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# THE PETERSSON CONJECTURE FOR THE WEIGHT ZERO. II

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ABSTRACT

The present paper is the completion of the proof of the Petersson conjecture for Hilbert spaces of automorphic functions of weight zero, invariant relative to the group $SL(2,\mathbb{Z})$. For the beginning of the proof, see [13].

## 7. Arithmetic sums

All arithmetic sums in (3.2) from [4] are denoted by a common symbol $R(w)$ whose exact meaning is defined in (1.4) from [4]. We need to estimate the double means $J$ and $S$ of this vector, together with the matrix coefficient $\mu_1^{-1}$ at the point $w = z_n \equiv z_{j_0,n} \equiv z_{j_0}$:

$$(\mu_1^{-1}R)(z_n) = \overset{\infty}{R}(z_n) + \overset{\infty}{V_l}(T, z_n) - \Gamma_1(z_n)(\mu_1^{-1}\mu_2)(z_n)\left[\overset{0}{R}(z_n) + \overset{0}{V_l}(T, z_n)\right]$$

$$+\left[\overset{\infty}{\Gamma}_1(z_n)\mu_1(z_n)\right]^{-1}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}[J(z_n) + r(z_n) + r_\tau(z_n)]. \tag{7.1}$$

On the right-hand side in (7.1), the vector-residues $\overset{\infty}{V_l}$ and $\overset{0}{V_l}$ that arise for the group SL$\,(2,\mathbb{Z})$ (as was noted at the end of [4]) are added to the remainders $\overset{\infty}{R}$ and $\overset{0}{R}$.

We recall that

$$\overset{\infty}{\Gamma}_1(w) = 2(2\pi)^{-2s_2} l^{1/2-w}\Gamma(1-w_1)\Gamma(1-w_2), \tag{7.2}$$

where

$$w_1 = w - \omega_1 + \frac{1}{2}; \quad w_2 = w - \omega_2 + \frac{1}{2}; \quad w_3 = w + \omega_3 - \frac{1}{2};$$

$$\omega_1 = s_1 + s_2 - \frac{1}{2}; \quad \omega_2 = s_2 - s_1 + \frac{1}{2}.$$

Moreover, from (2.21) of [1] we obtain:

$$\mu_1^{-1}(w) \equiv \mu^{-1}(w_1, w_2) = -\frac{\overline{\mu}(w_1, w_2)}{\sin\pi w_1 \sin\pi w_2}, \tag{7.3}$$

where the matrix $\overline{\mu}$ is defined by (2.20) in [1]. Therefore, from (7.2) and (7.3) we have

$$\left[\overset{\infty}{\Gamma}_1(w)\mu_1(w)\right]^{-1} = -\frac{(2\pi)^{2s_1-1}}{\pi} l^{w-1/2}\Gamma(w_1)\Gamma(w_2)\overline{\mu}(w_1, w_2). \tag{7.4}$$

In relation (0.4) of part I of the present paper, the assumption $\omega_2 = \overline{\omega}_1$ was introduced, which implies the conditions $\omega_1 = s_1$, $\omega_2 = \overline{s}_1 = 1 - s_1$, $s_2 = \frac{1}{2}$ (recall that $s_1 = \frac{1}{2} + it_1$). Consequently, from (7.2) we find

$$w_1 = w_3 = w - s_1 + \frac{1}{2}; \quad w_2 = w + s_1 - \frac{1}{2}.$$

As a result, from (2.20) of [1] the matrix $\overline{\mu}$ is expressed as

$$\overline{\mu}(w_1, w_2) = \begin{pmatrix}\cos\pi w, & \sin\pi s_1\\ \sin\pi s_1, & \cos\pi w\end{pmatrix} = \delta\sin\pi s_1 + E\cos\pi w, \tag{7.5}$$

$$E = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \delta = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Putting $w = z_n = \frac{1}{2} + i\varkappa_n$, where $\varkappa_n$ is fixed, $s_1 = \frac{1}{2} + it_1$, and $|t_1| \to \infty$, from (7.4) and (7.5) we obtain an asymptotics:

$$\left[\overset{\infty}{\Gamma}_1(z_n)\mu_1(z_n)\right]^{-1} = \frac{1}{2\pi} l^{i\varkappa_n}(e|t_1|)^{2i\varkappa_n}\left(\delta + O\left(\frac{1}{|s_1|}\right)\right). \tag{7.6}$$

In (7.6), the term $O(\frac{1}{|s_1|})$ describes the behavior of the matrix $(\alpha_{i,j})$ whose entries satisfy the estimate

$$\alpha_{i,j} \ll \frac{1}{|s_1|}, \quad i,\, j = 1, 2.$$

Recall that

$$\Gamma_1(w) = (2\pi)^{2\omega_2 - 1}\Gamma(1 - w_3)\Gamma^{-1}(1 - w_2)$$

and the precise value of $\Gamma_1(z_n)$ is given in (3.4) of part I of the present paper. The product of the matrices $(\mu_1^{-1}\mu_2)(z_n)$ was calculated in Sec. 3, part I (relation (3.2)).

The vectors-residues $\overset{\infty}{V_l}$ and $\overset{0}{V_l}$ in (7.1) under the action of the averaging operators $J$ and $S$ pass to the remainder for a trivial reason: they contain the factors $\Gamma(\frac{w_i}{k})$, $i = 1, 2, 3$, which exponentially decrease with respect to $t_1$ (recall that $k > 2$ is arbitrarily large but fixed). Among the remaining terms of the right-hand side of (7.1), the vectors $\overset{\infty}{R}$ and $\overset{0}{R}$ enjoy the most subtle properties. They can be estimated by one and the same method; for definiteness, we consider the vector $\overset{\infty}{R}$ in detail. Recall that it was defined by relation (3.24) in [1] as the vector $R^{(k)}(w)$. It was endowed by the superscript $\infty$ in [4] in order to distinguish the vectors of the vertices 0 and $\infty$. In addition, the first superscript "$k$" was omitted in order not to overload the notation of the vector by two superscripts. In the kernel of integral (3.24) from [1], of main importance is the vector-convolution $\zeta_\tau(\,\cdot\,, \chi)$ that is defined by (2.23) in [1]. For $\mathrm{SL}\,(2, \mathbb{Z})$, the congruence modulus is equal to 1, the multiplier $\chi \equiv 1$; moreover, by (0.4) and (0.5) of part I, $s_1 = \frac{1}{2}$, whence relation (2.29) from [1] takes the form

$$\zeta_\tau(\widetilde{w}, \chi)\big|_{\mathrm{SL}\,(2,\mathbb{Z})} = \zeta(w, s_1) + \zeta(0)\tau_{s_1}(l)l^{-w}\, {}^t(1, 0), \tag{7.7}$$

where $\zeta(w, s_1)$ is the initial vector-contvolution (0.4) from [1] for the group $\mathrm{SL}\,(2, \mathbb{Z})$:

$$\zeta(w, s_1) = {}^t(\zeta_l, \zeta_{-l})(w, s_1), \tag{7.8}$$

$$\zeta_{\pm l}(w, s_1) = \sum_{m \pm l \geq 1} \frac{\sigma_0(m)\tau_{s_1}(m \pm l)}{(m \pm l)^w};$$

in (7.7), $\zeta(0)$ is the value of the classical Riemann zeta function at the point $s = 0$. It is known that $\zeta(0) = -\frac{1}{2}$. By (7.7), the kernel of integral (3.24) in [1] and the integral itself split into two components. In addition, the second integral whose kernel contains the additional vector (7.7) is estimated by the magnitude $O(1)$ upon moving the contour of integration to the left to the straight line $\mathrm{Re}\, s = (\log l)^{-1}$.

Therefore, for the group $\mathrm{SL}\,(2, \mathbb{Z})$ relation (3.24) from [1] at the point $w = z_n$ takes the form

$$\overset{\infty}{R}(z_n) = \overset{\infty}{R}_1(z_n) + O(1), \tag{7.9}$$

where

$$\overset{\infty}{R}_1(z_n) = \frac{1}{2\pi} \int\limits_{\mathrm{Re}\, s = 1} \left(\frac{l}{T}\right)^{ks} \Gamma(-s)(\Gamma\overline{\mu})(w)\zeta(w, s_1)\, ds, \tag{7.10}$$

$$(\Gamma\overline{\mu})(w) = (2\pi)^{-w_1 - w_2}\Gamma(w_1)\Gamma(w_2)\overline{\mu}(w_1, w_2), \quad w = z_n + ks, \tag{7.11}$$

the matrix $\overline{\mu}$ is given in (7.5), $\zeta(w, s_1)$ is given in (7.8).

It should be noted that kernel (7.10) contains the function $\Gamma(-s)$ that decreases exponentially with respect to $t = \operatorname{Im} s$ on any contour $\operatorname{Re} s = \text{const}$, while the remaining part of the kernel increases with respect to $t$ as a power. Therefore, a nontrivial part of integral (7.10) is a part on the interval

$$|\operatorname{Im} s| \ll \log T. \tag{7.12}$$

The remaining part of integral (7.10) can be omitted for the trivial reason.

In integral (7.10) with condition (7.12), we use decomposition (7.5) of the matrix $\overline{\mu}$. Now we recall that the operator $J$ in relation (0.13) of part I of the present paper places the parameter $t_1 = \operatorname{Im} s_1$ in the interval

$$\frac{1}{2}T \le t_1 \le \frac{3}{2}T, \quad s_1 = \frac{1}{2} + it_1. \tag{7.13}$$

For the conjugate operator $\overline{J}$, the parameter $s_1$ is replaced by $\overline{s}_1 = 1 - s_1 = \frac{1}{2} - it_1$. Comparing (7.12) and (7.13), we see that integral (7.10) with the part $E \cos \pi w$ of the kernel from (7.5), where $w = z_n + ks$, will occur in the remainder with exponential decrease with respect to $T$. Consequently, the following relation is valid for integral (7.10)

$$\overset{\infty}{R}_1(z_n) = \overset{\infty}{R}_2(z_n) + O(1), \tag{7.14}$$

where

$$\overset{\infty}{R}_2(z_n) = \frac{1}{2\pi i} \int\limits_{\operatorname{Re} s = 1} \left(\frac{l}{T}\right)^{ks} \Gamma(w, s_1)\Gamma(-s)\overline{\zeta}(w, s_1)\, ds, \quad w = z_n + ks, \tag{7.15}$$

$$\overline{\zeta}(w, s_1) = \delta\zeta(w, s_1) = {}^{t}(\zeta_{-l}, \zeta_{+l})(w, s_1), \tag{7.16}$$

$$\Gamma(w, s_1) = (2\pi)^{-2w}\Gamma(w + it_1)\Gamma(w - it_1)\operatorname{ch} \pi t_1. \tag{7.17}$$

Now, the time is right to use the absolute convergence of convolution (7.16) in kernel (7.15) and to replace the infinite series for $\zeta_{\pm l}$ from (7.8) by finite sums. In order to understand how long these sums must be, we should take into account the estimate for $\Gamma(w, s_1)$ from (7.17):

$$\Gamma(w, s_1)\Big|_{w = ks + z_n} \ll T^{2k}. \tag{7.18}$$

which is valid for fixed $z_n = \frac{1}{2} + i\varkappa_n$ with conditions (7.12) and (7.13) imposed on $s$ and $s_1$, respectively.

Moreover, we recall that, by (0.17) of part I, $l = q\ell_1$, and, by (0.19), $l_1$ can be bounded as follows:

$$l_1 \ll N \log T, \tag{7.19}$$

because all the remaining $l_1$ will occur in the remainder for a trivial reason (exponential decay of the factor $\exp(-l_1/N)$).

Estimate (7.18) and condition (7.19), together with the relation $N = qT$ from (0.19) of part I, lead to an estimate of the part of kernel (7.15):

$$\left(\frac{l}{T}\right)^{ks} \Gamma(w, s_1) \ll (N \log T)^{2k}. \tag{7.20}$$

Therefore, when we replace the infinite series for $\zeta_{\pm l}$ from (7.8) by finite sums, the tails that we drop must compensate the order of magnitude (7.20). We can achieve this if we introduce a parameter

$$N_1 = N^{2(1+\varepsilon)}, \tag{7.21}$$

where $\varepsilon > 0$ is arbitrarily small but fixed, and subdivide the series for $\zeta_{\pm l}$ from (7.8) into two parts: $m \le N_1$ and $m > N_1$. Then the infinite tails for $m > N_1$ are of order of magnitude

$$N_1^{-k+1/2} \log^2 N_1 \tag{7.22}$$

for $\mathrm{Re}\, s = 1$. By condition, $k \ge 2$ is an arbitrarily large but fixed integer; therefore, for fiven $\varepsilon > 0$ we can always choose $k$ such that quantity (7.22) compensates the order of magnitude of (7.20). Any integer $k \ge 1/\varepsilon$ satisfies this condition. Therefore, in kernel (7.15) we can replace the infinite vector-convolution $\overline{\zeta}(w, s_1)$ by a finite part of it:

$$\overline{\zeta}(w, s_1; N_1) = {}^t(\zeta_{-l}, \zeta_{+l})(w, s_1; N_1), \tag{7.23}$$

where

$$\zeta_{\pm l}(w, s_1; N_1) = \sum_{\substack{m \pm l \ge 1 \\ m \le N_1}} \frac{\sigma_0(m) \tau_{s_1}(m \pm l)}{(m \pm l)^w}. \tag{7.24}$$

But in spectral expansions of finite convolutions, a discontinuous weight function of the type (1,0) is inconvenient for technical reasons, because we are forced to smooth it artificially. Therefore, in place of (7.24) it is helpful to consider an absolutely convergent series

$$\zeta_{\pm l}(\psi) = \sum_{m \pm l \ge 1} \frac{\sigma_0(m) \tau_{s_1}(m \pm l)}{(m \pm l)^w}\, \psi(m). \tag{7.25}$$

with a smooth weight function $\psi$:

$$\psi(m) = \exp\left[ - \left( \frac{m}{N_1} \right)^{2k} \right], \tag{7.26}$$

where $N_1$ and $k$ are the same parameters as in (7.21) and (7.22).

A rigorous passage from the series for $\zeta_{\pm l}$ from (7.8) to (7.25) is accomplished with the help of a decomposition of the unity:

$$1 = \psi(m) + [1 - \psi(m)], \tag{7.27}$$

which should be substituted into the infinite vector-convolution $\overline{\zeta}(w, s_1)$.

As a result, relation (7.14) takes the form

$$\overset{\infty}{R}_1(z_{j_0}) = \frac{1}{2\pi i} \int\limits_{\mathrm{Re}\, s = 0} \left( \frac{l}{T} \right)^{ks} \Gamma(w, s_1) \Gamma(-s) \overline{\zeta}(\psi)\, ds + O(1), \tag{7.28}$$

where

$$\overline{\zeta}(\psi) = {}^t(\zeta_{-l}, \zeta_{+l})(\psi).$$

We moved the contour of integration in (7.28) to the left, to the straight line $\mathrm{Re}\, s = 0$, bypassing the pole of $\Gamma(-s)$ at the point $s = 0$ along a right arc of arbitrarily small radius, because $\overline{\zeta}(\psi)$ converges and, at the same time, on the straight line $\mathrm{Re}\, s = 0$ there is no problem concerning the growth of kernel (7.20) as opposed to the straight line $\mathrm{Re}\, s = 1$.

Recall that

$$w = z_{j_0} + ks; \quad z_{j_0} = \frac{1}{2} + i\varkappa_{j_0} \equiv z_n. \tag{7.29}$$

Here we intentionally replaced the fixed subscript $n$ of the parameter $z_{j_0}$ by an individual subscript $j_0$ of an eigenfunction $f_{j_0}(z)$ from the $n$th box in order to use $n$ as the index of summation in the sequel. Moreover, in what follows, we may assume that the eigenfunction $f_{j_0}(z)$ with subscript $j_0$ is fixed.

## 8. Double convolutions

From representation (7.28) it is seen that the application of the operators $J$ and $S$ to the vector $\overset{\infty}{R}_1(z_{j_0})$ reduces to the application of them to the kernel of integral (7.28). Moreover, the operators $J$ and $S$ commute; therefore, ultimately, the problem reduces to the study of the integral mean $J$ (or $\overline{J}$) of the sum

$$S \otimes \left[ l^{ks} \zeta_{\pm l}(\psi) \right] = \frac{1}{N} \sum_{\substack{l_1=1 \\ (l_1,p)=1}}^{\infty} \overline{t_{j_0}(l_1)}\, \zeta_{\pm q l_1}(\psi)(q l_1)^{ks} \exp\left( -\frac{l_1}{N} \right). \tag{8.1}$$

The double convolutions may be studied in two way. On the one hand, the inner convolution $\zeta_{\pm q l_1}$ in (8.1) may be expanded in the spectrum of the group $\mathrm{SL}\,(2,\mathbb{Z})$ with fixed shift $q l_1$, and then it may be averaged over $l_1$. But in this case we return to the initial unknown $t_{j_0}(q)$ to be estimated.

The other way consists of interchanging the summations in (8.1). Then the double convolution is written in the form

$$\frac{1}{N} \sum_{-\infty \le m \le N_1} \sigma_0(m) \zeta_m^{(p)}(\pm q, N) \psi(m), \tag{8.2}$$

where

$$\zeta_m^{(p)}(\pm q, N) = \sum_{\substack{(l,p)=1 \\ m \pm q l \ge 1}} \overline{t_{j_0}(l)}\, \tau_{s_1}(m \pm q l) f_m(q l) \exp\left( -\frac{l}{N} \right), \tag{8.3}$$

$$f_m(x) = \frac{x^{ks}}{(x+m)^w}, \tag{8.4}$$

and $w$ is taken from (7.29). In (8.3), we omitted the subscript 1 of the index of summation $l$.

Next, we must expand the sum $\zeta_m^{(p)}$ in the spectrum of $\Gamma_0(q)$ with a fixed shift $m$ and average over $m$. As a result, we obtain a spectral representation that does not contain $t_{j_0}(q)$, because $q$ is the congruence modulus of the group $\Gamma_0(q)$. Moreover, the Selberg conjecture on the absence of small eigenvalues in the spectrum of the congruence groups is now proved (see [14]), and, thus, the estimate of double convolution (8.3) will be asymptotic. For these reasons, we choose the second way.

Note that the double convolution has already been used in [14]. Here, it differs essentially from that in [14]. The main distinction is in the eigenvalues $t_{j_0}(l)$ of Hecke operators that arose in (8.3) in place of the function of the number of divisors $\tau_s(l)$ in [14].

In a broad sense, these two functions are eigenvalues of Hecke operators; moreover, the first is related to a discrete spectrum and the second is associated with a continuous one. But the arithmetical structure of $t_{j_0}(l)$ is much more complicated and enigmatic than that of $\tau_s(l)$, therefore the passage to spectral representation (8.3) differs essentially from [14].

Convolution (8.3) has two types of shifts: $\{m + ql\}$ and $\{m - ql\}$. For definiteness, consider the first case of shifts $\{m + ql\}$. In this case, the restriction $m + ql \ge 1$ yields:

$$\{m + ql\}: \qquad -ql + 1 \le m \le \infty. \tag{8.5}$$

We split the interval for $m$ into the positive one

$$\{m + ql\}: \qquad 1 \le m \le \infty \tag{8.6}$$

and the negative one

$$\{m+ql\}: \qquad -ql+1 \le m \le -1. \tag{8.7}$$

The value $m=0$ is excluded because, by definition, $\sigma_s(0) \equiv 0$ for all $s \in \mathbb{C}$.

First we consider the positive interval (8.6) for $m$. For this interval, the part of double convolution (8.2) takes the form

$$\frac{1}{N}\sum_{m=1}^{\infty} \sigma_0(m)\zeta_m^{(p)}(q,N)\psi(m). \tag{8.8}$$

From (8.3), an explicit form of $\zeta_m^{(p)}$ follows:

$$\zeta_m^{(p)}(q,N) = \sum_{(l,p)=1} \overline{t_{j_0}(l)}\tau_{s_1}(ql+m)\, f_m(ql)\, \exp\left(-\frac{l}{N}\right). \tag{8.9}$$

The superscript $(p)$ of the convolution $\zeta_m^{(p)}$ means that the condition $(l,p)=1$ holds. We can get rid of this condition if we make use of the identity

$$\sum_{(l,p)=1} F(l) = \sum_{l=1}^{\infty} F(l) - \sum_{l=1}^{\infty} F(pl)$$

and the multiplicative structure of $t_{j_0}$:

$$t_{j_0}(pl) = \sum_{\delta|(p,l)} \mu(\delta)\, t_{j_0}\left(\frac{p}{\delta}\right) t_{j_0}\left(\frac{l}{\delta}\right).$$

As a result, we obtain a representation:

$$\zeta_m^{(p)}(q,N) = \zeta_m(q,N) - \overline{t_{j_0}(p)}\,\zeta_m\left(pq, \frac{N}{p}\right) + \zeta_m\left(p^2 q, \frac{N}{p^2}\right), \tag{8.10}$$

where $\zeta_m(q,N)$ means sum (8.9) over all positive integers:

$$\zeta_m(q,N) = \sum_{l=1}^{\infty} \overline{t_{j_0}(l)}\,\tau_{s_1}(ql+m)\, f_m(ql)\, \exp\left(-\frac{l}{N}\right). \tag{8.11}$$

If the shift $m$ in convolution (8.11) is relatively prime with $q$, then, from the arithmetical point of view, the spectral methods can be applied to such a convolution. Therefore, (8.11) with the condition

$$(m,q)=1 \tag{8.12}$$

is a model case, which we consider first. The last obstacle for the spectral expansion of (8.11) with condition (8.12) is the analytic behavior of the weight function

$$f_m(ql)\, \exp\left(-\frac{l}{N}\right) \tag{8.13}$$

at the zero point $l=0$.

The spectral methods work well when a weight function is sufficiently smooth and has zeros of high multiplicity at the ends of the interval of summation. Function (8.13) is analytic, exponentially decreases as $l \to \infty$, but it does not vanish at the point $l = 0$.

In order to correct it at this point, we multiply function (8.13) by the additional factor

$$\exp\Big(-\frac{1}{ql}\Big). \tag{8.14}$$

As a result, we obtain a new weight function:

$$f_m(ql)\,\exp\Big(-\frac{l}{N}-\frac{1}{ql}\Big). \tag{8.15}$$

If $l \geq 1$, then for factor (8.14) the following relation holds:

$$\exp\Big(-\frac{1}{ql}\Big) = 1 + O\Big(\frac{1}{ql}\Big),$$

which shows that the substitution in (8.11) of (8.15) for (8.13) generates a remainder whose average over $m$ is estimated by a quantity of order $0(1)$ and has no influence on the ultimate estimate of the required quantity $t_{j_0}(q)$.

Moreover, in order to transform (8.11) to a final convenient from, we need to use the factorization formula

$$\tau_{s_1}(n) = n^{s_1-1/2}\,\sigma_{1-2s_1}(n)$$

and to carry the factor $n^{s_1-1/2}$ (for $n = ql + m$) from here to the weight function of sum (8.11). As a result, instead of (8.11) we have

$$\zeta_m(\varphi) = \sum_{l=1}^{\infty} \overline{t_{j_0}(l)}\,\sigma_{1-2s_1}(ql+m)\,\varphi(ql), \tag{8.16}$$

where

$$\varphi(x) = \frac{x^{ks}}{(x+m)^{w-s_1+1/2}}\,\exp\Big(-\frac{x}{qN}-\frac{1}{x}\Big). \tag{8.17}$$

We recall that $w = z_{j_0} + ks$ (see (7.29)).

## 9. The spectral expansion of a convolution

Substitute the explicit form of the function of the number of divisors

$$\sigma_{1-2s_1}(n) = \sum_{d|n} d^{1-2s_1}, \quad n = ql + m,$$

into the right-hand side of (8.16) and change the orders of summation. As a result, (8.16) takes the form

$$\zeta_m(\varphi) = \sum_{(d,q)=1} d^{1-2s_1} \sum_{ql+m\equiv 0(d)} \overline{t_{j_0}(l)}\ \varphi(ql). \tag{9.1}$$

The condition $(d, q) = 1$ in (9.1) is a consequence of condition (8.12).

Subsitute the multiplier

$$\frac{1}{q}\sum_{a=1}^{d} e^{2\pi\frac{ql+m}{d}a} = \begin{cases} 1, & ql+m \equiv 0(d), \\ 0, & ql+m \not\equiv 0(d) \end{cases}$$

into (9.1) and pass from the complete system of residues $1 \le a \le d$ to the reduced systems over all divisors $c|d$ (i.e., $d = c\delta$); after this, we change the orders of summation. As a result, (9.1) is transformed to

$$\zeta_m(\varphi) = \sum_{(\delta,q)=1} \delta^{-2s_1} \sum_{(c,q)=1} c^{-2s_1} \sum_{(a,c)=1} e^{2\pi i\frac{am}{c}} \sum_{l=1}^{\infty} \overline{t_{j_0}(l)}\, e^{2\pi i\frac{aq}{c}l}\, \varphi(ql). \tag{9.2}$$

In the sequel, we assume that $\operatorname{Re} s_1 > 1$ in order to ensure the absolute convergence of the series with respect to $\delta$ and $c$. On obtaining the spectral expansion of (9.2), we return again to the line $\operatorname{Re} s_1 = \frac{1}{2}$ by the Sokhotsky–Plemejl theorem. In the inner sum in (9.2), we use the Kuznetsov lemma from [5]. For convenience, we repeat it again in a modified form.

**Lemma 5** (N. V. Kuznetsov)**.** *Let $\varphi(x)$ be an infinitely smooth function decreasing faster than any power of $x$ as $x \to 0$ and $x \to \infty$. Then, for any relatively prime numbers $c$ and $d$, $c \ge 1$, the following relation is valid ($d\,d' \equiv 1 \pmod{c}$):*

$$\sum_{l=1}^{\infty} \overline{t_{j_0}(l)}\, e^{2\pi i\frac{d}{c}l}\, \varphi(ql) = \frac{2\pi}{qc}\sum_{n=1}^{\infty} \overline{t_{j_0}(n)}\, e^{2\pi i\frac{d'}{c}n}\, f_{+m}\left(\frac{4\pi\sqrt{mn}}{c\sqrt{q}}\right)$$

$$+\frac{2\pi}{qc}\sum_{n=1}^{\infty} \overline{t_{j_0}(n)}\, e^{2\pi i\frac{d'}{c}n}\, f_{-m}\left(\frac{4\pi\sqrt{mn}}{c\sqrt{q}}\right), \tag{9.3}$$

*where*

$$f_{+m}(x) = \frac{2}{\pi}\operatorname{ch}\pi\varkappa_{j_0}\int_0^{\infty} K_{2i\varkappa_{j_0}}\left(x\sqrt{\frac{y}{m}}\right)\varphi(y)\,dy, \tag{9.4}$$

$$f_{-m}(x) = \int_0^{\infty} J_{\varkappa_{j_0}}\left(x\sqrt{\frac{y}{m}}\right)\varphi(y)\,dy, \tag{9.5}$$

$$J_t(z) = \frac{i}{2\operatorname{sh}\pi t}\left[J_{2it}(z) - J_{-2it}(z)\right]. \tag{9.6}$$

Relation (9.3) differs from a similar relation in [5], because in the present paper we use another normalization for the weight functions $\varphi$ and $f_{\pm m}$ that is more convenient for the congruence group $\Gamma_0(q)$ and for summation formulas for $\Gamma_0(q)$ from [8]. In addition, the classical MacDonald and Bessel functions from the Bessel theory of functions occur on the right-hand sides in (9.4) and (9.6). Substituting (9.3) into (9.2), we obtain a representation:

$$\zeta_m(\varphi) = a(s_1)\sum_{n=1}^{\infty} \overline{t_{j_0}(n)}\left(4\pi\sqrt{mn}\right)^{-2s_1}\left[\sum^{\gamma} \frac{S_{\infty,0}(m,n;\gamma)}{\gamma} f_{+m}(x)\,x^{2s_1}\right.$$

$$\left.+\sum^{\gamma}\frac{S_{\infty,0}(m,-n;\gamma)}{\gamma}\,f_{-m}(x)\,x^{2s_1}\right];\; x = \frac{4\pi\sqrt{mn}}{\gamma},\; \gamma = c\sqrt{q}, \tag{9.7}$$

where

$$a(s) = 2\pi q^{s-1/2}\left(1 - \frac{1}{p^{2s}}\right)\zeta(2s); \quad q = p^{\nu}. \tag{9.8}$$

On the right-hand side in (9.7), the symbol $S_{\infty,0}$ means the Kloosterman sum of the pair of vertices $(\infty, 0)$ of the group $\Gamma_0(q)$ in the sense of paper [8]. Here, we use the vertex $\alpha = 0$ in place of the vertex $\alpha = 1$ (equivalent to it) that was used in [8]. This was done in order to get rid of the additive character that relates the sum $S_{\infty,0}$ with the classical Kloosterman sum:

$$S_{\infty,1}(m, \pm n; \gamma) = e^{\pm 2\pi i \frac{n}{q}}\, S(mq', \pm n; c). \tag{9.9}$$

For the vertex $\alpha = 0$, a "pure" relation is valid:

$$S_{\infty,0}(m, \pm n; \gamma) = S(mq', \pm n; c). \tag{9.10}$$

The distinction between the right-hand sides in (9.10) and (9.9) lies in the fact that the vertex 0 is taken to $\infty$ by a multiplicative rotation, while the vertex 1 goes to $\infty$ with the help of an additive (left) shift. The multiplicative character in the group $\Gamma_0(q)$ is trivial, while the additive one is not trivial. Relation (9.10) gives an explanation to the absence of the additive character on the right in (9.7).

The weight functions $f_{\pm m}(x)\, x^{2s_1}$ ($\operatorname{Re} s_1 > 1$) satisfy the necessary requirements of smoothness and decrease at the ends in order that one can apply the summation formulas from [8] to (9.7). The subsequent calculation of the spectral expansion is standard, its details can be found in [6–11]; therefore, we restrict ourselves to brief comments on it.

The main distinction of the spectral expansion of convolution (9.7) from the convolution in [11] lies in the Rankin convolutions $R_{j,j_0}(s_1)$ that occur in place of the Hecke series in the components related to the discrete spectrum and classical spaces and in the generalized square of the Hecke series of index $j_0$ in place of the generalized fourth degree of the Riemann $\zeta$-function for the continuous component.

Substituting the spectral expansions of the three convolutions of the right-hand side in (8.10) into (8.8) and summing them over $m$, $(m, q) = 1$, we obtain the spectral expansion of the double convolution (8.8). The continuous components of each vertex are the principal parts of this expansion. The integral average $J$ (or $\overline{J}$) of the product of these components by $\Gamma(w, s_1)$ is a quantity of order

$$\sigma_0(q)\, \frac{\sqrt{N_1}}{N} \log^6 T, \tag{9.11}$$

where $\sigma_0(q)$ is the number of vertices, $N_1$ is from (7.21). Since $q = p^{\nu}$ and, thus, $\sigma_0(q) = \nu + 1$, for (9.11) we obtain the quantity

$$(\nu + 1)\, N^{\varepsilon} \log^6 T. \tag{9.12}$$

This is the principal order among all the estimates.

By condition, $N = qT$, $q = T^{\varepsilon_0}$, where $\varepsilon_0 \le \frac{1}{2}$ is a fixed number, therefore

$$N = q^{c_0}, \quad c_0 = 1 + 1/\varepsilon_0.$$

For definiteness, in the sequel we assume that $\varepsilon_0 = \frac{1}{2}$. Therefore, the order of (9.12) is no greater than

$$q^{4\varepsilon}. \tag{9.13}$$

The estimation of the discrete components is carried out with the help of the techniques of [6] and [8] for the second moments of the Rankin convolutions and for the fourth moments of the Hecke series from Lemmas 1–4, by analogy with Secs. 1 and 2. The only distinction is in the fact that, in Secs. 1 and 2, we considered the group $\mathrm{SL}(2,\mathbb{Z})$, but here we study $\Gamma_0(q)$ with increasing modulus $q$.

Paper [8] was written for the group $\Gamma_0(q)$, therefore carrying over the large sieve does not require clarification.

Paper [6] is written for the group $\mathrm{SL}(2,\mathbb{Z})$, and carrying over the results of it to $\Gamma_0(q)$ require an answer to the question as to why the estimates do not become worse under such a carry-over.

Here, the main remark is as follows. If we simplify Theorem 6 from [7] also obtained for $\mathrm{SL}(2,\mathbb{Z})$ and write it as an estimate:

$$\sum_{\varkappa_j \le X} |\rho_j(1)|^2/\operatorname{ch}\pi\varkappa_j \ll X^2, \tag{9.14}$$

then from [8] it follows that a similar estimate for $\Gamma_0(q)$ takes the form

$$\sum_{\varkappa_j \le N} |\rho_{j,\alpha}(1)|^2/\operatorname{ch}\pi\varkappa_j \ll X^2. \tag{9.15}$$

Estimate (9.15) is valid for any vertex $\alpha$ of the fundamental domain $F_q$ of the group $\Gamma_0(q)$. Comparing (9.14) and (9.15), we see that the estimate does not become worse under passage from $\mathrm{SL}(2,\mathbb{Z})$ to $\Gamma_0(q)$. Therefore, it is important to understand the reason for the stability of these estimates as compared with the classical asymptotics of Hermann Weil that depend on the volume of the fundamental domain $F_q$.

The answer is in the normalizing relation

$$\iint\limits_{F_q} \mid f_j(z) \mid^2 \, d\mu(z) = 1 \tag{9.16}$$

that is one and the same for all $q \ge 1$.

From relation (9.16) it follows that the normalizing coefficients $|\rho_j(1)|^2$ contain the volume $F_q$, which leads to the coincidence of estimates (9.14) and (9.15).

This phenomenon of the identity of estimates is carried over to a large class of sums that contain the normalizing coefficients $|\rho_j(1)|^2$, in particular, to the estimates from Lemmas 1–4 in Sec. 1. Moreover, Lemmas 2 and 4 are valid in the case of different vertices, and their analogs for finite-dimensional spaces remain also valid for $\Gamma_0(q)$. Therefore, for $\Gamma_0(q)$ these lemmas ensure the limit estimates of double convolution (8.8) with condition (8.12).

As a result, we conclude that the integral averages $J$ and $\overline{J}$ of the product of the discrete component by $\Gamma(w, s_1)$ also do not exceed the quantity (9.13). This remark also applies to its components related to finite-dimensional spaces.

## 10. The general case $(m,q) \ge 1$

Above we studied the spectral expansion of convolution (8.9) with condition (8.12). From the relation $q = p^\nu$ it follows that (8.12) is equivalent to the condition

$$(m,p) = 1. \tag{10.1}$$

In order to reduce convolution (8.9) for $(m,q)>1$ to the case of (10.1), we expand sum (8.8) in the spectrum over the greatest common divisors

$$(m,q)=p^r, \quad 0\le r\le \nu.$$

As a result, sum (8.8) takes the form

$$\sum_{r=0}^{\nu-1}\frac{1}{N}\sum_{(m,q)=p^r}(\,\cdot\,)+\tfrac{1}{N}\textstyle\sum_{m\equiv 0(q)}(\,\cdot\,). \tag{10.2}$$

If we put

$$m=p^r n, \quad (n,p)=1, \quad 0\le r\le \nu-1, \tag{10.3}$$

then the following relation is valid:

$$\tau_{s_1}(ql+m)=\tau_{s_1}(p^r)\tau_{s_1}(p^{\nu-r}\,l+n). \tag{10.4}$$

Moreover, from (8.4) and (7.26), under condition (10.3) it follows the relation

$$f_m(ql)=p^{-r z_{j_0}}\,f_n(p^{\nu-r}l). \tag{10.5}$$

Putting

$$q_r=p^{\nu-r}, \quad \text{i.e.,}\quad q_0=q \tag{10.6}$$

and substituting (10.4) and (10.5) into (8.9), we obtain

$$\zeta_m^{(p)}(q,N)=\frac{\tau_{s_1}(p^r)}{p^{r z_{j_0}}}\,\zeta_n^{(p)}(q_r,N). \tag{10.7}$$

In addition, for $m$ from (10.3) we have

$$\sigma_0(m)=\sigma_0(p^r)\,\sigma_0(n)=(r+1)\,\sigma_0(n). \tag{10.8}$$

Substituting (10.7) and (10.8) into (8.8), for the first term in (10.2) (the double sum) we obtain an explicit form:

$$\sum_{r=0}^{\nu-1}\frac{(r+1)\,\tau_{s_1}(p^r)}{N\,p^{r\,z_{j_0}}}\sum_{\substack{n=1\\(n,p)=1}}^{\infty}\sigma_0(n)\,\zeta_n^{(p)}(q_r,N)\,\psi_r(n), \tag{10.9}$$

where

$$\psi_r(n)=\exp\left[-\left(\frac{p^r n}{N_1}\right)^{2k}\right].$$

This definition of $\psi_r$ is a consequence of definition (7.26). The inner sum over $n$ in (10.9) is the general analog of sum (8.9) with condition (8.12) that is equivalent to (10.1). Its spectral expansion for the group $\Gamma_0(q_r)$ can be obtained in the same way as was obtained in Sec. 9 for the case $q_0=q$, only replacing $q_0$ by $q_r$. The additional function (8.14) must be of the form

$$\exp\left(-\frac{1}{q_r\,l}\right). \tag{10.10}$$

From the definition of the function $\tau_{s_1}$, the following estimate is valid:

$$\mid \tau_{s_1}(p^r) \mid \leq r+1, \quad \text{if } \operatorname{Re} s_1 = \frac{1}{2}.$$

Moreover, $z_{j_0} = \frac{1}{2} + i\varkappa_{j_0}$, therefore the coefficient dependent on $r$ in (10.10) in modulus is no greater than

$$\frac{(r+1)^2}{(\sqrt{p})^r}. \tag{10.11}$$

Consequently, all the remainders arising when we introduce the additional function (10.10) yield the common remainder of order $O(1)$.

The second term in (10.2) over $m \equiv 0(q)$ invites special consideration. In this case, in the relation $m = qn$, the condition $(n, p) = 1$ is absent contrary to the case of (10.9); therefore, the simple multiplicative relation (10.8) is absent here. It must be replaced by the general Möbius inversion relation

$$\sigma_0(qn) = \sum_{\delta|(q,n)} \mu(\delta)\, \sigma_0\left(\frac{q}{\delta}\right) \sigma_0\left(\frac{n}{\delta}\right). \tag{10.12}$$

With the help of this relation, the second term in (10.2) is transformed to the difference of two sums:

$$\frac{\sigma_0(q)}{N} \sum_{n=1}^{\infty} \sigma_0(n)\, \zeta_{qn}^{(p)}(q, N)\, \psi_\nu(n) - \frac{\sigma_0(qp^{-1})}{N} \sum_{n=1}^{\infty} \sigma_0(n)\, \zeta_{pqn}^{(p)}(q, N)\, \psi_{\nu+1}(n). \tag{10.13}$$

For $m = pqn$, relation (10.7) takes the form

$$\zeta_{pqn}^{(p)}(q, N) = \frac{\tau_{s_1}(q)}{q^{z_{j_0}}}\, \zeta_{pn}^{(p)}(1, N), \tag{10.14}$$

which is a consequence of the multiplicative relation

$$\tau_{s_1}\left(q(l+pn)\right) = \tau_{s_1}(q)\, \tau_{s_1}(l+pn), \tag{10.15}$$

because, by condition, $(l, p) = 1$.

Substituting (10.14) into the second sum of (10.13), we obtain

$$\frac{\nu \tau_{s_1}(q)}{N q^{z_{j_0}}} \sum_{n=1}^{\infty} \sigma_0(n)\, \zeta_{pn}^{(p)}(1, N)\, \psi_{\nu+1}(n). \tag{10.16}$$

Substituting (8.9) into (10.16) for $q = 1$ and $m = pn$ and changing the orders of summation, we have

$$\frac{\nu \tau_{s_1}(q)}{N q^{z_{j_0}}} \sum_{\substack{l=1 \\ (l,p)=1}}^{\infty} \overline{t_{j_0}(l)}\, \zeta_l(p, \psi_{\nu+1})\, l^{ks} \exp\left(-\frac{l}{N}\right), \tag{10.17}$$

where

$$\zeta_l(p, \psi_{\nu+1}) = \sum_{n=1}^{\infty} \frac{\sigma_0(n) \tau_{s_1}(pn+l)}{(pn+l)^w} \psi_{\nu+1}(n). \tag{10.18}$$

Thus, we have returned to the initial convolution (8.1) with congruence modulus $p$, but without multiplier $q$ of the shift $l$. Therefore, to (10.18) we can apply [11] in order to expand it in the spectrum of $\Gamma_0(p)$, because now the possibility of obtaining the unknown quantity $t_{j_0}(q)$, which was mentioned at the beggining of Sec. 8, is excluded. In place of it we have a well-known quantity $\tau_{s_1}(q)$ as a coefficient in (10.17).

But before applying the formulas from [11] to convolution (10.18), we should first multiply the function $\psi_{\nu+1}(n)$ by an additional factor

$$\exp\left(-\frac{1}{n}\right), \tag{10.19}$$

in order to obtain the necessary properties of the weight function in convolution (10.18) at zero. We used a similar device in Sec. 8 when we introduced the additional function (8.14). For $n \geq 1$, the following relation is valid:

$$\exp\left(-\frac{1}{n}\right) = 1 + 0\left(\frac{1}{n}\right),$$

therefore the difference between convolution (10.18) and a similar convolution where $\psi_{\nu+1}(n)$ is multiplied by (10.19) is in the remainder for a trivial reason.

For $\zeta_{qn}^{(p)}$ from the first sum in (10.13), the simple multiplicative relation (10.14) does not hold because for $\tau_{s_1}(q(l+n))$ there is no simple relation of the type (10.15). Therefore, here we must again use the general Möbius inversion of the type (10.12):

$$\tau_{s_1}(q(l+n)) = \sum_{\delta|(q,l+n)} \mu(\delta)\, \tau_{s_1}\left(\frac{q}{\delta}\right) \tau_{s_1}\left(\frac{l+n}{\delta}\right).$$

With the help of this relation, we have a representation:

$$\zeta_{qn}^{(p)}(q,N) = \frac{\tau_{s_1}(q)}{q^{z_{j_0}}}\, \zeta_n^{(p)}(1,N) - \frac{\tau_{s_1}(qp^{-1})}{q^{z_{j_0}}}\, \zeta_n^{(p)}(p^{-1},N), \tag{10.20}$$

where

$$\zeta_n^{(p)}(p^{-1},N) = \sum_{\substack{l=1\\(l,p)=1,\, l+n\equiv 0(p)}}^{\infty} \overline{t_{j_0}(l)}\tau_{s_1}\left(\frac{l+n}{p}\right) f_n(l) \exp\left(-\frac{l}{n}\right).$$

Substituting (10.20) into the first sum of (10.13) and changing the orders of summation, we have

$$\frac{\nu+1}{Nq^{z_{j_0}}} \sum_{\substack{l=1\\(l,p)=1}}^{\infty} \overline{t_{j_0}(l)}\left[\tau_{s_1}(q)\zeta_l(\psi_\nu) - \tau_{s_1}(qp^{-1})\zeta_{l/p}(\psi_\nu)\right] l^{ks} \exp\left(-\frac{l}{N}\right), \tag{10.21}$$

where

$$\zeta_{l/p}(\psi_\nu) = \sum_{\substack{n=1\\n+l\equiv 0(p)}}^{\infty} \frac{\sigma_0(n)\tau_{s_1}(\frac{n+l}{p})}{(n+l)^w}\psi_\nu(n). \tag{10.22}$$

Replacing the congruence $n+l \equiv 0(p)$ by the relation $n+l = p\cdot m$ and changing the variable of summation $n$ by $m$ with $n = pm - l$, we rewrite convolution (10.22) in the form

$$\zeta_{l/p}(\psi_\nu) \equiv \zeta_{-l)}(p,\psi_\nu) = \sum_{pm-l\geq 1} \frac{\sigma_0(pm-l)\tau_{s_1}(m)}{(pm)^w}\psi_\nu(pm-l). \tag{10.23}$$

This is an analog of convolution (10.18) with a negative shift $l$. The same remarks made for (10.18) applies to it. The only distinction is in the fact that the additional function (10.19) must be of the form

$$\exp\left(-\frac{1}{pm-l}\right). \tag{10.24}$$

The convolution $\zeta_l(\psi_\nu)$ from (10.21) is nothing but the convolution (10.18) for $p=1$:

$$\zeta_l(\psi_\nu)=\sum_{n=1}^{\infty}\frac{\sigma_0(n)\tau_{s_1}(n+l)}{(n+l)^w}\psi_\nu(n). \tag{10.25}$$

Its spectral expansion is studied in detail in [10]. First we need to multiply $\psi_\nu(n)$ by (10.19).

Substituting the spectral expansion of convolutions (10.24) and (10.23) into (10.21) and averaging them over $l$, for double convolution (10.21) we obtain a representation dependent on $s_1$, which we can consider on the line $\operatorname{Re}s_1=\frac{1}{2}$ by the Sokhotski–Plemejl theorem. Next, multiplying it by $\Gamma(w,s_1)$ and applying to the product obtained the averaging integral operators $J$ and $\overline{J}$, we arrive at an estimate of order $O(1)$.

## 11. The negative interval of averaging

In the preceding section we completed the study of interval (8.6) for the parameter of averaging $m$, which is the positive part of interval (8.5). It remains to study the negative part (8.7). For this interval, an analog of double convolution (8.8) is of the form

$$\frac{1}{N}\sum_{m=1}^{\infty}\sigma_0(m)\zeta_{-m}^{(p)}(q,N)\psi(m), \tag{11.1}$$

where

$$\zeta_{-m}^{(p)}(q,N)=\sum_{\substack{(l,p)=1,\\ lq-m\geq 1}}\overline{t_{j_0}(l)}\tau_{s_1}(ql-m)f_{-m}(ql)\exp\left(-\frac{l}{N}\right). \tag{11.2}$$

In sum (11.1), the sign of $m$ is changed only in the convolution $\zeta_{-m}^{(p)}$ because $\sigma_0(m)$ and $\psi(m)$ are even functions.

Convolution (11.2) differs from (8.9) by the condition $lq-m\geq 1$, which means that

$$l>\frac{m}{q}. \tag{11.3}$$

For the weight function $\exp(-\frac{l}{N})$, condition (11.3) leads to the inequality

$$\exp\left(-\frac{l}{N}\right)<\exp\left(-\frac{m}{qN}\right). \tag{11.4}$$

Hence it follows that the interval

$$m\leq qN\log^2 N \tag{11.5}$$

is the working interval of sum (11.1). The remaining part of the interval is taken into account in the remainder for trivial reasons: the right-hand side in (11.4) decays exponentially on it.

But in interval (11.5), the following relation is valid for the function $\psi(m)$ from (7.26):

$$\psi(m) = 1 + O\left(\frac{1}{N^k}\right), \tag{11.6}$$

where $k \geq 2$ is arbitrarily large but fixed. Therefore, to a high accuracy, the function $\psi(m)$ in (11.1) can be replaced by one. On the other hand, the condition $ql - m \geq 1$ enables us to use the identity

$$\exp\left(-\frac{l}{N}\right) = \exp\left(-\frac{m}{qN}\right)\exp\left(-\frac{ql-m}{qN}\right) \tag{11.7}$$

for the weight function $\exp(-\frac{l}{N})$. Substituting it into (11.2), we may replace series (11.1) by a modified series

$$\frac{1}{N}\sum_{m=1}^{\infty}\sigma_0(m)\widetilde{\zeta}^{(p)}_{-m}(q,N)\exp\left(-\frac{m}{qN}\right), \tag{11.8}$$

where

$$\widetilde{\zeta}^{(p)}_{-m}(q,N) = \sum_{\substack{(l,p)=1,\\ lq-m\geq 1}} \overline{t_{j_0}(l)}\tau_{s_1}(ql-m)f_{-m}(ql)\exp\exp\left(-\frac{ql-m}{qN}\right). \tag{11.9}$$

Further arguments concerning the estimation of sums (11.8) and (11.9) are the same as for sums (8.8) and (8.9), because in [10, 11] two kinds of shifts, positive and negative, were considered. The only distinction is related to additional function (8.14), which must be replaced by the function

$$\exp\left(-\frac{1}{ql-m}\right). \tag{11.10}$$

The corresponding change must be done in (10.10).

The estimates obtained do not exceed quantity (9.13), although this was intuitively clear from the outset, because the working length (11.5) of sum (11.8) is almost a square root of the working length of sum (8.8).

Now we recall that the convolutions $\zeta_{+l}(\psi)$ and $\zeta_{-l}(\psi)$ occur in sum (8.1). We considered only the case of $\zeta_{+l}$, which corresponds to set (8.5). It remains to consider the case of negative shifts $-ql$, which corresponds to the set

$$\{m-ql\} : m - ql \geq 1. \tag{11.11}$$

In this case, instead of (8.8) we have

$$\frac{1}{N}\sum_{m=q+1}^{\infty}\sigma_0(m)\zeta^{(p)}_{m}(-q,N)\psi(m), \tag{11.12}$$

where

$$\zeta^{(p)}_{m}(-q,N) = \sum_{\substack{(l,p)=1,\\ m-lq\geq 1}} \overline{t_{j_0}(l)}\tau_{s_1}(m-ql)f_{m}(-ql)\exp\left(-\frac{l}{N}\right). \tag{11.13}$$

The working interval of the parameter $m$ in (11.12) is again determined by the function $\psi(m)$ from (7.26), but there is a distinction in the intervals of summation of convolutions (11.13) and

(8.9). In (11.13), a pure splash $m - lq \geq 1$ is superimposed upon the smooth weight function $\exp(-\frac{l}{N})$, which means that

$$l < \frac{m}{q}. \tag{11.14}$$

But this restriction can be removed if we insert into (11.13) a discontinuous integral

$$\frac{1}{2\pi i} \int\limits_{-iN}^{+iN} \frac{x^{\omega}}{\omega}\, d\omega = \begin{cases} 1 + O\Big(\dfrac{1}{N \log x}\Big), & x > 1, \\ \dfrac{1}{2} + O\Big(\dfrac{1}{N}\Big), & x = 1, \\ 0 + O\Big(\dfrac{1}{N \log \frac{1}{x}}\Big), & x < 1, \end{cases} \tag{11.15}$$

with $x = \frac{m}{ql}$.

Moreover, we must somewhat change the definition of the function $f_m(x)$ in (8.4), replacing the parentheses in the denominator by the absolute value.

The finite contour $\mathrm{Re}\, \omega = 0$ in (11.15) bypasses the pole $\omega = 0$ from the right along an arc of radius $1/\log N$. Thus, sum (11.12) is replaced by an integral average of a similar sum into which the factor $m^{\omega}$ is inserted, and in convolution (11.13) we must remove restriction (11.14) and an additional factor $l^{-\omega}$ arises.

Note that the remainders in (11.15) when averaging over $l$ and $m$ are estimated in the standard way and their contribution in the total estimate is no greater than (9.13).

Further arguments concerning the estimation of the "changed" sum (11.12) are the same as for sums (8.8) and (8.9). The presence of a pure imaginary argument $\omega$ in the Hecke series and in the Rankin convolutions does not move them from the line $\mathrm{Re}\, s_1 = \frac{1}{2}$; therefore it is not dangerous because we need the integral average (11.15) over this parameter (Lemmas 1–4 from Sec. 1 carried over to the group $\Gamma_0(q)$).

Integral (11.15) gives the additional factor $\log^4 N$ to the estimate of sum (11.12) as compared with the estimate of sum (8.8), which also is not essential.

The only detail that should be noted consists of the following: we considered analogs of sums (8.8) and (8.9) with condition $(m, q) = 1$, which means the first sum in (10.2) in terms of Sec. 10. In the general case, $(m, q) = p^r$, $0 \leq r \leq \nu - 1$, the modulus $q = q_0$ should be replaced by $q_r = p^{\nu - r}$ in all formulas including (11.14) and (11.15).

At the same time, for the analog of the second sum in (10.2) for $m \equiv 0(q)$, integral operator (11.15) is not needed, because in this case the orders of summation are interchanged and the parameter $m$ is again an inner index of summation and $l$ is an outer index of summation. Therefore, the condition $m - lq > 0$ for $m = qn$ is converted into $n > l$, i.e., it determines a lower bound of the parameter $n$, and thus a zero point of the variable $(n - l) > 0$. Hence it follows that for the smoothness at the zero we need to interchange the additional functions (10.19) and (10.24).

Thus, we have completed the estimation of the double average $J$ and $S$ of the first arithmetic sum $\overset{\infty}{R}(z_{j_0})$ on the right in (7.1). The same estimate holds when we consider the conjugate operator $\overline{J}$ instead of the operator $J$.

The analog of this sum at the zero vertex, $\overset{0}{R}(z_{j_0})$, together with the matrix coefficient, is studied in the same way. To this end, we must use relation (3.2) for the matrix $(M_1^{-1} \cdot M_2)(z_{j_0})$

and relation (3.4) for $\Gamma_1(z_{j_0})$. The triviality of the estimates of the vector residues $\overset{\infty}{V_l}$ and $\overset{0}{V_l}$ was mentioned at the beginning of Sec. 7.

It remains to estimate three types of sums in (7.1) supplied with the matrix coefficient (7.6). We turn to the estimation of these sums.

## 12. The other arithmetic sums

The last term on the right in (7.1) consists of three vectors (without coefficient)

$$J(z_{j_0}) + r(z_{j_0}) + r_\tau(z_{j_0}). \tag{12.1}$$

In this sum we again replaced the fixed index of a "box" $n$ by the index $j_0$ of a fixed eigenfunction from this box whose coefficient $t_{j_0}(q)$ we study.

Consider the first vector in (12.1). It was defined in the introduction (see [4]):

$$J(w) = {}^t\Big(\overset{0}{J}_{+l}, \overset{\infty}{J}_{-l}\Big)(w) = \frac{1}{2\pi i}\int\limits_{\operatorname{Re} s=1} \frac{\Gamma(-s)\Gamma(w_1+s)}{\Gamma(w_1)l^{-s}}\zeta(w+s)ds, \tag{12.2}$$

where

$$w_1 = w - s_2 - s_1 + 1 = w - s_1 + \frac{1}{2}, \quad \text{because} \quad s = \frac{1}{2},$$

$$\zeta(w) = {}^t\Big(\overset{0}{\zeta}_{+l}, \overset{\infty}{\zeta}_{-l}\Big)(w), \tag{12.3}$$

$$\overset{0}{\zeta}_{+l}(w) = \sum_{m=1}^{\infty} \frac{\sigma_0(m)\sigma_{1-2s_1}(m+l)}{m^{w-s_1+1/2}}, \tag{12.4}$$

$$\overset{\infty}{\zeta}_{-l}(w) = \sum_{m-l\geq 1}^{\infty} \frac{\sigma_0(m)\sigma_{1-2s_1}(m-l)}{(m-l)^{w-s_1+1/2}}. \tag{12.5}$$

Note that convolution (12.5) coincides with the second convolution $\zeta_{-l}(w, s_1)$ from (7.8). Convolution (12.4) differs not significantly from the first convolution $\zeta_{+l}(w, s_1)$ from (7.8). In the kernels of integrals (12.2) and (7.15) there is one and the same factor $\Gamma(-s)$ which governs the poles of the kernels in the half-plane $\operatorname{Re} s \geq 0$. The other factors of the kernel in integral (7.15) contain a multiple parameter of integration $ks$ unlike the kernel of integral (12.2). This distinction leads to some distinctions in studies of integral (12.2) as compared with (7.15), although these distinctions are not fundamental. The essence of them lies in the fact that we cannot trivially replace the infinite convolution (12.3) by a finite sum of length $N_1$ as in (7.23) because of a slow convergence of series (12.4) and (12.5). If we estimate them trivially, in modulus, as in (7.23), then the remainder is of order $N^{-1}$ and it does not compensate the growth of order $N^2$ of the kernel in integral (12.2). Therefore, the "tail," which we discard trivially in (7.23), must be estimated more carefully with the help of spectral methods.

First we use again the function $\psi(m)$ from (7.26), and with the help of the decomposition (7.27) of unity, we decompose (12.3) into two parts

$$\zeta(w) = \zeta(w; \psi) + \zeta(w; 1 - \psi). \tag{12.6}$$

In accordance with this decomposition, we decompose integral (12.2) into two parts

$$J(w) = J(w; \psi) + J(w; 1 - \psi). \tag{12.7}$$

After that, we move the contour of the integral $J(w,\psi)$ to the left, to the line $\mathrm{Re}\, s = 0$, bypassing the pole $s = 0$ from the right along an arc of arbitrarily small radius:

$$J(w;\psi) = \frac{1}{2\pi i} \int\limits_{\mathrm{Re}\, s=0} \frac{\Gamma(w_1+s)\Gamma(-s)}{\Gamma(w_1)l^{-s}} \zeta(w+s;\psi)\, ds. \tag{12.8}$$

Further estimates of the double average $S$ and $J$ (or $S$ and $\overline{J}$) of the product of integral (12.8) at the point $w = z_{j_0}$ and the matrix coefficient (7.6) are similar to the estimates of integral (7.28).

The second integral in (12.7) is as follows:

$$J(w;1-\psi) = \frac{1}{2\pi i} \int\limits_{\mathrm{Re}\, s=1} \frac{\Gamma(w_1+s)\Gamma(-s)}{\Gamma(w_1)l^{-s}} \zeta(w_1+s;1-\psi) ds. \tag{12.9}$$

The analog of this integral in Sec. 7 is in the remainder for a trivial reason, but here we do not have such a triviality, because the kernel does not contain the large parameter $k$.

In order to estimate integral (12.9) correctly, we recall that, by definition, the contour $\mathrm{Re}\, s = 1$ bypasses the pole of $\Gamma(-s)$ at the point $s = 1$ from the left along an arc of arbitrarily small radius. Therefore, moving the contour of (12.9) to the right, to the line $\mathrm{Re}\, s = k$, where $k > 2$ is the same parameter as in the definition of the function $\psi$ form (7.26), we obtain the relation

$$J(w;1-\psi) = -\sum_{n=1}^{k-1} \frac{(-1)^n}{n!} \frac{\Gamma(w_1+n)}{\Gamma(w_1)}\, l^n \zeta(w+n;1-\psi) + J_k(w;1-\psi). \tag{12.10}$$

In (12.10), $J_k$ is the integral (12.9) over the contour $\mathrm{Re}\, s = k$ with the same bypass of the pole $s = k$ from the left along an arbitrarily small arc (recall that $k$ is an integer). For the kernel of the integral $J_k(w;1-\psi)$, an analog of estimate (7.22) is valid. Therefore, the double average of it at the point $w = z_{j_0}$ multiplied by coefficient (7.6) is no greater than $0(1)$.

The double average $S$ and $J$ of the product of the principal part of (12.10) by coefficient (7.6) also has order $0(1)$, but we should estimate it more carefully, using the spectral expansion of each component of $\zeta(w+n;1-\psi)$ (which we did not do in Sec. 7).

From (12.3) it follows that

$$\zeta(w+n;1-\psi) = {}^t\!\left( \overset{0}{\zeta}_{+l},\, \overset{\infty}{\zeta}_{-l} \right)(w+n;1-\psi). \tag{12.11}$$

The components of (12.11) are absolutely convergent series (12.4) and (12.5) supplied with the weight function $[1-\psi(m)]$, i.e., "empty" at the beginning. In order to apply spectral methods to them, we need to strengthen the convergence properties of the weight functions of series (12.11) at infinity. To this end, we call attention to the fact that for $w = z_{j_0} = \frac{1}{2} + i\varkappa_{j_0}$, the series of vector (12.11) can be cut off for $m \geq \mu_n$, where

$$\mu_n = N_1^{\frac{n}{n-1/2}}, \quad n \geq 1. \tag{12.12}$$

But a simple cut-off is inconvenient for the spectral technique, which works well for smooth functions. Therefore, we again use the decomposition of unity:

$$1 = \varphi_n(m) + [1-\varphi_n(m)], \tag{12.13}$$

where

$$\varphi_n(m) = \exp\left[-\left(\frac{m}{\mu_n}\right)^{2k}\right],$$

$\mu_n$ is in (12.12), $k > 2$ is the same parameter as in (12.10).

With the help of (12.13), we can write vector (12.11) in the form of a sum

$$\zeta(w+n; 1-\psi) = \zeta\big(w+n; (1-\psi)\varphi_n\big) + \zeta\big(w+n; (1-\psi)(1-\varphi_n)\big). \tag{12.14}$$

Substituting (12.14) into (12.10) for $w = z_{j_0}$, we have the possibility of discarding the sum over $n$ with the second summand from (12.14) for a trivial reason. After that, (12.10) takes the form

$$J(w; 1-\psi) = -\sum_{n=1}^{k-1} \frac{(-1)^n}{n!} \frac{\Gamma(w_1+n)}{\Gamma(w_1)} l^n \zeta(w+n; (1-\psi)\varphi_n) + 0(1). \tag{12.15}$$

In order to estimate the double average $J$ and $S$ (or $\overline{J}$ and $S$) of the product of each term in (12.15) at the point $w = z_{j_0}$ by coefficient (7.6), we can use the spectral expansion in the same way as was done in Secs. 7–11 with the total estimate $0(1)$. Moreover, the accuracy of the square root of the principal term is needed only for the first term of sum (12.15) for $n = 1$; with increasing $n$, we need a lower accuracy of spectral estimates and for $n = k$ a trivial estimate in the moduli of the summands is sufficient.

Consider the second vector $r(w)$ of sum (12.1). From the definition (0.20) in [4], it follows that

$$r(w) = {}^t\Big(\overset{\infty}{r}_{+l}, \overset{0}{r}_{-l}\Big)(w), \tag{12.16}$$

where

$$\overset{\infty}{r}_{+l}(w) = \sum_{m=1}^{l-1} \frac{\sigma_0(m)\sigma_{1-2s_1}(l-m)}{(l-m)^{w-s_1+1/2}},$$

$$\overset{0}{r}_{-l}(w) = \sum_{m=1}^{l-1} \frac{\sigma_0(m)\sigma_{1-2s_1}(l-m)}{(m)^{w-s_1+1/2}}.$$

The double average $J$ and $S$ of the product of convolutions (12.16) and coefficient (7.6) is studied by the method from Secs. 7–11 with a total estimate $0(q^\varepsilon)$. The only distinction is related to the analog of the second sum in (10.2) with condition $m \equiv 0(q)$, which requires smoothing the sum at the end $n = l$ with the help of a compactly supported function on a short interval of length $\sqrt{N}$ instead of the standard length $N^{2/3}$. Nevetheless, on the average, we obtain the limit estimate by the method from [10], because the right end of sum (12.16) is equal to the parameter $l$ over which averaging proceeds.

It remains to estimate the last vector $r_\tau(w)$ in (12.1). The explicit form of it follows from (0.21) in [4] for $q = 1$, $\chi \equiv 1$, and $s_2 = 1/2$:

$$r_\tau(w) = -\frac{1}{2} l^{-w} \tau_{s_1}(l)\, {}^t(1,0); \quad w = z_{j_0}. \tag{12.17}$$

From the right-hand side in (12.17), it follows that the double average $J$ and $S$ (or $\overline{J}$ and $S$) of the product of $r_\tau(z_{j_0})$ and coefficient (7.6) is of order

$$\frac{\log N}{\sqrt{N}}.$$

Thus, we have completed the proof of the estimate

$$t_{j_0}(q) \ll q^{\varepsilon}, \tag{12.18}$$

where $\varepsilon > 0$ is arbitrarily small but fixed, $q = p^{\nu}$, $\nu \to \infty$. The Petersson conjecture

$$| \, t_{j_0}(p) \, | \leq 2$$

follows from (12.18) in the standard way (see [7]).

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