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BOUNDARY CONTROL OF THE MAXWELL DYNAMICAL SYSTEM: LACK OF CONTROLLABILITY BY TOPOLOGICAL REASONS

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ABSTRACT

The paper deals with a boundary control problem for the Maxwell dynamical system in a bounbed domain $\Omega \subset \mathbf{R}^3$. Let $\Omega^T \subset \Omega$ be a subdomain filled by waves at the moment T, T_* the moment at which the waves fill the whole of Ω. The following effect occurs: for small enough T the system is approximately controllable in Ω^T whereas for large $T < T_*$ a lack of controllability is possible. The subspace of unreachable states is of finite dimension determined by topological characteristics of Ω^T.

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0. Introduction

Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with a smooth boundary Γ. We consider the Maxwell system

$$\begin{aligned} &\varepsilon e_t = \operatorname{rot} h; \quad \mu h_t = -\operatorname{rot} e \quad \text{in } \Omega \times (0,T); \\ &\operatorname{div} \varepsilon e = 0, \quad \operatorname{div} \mu h = 0 \quad \text{in } \Omega; \\ &e|_{t=0} = 0, \quad h|_{t=0} = 0; \\ &\nu \times e|_{\Gamma \times [0,T]} = f, \end{aligned}$$

where ε, μ are smooth positive scalar functions (permeabilities) given in $\overline{\Omega}$, ν is a normal on Γ, f is a boundary control; let $\{e^f(x,t),\, h^f(x,t)\}$ be a solution (wave).

Permeabilities determine the velocity $c = (\varepsilon\mu)^{1/2}$ and the optical metric

$$d\tau^2 = \frac{|dx|^2}{c^2},$$

which turns Ω into a Riemannian manifold; we denote dist_c the corresponding distance. Let

$$\Omega^T := \{x \in \Omega \,|\, \operatorname{dist}_c(x,\Gamma) < T\}, \quad T > 0$$

be a near-boundary layer of optical thickness T; the surface

$$\Gamma^T := \{x \in \Gamma \,|\, \operatorname{dist}_c(x,\Gamma) = T\}$$

is an inner component of $\partial\Omega^T$. The magnitude

$$T_* := \inf\{T > 0 \,|\, \Omega^T = \Omega\}$$

coincides with time needed for waves moving into Ω from Γ to fill the whole of the domain.

Introduce the (electric) reahable set

$$\mathcal{E}^T := \left\{e^f(\cdot,T) \,|\, f \in L_2((0,T);\, H^1(\Gamma)),\ f \cdot \nu = 0 \, on \, \Gamma\right\}$$

($H^1(\dots)$ is the Sobolev class); let

$$J^T := \left\{y \in L_2(\Omega) \,|\, \operatorname{div} \varepsilon y = 0 \text{ in } \Omega,\ \operatorname{supp} y \subset \overline{\Omega}^T\right\}$$

be the space of ε-solenoidal fields localized in $\overline{\Omega}^T$. By finiteness of c, the embedding

$$\mathcal{E}^T \subset J^T, \quad T > 0$$

occures. The main question under consideration is a density of this embedding. Our results are the following.

Let us say that Ω^T satisfies the EP-condition (existence of potential) if any cycle (simple smooth closed curve) lying in Ω^T may be continuously deformed into a cycle lying on Γ.

Theorem 1. *If Ω^T satisfies the EP-condition the equality*

$$\operatorname{clos} \mathcal{E}^T = J^T \tag{$*$}$$

holds.

In particular, for small enough T relation $(*)$ is valid, i.e., the electric component of the Maxwell system is approximately controllable.

If the EP-condition is violated the unreachable subspace

$$\mathcal{N}^T = J^T \ominus \mathcal{E}^T,$$

turns out to be nontrivial, i.e. the system is not controllable. The subspace $\mathcal{N}^T$ is of a finite dimension determined by topological characteristics of Ω^T. For example, if Ω is homeomorphic to a ball and $\Omega \setminus \overline{\Omega}^T$ is homeomorphic to a ball with n handles then $\dim \mathcal{N}^T = n$.

A lack of controllability described above is of purely topological nature: it is not connected with a presence of real obstacles in Ω. In particular, if the system is not controllable at the moment $t = T_0$, however, the equality $(*)$ may be restored later for some $T > T_0$.

1. Domains and spaces

Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with a boundary $\Gamma \in C^\infty$, $\varepsilon, \mu \in C^\infty(\overline{\Omega})$ strictly positive functions (permeabilities); denote $c := (\varepsilon\mu)^{-1/2}$.

Equipe Ω with the optical metric

$$d\tau^2 = \frac{|dx|^2}{c^2};$$

let dist_c be the corresponding distance; introduce the eikonal $\tau(x) := \operatorname{dist}_c(x, \Gamma)$, $x \in \overline{\Omega}$. The eikonal determines an increasing family of subdomains

$$\Omega^T := \{x \in \Omega \,|\, \tau(x) < T\}, \quad T > 0$$

and the level surfaces

$$\Gamma^T := \{x \in \Omega \,|\, \tau(x) = T\}, \quad T \geq 0$$

$(\Gamma^0 = \Gamma)$; denote

$$T_* := \inf \left\{T > 0 \,|\, \Omega^T = \Omega\right\} = \max_{\Omega} \tau(\cdot).$$

Let us introduce spaces and classes of $\mathbf{R}^3$-valued functions (fields) used in the paper:

the Sobolev classes $H^s(\dots)$;

the space of ε-solenoidal fields $J := \{y \in L_{2,\varepsilon}(\Omega) \mid \operatorname{div} \varepsilon y = 0 \; in\, \Omega\}$ (with measure εdx);

the subspace $J^T := \left\{y \in J \mid \operatorname{supp} y \subset \overline{\Omega}^T\right\}$ of fields localized in $\overline{\Omega}^T$;

the class $J_+ := J \cap H^1(\Omega)$ (with H^1-topology) and its dual $J_- := (J_+)'$ with respect to J;

the space of tangent fields $\mathcal{T} := \{g \in L_2(\Gamma) \mid \nu \cdot g = 0 \; on\, \Gamma\}$ (ν is a normal);

the class $\mathcal{T}_+ := \mathcal{T} \cap H^1(\Gamma)$ (with H^1-topology) and its dual $\mathcal{T}_- := (\mathcal{T}_+)'$ with respect to $L_2(\Gamma)$;

the space of controls $\mathcal{F}^T := L_2\left([0, T]; \mathcal{T}\right)$;

the class $\mathcal{F}^T_+ := L_2((0,T); \mathcal{T}_+)$ and its dual $\mathcal{F}^T_- := (\mathcal{F}^T_+)' = L_2((0,T); \mathcal{T}_-)$ with respect to $\mathcal{F}^T$.

2. The Maxwell system with boundary control. Electric subsystem

Denote $Q^T := \Omega \times (0,T)$, $\Sigma^T := \Gamma \times [0,T]$ and consider the system

$$\varepsilon e_t = \operatorname{rot} h, \;\; \mu h_t = -\operatorname{rot} e \quad \text{in } Q^T; \tag{2.1}$$

$$e|_{t=0} = 0, \;\; h|_{t=0} = 0; \tag{2.2}$$

$$\nu \times e|_{\Sigma^T} = f, \tag{2.3}$$

with (electric) boundary control f; let $\{e^f(x,t),\, h^f(x,t)\}$ be its solution. Note that (2.1), (2.2) imply

$$\operatorname{div} \varepsilon e = 0, \quad \operatorname{div} \mu h = 0 \quad \text{in } \Omega.$$

For $f \in \mathcal{F}^T_+$ problem (2.1)–(2.3) is uniquely solvable in an appropriate class (see [7, 10]). The well known fact is that solutions (waves) propagate with velocity c:

$$\operatorname{supp}\{e^f,\, h^f\} \subset \left\{(x,t) \in \overline{Q}^T \,|\, t \geq \tau(x)\right\}. \tag{2.4}$$

The electric component satisfies

$$e_{tt} + \frac{1}{\varepsilon}\operatorname{rot}\frac{1}{\mu}\operatorname{rot} e = 0 \quad \text{in } Q^T; \tag{2.5}$$

$$e|_{t=0} = e_t|_{t=0} = 0 \quad \text{in } \Omega; \tag{2.6}$$

$$\nu \times e|_{\Sigma^T} = f. \tag{2.7}$$

For $f \in \mathcal{F}^T_+$ the inclusion $e^f \in C([0,T];\, J)$ holds, and the map $f \to e^f$ is continuous in corresponding norms; this property ensures a continuity of the map $W^T : f \to e^f(\cdot\,, T)$ from $\mathcal{F}^T_+$ into J.

Theorem 2. *For times $T < T_*$ the map W^T is injective.*

Proof. Choose $g \in \operatorname{Ker} W^T$; let e^g, h^g be the solution of (2.1)–(2.3). Consider the extensions:

$$e(\cdot,t) := \begin{cases} 0 & -\infty < t < 0; \\ e^g(\cdot,t) & 0 \leq t < T; \\ -e^g(\cdot, 2T-t) & T \leq t < 2T; \\ 0 & 2T \leq t < \infty \end{cases}$$

and

$$h(\cdot,t) := \begin{cases} 0 & -\infty < t < 0; \\ h^g(\cdot,t) & 0 \leq t < T; \\ h^g(\cdot, 2T-t) & T \leq t < 2T; \\ 0 & 2T \leq t < \infty. \end{cases}$$

By virtue of $e^g(\cdot,T) = 0$, extending by oddness one doesn't violate a continuity of e^g and the pair $\{e,h\}$ turns out to be a solution of the system

$$\varepsilon e_t = \operatorname{rot} h, \quad \mu h_t = -\operatorname{rot} e \quad \text{in } \Omega \times (-\infty,\infty) \tag{2.8}$$

Relation (2.4) implies $\operatorname{supp}\{e(\cdot,\,t),\,h(\cdot,t)\}\subset\overline{\Omega}^T$ for any t that leads to

$$e=0, \quad h=0 \qquad \text{in } (\Omega\setminus\overline{\Omega}^T)\times(-\infty,\infty). \tag{2.9}$$

Applying the Fourier transform on time to (2.8), (2.9) we get

$$ik\varepsilon\,\tilde e(\cdot,\,k)=\operatorname{rot}\tilde h(\cdot,\,k), \quad -ik\mu\,\tilde h(\cdot,\,k)=\operatorname{rot}\tilde e(\cdot,\,k) \quad \text{in } \Omega; \tag{2.10}$$

$$\tilde e(\cdot,\,k)=0, \quad \tilde h(\cdot,\,k)=0 \quad \text{in } \Omega\setminus\overline{\Omega}^T \tag{2.11}$$

for all $k\in(-\infty,\infty)$. By virtue of $\operatorname{div}\varepsilon\tilde e(\cdot,\,k)=0$, $\operatorname{div}\mu\tilde h(\cdot,\,k)=0$, system (2.10) turns out to be elliptic, its solution vanishing on a nonvoid open subset (see (2.11)). By known uniqueness theorem (see [11] , Th. 8.17) the solution vanishes in Ω identically that implies $\tilde e=0$, then $e=0$, $e^g=0$, and, finally, $g=0$. Thus, $\operatorname{Ker}W^T=\{0\}$; the theorem is proved.

A simple generalization of the proof enables to obtain the following interesting result. Let us say that a subset $\omega\subset\Omega^T$ belongs to the class $\mathcal{D}^T$ if $\operatorname{dist}_c(\omega,\,\partial\Omega^T)>0$, i.e., ω is separated from $\Gamma\cup\Gamma^T$, and the (open) set $\Omega^T\setminus\overline{\omega}$ is connected. Put also $\emptyset\in\mathcal{D}^T$ by definition.

Lemma 1. *Let $T<T_*$, $\{e^f,\,h^f\}$ satisfy (2.1)–(2.3) for $f\in\mathcal{F}^T_+$. If $\operatorname{supp}e^f(\cdot,\,T)\in\mathcal{D}^T$ then $f=0$ and $e^f=0$, $h^f=0$.*

The analogous result for the scalar wave equation was established in [1]. Notice that *Theorem 2* is a simple corollary of *Lemma 1.*

3. Boundary control problem

Let us return back to the system (2.1)–(2.3). As *Theorem 2* shows, for times $T<T_*$ electric component $e^f(\cdot,T)$ determines uniquely control f which, in turn, determines magnetic component $h^f(\cdot,T)$. Therefore, managing f one cann't control both of the components simultaneously. Thus, in the case $T<T_*$, the following statement of the boundary control problem (BCP) turns out to be natural: given $y\in J^T$ to find control $f\in\mathcal{F}^T_+$ such that the equality

$$e^f(\cdot,T)=y$$

holds. By virtue of *Theorem 2* the BCP has no more than one solution.

The operator $W^T:\mathcal{F}^T\to J$, $\operatorname{Dom}W^T=\mathcal{F}^T_+$, $W^Tf:=e^f(\cdot,T)$ is well defined due to sec.2; it is injective for $T<T_*$. By virtue of (2.4), W^T acts to the subspace J^T. The set

$$\mathcal{E}^T:=\operatorname{Ran}W^T=\left\{\,e^f(\cdot,T)\middle|\;f\in\mathcal{F}^T_+\right\}$$

is said to be reachable (at the moment $t=T$). The goal of the paper is to treat the embedding $\mathcal{E}^T\subset J^T$.

In the case of $T<T_*$ *Lemma 1* shows that any nonzero $y\in J^T$, $\operatorname{supp}y\in\mathcal{D}^T$ doesn't belong to $\mathcal{E}^T$. Thus, the set $J^T\backslash\mathcal{E}^T$ is rich enough and the equality $\mathcal{E}^T=J^T$ (exact controllability) certainly doesn't hold. This raises the question of whether the equality $\operatorname{clos}\mathcal{E}^T=J^T$ (approximate controllability) holds, which is main subject of the paper.

4. Dual system

The system

$$\varepsilon\varphi_t = \operatorname{rot}\psi, \quad \mu\psi_t = -\operatorname{rot}\varphi \quad \text{in } Q^T; \tag{4.1}$$

$$\varphi|_{t=T} = y, \quad \psi|_{t=T} = 0; \tag{4.2}$$

$$\nu \times \varphi|_{\Sigma^T} = 0; \tag{4.3}$$

is called dual to system (2.1)–(2.3); let $\varphi = \varphi^y(x,t)$, $\psi = \psi^y(x,t)$ be its solution. The following is something of the properties of $\{\varphi^y,\, \psi^y\}$ (see [9, 10]):

(i) for $y \in J$ one has $\varphi^y \in C([0,T];\, J)$; $\psi^y \in C([0,T];\, L_2(\Omega))$; $\operatorname{div}\mu\psi^y = 0$; $\nu \cdot \psi^y = 0$ on Σ^T;

(ii) the map $y \to \nu \cdot \psi^y|_{\Sigma^T}$ acts continiously from J into $\mathcal{F}^T_-$;

(iii) by finiteness of velocity of wave propagation, solution $\{\varphi^y,\, \psi^y\}$ in the subdomain $\{(x,t) \in Q^T \,|\, t > \tau(x)\}$ is determined by $y|_{\Omega^T}$ (doesn't depend on $y|_{\Omega\setminus\Omega^T}$);

(iv) the duality relation

$$\left(e^f(\cdot,T), y\right)_J = -\left(\, f,\, \psi^y|_{\Sigma^T}\right)_{\mathcal{F}^T} \tag{4.4}$$

holds for any $f \in \mathcal{F}^T_+$, $y \in J$.

5. Unreachable states

The subspace

$$\mathcal{N}^T := J^T \ominus \operatorname{clos}\mathcal{E}^T$$

is said to be unreachable. To describe $\mathcal{N}^T$ let us introduce the set $\mathcal{N}^T_*$ of $y \in J^T$ such that:

1) y is C^∞-smooth in $\Omega^T \cup \Gamma$;
2) $\nu \times y = 0$ on Γ;
3) $\operatorname{rot} y = 0$ in Ω^T.

Theorem 3. *For any $T > 0$ the equality*

$$\mathcal{N}^T = \mathcal{N}^T_* \tag{5.0}$$

holds.

Proof. (i) Choose $y \in \mathcal{N}^T_*$; As is easy to check, the pair $\{y(x), 0\}$ satisfies (4.1)–(4.3) for $t > \tau(x)$ (see (iii), sec. 4). Therefore, by uniqueness of solution of the dual system one has

$$\varphi^y(x,t) = y(x), \quad \psi^y(x,t) = 0 \quad \text{in } \{(x,t) \in Q^T \,|\, t > \tau(x)\};$$

in particular, $\psi^y = 0$ holds on Σ^T. Duality (4.4) leads to $(e^f(\cdot,T),\, y)_J = 0$ for any $f \in \mathcal{F}^T_+$; hence $y \perp \mathcal{E}^T$, i.e., $y \in \mathcal{N}^T$ and we get $\mathcal{N}^T_* \subset \mathcal{N}^T$. To prove the theorem one needs to check the opposite inclusion $\mathcal{N}^T_* \supset \mathcal{N}^T$.

(ii) Choose $y \in \mathcal{N}^T$; let $\{\varphi^y,\, \psi^y\}$ be the corresponding solution of (4.1)–(4.3). Boundary condition (4.3), duality (4.4) and property (i), sec.4 lead to

$$\nu \times \varphi^y = 0, \quad \psi^y = 0 \qquad \text{on } \Sigma^T, \tag{5.1}$$

the latter equality being understood in accordance with (ii), sec.4.

Extending the solution as follows

$$\varphi(\cdot,t) := \begin{cases} \varphi^y(\cdot,t), & 0 \le t < T, \\ \varphi^y(\cdot,2T-t), & T \le t < 2T; \end{cases}$$

$$\psi(\cdot,t) := \begin{cases} \psi^y(\cdot,t), & 0 \le t < T, \\ -\psi^y(\cdot,2T-t), & T \le t < 2T; \end{cases}$$

and taking into account (5.1) one can check that φ, ψ satisfy

$$\varepsilon\,\varphi_t = \mathrm{rot}\,\psi, \quad \mu\,\psi_t = -\mathrm{rot}\,\varphi \qquad \text{in } \; Q^{2T}; \tag{5.2}$$

$$\nu \times \varphi = 0, \quad \psi = 0 \qquad \text{on } \Sigma^{2T}. \tag{5.3}$$

(iii) To deal with classical solutions we apply smoothing with respect to time. Choose a scalar function $\chi \in C_0^\infty(-\infty,\infty)$:

$$\chi(-t) = \chi(t), \;\; \chi(t) \ge 0, \;\; \mathrm{supp}\,\chi \subset [-1,1], \;\; \int_{-1}^{1} \chi(t)dt = 1,$$

and denote $\chi_\delta(t) := \frac{1}{\delta}\chi\left(\frac{t}{\delta}\right)$ $(\delta > 0)$, so that χ_δ converges to the Dirac function as δ tends to zero. The vector valued functions

$$\varphi^\delta(\cdot,t) := \chi_\delta(t) * \varphi(\cdot,t), \quad \psi^\delta(\cdot,t) := \chi_\delta(t) * \psi(\cdot,t)$$

are defined in $Q_\delta^{2T} := \Omega \times (\delta, 2T-\delta)$ and satisfy

$$\varepsilon\,\varphi_t^\delta = \mathrm{rot}\,\psi^\delta, \quad \mu\,\psi_t^\delta = -\mathrm{rot}\,\varphi^\delta \qquad \text{in } Q_\delta^{2T}; \tag{5.4}$$

$$\nu \times \varphi^\delta = 0 \quad \text{on } \Sigma_\delta^{2T} \tag{5.5}$$

$$\psi^\delta = 0 \quad \text{on } \Sigma_\delta^{2T} \tag{5.6}$$

where $\Sigma_\delta^{2T} := \Gamma \times [\delta, 2T-\delta]$. A pecular feature of the Maxwell system is that time smoothing leads to smoothing with respect to space variables. This may be justified, for instance, by means of the Fourier method expanding $\varphi(\cdot,t)$, $\psi(\cdot,t)$ over the eigenbasis of the Maxwell operator associated with system (2.1)–(2.3) (see [11]). Smoothed solutions turns out to be classical: φ^δ, $\psi^\delta \in C^\infty(\overline{Q}_\delta^{2T})$.

(iv) A simple fact of the vector analysis is that relation (5.6) implies

$$\nu \cdot \mathrm{rot}\,\psi^\delta = 0 \qquad \text{on } \Sigma_\delta^{2T}. \tag{5.7}$$

Multiplying (5.4) by ν on Γ we get

$$\nu \cdot \varphi_t^\delta = \frac{1}{\varepsilon}\nu \cdot \mathrm{rot}\,\psi^\delta = 0 \quad \text{on } \Sigma_\delta^{2T} \tag{5.8}$$

in view of (5.7). Relations (5.5), (5.8) lead to

$$\varphi_t^\delta = 0 \quad \text{on } \Sigma_\delta^{2T}; \tag{5.9}$$

(5.9) and (5.4) give

$$\mathrm{rot}\,\psi^\delta = 0 \quad \text{on } \Sigma_\delta^{2T}. \tag{5.10}$$

We omit the proof of the following auxiliary result.

Proposition 5.1. *If* $\eta \in C^1(\overline{\Omega}^\xi)$ *satisfies* $\operatorname{div} \mu\eta = 0$ *in* Ω^ξ *and* $\eta = \operatorname{rot} \eta = 0$ *on* Γ *then* $\frac{\partial\eta}{\partial\nu} = 0$ *on* Γ.

The equality

$$\frac{\partial\psi^\delta}{\partial\nu} = 0 \quad \text{on} \quad \Sigma_\delta^{2T} \tag{5.11}$$

follows from (5.6), (5.10) and the proposition.

(v) Separating ψ^δ in (5.4) one obtains the equation

$$\psi_{tt}^\delta + \frac{1}{\mu}\operatorname{rot}\frac{1}{\varepsilon}\operatorname{rot}\psi^\delta = 0 \quad \text{in} \quad Q_\delta^{2T}$$

that may be written in the form

$$\psi_{tt}^\delta - \frac{1}{c^2}\Delta\psi^\delta + \ldots = 0 \quad \text{in} \quad Q_\delta^{2T} \tag{5.12}$$

taking into account $\operatorname{div} \mu\psi^\delta = 0$ (the low order terms are omitted). So ψ^δ turns out to be a solution of the hyperbolic system (5.12) with zero Cauchy data (5.6), (5.11) on the time-like noncharacteristic hypersurface Σ_δ^{2T}. Applying the vectorial version [8] of the Holmgren-John-Tataru uniqueness theorem [16] and using the D.Russell's scheme [14] (see also [2]) one can conclude that ψ^δ is continued by zero from Σ_δ^{2T} into the subdomain

$$K_\delta^{2T} := \left\{ (x,t) \in Q_\delta^{2T} \,\middle|\, \tau(x) + \delta < t < 2T - \tau(x) - \delta \right\}$$

bounded by characteristic surfaces:

$$\psi^\delta = 0 \quad \text{in} \quad K_\delta^{2T}.$$

Therefore, by (5.4) we get

$$\operatorname{rot}\varphi^\delta = 0 \quad \text{in} \quad K_\delta^{2T}. \tag{5.13}$$

(vi) As $\delta \to 0$, the convergence $\varphi^\delta \to \varphi$ occures in $C([\delta_0, 2T-\delta_0];\ J)$ for any fixed $\delta_0 > 0$; in particular, one has $\varphi^\delta(\cdot, T) \to \varphi(\cdot, T) = y$ in J.

Choose any field $\rho \in C^\infty(\overline{\Omega})$, $\operatorname{supp}\rho \subset \overline{\Omega}^\xi$ for $\xi < T$. By virtue of (5.5) and (5.13) the equalities

$$0 = (\operatorname{rot}\varphi^\delta(\cdot, T),\, \rho)_{L_2(\Omega)} = (\varphi^\delta,\, \operatorname{rot}\rho)_{L_2(\Omega)} \tag{5.14}$$

are valid. The limit passage $\delta \to 0$ gives

$$(y,\, \operatorname{rot}\rho)_{L_2(\Omega)} = 0,$$

which means that y satisfies

$$\operatorname{rot} y = 0 \text{ in } \Omega^T, \quad \nu \times y = 0 \text{ on } \Gamma \tag{5.15}$$

in a weak sense (see e.g. [7]). Since the boundary Γ is smooth, (5.15) and $\operatorname{div}\varepsilon y = 0$ lead to C^∞-smoothness of y in Ω^T up to Γ by standard elliptic theory. Thus, we get $y \in \mathcal{N}^T$ that proves the theorem.

6. Approximate controllability

We continue to study the subspace $\mathcal{N}^T$. Let us say that subdomain Ω^T satisfies the EP-condition (existence of potential) if any cycle (a simple smooth closed curve) in Ω^T may be continuously deformed into a cycle lying on Γ.

Theorem 4. *If time $T > 0$ is such that Ω^T satisfies the EP-condition then*

$$\mathcal{N}^T = \{0\}.$$

Proof. Choose $y \in \mathcal{N}^T$. In accordance with *Theorem 3* one has

$$\operatorname{rot} y = 0 \text{ in } \Omega^T; \quad \nu \times y = 0 \text{ on } \Gamma. \tag{6.1}$$

Due to the EP-condition, (6.1) ensures existence of a scalar function (potential) p, such that

$$\nabla p = y \quad \text{in} \quad \Omega^T; \qquad p = 0 \quad \text{on } \Gamma; \tag{6.2}$$

the inclusion $y \in J^T$ implies

$$\operatorname{div} \varepsilon \nabla p = 0 \quad \text{in } \Omega^T. \tag{6.3}$$

Since $\operatorname{supp} y \subset \overline{\Omega}^T$ and $\operatorname{div} \varepsilon y = 0$ in the whole of Ω, the equality $\nu \cdot y = 0$ holds on Γ^T in appropriate (weak) sence that implies

$$\frac{\partial p}{\partial \nu} = 0 \quad \text{on } \Gamma^T. \tag{6.4}$$

Let $B_r(x_0) := \left\{x \in \overline{\Omega} \,|\, \operatorname{dist}_c(x,\, x_0) \leq r\right\}$ be a 'ball'; representing

$$\Omega^T = \bigcup_{\gamma \in \Gamma} B_T(\gamma)$$

one can easily show that subdomain Ω^T satisfies the cone condition (see e.g. [12]). In this case the elliptic equation (6.3) has a unique solution $p \in H^1(\Omega^T)$ satisfying boundary conditions (6.2), (6.4). Hence, $p = 0$ and $y = \nabla p = 0$ that proves the theorem.

As a corollary, we conclude: for time $T > 0$ such that Ω^T satisfies the EP-condition the relation

$$\operatorname{clos} \mathcal{E}^T = J^T \tag{6.6}$$

holds, i.e., electric subsystem of the Maxwell system turns out to be approximately controllable.

The EP-condition is realized for small enough T or in the case of $\Omega \setminus \Omega^T = \bigcup_{j=1}^{m} B_j$ where $B_i \cap B_j = \emptyset$, each B_j is homeomorphic to a closed ball. In both cases approximate controllability occures.

7. Lack of controllability

Comparing controllability properties of the Maxwell system with ones of the system gouverned by the wave equation [2] the following pecularity could be noted. In the case of the wave equation, the Holmgren-John-Tataru uniqueness theorem gives the implication

$$y \in \{ \text{ unreachable subspace } \} \Rightarrow y = 0,$$

whereas for system (2.1)–(2.3) it leads to conditions

$$\operatorname{rot} y = 0, \quad \operatorname{div} \varepsilon y = 0 \quad \text{ in } \Omega^T; \tag{7.1}$$

$$\nu \times y = 0 \quad \text{ on } \Gamma; \tag{7.2}$$

$$\nu \cdot y = 0 \quad \text{ on } \Gamma^T. \tag{7.3}$$

The known fact is that, depending on topology of Ω^T, problem (7.1)–(7.3) may have non-trivial solutions (see [6, 15]). Consider an example, assuming for simplicity $\varepsilon = \mu = 1$.

Lemma 2. *Let Ω be homeomorphic to a ball, $\Omega \setminus \Omega^T$ homeomorphic to a torus; then*

$$\dim \mathcal{N}^T = 1.$$

Proof. At first, let us note that the case under consideration is realizable. As example, one can consider a rotation body Ω having dumbbell shaped cross-section and take large enough T.

Denote $D := \{(x^1, x^2) \in \mathbf{R}^2 \,|\, (x^1)^2 + (x^2)^2 \le 1\}$, $S := \partial D$; let $D \times S$ be the torus, φ a homeomorphism from $D \times S$ onto $\Omega \setminus \Omega^T$. The curve $\gamma := \varphi[\{(0,0)\} \times S]$ is a cycle lying in $\Omega \setminus \Omega^T$.

Choose a cycle $l \subset \Omega^T$ which envelopes $\Omega \setminus \overline{\Omega}^T$ and cann't be deformed into a cycle lying on Γ. Define the circulation of a field y:

$$C_l[y] := \int_l y \cdot dl.$$

The Biot-Savart field

$$b(x) := \alpha \int_\gamma \frac{(x - \xi) \times dl_\xi}{|x - \xi|^3}$$

($\alpha =$ const) satisfies (7.1) and has nonzero circulation; assume α to be such that

$$C_l[b] = 1. \tag{7.4}$$

For any cycle $\lambda \subset \Gamma$ one has

$$C_\lambda[b] = C_\lambda[b_\theta] = 0,$$

where $b_\theta := b - (b \cdot \nu)\nu$; therefore, the field b_θ has a surface potential on Γ: there exists smooth π such that $\nabla_\Gamma \pi = b_\theta$ on Γ.

Find p as a (unique) solution of the Neumann-Dirichlet problem:

$$\Delta p = 0 \quad \text{in } \Omega^T;$$

$$\frac{\partial p}{\partial \nu} = -b \cdot \nu \quad \text{on } \Gamma^T;$$

$$p = \pi \quad \text{on } \Gamma.$$

As is easy to check, the field

$$a := b - \nabla p$$

satisfies (7.1)–(7.3) and is nontrivial due to (7.4); thus, $a \in \mathcal{N}^T$, and $\dim \mathcal{N}^T \geq 1$.

Take $y \in \mathcal{N}^T$ and denote $g = y - C_l[y]a$. For any cycle l lying in Ω^T one has $C_l[g] = 0$. This, together with $\operatorname{rot} g = 0$, leads to existence of a potential q: $\nabla q = g$ in Ω^T. By (7.1)–(7.3), we obtain:

$$\begin{aligned} &\Delta q = 0 \quad \text{in } \Omega^T; \\ &\frac{\partial q}{\partial \nu} = 0 \quad \text{on } \Gamma^T; \\ &q = \text{const} \quad \text{on } \Gamma, \end{aligned}$$

that implies $q = \text{const}$ and $g = 0$. Hence, $y = C_l[y]a$ that leads to $\dim \mathcal{N}^T = 1$. The lemma is proved.

Note that idea of the proof is taken from [6].

The obtained result may be simply generalized as follows: if Ω is homeomorphic to a ball whereas $\Omega \setminus \Omega^T$ is homeomorphic to a ball with n handles then $\dim \mathcal{N}^T = n$.

Denote $\mathcal{H}^T := \{y \in J^T \,|\, \operatorname{rot} y = 0 \; in \; \Omega^T\}$, $\mathcal{G}^T := \{y \in J^T \,|\, y = \nabla p,\, p \in H^1(\Omega^T)\}$. A simple analisys of the proof of *Lemma 3* and its generalization mentioned above leads to the equality

$$\dim \mathcal{N}^T = \dim \mathcal{H}^T / \mathcal{G}^T;$$

relations of this kind are well-known in the Hodge Theory (see [15]).

In conclusion let us consider an example demonstrating a curious behaviour of subspace $\mathcal{N}^T$. Let Ω_1 be a rotation body with a dumbbell crossection, Ω_2 a big ball, Ω_3 a narrow cylindric channel connecting Ω_1 with Ω_2, so that the domain $\Omega := \Omega_1 \cup \Omega_2 \cup \Omega_3$ is homeomorphic to a ball.

(i) If T is small enough, Ω^T satisfies the EP-condition; hence $\mathcal{N}^T = \{0\}$;

(ii) if T is such that $\Omega_3 \subset \Omega^T$ (the channel is captured by waves) but $\Omega \setminus \overline{\Omega}^T$ contains a torus lying in Ω_1, we have $\mathcal{N}^T \neq \{0\}$;

(iii) for large enough $T < T_*$ one has $\Omega_1 \cup \Omega_3 \subset \Omega^T$ (Ω_1 and the channel are captured) whereas Ω^T turns out to be homeomorphic to a spherical layer satisfying the EP-condition; hence $\mathcal{N}^T = \{0\}$ holds again.

8. Remarks and Aknowledgments

(i) The version of the BCP studied in the paper differs from traditional ones (see e.g. [9, 13, 17]). A reason of our interest is that it is the version which works in an approach to the inverse problems based upon their relations to the boundary control theory (the BC-method [2, 3, 5]).

(ii) Lack of controllability discussed above was first noticed in [4] and mentioned in [3] in connection with the inverse problem for system (2.1)–(2.3): the presence of nontrivial $\mathcal{N}^T$ creates complications there.

(iii) We are grateful to our colleagues for fruitful discussions and kind help:

control problems for the Maxwell system were discussed with C. Bardos;

S. Kichenassamy explains the relationship between problem (7.1)–(7.3) and the Hodge theory.

(iv) This paper was submitted to the Journal ESAIM COCV. We would like to thank the Referee of the Journal for very useful criticism: the paper has been thoroughly revised under his recommendations.

(v) In the paper [17] by N. Weck an analogous effect (lack of controllability) is exhibited and studied. The author deals with more delicate problem of exact controllability, a description of an unreachable subspace being given in natural topological terms (the Betti numbers of Ω).

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