POMI PREPRINT — 19 /1999

SPECTRAL GEOMETRIES

ON A COMPACT METRIC SPACE

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October 1999

Abstract

The notion of a spectral geometry on a compact metric space X is introduced. This notion serves as a discrete approximation of X motivated by the notion of a spectral triple from noncommutative geometry. A set of axioms characterizing spectral geometries is given. Bounded deformations of spectral geometries are studied and relationship between the dimension of a spectral geometry and more traditional dimensions of metric spaces is investigated.

*The author was supported by RFFI Grant 99-01-00104

§0. Introduction

The notion of a spectral geometry on a compact metric space X is introduced in [Bu] as a tool of discrete approximation of X motivated by the notion of a spectral triple from noncommutative geometry, see [Co]. Each spectral geometry M on X defines a Connes metric d_M on X. In §1, we study relations between d_M and the initial metric d of X, and also give a set of simple axioms, which characterize spectral geometries. In particular, we prove the existence of spectral geometries M on X such that the identity map $(X, d) \to (X, d_M)$ is a bi-Lipschitz homeomorphism with the Lipschitz constant arbitrarily close to 1.

Spectral geometries seem to be a convenient tool for construction of metrics with prescribed properties. A corresponding construction called *a bounded deformation* of spectral geometry is considered in §2. Here we study relations between different types of convergence on the space of deformations and topologies on the space of metrics.

In the last §3, we study relations between the dimension of a spectral geometry M on X introduced in [Bu] and traditional notions such as the Hausdorff dimension, the upper and lower box-counting dimension, the Minkowski–Bouligand dimension.

§1. Axioms of spectral geometry

1. The main construction. Assume that the set $B \subset X^2 \setminus \Delta$ possesses the following properties

(a) B is symmetric, i.e., $(y, y') \in B \Leftrightarrow (y', y) \in B$;
(b) the set $B_t = \{(y, y') \in B \mid |yy'| \geq t\}$ is finite for each $t > 0$.

Then we define a spectral triple $M(B) = \{\mathcal{A}, \mathcal{H}, ds\}$ as follows. The Hilbert space $\mathcal{H}$ consists of functions $\xi : B \to \mathbb{C}$ satisfying the condition

$$\|\xi\|^2 = \sum_{(y,y') \in B} |\xi(y, y')|^2 < \infty.$$

The representation π of $\mathcal{A}$ in $\mathcal{H}$ is given by

$$(\pi(a)\xi)\,(y, y') = a(y)\xi(y, y')$$

for $a \in \mathcal{A}$, $\xi \in \mathcal{H}$, $(y, y') \in B$. Clearly, this representation is involutive, $\pi(a^*) = (\pi(a))^*$, and

$$\|\pi(a)\| = \sup_{y \in Y} |a(y)| \leq \max_{x \in X} |a(x)|,$$

where Y is the projection of B on X (by (a) it is the same whether one takes the projection on the first or on the second factor).

The unit length operator ds is defined by

$$ds(\xi)(y, y') = |yy'|\xi(y', y).$$

This operator is injective and self-adjoint. Its eigenvalues are $\pm|yy'|$, $(y, y') \in B$. It follows from (b) that ds is compact.

The operator $\mathcal{D} = ds^{-1}$ is defined on

$$\operatorname{dom} \mathcal{D} = \left\{ \xi \in \mathcal{H} \mid \sum_{(y,y') \in B} \|\xi(y, y')\|^2 / |yy'|^2 < \infty \right\}$$

and called the *Dirac* operator. The set $\operatorname{dom}\mathcal{D}$ contains all functions $\xi \in \mathcal{H}$ with finite support and whence is dense in $\mathcal{H}$. Furthermore, $\operatorname{dom}\mathcal{D} = \operatorname{dom}\mathcal{D}^*$ and $\mathcal{D} = \mathcal{D}^*$. Thus $\mathcal{D}$ is an unbounded self-adjoint operator with a discrete spectrum

$$\big\{\pm|yy'|^{-1} \,|\, (y,y') \in B\big\} \subset \mathbb{R}.$$

The set $\operatorname{dom}\mathcal{D}$ is invariant for every $\pi(a)$, $a \in \mathcal{A}$, and we have

$$([\mathcal{D}, \pi(a)]\xi)\,(y,y') = \frac{a(y') - a(y)}{|yy'|}\xi(y',y).$$

Then

$$\|[\mathcal{D}, \pi(a)]\| = \sup_{(y,y')\in B} \frac{|a(y) - a(y')|}{|yy'|} \le \mathrm{Lip}\ (a).$$

Thus the operator $[\mathcal{D}, \pi(a)]$ can be extended to a bounded operator on $\mathcal{H}$. Now the formula

$$|xx'|_M = \sup\left\{|a(x) - a(x')| \mid \|[\mathcal{D}, \pi(a)]\| \le 1\right\}$$

defines the Connes metric $(x,x') \mapsto |xx'|_M$ on X.

2. Density of a spectral triple. We use the metric $|zz'| = \max\{|xx'|, |yy'|\}$ on $X^2 = X \times X$, where $z = (x,y)$, $z' = (x',y')$. We denote by $|z| = |xy|$ the distance between x, $y \in X$. *The density* of a subset $B \subset X^2$ is defined by

$$\mathrm{dens}(B) = \inf_{z\in X^2\setminus\Delta}\ \sup_{b\in B} |z|/|zb|.$$

The number $\mathrm{dens}(M) := \mathrm{dens}(B)$ is called *the density* of the spectral triple $M(B)$. If the set X is infinite, then it follows from (b) that $\mathrm{dens}(M) < \infty$. If $\mathrm{dens}(M) > 0$, then for each $\epsilon \in (0, \mathrm{dens}(M))$ and $(x,x') \in X^2 \setminus \Delta$ there is a point $(y,y') \in B$ with

$$|xy|, |x'y'| \le \alpha|xx'|,$$

where $\alpha = 1/(\mathrm{dens}(M) - \epsilon)$. Thus the set Y is dense in the set of nonisolated points of X. If $\mathrm{dens}(M) > 1$, then Y is everywhere dense in X, in particular,

$$\|\pi(a)\| = \max_{x\in X} |a(x)|$$

for each function $a \in \mathcal{A}$. In the case $\mathrm{dens}(M) > 2$, one can take $\alpha = 1/\mathrm{dens}(M)$ in the inequality above. This easily follows from (b).

1.1. Lemma. *Assume that the set $B \subset X^2 \setminus \Delta$ has the density $D = \mathrm{dens}(B) > 2$. Then for every function $a \in \mathcal{A}$ we have*

$$\sup_{(y,y')\in B} \frac{|a(y) - a(y')|}{|yy'|} \ge \frac{D-2}{D+2} \cdot \mathrm{Lip}\ (a).$$

Proof. Fix an $\epsilon \in (0, D-2)$. Then $\alpha := 1/(D - \epsilon) < 1/2$, and for each point $(x,x') \in X^2 \setminus \Delta$ there is a point $(y,y') \in B$ with $|xy|$, $|x'y'| \le \alpha|xx'|$. In particular, this is true for $(x,x') \in X^2 \setminus \Delta$ with $|a(x) - a(x')|\,/|xx'| \ge L - \epsilon$, where $L = \mathrm{Lip}\ (a)$. Then $|yy'| \le (1 + 2\alpha)|xx'|$, and we have

$$\begin{aligned}|a(x) - a(x')|\,/|xx'| &\le \big(|a(y) - a(y')|\,/|yy'|\big)(1 + 2\alpha) + L\big(|xy| + |x'y'|\big)/|xx'|\\ &\le (1 + 2\alpha)\,|a(y) - a(y')|\,/|yy'| + 2\alpha L,\end{aligned}$$

which implies $|a(y) - a(y')|\,/|yy'| \ge L \cdot (1 - 2\alpha)/(1 + 2\alpha) - \epsilon/(1 + 2\alpha)$. The claim follows from that the $\epsilon > 0$ was chosen arbitrary. $\square$

1.2. Proposition. *Let $M = M(B)$ be a spectral triple of density $D = \mathrm{dens}(M) > 2$ on a compact metric space X. Then for every $x, x' \in X$ we have*

$$|xx'| \le |xx'|_M \le \frac{D+2}{D-2}|xx'|.$$

Moreover, for each $(y, y') \in B$ we have $|yy'| = |yy'|_M$, where $|\ |_M$ is the Connes metric associated with M.

Proof. If $a \in \mathcal{A}$ and Lip $(a) \le 1$, then $\|[\mathcal{D}, \pi(a)]\| \le 1$, where $\mathcal{D} = ds^{-1}$. Choosing $a = a_x : X \to \mathbb{R}$, $a_x(x') = |xx'|$, we obtain $|xx'|_M \ge |xx'|$. If $\|[\mathcal{D}, \pi(a)]\| \le 1$, then by Lemma 1.1 we have Lip $(a) \le (D+2)/(D-2)$. This gives

$$|xx'|_M \le \frac{D+2}{D-2}|xx'|.$$

If $(y, y') \in B$ and $\|[\mathcal{D}, \pi(a)]\| \le 1$, then $|a(y) - a(y')| \le \|[\mathcal{D}, \pi(a)]\| \cdot |yy'| \le |yy'|$, whence $|yy'|_M \le |yy'|$. $\square$

1.3. Corollary. *Let $M = M(B)$ be a spectral triple with density $D = \mathrm{dens}(M) > 2$ on a compact metric space X. Then M is a spectral triple M' with density $D' = \mathrm{dens}(M')$ such that*

$$D\frac{D-2}{D+2} \le D' \le D\frac{D+2}{D-2}$$

for the compact metric space (X, dist_M), where dist_M is the Connes metric associated with M. Furthermore, $\mathrm{dist}_{M'} = \mathrm{dist}_M$. $\square$

Proof. Using Proposition 1.2, we obtain

$$\frac{D-2}{D+2}\frac{|z|}{|zb|} \le \frac{|z|_M}{|zb|_M} \le \frac{D+2}{D-2}\frac{|z|}{|zb|}$$

for each $z \in X^2 \setminus \Delta$, $b \in B$. This immediately implies the required estimate for $\mathrm{dens}(M')$. Since $|yy'|_M = |yy'|$ for every $(y, y') \in B$, the unit length and Dirac operators of the triples M and M' coincide. Hence $\mathrm{dist}_{M'} = \mathrm{dist}_M$. $\square$

The property $\mathrm{dist}_{M'} = \mathrm{dist}_M$ of Connes metrics is similar to the well known property of intrinsic metrics to be invariant under the standard procedure of getting an intrinsic metric from the given one.

Local density. Let $\mathrm{dens}(B, U_t)$ be the density of the set $B \subset X^2$ in the neighborhood $U_t = \{z \in X^2 \mid |z| \le t\}$ of the diagonal Δ. This function does not increase in t, and thus there exists the limit

$$\mathrm{dens}_{\mathrm{loc}}(B) = \lim_{t \to 0} \mathrm{dens}(B, U_t),$$

which is called *the local density* of B. Clearly, $\mathrm{dens}_{\mathrm{loc}}(B) \ge \mathrm{dens}(B)$. Thus in contrast to the density $\mathrm{dens}(M)$, the local density $\mathrm{dens}_{\mathrm{loc}}(M) = \mathrm{dens}_{\mathrm{loc}}(B)$ of a spectral triple $M = M(B)$ might be infinite even for infinite X (see Sec. 4).

If X is a space with intrinsic metric, then for each Lipschitz function a on X we have

$$\mathrm{Lip}\ (a) = \sup_{x \in X} \mathrm{dil}_x(a),$$

where $\mathrm{dil}_x(a) = \lim_{\epsilon \to 0} \mathrm{Lip}\ (a|D_\epsilon(x))$ is the dilatation of a at the point x, $D_\epsilon(x) = \{x' \in X \mid |xx'| \le \epsilon\}$. In that case, Lemma 1.1 and Proposition 1.2 remain true for $D = \mathrm{dens}_{\mathrm{loc}}(M)$. In particular, we have

1.4. Proposition. *Let $M = M(B)$ be a spectral triple of infinite local density on a space X with intrinsic metric. Then its Connes metric coincides with the metric of X,*

$$|xx'|_M = |xx'|$$

for every x, $x' \in X$. $\square$

One can slightly modify the main construction additionally introducing a symmetric function $m : B \to \mathbb{N}$, $m(y, y') = m(y', y)$ for all $(y, y') \in B$, *a multiplicity* function. The space $\mathcal{H}$ is then defined as the space of sections of the bundle $p : \mathbb{C}^m \to B$, satisfying the condition

$$\|\xi\|^2 = \sum_{(y,y') \in B} \|\xi(y, y')\|^2 < \infty,$$

where the fiber over a point $(y, y') \in B$ is a finite dimensional Hilbert space $p^{-1}(y, y') \simeq \mathbb{C}^{m(y,y')}$. The definitions of the representation π and the operator ds remain the same as above. The fact that ds is well defined follows from that the function m is symmetric. So modified spectral triple $\{\mathcal{A}, \mathcal{H}, ds\}$ we denote by $M(B, m)$. Obviously, $M(B) = M(B, 1)$. It is easy to see that Propositions 1.2 and 1.4, and Corollary 1.3 are true for triples $M(B, m)$ as well.

3. Axioms of spectral geometry. We give here a set of axioms which leads to spectral triples of form $M(B, m)$.

Recall (see [N, Chapter IV]) that a spectral measure on a locally compact space T is any operator-valued function P, defined on the Borel subsets of T and having the properties

- given a Borel subset $A \subset T$, the operator $P(A)$ is an orthogonal projector in a fixed Hilbert space $\mathcal{H}$;
- given $\xi \in \mathcal{H}$, the function $(P(A)\xi, \xi)$ is a Borel measure on T.

In particular, $P(A_1 \cup A_2) = P(A_1) + P(A_2)$ and $P(A_1)P(A_2) = 0$ for $A_1 \cap A_2 = \emptyset$.

A spectral geometry on a compact metric space X is given by an involutive representation π of the algebra $\mathcal{A}$ of Lipschitz functions on X in a Hilbert space $\mathcal{H}$ and an injective compact self-adjoint operator ds on $\mathcal{H}$ such that the following conditions (i)–(iv) are satisfied

(i) the representation π is the restriction on $\mathcal{A}$ of an involutive representation $\widetilde{\pi}$ of C^*-algebra $\mathcal{C}(X^2)$ of continuous functions on X^2, where a function $a \in \mathcal{A}$ is considered as the continuous function $a(x, x') = a(x)$ on X^2.

According to a classical result of Dunford (see [N, Chapter IV]), the knowledge of the representation $\widetilde{\pi}$ is equivalent to the knowledge of a spectral measure P on X^2 such that

$$\widetilde{\pi}(a) = \int_{X^2} a(x, x') dP(x, x')$$

for every $a \in \mathcal{C}(X^2)$, in particular, $\pi(a) = \int_{X^2} a(x) dP(x, x')$ for every $a \in \mathcal{A}$.

(ii) $ds \circ P(A) = P(\gamma(A)) \circ ds$ for every Borel subset $A \subset X^2$, where $\gamma(x, x') = (x', x)$ for $(x, x') \in X^2$.

It follows from (ii) that the operator ds^2 commutes with P, in particular, $[ds^2, \pi(a)] = 0$ for every $a \in \mathcal{A}$.

Let L_λ be the eigenspace of ds^2 corresponding to an eigenvalue λ^2, $A_\lambda = A_{-\lambda} \subset X^2$ the union of all open in X^2 subsets A for which $P(A)$ is the projector on a subspace orthogonal to L_λ (i.e., lying in the orthogonal complement $L_\lambda^\perp$ to L_λ). Then, as it is easy to see (Lemma 1.7), $P(A_\lambda)$

is also the projector on a subspace orthogonal to L_λ, and A_λ is a (unique) maximal open subset with this property.

(iii) $P(A_\lambda)$ is the projector onto the orthogonal complement $L_\lambda^\perp$ of L_λ for each eigenvalue λ of ds.

We put $\Lambda_\lambda = X^2 \setminus A_\lambda$.

(iv) $|yy'| = |\lambda|$ for every $(y, y') \in \Lambda_\lambda$.

The set $B = \cup_\lambda \Lambda_\lambda$, where λ runs over all eigenvalues of ds, is called *the support* of the spectral geometry $M = \{\mathcal{A}, \mathcal{H}, ds\}$. This terminology is justified by the fact that the closure $\overline{B}$ coincides with the support of the operator ds (see Corollary 1.8).

1.5. Proposition. *Let X be a compact metric space, a subset $B \subset X^2 \setminus \Delta$ satisfies the conditions* (a), (b) *of the main construction, the function $m : B \to \mathbb{N}$ is symmetric. Then the spectral triple $M(B, m) = \{\mathcal{A}, \mathcal{H}, ds\}$ is a spectral geometry on X.*

Proof. The representation π is the restriction on $\mathcal{A}$ of the representation $\widetilde{\pi}$,

$$(\widetilde{\pi}(a)\xi)(y, y') = a(y, y')\xi(y, y'), \quad (y, y') \in B$$

of C^*-algebra $\mathcal{C}(X^2)$ in $\mathcal{H}$.

Given $(y, y') \in B$, we define the projector $P(y, y')$ as

$$(P(y, y')\xi)(z, z') = \begin{cases} \xi(y, y'), & z = y, \ z' = y'; \\ 0, & \text{otherwise}, \end{cases}$$

$\xi \in \mathcal{H}$. Then the spectral measure P associated with $\widetilde{\pi}$ is defined by

$$P(A) = \sum_{(y,y') \in A \cap B} P(y, y')$$

for every Borel $A \subset X^2$. It follows from the definitions of operators ds and $P(y, y')$ that $ds \circ P(y, y') = P(y', y) \circ ds$ for every $(y, y') \in B$, which implies the condition (ii) of the spectral geometry.

The set $\Lambda_\lambda = \{(y, y') \in B \,|\, |yy'| = |\lambda|\}$ is finite according to (b), and $P(\Lambda_\lambda)$ is the projector on the λ^2-eigenspace L_λ of ds^2 for each eigenvalue λ of ds. Thus the conditions (iii) and (iv) from the definition of spectral geometry are fulfilled. $\square$

The main result of this section is that any spectral geometry can be obtained by using the main construction (and taking into account the multiplicity function m of eigenvalues of the unit length operator).

1.6. Theorem. *Let $\{\mathcal{A}, \mathcal{H}, ds\}$ be a spectral geometry on a compact metric space X. Then its support $B \subset X^2 \setminus \Delta$ satisties the conditions (a), (b), and there exists a symmetric function $m : B \to \mathbb{N}$, for which $\{\mathcal{A}, \mathcal{H}, ds\}$ is isomorphic to the spectral geometry $M(B, m) = \{\mathcal{A}, \mathcal{H}', ds'\}$ by an isometry $U : \mathcal{H} \to \mathcal{H}'$ conjugating the representations of $\mathcal{A}$ and unit length operators ds, ds'.*

The proof is based on the following lemma.

1.7. Lemma. *Assume that a spectral triple $\{\mathcal{A}, \mathcal{H}, ds\}$ on a compact metric space X satisfies the axioms (i), (ii) of spectral geometry. Let L_λ be the eigenspace of ds^2 corresponding to an eigenvalue λ^2. Then there exists a unique maximal open subset $A_\lambda \subset X^2$, for which $P(A_\lambda)$ is the projector on a subspace orthogonal L_λ. Furthermore, its complement $\Lambda_\lambda = X^2 \setminus A_\lambda$ is finite.*

Proof. Let $\{A_\sigma\}$ be the collection of all open sets in X^2, for which $P(A_\sigma)$ is the projector on a subspace orthogonal to L_λ, $A_\lambda = \cup_\sigma A_\sigma$. We show that $P(A_\lambda)$ is also the projector on a subspace Q_λ, which is orthogonal to L_λ. Let P_λ be the projector on L_λ. Then since the spectral measure P commutes with P_λ the operator $P(A_\lambda) \circ P_\lambda$ is the projector on $Q_\lambda \cap L_\lambda$. Thus if the assertion is not true, then there exists a vector $\xi \in L_\lambda$, $\|\xi\| = 1$, for which $P(A_\lambda)\xi = \xi$. Then the Borel measure $\mu = (P\xi, \xi)$ possesses the properties: $\mu(A_\sigma) = 0$ for all σ and $\mu(A_\lambda) = 1$. This, obviously, is impossible, because for every $\epsilon > 0$ one can find a compact subset $K \subset A_\lambda$ with $\mu(K) \geq 1 - \epsilon$. On the other hand, one can cover K by a finite number of subsets from $\{A_\sigma\}$ and consequently $\mu(K) = 0$.

Therefore, A_λ is a maximal open subset, for which $P(A_\lambda)$ is the projector on a subspace orthogonal to L_λ. Obviously, it is unique with this property. It remains to show that its complement Λ_λ is finite.

The operator ds^2 is compact, thus the space L_λ is finite dimensional. We show that Λ_λ consists of at most $\dim L_\lambda (= \dim_{\mathbb{C}} L_\lambda)$ points. If this is not the case, then choosing disjoint neighborhoods U_i of points $z_1, \dots, z_n \in \Lambda_\lambda$, $n > \dim L_\lambda$, we find that at least one of the projectors $P(U_i)$ is the projector on a subspace orthogonal to L_λ. The set $A'_\lambda = A_\lambda \cup U_i$ is open in X^2 and contains A_λ as a proper subset. Furthermore, $P(A'_\lambda) = P(A_\lambda) + P(U_i \cap \Lambda_\lambda)$ is the projector on a subspace orthogonal to L_λ. This contradicts the fact that A_λ is maximal. $\square$

Proof of Theorem 1.6. By axiom (iv), given $(y, y') \in B$ there exists an eigenvalue λ of ds with $|yy'| = |\lambda|$. Since ds is injective, it follows that $B \subset X^2 \setminus \Delta$. The subspace L_λ is invariant for ds, thus it follows from (ii) that A_λ is symmetric. Thus B is symmetric, $\gamma(B) = B$. Using (iv), we obtain that $B_t = \{(y, y') \in B \,|\, |yy'| \geq t\}$ coincides with $\cup_{|\lambda| \geq t} \Lambda_\lambda$, where the union is taken over all eigenvaues λ of ds with $|\lambda| \geq t$. By Lemma 1.7, each Λ_λ is finite. Then B_t is finite for each $t > 0$, since ds is compact. Therefore, B satisfies the conditions (a), (b) of the main construction.

By Lemma 1.7 and axiom (iii), the operator $P(y, y')$ is a nonzero projector on a subspace $L_{y,y'} \subset L_\lambda$ for every point $(y, y') \in \Lambda_\lambda$. We define a function $m : B \to \mathbb{N}$ by $m(y, y') = \dim L_{y,y'}$. It follows from (ii) that

$$ds(L_{y,y'}) = ds \circ P(y, y')(\mathcal{H}) = P(y', y) \circ ds(\mathcal{H}) = L_{y',y},$$

thus m is symmetric.

Using injectivity of ds and axiom (i), now it is easy to construct an isometry $U : \mathcal{H} \to \mathcal{H}'$ establishing an isomorphism between $\{\mathcal{A}, \mathcal{H}, ds\}$ and $M(B, m) = \{\mathcal{A}, \mathcal{H}', ds'\}$. $\square$

The support of an operator A in $\mathcal{H}$ is defined as the complement to the union of all open sets $U \times U' \subset X^2$, for which $\pi(a) \circ A \circ \pi(a') = 0$ for all a, $a' \in \mathcal{A}$ with $\operatorname{supp}(a) \subset U$, $\operatorname{supp}(a') \subset U'$.

1.8. Corollary. *Let $M = \{\mathcal{A}, \mathcal{H}, ds\}$ be a spectral geometry on compact metric space X. Then the support of ds coincides with the closure of its support B, $\operatorname{supp} ds = \overline{B}$.*

Proof. By Theorem 1.6, the spectral geometry M coincides with $M(B, m)$, where m is the multiplicity function. For every a, $a' \in \mathcal{A}$, $\xi \in \mathcal{H}$ and $(y, y') \in B$ we have

$$(\pi(a) \circ ds \circ \pi(a')\xi)\,(y, y') = a(y) \cdot |yy'| \cdot a'(y') \cdot \xi(y', y).$$

Thus $\pi(a) \circ ds \circ \pi(a') = 0 \Leftrightarrow a(y) \cdot a'(y') = 0$ for all $(y, y') \in B$. Hence the claim. $\square$

4. Existence of spectral geometries with nonzero density. Given a compact metric space X we shall show how to construct a spectral geometry with an arbitrary large density. By Theorem 1.6, it suffices to find a subset $B \subset X^2 \setminus \Delta$ satisfying the conditions (a), (b) of the main construction and having the density bigger than a given number.

1.9. Proposition. *Given $D \in (2, \infty]$, let $\{Y_k\}$ be a sequence of finite δ_k-nets in X such that $\delta_k \searrow 0$ as $k \to \infty$. We put $B = \cup_k B(k)$, where*

$$B(k) = \left\{(y, y') \in Y_k^2 \setminus \Delta \,|\, |yy'| \le \delta_{k-1}\phi(\delta_{k-1})\right\},$$

$\phi(t) = 2D$ if $D < \infty$ and if $D = \infty$, then $\phi(t) \ge 4$ be a function with $\phi(t) \to \infty$ and $t\phi(t) \searrow 0$ as $t \to 0$.

Then the set B satisfies the conditions (a), (b) of the main construction and has the density $\mathrm{dens}(B) \ge D$, *is $D < \infty$, and local density* $\mathrm{dens}_{\mathrm{loc}}(B) = \infty$, *if $D = \infty$.*

Proof. The set B is obviously symmetric. Let $t > 0$. Then for $(y, y') \in B_t$ we have $(y, y') \in B(k)$ and $t \le |yy'| \le \delta_{k-1}\phi(\delta_{k-1})$. Hence $k \le k(t)$ for all $(y, y') \in B_t$. Thus B_t is finite.

One can assme that $\operatorname{diam} X \le \delta_1\phi(\delta_1)/2$. Given $x \ne x' \in X$ there exists $k \ge 2$ such that $\delta_k\phi(\delta_k)/2 \le |xx'| \le \delta_{k-1}\phi(\delta_{k-1})/2$. Since Y_k is a δ_k-net, one can find y, $y' \in Y_k$ with $|xy|$, $|x'y'| \le \delta_k \le 2|xx'|/\phi(\delta_k)$. Furthermore, $|yy'| \ge |xx'| - 2\delta_k \ge \delta_k(\phi(\delta_k)/2 - 2) > 0$ and $|yy'| \le |xx'| + 2\delta_k \le \delta_{k-1}\phi(\delta_{k-1})/2 + 2\delta_k \le \delta_{k-1}\phi(\delta_{k-1})$. Thus $(y, y') \in B$. Therefore, given $z = (x, x') \in X^2 \setminus \Delta$ we have found $b = (y, y') \in B$ with $|z|/|zb| \ge \phi(\delta_k)/2$. Thus $\mathrm{dens}(B) \ge D$, if $D < \infty$. For $D = \infty$ this argument shows that a point $z \in X^2 \setminus \Delta$ can be approximated by points of B with a prescribed arbitrarily small error, if z lies in a sufficiently small neighborhood of the diagonal Δ. Thus $\mathrm{dens}_{\mathrm{loc}}(B) = \infty$. $\square$

§2. Bounded deformations of a spectral geometry

If for a spectral triple the axioms (i)–(iii) are fulfilled, then axiom (iv) uniquely defines a spectral geometry whose unit length operator differs from that one of the spectral triple only by the spectrum. Forgetting about axiom (iv) we come up to the notion of deformation of a spectral geometry.

Let $M = \{\mathcal{A}, \mathcal{H}, ds\}$ be a spectral geometry on compact metric space X. Its *deformation* is given by a compact self-adjoint operator ds' in $\mathcal{H}$ of form

$$ds'(\xi)(y, y') = \rho(y, y')\xi(y', y),$$

where $\rho : B \to \mathbb{R}$ is a symmetric positive function on the support B of M. Notation: $ds' = ds_\rho$. Therefore, the deformation ds_ρ of M can be identified with a positive symmetric function $\rho : B \to \mathbb{R}$, which is called *the marked spectrum* of the deformation ds_ρ.

A deformation ds_ρ is said to be *bounded*, if the ratio $\rho(y, y')/|yy'|$ is bounded and separated away from zero on B.

Every deformation ds_ρ of a spectral geometry $\{\mathcal{A}, \mathcal{H}, ds\}$ defines the Connes metric on X,

$$|xx'|_\rho = \sup\left\{|a(x') - a(x)| \,|\, a \in \mathcal{A}, \|[\mathcal{D}_\rho, \pi(a)]\| \le 1\right\},$$

where $\mathcal{D}_\rho$ is the operator inverse to ds_ρ. The following proposition immediately follows from definitions.

2.1. Proposition. *Let ds_ρ be a bounded deformation of a spectral geometry M on a compact metric space X. Then its Connes metric $|\ |_\rho$ is biLipschitz equivalent to the Connes metric $|\ |_M$ of M. More precisely,*

$$c|xx'|_M \leq |xx'|_\rho \leq C|xx'|_M \quad \text{for all} \quad x \neq x' \in X$$

and any constants C, $c > 0$, satisfying the condition $c \leq \rho(y, y')/|yy'| \leq C$ for all $(y, y') \in$ $\operatorname{supp} M$. $\square$

1. Bounded deformations of a connected spectral geometry.
Let $M = \{\mathcal{A}, \mathcal{H}, ds\}$ be a spectral geometry on a compact metric space X, $B \subset X^2 \setminus \Delta$ its support. Given $t > 0$ we define Γ_t as a graph whose set of oriented edges is $B_t = \{b \in B \mid |b| \geq t\}$ and the vertex set Y_t is the projection of B_t on X. The graph Γ_t is called *the incidence graph* of level t of M. It follows from the axioms that the incidence graph Γ_t is finite and has no isolated vertices for every $t > 0$.

A spectral geometry M is said to be *connected* if its incidence graph Γ_t is connected for each sufficiently small $t > 0$. The property of a spectral geometry to be connected is not very much restrictive, and the construction described in Proposition 1.9 gives usually a connected spectral geometry.

Let $Y \subset X$ be the projection of the support $B \subset X^2$ of a spectral geometry M on the factor X. Then points y, $y' \in Y$ are vertices of the incidence graph Γ_t for every sufficiently small level $t > 0$.

Let ρ be the marked spectrum of a bounded deformation ds_ρ of M. Its restriction to Γ_t defines an intrinsic metric $\overline{\rho}_t$ on each connected component of the graph Γ_t, i.e., we first extend ρ to a locally intrinsic metric on Γ_t, for which the length of each edge $(y, y') \in B$ is equal to $\rho(y, y')$ and then we let $\overline{\rho}_t(y, y')$ be the length of a shortest path in Γ_t between y and y' with respect to this metric. The function $\overline{\rho}_t(y, y')$ does not decreases in t, thus there exists a limit

$$\overline{\rho}(y, y') = \lim_{t \to 0} \overline{\rho}_t(y, y').$$

If $(y, y') \in B$ or if the geometry M is connected, then this limit is finite. Furthermore, $\overline{\rho}(y, y') = 0 \Leftrightarrow y = y'$ since ds_ρ is a bounded deformation.

2.2. Proposition. *Let d_ρ be the Connes metric associated with a bounded deformation ρ of a connected spectral geometry M on a compact metric space X, $D = \operatorname{dens}(M) > 2$. Then $d_\rho(y, y') = \overline{\rho}(y, y')$ for every y, $y' \in Y$.*

Proof. Assume that a function $a \in \mathcal{A}$ satisfies the condition $\|[\mathcal{D}_\rho, \pi(a)]\| \leq 1$. This means that $|a(y) - a(y')| \leq \rho(y, y')$ for each point $(y, y') \in B$. Thus

$$|a(y) - a(y')| \leq \sum_{i=0}^{k-1} |a(y_{i+1}) - a(y_i)| \leq \sum_{i=0}^{k-1} \rho(y_{i+1}, y_i)$$

for every sequence of edges with consecutive vertices $(y_0, \ldots, y_k) \in Y$ between $y = y_0$ and $y' = y_k$. In particular, $|a(y) - a(y')| \leq \overline{\rho}(y, y')$ and whence $d_\rho(y, y') \leq \overline{\rho}(y, y')$.

For the proof of the opposite inequality we recall that the set Y is everywhere dense in X due to the condition $D > 2$ (see §1). We fix $y \in Y$ and consider the function $a_y : Y \to \mathbb{R}$, $a_y(y') = \overline{\rho}(y, y')$. Though this function is not defined on all over X and a priori is not necessarily

Lipschitz on Y, nevertherless, one can apply to it the arguments from Lemma 1.1 and obtain that

$$\text{Lip }(a_y) \le \frac{D+2}{D-2} \sup_{(z,z')\in B} \frac{|a_y(z') - a_y(z)|}{|zz'|}$$

(the Lipschitz constant Lip (a_y) is not supposed to be finite). Since

$$|a_y(z') - a_y(z)| \le \overline{\rho}(z,z') \le \rho(z,z') \le C|zz'|$$

for all $(z,z') \in B$, we have Lip $(a_y) \le C(D+2)/(D-2) < \infty$. Thus the function a_y can be extended to a Lipschitz function on X, for which we use the same notation. Furthermore, for the operator $\mathcal{D}_\rho = ds_\rho^{-1}$ we have

$$\|[\mathcal{D}_\rho, \pi(a_y)]\| = \sup_{(z,z')\in B} \frac{|a_y(z') - a_y(z)|}{\rho(z,z')} \le 1$$

and $|a_y(y) - a_y(y')| = \overline{\rho}(y,y')$. Therefore, $d_\rho(y,y') \ge \overline{\rho}(y,y')$. $\square$

2. Bounded deformations and Hausdorff–Gromov convergence.

Deformations of a spectral geometry seem to be a convenient tool for constructions of metrics with prescribed properties. Thus it is important to understand relations between different topologies on the sets of deformations and Connes metrics associated with these deformations. We start with the compact-open topology on the first set and Hausdorff–Gromov topology on the second one.

The compact-open topology on the set Ξ of bounded deformations is defined as the compact-open topology on the set of maps of form $1/\rho : B \to \mathbb{R}$, $\rho \in \Xi$. In other words, the base of neighborhoods of a marked spectrum $\rho_0 \in \Xi$ with respect to this topology consists of the sets

$$U_{t,\epsilon}(\rho_0) = \left\{ \rho \in \Xi \mid \left| \frac{1}{\rho(y,y')} - \frac{1}{\rho_0(y,y')} \right| < \epsilon \quad \text{for all} \quad (y,y') \in B_t \right\},$$

t, $\epsilon > 0$.

The condition that deformations are close w.r.t. compact-open topology is too weak for the associated Connes metrics to be close even in such rough topology as the Hausdorff–Gromov one. Nevertherless, they are close under some additional restrictions.

2.3. Theorem. *Let $M = \{\mathcal{A}, \mathcal{H}, ds\}$ be a connected spectral geometry of density $D > 2$ on a compact metric space X. If a sequence $\{\rho_i\}$ of its deformations converges to $\rho_0 \in \Xi$ in the compact-open topology and $\rho_0(y,y') \le \rho_i(y,y') \le C|yy'|$ for all $(y,y') \in \operatorname{supp} M$, $i \ge 1$ and some constant $C > 0$, then the associated Connes metrics $d_i = d_{\rho_i}$ on X Hausdorff–Gromov converge to the Connes metric d_0, associated with ρ_0.*

.

We first briefly describe the idea of the proof. By the condtion on the density D of M, the vertex set Y_t of its incidence graph is a $\delta(t)$-net in X with $\delta(t) \le Dt/(D-2)$ for every level $t > 0$. This is established in Lemma 2.4. Thus it suffices to prove the uniform convergence of the Connes metrics on sets Y_t. By Proposition 2.2, we know how to find distances in the Connes metric d_ρ, associated with a bounded deformation $\rho \in \Xi$, between points of Y_t. The convergence $\rho_i \to \rho_0$ in the compact-open topology implies the convergence of intrinsic metrics $(\overline{\rho}_i)_t \to (\overline{\rho}_0)_t$ on the incidence graphs of every level $t > 0$. The required convergence $\overline{\rho}_i \to \overline{\rho}_0$ on Y_t then follows from the condition $\rho_i \ge \rho_0$.

The number $\delta(Y) = \inf\{\delta > 0 \mid X \subset \cup_{y\in Y} D_\delta(y)\}$, where $D_r(x)$ be the ball $\{x' \in X \mid |x'x| \le r\}$, is called *the bounding radius* of $Y \subset X$.

2.4. Lemma. *Assume that a subset $B \subset X^2$ is symmetric and has the density $D > 2$. For $t > 0$ let $Y_t \subset X$ be the projection on X of $B_t = \{(y, y') \in B \mid |yy'| \geq t\}$, $\delta(t)$ the bounding radius of Y_t. Then*

$$\delta(t) \leq \frac{D}{D-2} \cdot t.$$

Proof. Suppose that $\delta(t) > Dt/(D-2)$. Since X is compact, there exists $x \in X$ with $|xY_t| = \delta(t)$ and respectively $x' \in X$ with $|xx'| = \delta(t)$. We fix $\epsilon \in (0, D-2)$ such that $\delta(t)/t \geq (D-\epsilon)/(D-2-\epsilon)$, and approximate the points x, x' by a pair $(y, y') \in B$ with $|xy|$, $|x'y'| \leq \alpha|xx'|$, where $\alpha = 1/(D-\epsilon) < 1/2$. Then $(1-2\alpha)\delta(t) \geq t$ by the choice of ϵ, and $|yy'| \geq (1-2\alpha)|xx'| \geq t$. Thus $(y, y') \in B_t$ and $y \in Y_t$. Furthermore, $|xy| \leq \alpha\delta(t) < \delta(t)$, which contradicts the condition $|xY_t| = \delta(t)$. $\square$

Proof of Theorem 2.3. Let $\delta > 0$. By Lemma 2.4, the set Y_t is a finite δ-net in X for all sufficiently small $t > 0$. It follows from the condition of the theorem that

$$0 < c \leq \rho_i(y, y')/|yy'| \leq C$$

for all $(y, y') \in B = \operatorname{supp} M$ and $i \geq 0$. Thus in view of Proposition 2.1, one can assume that Y_t is a δ-net also with respect to the Connes metrics d_0, d_i, $i \geq 1$ on X. Therefore, it suffices to show that $d_i(y, y') \to d_0(y, y')$ as $i \to \infty$ for every y, $y' \in Y_t$.

By Proposition 2.2, we have $d_i(y, y') = \overline{\rho}_i(y, y')$, $i \geq 0$, where

$$\overline{\rho}_i(y, y') = \lim_{\tau \to 0} (\overline{\rho}_i)_\tau(y, y')$$

and $(\overline{\rho}_i)_\tau$ is an intrinsic metric on the incidence graph Γ_τ, constructed by the restriction $\rho_i|\Gamma_\tau$.

It follows from the condition $\rho_0 \leq \rho_i$, $i \geq 1$ that $\overline{\rho}_0 \leq \overline{\rho}_i$ and hence

$$\overline{\rho}_0(y, y') \leq \liminf_{i \to \infty} \overline{\rho}_i(y, y').$$

To prove the opposite inequality, we approximate the distance $d_0(y, y') = \overline{\rho}_0(y, y')$ by the distances $(\overline{\rho}_0)_\tau(y, y')$ as $\tau \to 0$, and for every fixed $\tau > 0$ the distance $(\overline{\rho}_0)_\tau(y, y')$ by the distances $(\overline{\rho}_i)_\tau(y, y')$ as $i \to \infty$ using the convergence $\rho_i \to \rho_0$ in the compact-open topology. Since $\overline{\rho}_i(y, y') \leq (\overline{\rho}_i)_\tau(y, y')$, we obtain

$$\overline{\rho}_0(y, y') \geq \limsup_{i \to \infty} \overline{\rho}_i(y, y'). \quad \square$$

3. Weak topology on bounded deformations. Let $\mathcal{D}_\rho = ds_\rho^{-1}$ be the Dirac operator associated with a deformation ρ. The compact-open topology on the set Ξ of bounded deformations (of geometry M) can be described as the topology of pointwise convergence of operators $\mathcal{D}_\rho$, $\rho \in \Xi$ on functions with *finite* supports. In other words, $\rho_i \to \rho$ in the compact-open topology if and only if $\mathcal{D}\rho_i(\xi) \to \mathcal{D}_\rho(\xi)$ for each $\xi \in \mathcal{H}$ with finite support.

The weak topology on Ξ is defined as the topology of pointwise convergence of operators $\mathcal{D}_\rho$, $\rho \in \Xi$ on arbitrary elements $\xi \in \mathcal{H}$. This simultaneously defines a corresponding topology on the set of the associated Connes metrics. d_ρ. The weak topology is finer than the compact-open one. Still, it is not clear whether the convergence of deformations in former topology implies the Hausdorff–Gromov convergence of the associated Connes metrics. d_ρ.

Question. Assume that the marked spectra $\rho_i \to \rho_0$ with respect to the weak topology. Is it true that the associated Connes metrics d_i converge to d_0 uniformly or at least by Hausdorff–Gromov? (See Proposition 3.8 below.)

4. Uniform topology and the Lipschitz convergence. *The uniform* topology on Ξ is defined as the topology of the norm convergence of operators $\mathcal{D}_\rho$, $\rho \in \Xi$. In other words, the base of neighborhoods of a marked spectrum $\rho_0 \in \Xi$ in this topology is formed by the sets

$$U_\epsilon(\rho_0) = \left\{ \rho \in \Xi \mid \left| \frac{1}{\rho(y,y')} - \frac{1}{\rho_0(y,y')} \right| < \epsilon \quad \text{for all} \quad (y,y') \in \operatorname{supp} M \right\}, \quad \epsilon > 0.$$

Recall that the Lipschitz distance between metrics d, d' on X is defined as

$$|dd'|_L = \sup_{x \neq y} \left| \log \frac{d'(x,y)}{d(x,y)} \right| .$$

The next statement follows almost immediately from definitions.

2.5. Proposition. *Let $M = \{\mathcal{A}, \mathcal{H}, ds\}$ be a spectral geometry on a compact metric space X. Assume that a sequence of marked spectra $\{\rho_i\} \in \Xi$ of its bounded deformations converges to $\rho_0 \in \Xi$ in the uniform topology. Then for the associated Connes metrics on X we have*

$$|d_{\rho_i} d_{\rho_0}|_L \to 0 \quad \text{as} \quad i \to \infty. \quad \square$$

5. Pertubations of degenerations. A deformation ds_ρ of a spectral geometry M is said to be *regular*, if $\rho(y,y') = \overline{\rho}(y,y')$ for all $(y,y') \in \operatorname{supp} M$ (see Sec. 1). To a given deformation ρ it canonically corresponds a regular deformation $\overline{\rho}$. A deformation ρ is said to be *simple*, if for every (y,y'), $(z,z') \in \operatorname{supp} M$ the condition $\rho(y,y') = \rho(z,z')$ implies $(y,y') = (z,z')$ or $(y,y') = (z',z)$.

2.6. Lemma. *Every spectral geometry $M = \{\mathcal{A}, \mathcal{H}, ds\}$ on a compact metric space X can be approximated in the uniform topology by its simple regular deformations.*

Proof. Given $z \in B = \operatorname{supp} M$ we put $\rho(z) = |z|$. Obviously, $\overline{\rho} = \rho$, i.e., the geometry M itself is regular. The support B can be represented as a sequence $B = \{z_1, z_2, \dots\}$ in such a way that $\rho(z_{i+1}) \le \rho(z_i)$ for every $i \ge 1$. We fix a positive sequence $\{\epsilon_i\}$ satisying the following conditions

(1) $\epsilon_i < \rho^2(z_i)/(1 + \rho(z_i))$ for all $i \ge 1$;
(2) $\sum_{i>j} \epsilon_i < \epsilon_j$ for all $j \ge 1$;
(3) if $\rho(z_{i+1}) < \rho(z_i)$, then $\epsilon_k < \rho(z_i) - \rho(z_{i+1})$ for $k = \min\{j \mid \rho(z_j) = \rho(z_i)\}$.

Given $h \in (0,1)$ and $i \ge 1$ we put $\rho_h(z_i) = \rho(z_i) - h\epsilon_i$. Then $\rho_h(z_i) > 0$, since $\epsilon_i < \rho(z_i)$. Furthermore,

$$\frac{1}{\rho_h(z_i)} - \frac{1}{\rho(z_i)} = \frac{h\epsilon_i}{\rho(z_i)\,(\rho(z_i) - h\epsilon_i)} < h$$

by condition (1). Thus $\rho_h \to \rho$ as $h \to 0$ in the uniform topology. It follows from the regularity of ρ and condition (2) that the deformations ρ_h are regular for all $h \in (0,1)$. Conditions (2) and (3) imply that all deformations ρ_h are simple. $\square$

§3. Dimension of spectral geometry

1. Definition and properties of dimension of spectral geometries.

The dimension of a spectral geometry is defined in [Bu] via the notion of the Dixmier trace Tr_ω (see [Di], [Co]). Let $M = \{\mathcal{A}, \mathcal{H}, ds\}$ be a spectral geometry on a compact metric space X. Its *dimension* is defined as

$$\dim M := \sup\{p \ge 0 \mid \operatorname{Tr}_\omega(|ds|^p) = \infty\} = \inf\{p \ge 0 \mid \operatorname{Tr}_\omega(|ds|^p) = 0\}.$$

The following lemma shows that this notion is well defined.

3.1. Lemma. *Let $M = \{\mathcal{A}, \mathcal{H}, ds\}$ be a spectral geometry on a compact metric space X. Put $p_- = \sup\{p \geq 0 \mid \mathrm{Tr}_\omega(|ds|^p) = \infty\}$, $p_+ = \inf\{p \geq 0 \mid \mathrm{Tr}_\omega(|ds|^p) = 0\}$. Then $p_- = p_+$.*

Proof. Obviously, $p_- \leq p_+$. Suppose that $p_- < p_+$. Then

$$0 < \mathrm{Tr}_\omega(|ds|^p) < \infty$$

for every $p \in (p_-, p_+)$. Let $\mu_0 \geq \mu_1 \geq \dots$ be the sequence of the eigenvalues of $|ds|$, written in nonincreasing order according to multiplicity. We choose p, $p' \in (p_-, p_+)$, $p' > p$ and fix an $\epsilon > 0$. There is a $n_0 > 0$ such that $\mu_n^{p'-p} < \epsilon$ for all $n \geq n_0$ and, furthermore, $\sigma_N(|ds|^p) = \sum_{n=0}^{N-1} \mu_n^p \leq C \log N$ for all sufficiently large N, where the constant $C > 0$ does not depend on N. Then

$$\sigma_N\left(|ds|^{p'}\right) \leq \sum_{n=0}^{n_0-1} \mu_n^{p'} + \epsilon \sum_{n=n_0}^{N-1} \mu_n^p \leq \epsilon(1+C)\log N$$

for all sufficiently large N. Thus $\mathrm{Tr}_\omega\left(|ds|^{p'}\right) = 0$. This contradicts the assumption $p' < p_+$. $\square$

Let E_t be the maximal subspace in $\mathcal{H}$ with $\|ds^{-1}|E_t\| \leq t^{-1}$. In the case when the multiplicity function $m \equiv 1$, we have $\dim E_t = |B_t|$. The following proposition is proved in [Bu].

3.2. Proposition. *The dimension* $\dim M$ *coicides with*

$$\dim{}_2 M = \limsup_{t \to 0} \frac{\log \dim E_t}{\log 1/t} \qquad \textit{and} \qquad \dim{}_3 M = \inf\{p \geq 0 \mid \mathrm{Trace}(|ds|^p) < \infty\}. \quad \square$$

Furthermore, we have

3.3. Lemma. *For every $p > \dim M$ the trace of the operator $|ds|^p$ is finite,*

$$\mathrm{Trace}(|ds|^p) = \sum_n \mu_n^p < \infty.$$

Proof. As usual, let $\mu_0 \geq \mu_1 \geq \dots$ be the sequence of the eigenvalues of the operator $|ds|$ written in nonincreasing order according to multiplicity. By Proposition 3.2, we have

$$\dim M = \limsup_{t \to 0} \frac{\log \dim E_t}{\log t^{-1}}.$$

Take an $\epsilon > 0$ such that $d = \dim M < p - \epsilon$. Then for every sufficiently small $t > 0$ we have $\dim E_t \leq t^{-(d+\epsilon)}$, where $\dim E_t = \sum_{(y,y') \in B_t} m(y,y')$ coincides with the number of eigenvalues (with multiplicity) of $|ds|$, which $\geq t$. Therefore, if $\mu_n = t$, then $n \leq t^{-(d+\epsilon)} + C$ and $\mu_n^p \leq (n-C)^{-p/(d+\epsilon)}$, where the constant C does not depend on n. Hence, the claim. $\square$

2. Dixmier trace and Hausdorff measures. Let $M = \{\mathcal{A}, \mathcal{H}, ds\}$ be a spectral geometry on X, $B \subset X^2 \setminus \Delta$ its support. As usual, for $t > 0$ we put $B_t = \{b \in B \mid |b| \geq t\}$, $Y_t \subset X$ the projection of B_t on the factor X, $\delta(t)$ the bounding radius of Y_t (see §2). *The relative bounding radius* of the support B of the geometry M is defined as

$$\nu = \nu(B) = \limsup_{t \to 0} \delta(t)/t.$$

In what follows, we assume that $\nu < \infty$. This condition is fulfilled, if, for example, $D = \mathrm{dens} M > 2$ (see Lemma 2.4). We use notation $H^p(X)$ for the p-dimensional Hausdorff measure of X. Recall that $H^p(X) = \lim_{t \to 0} H_t^p(X)$, where

$$H_t^p(X) = \inf \sum_{\mathrm{diam}\, F \leq t} (\mathrm{diam}\, F)^p$$

and the infimum is taken over all covers of X by nonempty closed subset of diameter $\leq t$.

3.4. Theorem. *Given $p > 0$, we have*

$$(2\nu^p)\operatorname{Tr}_\omega(|ds|^p) \geq H^p(X)$$

(in the case $\nu = 0$, $H^p(X) > 0$ it means that $\operatorname{Tr}_\omega(|ds|^p) = \infty$).

Proof. We can assume that $H^p(X) > 0$, since otherwise it is nothing to prove.

(1) The set Y_t is a $\delta(t)$-net in X. Thus

$$\sum_{y \in Y_t} \left(\operatorname{diam} D_{\delta(t)}(y)\right)^p \geq H^p_{2\delta(t)}(X).$$

On the other hand,

$$\sum_{y \in Y_t} \left(\operatorname{diam} D_{\delta(t)}(y)\right)^p \leq (2\delta(t))^p\,|Y_t|,$$

whence

$$|Y_t| \geq \frac{H^p_{2\delta(t)}(X)}{(2\delta(t))^p}.$$

Fix an $\epsilon > 0$. Then $\delta(t)/t \leq \nu + \epsilon$ for all sufficiently small t. Furthermore, we can assume that $H^p_{2\delta(t)}(X) \geq H^p(X) - \epsilon$ (if $H^p(X) = \infty$, then $H^p_{2\delta(t)}t(X) \geq 1/\epsilon$). For such a t we have

$$|Y_t| \geq \frac{H^p(X) - \epsilon}{(2(\nu + \epsilon))^p t^p}.$$

(2) We consider first the case when the marked spectrum of ds is simple (see §2, Sec. 5). One can also assume that the multiplicity function $m : B \to \mathbb{N}$ is constant and equals 1, since otherwise the required inequality becomes only stronger. Therefore, the multiplicity of every eigenvalue of $|ds|$ equals 2. Let $\mu = \{\mu_1 = \mu_2 > \mu_3 = \mu_4 > \dots\}$ be the sequence of the eigenvalues of $|ds|$ written in nonincreasing order according to multiplicity. We put $\mu_{k-1} = \mu_k = t$. Then for all sufficiently large k, say $k \geq k_0$ we have due to (1)

$$k = |B_t| \geq |Y_t| \geq \frac{H^p(X) - \epsilon}{(2(\nu + \epsilon))^p t^p}.$$

Hence

$$\mu^p_{k-1} = \mu^p_k \geq \frac{H^p(X) - \epsilon}{(2(\nu + \epsilon))^p k}.$$

Recalling the definition of the Dixmier trace, we immediately obtain

$$\operatorname{Tr}_\omega(|ds|^p) \geq \frac{1}{(2\nu)^p} H^p(X),$$

and in the case of the simple spectrum the theorem is proved.

(3) Now we consider the general case. By Lemma 2.6, the operator ds can be approximated in the uniform topology by simple regular deformations. For every such a deformation ρ, we have already proved that

$$(2\nu_\rho)^p \operatorname{Tr}_\omega(|ds_\rho|^p) \geq H^p((X, d_\rho)),$$

where d_ρ is the Connes metric on X associated with the deformation. By Proposition 2.5, these Connes metrics approximate the initial metric of X in the Lipschitz metric. Thus $H^p((X, d_\rho)) \to H^p(X)$. On the other hand, the convergence $\rho \to \mu$ in the uniform topology, obviously, implies that $\mathrm{Tr}_\omega(|ds_\rho|^p) \to \mathrm{Tr}_\omega(|ds|^p)$. Thus to complete the proof, it suffices to show that $\limsup_{\rho\to\mu} \nu_\rho \le \nu$.

Fix an $\epsilon > 0$. Then for all deformations ρ sufficiently close to μ we have $1-\epsilon \le |xx'|_\rho/|xx'| \le 1+\epsilon$ for each x, $x' \in X$. Thus $B_t \subset B_{\rho,(1-\epsilon)t}$ and $Y_t \subset Y_{\rho,(1-\epsilon)t}$, where $B_{\rho,t} = \{b \in B \mid |b|_\rho \ge t\}$ and $Y_{\rho,t}$ be the projection of $B_{\rho,t} \subset X^2$ on the factor X. Then the bounding radius (see §2) we have

$$\delta(t) = \delta(Y_t) \ge \delta(Y_{\rho,(1-\epsilon)t}) \ge (1+\epsilon)^{-1}\delta_\rho(Y_{\rho,(1-\epsilon)t}) = \delta_\rho\left((1-\epsilon)t\right)/(1+\epsilon).$$

Consequently

$$\nu = \limsup_{t\to 0} \delta(t)/t \ge \nu_\rho(1-\epsilon)/(1+\epsilon),$$

which implies the required inequality. Note that similar argument shows that actually $\nu_\rho \to \nu$. $\square$

3.5. Remark. The appearence of the factor ν in the inequality above, which is closely related to the density of spectral geometry, is not incidental. There is a similar factor in the Connes-Sullivan formula ([Co, IV, Theorem 17]) and others formulars from [Co].

Here we consider this factor in more details. If the density of a spectral geometry $D = \mathrm{dens}M > 2$, then by Lemma 2.4, $\nu \le D/(D-2)$. One can give more precise estimate for sufficiently "regular" spaces.

A metric space X is called *R-uniformly perfect*, $R \ge 1$, if there is an $\epsilon > 0$ such that for every $x \in X$ the set $D_r(x) \setminus D_{r/R}(x)$ is not empty for all $0 < r \le \epsilon$.

The following lemma is a version of Lemma 2.4.

3.6. Lemma. *Assume that a metric space X is R-uniformly perfect, $R \ge 1$, a subset $B \subset X^2 \setminus \Delta$ is symmetric and has local density* $\mathrm{dens}_{\mathrm{loc}}(B) = D > 2$. *Given $t > 0$ let $Y_t \subset X$ be the projection on X of $B_t = \{(y, y') \in B \mid |yy'| \ge t\}$, $\delta(t)$ the bounding radius of Y_t. Then*

$$\nu(B) = \limsup_{t\to 0} \delta(t)/t \le R/(D-2).$$

In particular, $\nu(B) = 0$, if $\mathrm{dens}_{\mathrm{loc}}(B) = \infty$.

Proof. We can assume that $D < \infty$. Suppose that $\nu = \nu(B) > R/(D-2)$ and fix an $\epsilon \in (0, D-2-R/\nu)$. Then $\nu > R/(D-2-\epsilon)$, and there is an arbitrary small $t > 0$ with

(1) $\delta(t)/t \ge R/(D-2-\epsilon)$;
(2) for every pair $(x, x') \in X^2 \setminus \Delta$ with $|xx'| \le t$ there is a pair $(y, y') \in B$ with $|xy|$, $|x'y'| \le \alpha|xx'|$, where $\alpha = 1/(D-\epsilon) < 1/2$.

Then for $r = (D-\epsilon)\delta(t)$ and $r' = (D-\epsilon)t/(D-2-\epsilon)$ we have $r/r' \ge R$ by (1). Now we choose a point $x \in X$ with $|xY_t| = \delta(t)$. Since the space is uniformly perfect, there is a point $x' \in X$ with

$$\frac{(D-\epsilon)t}{D-2-\epsilon} \le |xx'| < (D-\epsilon)\delta(t).$$

We approximate the pair (x, x') according (2) by a pair $(y, y') \in B$ with $|xy|$, $|x'y'| \le \alpha|xx'|$. Then $|yy'| \ge (1-2\alpha)|xx'| \ge t$, thus $(y, y') \in B_t$ and $y \in Y_t$. Furthermore, $|xy| \le \alpha|xx'| < \delta(t)$, which is a contradiction with the choice of x. $\square$

The next statement directly follows from Theorem 3.4.

3.7. Corollary. *Let M be a spectral geometry o a compact metric space X, and assume that $\nu(B) < \infty$. Then*

$$\dim M \geq \dim_{\mathrm{H}} X,$$

where $\dim_{\mathrm{H}} X$ is the Hausdorff dimension of X. $\square$

3. Weak L^p-topology on bounded deformations.

Let $M = \{\mathcal{A}, \mathcal{H}, ds\}$ be a spectral geometry on a compact metric space X. For $p > 0$ we denote by $L^p(B)$ the Banach space of sections $\xi : B \to \mathbb{C}^m$ satisfying the condition

$$\|\xi\|^p = \sum_{(y,y')\in B} \|\xi(y,y')\|^p < \infty,$$

where $B \subset X^2 \setminus \Delta$ is the support of M, $m : B \to \mathbb{N}$ the multiplicity function (cp. §1). Given a bounded deformation $\rho \in \Xi$ of M (see §2), the operator $\mathcal{D}_\rho = ds_\rho^{-1}$ is defined on a dense subset in $L^p(B)$. *The weak L^p-topology* on the set Ξ of bounded deformations is defined as the topology of pointwise convergence of operators $\mathcal{D}_\rho$, $\rho \in \Xi$ on arbitrary elements $\xi \in L^p(B)$. For $p = 2$, this definition coincides with one given in §2, Sec. 3.

3.8. Proposition. *Let $M = \{\mathcal{A}, \mathcal{H}, ds\}$ be a spectral geometry on a compact metric space X. Assume that a sequence of marked spectra $\{\rho_i\} \in \Xi$ converges in the weak L^p-topology to $\rho_0 \in \Xi$. If $p > \dim M$, then the associated Connes metrics d_{ρ_i} on X converge in the Lipschitz topology to d_{ρ_0},*

$$|d_{\rho_i} d_{\rho_0}|_L \to 0 \quad \text{as} \quad i \to \infty.$$

In particular, if $\dim M < 2$, then the convergence of marked spectra in the weak topology implies the convergence of the associated Connes metrics in the Lipschitz topology.

Proof. Using Lemma 3.3 and the fact that every deformation $\rho \in \Xi$ is bounded, we obtain

$$\sum_{(y,y')\in B} (\rho(y,y'))^p \leq C \sum_{(y,y')\in B} |yy'|^p \leq C\,\mathrm{Trace}\,(|ds|^p) < \infty.$$

It is easy to construct a section $\xi_0 : B \to \mathbb{C}^m$ with $\|\xi_0(y,y')\| = \rho_0(y,y')$ for all $(y,y') \in B$. Then $\xi_0 \in L^p(B)$ and, by the condition,

$$\sum_{(y,y')\in B} \left| \frac{1}{\rho_i(y,y')} - \frac{1}{\rho_0(y,y')} \right|^p \|\xi_0(y,y')\|^p \to 0 \quad \text{as} \quad i \to \infty.$$

By the choice of ξ_0 this implies that for every $\epsilon > 0$ we have

$$1 - \epsilon \leq \frac{\rho_0(y,y')}{\rho_i(y,y')} \leq 1 + \epsilon$$

for all $(y,y') \in B$ and all sufficiently large i. It follows that for such i we have

$$(1-\epsilon) d_{\rho_i}(x,x') \leq d_{\rho_0}(x,x') \leq (1+\epsilon) d_{\rho_i}(x,x')$$

for every x, $x' \in X$. This means that

$$|d_{\rho_i} d_{\rho_0}|_L \to 0 \quad \text{as} \quad i \to \infty. \quad \square$$

4. Estimate from above for the spectral dimension. *The spectral dimension* of a compact metric space X is defined as

$$\dim_{\mathrm{S}} X := \liminf\{\dim M \mid \mathrm{dens}_{\mathrm{loc}}(M) \to \infty\},$$

where the lower limit is taken over all spectral geometries M on X, whose local density unboundedly increases. It is easy to see that the spectral dimesion is a biLipschitz invariant of X. By Lemma 3.6, the relative bounding radius of M is finite, if $\mathrm{dens}_{\mathrm{loc}}(M) > 2$. Thus $\dim_{\mathrm{S}} X \geq \dim_{\mathrm{H}} X$ by Corollary 3.7.

For sufficiently homogeneous spaces, the spectral dimension coincides with the Hausdorff dimension. A metric space X is said to be *q-quasi-homogeneous*, (q-q.h. for brevity), $q > 0$, if for every $\delta > 0$ there exists a δ-net $Y_\delta \subset X$ such that the ball $D_\rho(x)$ contains at most $C(\rho/\delta)^q$ points of Y_δ for every $\rho > 0$ and $x \in X$, where the constant $C > 0$ does not depend on δ, ρ and x.

3.9. Theorem. *Assume that a compact metric space X is q-q.h. Then there exists on X a spectral geometry M with the infinite local density and the dimension $\dim M \leq q$. In particular, $\dim_{\mathrm{S}} X \leq q$.*

The proof is based on the following proposition.

3.10. Proposition. *Assume that a compact metric space X is q-q.h. Then there exists a symmetric subset $B \subset X^2 \setminus \Delta$ having the infinite local density such that the set $B_t = \{b \in B \mid |b| \geq t\}$ contains at most $C(\log 1/t)^{2q+1}/t^q$ elements for every (sufficiently small) $t > 0$, where the constant $C > 0$ does not depend on t.*

Proof. We put $\delta_k = 2^{-k}$ and for every $k \geq 5$ consider a δ_k-net $Y_k = Y_{\delta_k}$ from the definition of quasi-homogeneity. Due to that property, we have $|Y_k| \leq C\delta_k^{-q}$. Let $B = \cup_k B(k)$, where $B(k) = \big\{(y, y') \in Y_k^2 \setminus \Delta \mid |yy'| \leq R_k\big\}$, $R_k = (k-1)\delta_{k-1}$. The set B is, obviously, symmetric and, by Proposition 1.9, has the infinite local density. Let $t > 0$. If $(y, y') \in B_t$, then $(y, y') \in B(k)$ with $R_k \geq |yy'| \geq t$. Thus $(k-1)/2^{k-1} \geq t$. Let k_0 be the maximal integer satisfying this inequality. Then

$$|B_t| \leq \sum_{k=5}^{k_0} \sum_{y \in Y_k} |D_{R_k}(y) \cap Y_k| \leq C \sum_{k=5}^{k_0} |Y_k|(R_k/\delta_k)^q \leq C k_0^{q+1} 2^{(k_0+1)q}.$$

Using the condition $k_0 - 1 \geq t2^{k_0-1}$, we obtain that $|B_t| \leq C'(\log 1/t)^{2q+1}/t^q$, where the constant $C' > 0$ does not depend on t. $\square$

Proof of Theorem 3.9. We consider the spectral geometry $M = M(B, 1)$ on X, where the set B is constructed in Proposition 3.10. For the subspace $E_t \subset \mathcal{H}$ and every $t > 0$, we, obviously, have $\dim E_t = |B_t|$. Computing the dimension of M with help of Proposition 3.2 and using the estimate from Proposition 3.10, we obtain $\dim M \leq q$. $\square$

Every compact Riemannian manifold X is, obviously, q-q.h. with $q = \dim X$. Thus using Theorem 3.9 and Corollary 3.7, we obtain

3.11. Corollary. *Every compact Riemannian manifold X possesses a spectral geometry M with the infinite local density and the dimension $\dim M = \dim X$. In particular, $\dim_{\mathrm{S}} X = \dim X$.* $\square$

We give another example of a compact Hausdorff space, whose spectral dimension coincides with the Hausdorff one.

Given $n \in \mathbb{N}$ and $p > 1$, the set $\mathcal{P}_{n,p} = \{0, \dots, n\}^{\mathbb{N}}$ with metric

$$|x\, x'| = \frac{1}{p^k},$$

where $k = \min\{N \mid x(N+1) \neq x'(N+1)\}$, is a compact Cantor space of diameter $\operatorname{diam}(\mathcal{P}_{n,p}) = 1$.

3.12. Lemma. *The compact space $\mathcal{P}_{n,p}$ is q-quasi-homogeneous with $q = \log_p(n+1)$.*

Proof. For every $N \in \mathbb{N}$, the set

$$\mathcal{P}_{n,p}^N = \{x \in \mathcal{P}_{n,p} \mid x(k) = x(N) \text{ for all } k \geq N\}$$

is a δ-net in $\mathcal{P}_{n,p}$ with $\delta = p^{-N}$ and contains $(n+1)^N$ elements. Therefore, for $Y_\delta = \mathcal{P}_{n,p}^N$ we have $|Y_\delta| = (n+1)^N = \delta^{-q}$. Given $\rho > 0$ and $x \in \mathcal{P}_{n,p}$, the ball $B_\rho(x)$ coincides with $\{x' \in \mathcal{P}_{n,p} \mid p^{-Q} \leq \rho\}$, where $Q = Q(x')$ and $Q+1$ is the minimal integer k with $x'(k) \neq x(k)$. Thus for the minimal Q satisfying the inequality $p^{-Q} \leq \rho$, we have

$$|\,(B_\rho(x) \cap Y_\delta)\,| \leq (n+1)^{N-Q} \leq \delta^{-q} p^{-qQ} \leq C(\frac{\rho}{\delta})^q,$$

where the constant $C > 0$ depends only on p and q. $\square$

The following statement is well known (at least for $p = 2$), and we give its proof for sake of completeness only.

3.13. Lemma. $\dim_{\mathrm{H}} \mathcal{P}_{n,p} \geq \log_p(n+1)$.

Proof. We introduce a probability measure mes on $\mathcal{P}_{n,p}$ defined by the condition that the numbers $\{0, \dots, n\}$ appear as $x(k)$ with probability $1/(n+1)$ for every $x \in \mathcal{P}_{n,p}$, $k \in \mathbb{N}$. In other words, for every $x \in \mathcal{P}_{n,p}$, $N \in \mathbb{N}$ we have

$$\operatorname{mes}\big(B_{p^{-N}}(x)\big) = \frac{1}{(n+1)^N}.$$

Thus for $q = \log_p(n+1)$ we have $\operatorname{mes}\big(B_{p^{-N}}(x)\big) = \big(\operatorname{diam}\,\big(B_{p^{-N}}(x)\big)\big)^q = p^{-qN}$.

Given a closed subset $F \subset \mathcal{P}_{n,p}$, there are $x \in \mathcal{P}_{n,p}$, $N \in \mathbb{N}$ such that $F \subset B_{p^{-N}}(x)$ and $\operatorname{diam}(F) = \operatorname{diam}\,\big(B_{p^{-N}}(x)\big)$.

Indeed, F is compact, thus there exist x, $x' \in F$ with

$$|x\, x'| = \operatorname{diam}(F) = p^{-N}$$

for some $N \in \mathbb{N}$. Hence $F \subset B_{p^{-N}}(x)$ and $\operatorname{diam}(F) = \operatorname{diam}\,\big(B_{p^{-N}}(x)\big)$.

Therefore,

$$\operatorname{mes}(F) \leq (\operatorname{diam}(F))^q$$

for every closed $F \subset \mathcal{P}_{n,p}$. Thus given a covering $\{F\}$ of $\mathcal{P}_{n,p}$ by closed subsets, we have

$$\sum (\operatorname{diam} F)^q \geq \sum \operatorname{mes}(F) \geq \operatorname{mes}(\mathcal{P}_{n,p}) = 1.$$

For the q-dimensional Hausdorff measure this gives $H_q(\mathcal{P}_{n,p}) \geq 1$, and $\dim_{\mathrm{H}} \mathcal{P}_{n,p} \geq \log_p(n+1)$. $\square$

From Corollary 3.7, Theorem 3.9, and Lemmas 3.12 and 3.13, we obtain

3.14. Corollary. *For every $n \in \mathbb{N}$, $p > 1$, there exists a spectral geometry M with the infinite local density and the dimension $\dim M = \log_p(n+1) = \dim_{\mathrm{H}} \mathcal{P}_{n,p}$ on the compact metric space $\mathcal{P}_{n,p}$. In particular, $\dim_{\mathrm{S}} \mathcal{P}_{n,p} = \dim_{\mathrm{H}} \mathcal{P}_{n,p}$.* $\square$

5. Comparison with other definitions of dimension. A number of definitions of dimension suitable for subsets of metric spaces is known (see, for example [KT], [Tr]), especially in the literature on fractal geometry [Fa1], [Fa2], which give different results for sufficiently pathological spaces. We mention here besides the Hausdorff's also packing dimension, upper and lower box dimension, the Minkowski–Bouligand dimension [Bo] (the last is defined for subsets of Euclidean spaces). It is well known that the upper box dimension dominates all dimensions from this list. Thus we compare the spectral dimension $\dim_{\mathrm{S}}$ only with the upper box dimension $\overline{\dim}_{\mathrm{B}}$, which is defined as follows.

Let $N_t(X)$ be the least number of elements, which is needed to cover X by sets of diameter $\leq 2t$. We put

$$\overline{\dim}_{\mathrm{B}} X = \limsup_{t \to 0} \frac{\log N_t(X)}{\log t^{-1}}.$$

In [KT], this quantity is called *the upper metric dimension* of X.

3.15. Theorem. *For each compact metric space X it holds*

$$\dim_{\mathrm{S}} X \geq \overline{\dim}_{\mathrm{B}} X.$$

Proof. It is easy to see that $N_t(X)$ is at most the maximal number of t-separated points in X and is at least the maximal number of $2t$-separated points in X (see [KT, Theorem IV]). Thus in the definition of the upper box dimension, one can take as $N_t(X)$ the maximal number of t-separated points in X.

We fix $\epsilon > 0$ and $D \geq 4$. Since

$$\dim_{\mathrm{S}} X \geq \inf\{\dim M \mid \mathrm{dens}_{\mathrm{loc}}(M) \geq D + \epsilon\},$$

there exists a spectral geometry M on X with $\mathrm{dens}_{\mathrm{loc}}(M) \geq D+\epsilon$, for which $\dim_{\mathrm{S}} X + \epsilon \geq \dim M$. Using the definition of the local density $\mathrm{dens}_{\mathrm{loc}}(M)$, we find $t > 0$ such that for each x, $x' \in X$ with $0 < |xx'| \leq t$ there is $b = (y, y') \in \mathrm{supp}\, M$ with $\max\{|xy|, |x'y'|\} \leq |xx'|/D$.

The set of $t/2$-isolated points of X (i.e., separated from any other point by the distance $> t/2$) is finite because X is compact. Removing this set from X, we change neither the spectral nor the upper box dimension of X. Thus in what follows, we assume that X contains no $t/2$-isolated point.

Any subset in X of diameter $\leq t/10$, separated from its complement by the distance $> t/2$, is called *t-cluster*. Obviously, the number of t-clusters in X is finite. Let δ_0 be the minimum of their diameters. Then $\delta_0 > 0$, since X contains no $t/2$-isolated point.

For each $\delta \in (0, \delta_0/2)$, there is $\tau \in (0, \delta)$ with

$$\frac{\log N_\tau(X)}{\log \tau^{-1}} \geq \overline{\dim}_{\mathrm{B}} X - \epsilon.$$

Let A_τ be a subset in X, consisting from the maximal number of τ-separated points, $|A_\tau| = N_\tau(X)$. For each point $a \in A_\tau$, we associate the pair $z_a = (a, a') \in X^2 \setminus \Delta$, where $a' \in A_\tau$ be (some) closest to a points. Then $|aa'| \geq \tau$. We show that $|aa'| \leq t$.

Assume that $|aa'| > t$. There is $x \in X$ different from a with $|xa| \le t/2$, since there is no $t/2$-isolated point. Then $|xa'| > t/2 \ge 5\delta_0 \ge 5\tau$. Moreover, by the definition of δ_0, one can choose x so that $|xa| \ge \delta_0/2 \ge \tau$. However, this contradicts the maximality of A_τ. Therefore, $|aa'| \le t$.

For different a, $a' \in A_\tau$ (nonordered) pairs z_a, $z_{a'}$ coincide or $|z_a z_{a'}| \ge \tau$. Hence, we obtained at least $N_\tau(X)/2$ such pairs. Each pair $z_a = (a, a')$ can be approximated by $b_a = (y, y') \in \operatorname{supp} M$ so that $\max\{|ay|, |a'y'|\} \le |aa'|/D$, because $|aa'| \le t$. Since $D \ge 4$, for different pairs such points are different and $|yy'| \ge (1 - 2/D)|aa'| \ge \tau/2$. Thus for the spectral geometry M we have $\dim E_{\tau/2} \ge N_\tau(X)/2$ and

$$\frac{\log \dim E_{\tau/2}}{\log(\tau/2)^{-1}} \ge (\overline{\dim}_{\mathrm{B}} X - \epsilon)\frac{1 - \log 2/\log N_\tau(X)}{1 + \log 2/\log \tau^{-1}}.$$

Taking $\delta \to 0$, we obtain

$$\dim_{\mathrm{S}} X + \epsilon \ge \dim M \ge \overline{\dim}_{\mathrm{B}} X - \epsilon.$$

Now the required inequality follows since ϵ is arbitrary. $\square$

3.16. Remark. We can see from the proof that the following more precise inequality

$$\overline{\dim}_{\mathrm{B}} X \le \inf\{\dim M \mid \mathrm{dens}_{\mathrm{loc}}(M) \ge 4\}$$

holds.

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