

First of all, if $L_p, .., L_q$ are oriented links which have diagrams as above, then these links are isotopic to the closures of the braids $\eta, \sigma_1\eta, .., {\sigma_1}^{q-p}\eta$, for some mixed braid η. Then

$$T_{\mathfrak{S}}(L_i) = T_{\mathfrak{S}}({\sigma_1}^i\eta) = \alpha^{-i-w(\eta)}\beta^{-n} Tr[(R_1)^i) \circ b_{R,K}(\eta) \circ \mu^{\otimes^{l+n}}],$$

where n is the number of the strings of η. Therefore,

$$\sum_{i=p}^{q} k_i \alpha^i T_{\mathfrak{S}}(L_i) = \alpha^{-w(\eta)} \beta^{-n} Tr[(\sum_{i=p}^{q} k_i R_1^i) \circ b_{R,K}(\eta) \circ \mu^{\otimes^{l+n}}] = 0.$$

Secondly, if M_1, M_0, M_1 are oriented links, which have diagrams as above, then these links are isotopic to the closures of the braids $\tau\xi, \xi, \tau^{-1}\xi$, for some mixed braid ξ. Then

$$T_{\mathfrak{S}}(M_i) = T_{\mathfrak{S}}(\tau^j \xi) = \alpha^{-w(\xi)} \beta^{-m} Tr[(K^j) \circ b_{R,K}(\xi) \circ \mu^{\otimes^{l+m}}],$$

where m is the number of strings of ξ (with $j = (1, 0, -1)$). Therefore,

$$l_1 T_{\mathfrak{S}}(M_1) + l_0 T_{\mathfrak{S}}(M_0) + l_{-1} T_{\mathfrak{S}}(M_{-1}) =$$

$$\alpha^{-w(\xi)} \beta^{-m} Tr[(l_1 K + l_0 + l_{-1} K^{-1}) \circ b_{R,K}(\xi) \circ \mu^{\otimes^{l+m}}] = 0.$$

This completes the proof of Theorem 2. ♣

3. Jones Type Polynomial for B–Type Links

In this Section we give some examples of the application of Theorem 1 relating to the quantum group $U_q(sl(2))$. For usual (or: for A–type) links this group corresponds to the case of the Jones polynomial of links. An extension of this invariant for the case of B–type links could be regarded as Jones type polynomial for B–type links. Recall R matrix for $U_q(sl(2))$ (see [7]):

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \omega & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}; \quad \omega := q - q^{-1}. \tag{3.1}$$

Choose a case of $l = 1$, i. e. K is an operator acting in $V^{\otimes 2}$. By fixing a base in V we can represent $K : V^{\otimes 2} \to V^{\otimes 2}$ by a matrix $K = (\begin{smallmatrix} u & x \\ y & z \end{smallmatrix})$, where u, x, y, z are the operator–valued entries. Now, substitute R and K matrices into the reflection equation which is equivalent to [15], [16]:

$$RK_1\widetilde{R}K_2 = K_2RK_1\widetilde{R}, \tag{3.2}$$

where $\widetilde{R} := \mathfrak{P}R\mathfrak{P}$ and $\mathfrak{P}$ is the permutation operator of the tensor product of the two vector spaces $V_1 \otimes V_2 (V_1 \simeq V_2 \simeq V)$. (3.2) is the equation for matrices acting in the tensor product $V_1 \otimes V_2$, and $K_1 := K \otimes Id, K_2 := Id \otimes K$. Solving (3.2) in terms of entries of $K = (\begin{smallmatrix} u & x \\ y & z \end{smallmatrix})$ one can obtain the following commutation rules for u, x, y, z:

$$ux = q^{-2}xu, \;\; uy = q^2yu, \;\; [u, z] = 0, \;\; [x, y] = q^{-1}(uz - u^2),$$

$$[x,z]=-q^{-1}\omega ux,\ \ [y,z]=q^{-1}\omega yu. \tag{3.3}$$

Below we discuss the two different regimes for the solution K. (We have to take in mind that $[K,\mu\otimes\mu]=0$).

1. $\mathbf{R}^2$–matrix solution

This solution corresponds to the embedding $B_{1,n}\subset B_{n+1}$. In this case the K–matrix can be written explicitly:

$$K=\begin{pmatrix} q^2 & 0 & 0 & 0\\ 0 & 1 & \omega & 0\\ 0 & \omega & 1+\omega^2 & 0\\ 0 & 0 & 0 & q^2\end{pmatrix}.$$

In this case $u=\begin{pmatrix} q^2 & 0\\ 0 & 1\end{pmatrix}$, $z=\begin{pmatrix} 1+\omega^2 & 0\\ 0 & q^2\end{pmatrix}$, $x=\begin{pmatrix} 0 & 0\\ \omega & 0\end{pmatrix}$, $y=\begin{pmatrix} 0 & \omega\\ 0 & 0\end{pmatrix}$. Clearly, $[K,\mu\otimes\mu]=0$, therefore, we have an example of application of Theorem 1. Of course, such example is not very interesting, since the corresponding link polynomial coincides with the Jones polynomial of a "bigger" link obtaining from the initial link by a replacement of the singularity to an additional component of such "bigger" link.

2. Diagonal solution

If we want to find the K–matrix in the following form $K=\begin{pmatrix} u & 0\\ 0 & z\end{pmatrix}$, then the commutation rules (3.3) imply that $u=z$. Now, $[K,\mu\otimes\mu]=0$, where $\mu=\begin{pmatrix} q^{-1} & 0\\ 0 & q\end{pmatrix}$, and we find a solution in the form $u=z=\begin{pmatrix} \lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$, and $\lambda_1,\lambda_2\in\mathbf{C}^*$. ($\mathbf{C}^*$ is the multiplicative group of the field of the complex numbers $\mathbf{C}$.) In this case the solution of the reflection equation is diagonal: $K=diag(\lambda_1,\lambda_2,\lambda_1,\lambda_2)$. Therefore we obtain another Jones type link invariant for solid torus links which has λ_1,λ_2 as the additional parameters.

We finished this Section by remarking on clasp relation relating to the solution of the second type. Clearly,

$$\frac{1}{\sqrt{\lambda_1\lambda_2}}K+\sqrt{\lambda_1\lambda_2}K^{-1}=\left\{\sqrt{\frac{\lambda_1}{\lambda_2}}+\sqrt{\frac{\lambda_2}{\lambda_1}}\right\}I. \tag{3.4}$$

By Theorem 2 this quadratic relation on K implies the clasp relation on $T_{\mathfrak{S}}$:

$$\frac{1}{\sqrt{\lambda_1\lambda_2}}T_{\mathfrak{S}}(M_1)+\sqrt{\lambda_1\lambda_2}T_{\mathfrak{S}}(M_{-1})=\left\{\sqrt{\frac{\lambda_1}{\lambda_2}}+\sqrt{\frac{\lambda_2}{\lambda_1}}\right\}T_{\mathfrak{S}}(M_0), \tag{3.5}$$

which is reminiscent to the clasp rule in [17]. With respect to the usual over– and undercrossings the invariant $T_{\mathfrak{S}}$ obtaining by using the diagonal K–matrix has the same properties as Jones polynomial. Of course, these properties reflect the quadratic nature of R–matrix.

4. Discussion

In this paper we have seen that constructing the invariants of **B**–type links can be considered within the general Turaev approach to links invariants [23]. The new additional feature of this approach is the notion of "coupling" of the reflection equation operator K to the Yang–Baxter operator R. This means that K–operator which used in constructing a representation of solid torus braid group $B_{1,n}$ cannnot be arbitrary solution of the reflection equation, but must have a special behaviour with the respect to the operator of the quantum trace. Another additional feature in our approach is the presence of the discrete parameter "l" reflecting the nature of the "inner" states of the boundary.

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