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COMPARISON OF SPECTRA. III

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ABSTRACT

The Selberg spectral conjecture is proved for spaces of automorphic functions of weight 1/2.

Dedicated to the 75-th
birthday of I. R. Shafarevich

Introduction

In [1], I, the Selberg spectral conjecture was proved for congruence groups $\Gamma_0(q)$ without a multiplier; in [1], II, this conjecture was proved for congruence groups with a nontrivial multiplier.

In the present paper, we prove one more version of this conjecture, namely, for spaces of weight 1/2. Two previous versions were related to spaces of weight 0. The main idea of the proof is the same as in [1], I, II, namely, we compare two spectral expansions of one and the same convolution. However, in this case we consider an esentially different convolution:

$$\zeta_{q,l}(w,s)=\sum_{n=1}^{\infty}\frac{\tau_s(n^2+l;\Gamma)}{(\sqrt{n^2+l})^w}, \tag{0.1}$$

where $l\ge 1$ is a fixed integer; $q\equiv 0 \pmod 4$ is the congruence modulus of the group $\Gamma=\Gamma_0(q)$; $\tau_s(m,\Gamma)$ is an arithmetic Fourier coefficient of the Eisentein series of weight 0 of the group $\Gamma=\Gamma_0(q)$:

$$E(z,s)=y^s+\frac{\xi(2s-1)}{\xi(2s)}\,\sigma_{1-2s}(0,\Gamma)y^{1-s}+$$
$$+\frac{2\sqrt{y}}{\xi(2s)}\sum_{m\ne 0}\tau_s(m,\Gamma)\,K_{s-\frac12}(2\pi|m|y)\,e^{2\pi imx} \tag{0.2}$$

For $m\ne 0$, the explicit form of $\tau_s(m,\Gamma)$ is as follows:

$$\tau_s(m,\Gamma)=|m|^{s-\frac12}\,\sigma_{1-zs}(m,\Gamma) \tag{0.3}$$

$$\sigma_{1-2s}(m,\Gamma)=\frac{1}{q^{2s}}\sum_{k|(q,m)}k\mu\left(\frac{q}{k}\right)\prod_{p|\frac{q}{k}}\left(1-\frac{1}{p^{2s}}\right)^{-1}\sigma_{1-2s}\left(m,\frac{q}{k}\right) \tag{0.4}$$

$$\sigma_s(m,q)=\sum_{\substack{d|m,\\(d,q)=1,\\d>0}}d^s.$$

For $m=0$,

$$\sigma_{1-2s}(0,\Gamma)=\frac{\varphi(q)}{q^{2s}}\prod_{p|q}\left(1-\frac{1}{p^{2s}}\right)^{-1}. \tag{0.5}$$

For details of expansion (0.2), see paper [6].

The parameters s and w in (0.1) are arbitrary complex numbers satisfying the condition

$$\operatorname{Re}s\ge 0,\quad \operatorname{Re}w>\operatorname{Re}s+1. \tag{0.6}$$

They ensure the absolute convergence of series (0.1).

We also recall that $\xi(s)$ is a normalized Riemann zeta function:

$$\xi(s)=\pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

which is invariant under the change $s \to (1-s)$:

$$\xi(s) = \xi(1-s).$$

The analytic extension of convolution (0.1) into the critical strip with respect to the parameter w for $\operatorname{Re} s = \frac{1}{2}$ is studied in detail in paper [2], which is published as a PDMI preprint. In [2], only the case of the group $SL(2, \mathbb{Z})$ and the classical function of the number of divisors $\tau(m) = \sum\limits_{d|m} 1$ is considered. But from the text of [2], it is clear that all ideas can be carried over without difficulties to the general function $\tau_s(m, q)$ from definition (0.4) and to the congruence groups $\Gamma_0(q)$. In [2], the spectral expansion in a space of weight zero is used, where the Selberg conjecture is proved (see [1]). Therefore, for any fixed s on the line $\operatorname{Re} s = \frac{1}{2}$, the meromorphic expansion of convolution (0.1) in the critical strip $\operatorname{Re} w \in (0, 1)$ has singularities only on the line $\operatorname{Re} w = \frac{1}{2}$, excluding the trivial poles on the line $\operatorname{Re} w = 1$ generated by the continuous spectrum, i.e., by the Eisentein series of all vertices of the fundamental domain F_q of the group $\Gamma_0(q)$. Consequently, it remains to us to obtain the spectral expansion of convolution (0.1) in a space of weight 1/2 and to compare its singularities with the expansion from [2]. This will imply the Selberg conjecture for the weight 1/2.

1. An integral convolution

In order to obtain the spectral expansion of convolution (0.1) of weight 1/2, we need to express it in terms of the integral convolution over the unit strip $\Pi = \{z = x + iy,\ |x| \le \frac{1}{2},\ y > 0\}$ of the upper half-plane $H = \{z = x + iy,\ y > 0\}$:

$$\zeta^*_{q,l}(w, s) = \iint_{\Pi} E(z, s)\theta(z) y^{\frac{w+1}{2}}\, e^{2\pi i l z}\, d\mu(z), \tag{1.1}$$

where $E(z, s)$ is the Eisentein series from (0.2), $d\mu(z) = \frac{dx\,dy}{y^2}$ is an invariant differential, $\theta(z)$ is the Jacobi function:

$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{2\pi i z n^2}. \tag{1.2}$$

Note that this is the classical Jacobi theta function of the doubled argument: $\theta(z) = \theta_1(2z)$, where

$$\theta_1(z) = \sum_{n=-\infty}^{\infty} e^{\pi i z n^2}. \tag{1.3}$$

Substituting series (0.2) and (1.2) into (1.1) and changing the orders of summation and integration (which is valid in domain (0.6)), we obtain an explicit expression for integral convolution (1.1):

$$\zeta^*_{q,l}(w, s) = 2\Phi_q(w, s) \left[\zeta_{q,l}(w, s) + \frac{1}{2} \frac{\tau_s(l, \Gamma)}{(\sqrt{l})^w} \right], \tag{1.4}$$

where $\zeta_{q,l}(w, s)$ is the initial convolution (0.1) and $\Phi_q(w, s)$ is a normalizing coefficient:

$$\Phi_q(w, s) = \frac{2q^s}{\varphi_{2s}(q)} \frac{\Gamma\left(\frac{w-1}{2} + s\right)\Gamma\left(\frac{w+1}{2} - s\right)}{(\sqrt{2})^w \xi(2s) \Gamma\left(\frac{w+1}{2}\right)} \tag{1.5}$$

$$\varphi_s(q) = q^s \prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

Inverting relation (1.4), we derive an integral representation of the initial convolution:

$$\zeta_{q,l}(w,s) = \frac{1}{2}\,\Phi_q^{-1}(w,s)\zeta^*_{q,l}(w,s) - \frac{1}{2}\,\frac{\tau_s(l,\Gamma)}{(\sqrt{l})^w}. \tag{1.6}$$

This relation enables us to extend analytically our convolution to the left half-plane $\operatorname{Re} w \le 1$, using automorphic properties of the kernel of the integral representation (1.1). To this end, we must transform the unit strip Π onto the fundamental domain F_q of the group $\Gamma = \Gamma_0(q)$ and obtain from (1.1) a new relation:

$$\zeta^*_{q,l}(w,s) = \iint_{F_q} E(z,s)\theta(z)P_l^*(z,w)d\mu(z), \tag{1.7}$$

where $P_l^*(z,w)$ is a Poincaré series of weight 1/2:

$$P_l^*(z,w) = \sum_{\sigma\in\Gamma_\infty\setminus\Gamma} y^{\frac{w+1}{2}}(\sigma z)\,\frac{\theta(\sigma z)}{\theta(z)}\,e^{2\pi i l\sigma(z)}. \tag{1.8}$$

The ratio of the theta functions in sum (1.8) is the symbol of the Jacobi modular coefficient in the relation

$$\theta(\sigma z) = \varepsilon(\sigma)\sqrt{cz+d}\,\theta(z) \tag{1.9}$$

for

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \setminus \Gamma,$$

where $\varepsilon(\sigma)$ is an arithmetic multiplier whose explicit form can be found in papers [4, 5]. We do not give it here because of its cumbersome form, but, in main, this is the sign of the Gauss sum of the modulus $c \equiv 0 \pmod q$ and the parameter $a = a(\sigma)$. But the square of it is very simple:

$$\varepsilon^2(\sigma) = \chi_4(d).$$

In order that the factor $\sqrt{cz+d}$ in (1.9) be unique, we introduce the additive normalization

$$-\pi < \arg(cz+d) \le \pi. \tag{1.10}$$

This condition means that we cut the complex plane along the real negative semiaxis.

On such a plane with a cut, the following relation is valid:

$$\sqrt{cz+d} = \exp\left[\frac{i}{2}\,\arg(cz+d)\right]\sqrt{|cz+d|}. \tag{1.11}$$

Moreover, we imply that condition (1.10) is equivalent to the condition

$$-\frac{\pi}{2} < \arg\sqrt{cz+d} < \frac{\pi}{2}.$$

Substituting (1.11) into (1.9), we obtain

$$\frac{\theta(\sigma z)}{\theta(z)} = \varepsilon(\sigma, z)\sqrt{|cz+d|}, \tag{1.12}$$

where

$$\varepsilon(\sigma, z) = \varepsilon(\sigma)\exp\left[\frac{i}{2}\arg(cz+d)\right]. \tag{1.13}$$

In order to express $\varepsilon(\sigma, z)$ in (1.12), we recall that

$$y(\sigma z) = \frac{y}{|cz+d|^2}, \quad \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \setminus \Gamma.$$

From this relation, we have a consequence:

$$\left(\frac{y(\sigma z)}{y}\right)^{\frac{1}{4}} = \frac{1}{\sqrt{|cz+d|}}.$$

Multiplying this relation with (1.12) termwise, we finally obtain

$$\frac{\theta(\sigma z)}{\theta(z)}\left(\frac{y(\sigma z)}{y}\right)^{\frac{1}{4}} = \varepsilon(\sigma, z). \tag{1.14}$$

This is the normalized multiplier generated by the Jacobi theta function, which we shall deal with in the sequel, in the space of weight 1/2.

Relation (1.14) immediately implies the following modular relation for $\varepsilon(\sigma, \sigma_0 z)$, for any $\sigma_0 \in \Gamma_\infty \setminus \Gamma$:

$$\varepsilon(\sigma, \sigma_0 z) = \bar{\varepsilon}(\sigma_0, z)\varepsilon(\sigma\sigma_0, z). \tag{1.15}$$

The bar means complex conjugation.

Now we use relation (1.14) to normalize the Poincaré series (1.8). First we represent it in the form

$$y^{\frac{w+1}{2}}(\sigma z) = y^{\frac{2w+1}{4}}(\sigma z)y^{\frac{1}{4}}(\sigma z)$$

and combine the factor $y^{\frac{1}{4}}(\sigma z)$ with the ratio of theta functions in sum (1.8), dividing it preliminary by $\sqrt[4]{y}$. As a result, the Poincaré series (1.8) takes a new form:

$$P_l^*(z, w) = \sqrt[4]{y}\, P_l(z, w_1), \tag{1.16}$$

where

$$P_l(z, w_1) = \sum_{\sigma \in \Gamma_\infty \setminus \Gamma} y^{w_1}(\sigma z)\varepsilon(\sigma, z)e^{2\pi i l \sigma(z)}, \quad w_1 = \frac{2w+1}{4}. \tag{1.17}$$

From this definition and relation (1.15), we obtain an invariant relation for the new Poincaré series:

$$P_l(\sigma_0 z, w_1) = \bar{\varepsilon}(\sigma_0, z)P_l(z, w_1) \tag{1.18}$$

for any $\sigma_0 \in \Gamma_\infty \setminus \Gamma$.

Substituting relation (1.16) into (1.7), we arrive at an integral representation of the convolution with automorphic kernel:

$$\zeta^*_{q,l}(w,s) = \iint_{F_q} E(s,z)f(z)P_l(z,w_1)d\mu(z), \tag{1.19}$$

where

$$f(z) = \sqrt[4]{y}\,\theta(z). \tag{1.20}$$

We use this representation in order to expand convolution (0.1) with respect to the spectrum of the space of weight 1/2, and, thus, this enables us to extend (0.1) into the critical strip in a new way different from the expansion in [2]. Comparing the singularities of these two expansions of convolution (0.1), we obtain the Selberg conjecture for the weight 1/2.

At the end of this section, we note that we simplified the symbols of multipliers from [3], regarding the Jacobi modular coefficient as a whole. We took its normalized value (1.14) as a total multiplier without subdividing it into two parts as is done in [3, relation (5)]; such a definition of a multiplier enables us to exclude a cumbersome symbol (3) in [3]. But this simplification holds only for the weight 1/2. For a general weight, the Jacobi transform is absent; therefore, the complicated symbols of a multiplier in [3] remain valid for a general weight.

2. Spectral expansion of the Poincaré series

Our next step is the expansion of the Poincaré series from (1.17) and (1.18) in eigenfunctions of an invariant differential operator in a space of weight 1/2. The details of this theory can be found in papers [7–9]. All necessary results of these papers are contained in the preprint [3]. Therefore, we stop here only on one detail associated with the invariant relation (1.18).

A "valid" invariant relation for a generic function $f(z)$ of weight 1/2 is the relation

$$f(\sigma z) = \varepsilon(\sigma,z)f(z). \tag{2.1}$$

The Jacobi function $\theta(z)$ normalized by the factor $\sqrt[4]{y}$ in (1.20) satisfies such a relation. Therefore, relation (1.18) is "invalid" because it contains the conjugate factor $\bar{\varepsilon}(\sigma,z)$ in place of $\varepsilon(\sigma,z)$. This means that the function $f(z)$ and the Poincaré series $P_l(z,w_1)$ belong to different spaces generated by the complex conjugate operators

$$\Delta_{\pm k} = \Delta_0 \pm ik\,y\,\frac{\partial}{\partial x}, \quad k = \frac{1}{2}, \tag{2.2}$$

where Δ_0 is the classical Laplace operator

$$\Delta_0 = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \tag{2.3}$$

Both operators (2.2) are selfadjoint and, in principle, they are equivalent. But conventionally, Δ_{+k} is regarded as a direct ("proper") operator, and Δ_{-k} is considered as a complex conjugate to it.

Since Δ_{+k} is selfaddjoint, it follows that all eigenvalues of Δ_{+k} are real and positive (for Laplace operator (2.3)):

$$\Delta_{+k}\,f_j(z) = \lambda_j\,f_j(z), \quad \lambda_j > 0. \tag{2.4}$$

We do not consider the trivial case $\lambda_j = 0$ and $f_j(z) \equiv 0$.

From the complex conjugation of relation (2.4):

$$\Delta_{-k}\,\overline{f_j(z)} = \lambda_j \overline{f_j(z)}, \quad \lambda_j > 0, \tag{2.5}$$

if follows that the space of the operator Δ_{-k} consists of the same functions as the space of Δ_{+k}, but they are complex conjugate with respect to the parameter z. Therefore, the Poincaré series from (1.18) has the same expansion in eigenfunctions that can be found in [8] (see also [3]), but all eigenfunction are complex conjugate:

$$P_l(z, w_1) = \frac{1}{2}\sum_{\alpha}\frac{1}{2\pi i}\int_{\mathrm{Re}\,\omega=\frac{1}{2}} a_{\alpha,l}(\omega)\,\overline{E}_{\alpha,\frac{1}{2}}(z,\omega)d\omega + \sum_{j=1}^{\infty} a_{j,l}(w_1)\overline{f_j(z)}, \tag{2.6}$$

where $E_{\alpha,\frac{1}{2}}(z,\omega)$ is the Eisentein series of weight 1/2 of the vertex α. We endow it with the subscript 1/2 in order to distinguish it from the Eisentein series of weight zero in (0.2). For details of the structure of the series $E_{\alpha,\frac{1}{2}}$, see [3, 8, 9].

The coefficient $a_{\alpha,l}(\omega)$ in the kernel of integrals (2.6) has the form

$$a_{\alpha,l}(\omega) = \iint_{F_q} E_{\alpha,\frac{1}{2}}(z,\omega)\,P_l(z,w_1)d\mu(z). \tag{2.7}$$

The coefficient $a_{j,l}(w_1)$ of the discrete part of the spectral expansion has the form

$$a_{j,l}(w_1) = \iint_{F_q} f_j(z)\,P_l(z,w_1)d\mu(z). \tag{2.8}$$

These relations are consequences of the orthonormality of our basis:

$$\iint_{F_q} \overline{f_{j_1}(z)}\,f_{j_2}(z)\,d\mu(z) = \begin{cases} 1, & j_1 = j_2 \\ 0, & j_1 \neq j_2 \end{cases},$$

which is the same for the operators $\Delta_{\pm k}$, $k = \frac{1}{2}$.

The eigenfunction, in kernels (2.7) and (2.8) satisfy the "direct" invariant relation (2.1), while the Poincaré series $P_l(z, w_1)$ satisfies the inverse of it from (1.18). Therefore, both integrals can be "un folded" from F_q to the unit strip Π, using the Poincaré series as a "protector." As a result, (2.8) can be transformed to the form

$$a_{j,l}(w_1) = \iint_{\Pi} y^{w_1}\,f_j(z)\,e^{2\pi i l z}\,d\mu(z). \tag{2.9}$$

A similar transformation is valid for integral (2.7), which we omit for brevity.

We consider the simplest case where a space of weight 1/2 is generated by a system of multipliers of the Jacobi theta function on the congruence group $\Gamma_0(q)$. Therefore, on the group of shifts $\Gamma_\infty\left\{\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}, n\in\mathbb{Z}\right\}$, the multiplier is trivial (see [3], relation (8) with $\varkappa_\infty = 0$). Therefore, the Fourier series for an eigenfunction $f_j(z)$ relative to any vertex α from F_q has the form

$$f_j(\sigma_\alpha z) = \sum_{n\neq 0}\rho_{j,\alpha}(n)\,\sqrt{y}\,W_{\frac{n}{|n|}\frac{1}{4};i\varkappa_j}(4\pi|n|y)\,e^{2\pi i n x} + \delta_j\,\rho_{j,\alpha}(0)\,y^{\frac{1}{2}+i\varkappa_j}, \tag{2.10}$$

where

$$\delta_j = \begin{cases} 1, & \lambda_j < 1/4 \\ 0, & \lambda_j \geq 1/4 \end{cases}.$$

The presence of the zeroth term in (2.10) for small eigenvalues $\lambda_j < 1/4$ is explained by the fact that, in this case, the quantity

$$i\varkappa_j = i\sqrt{\lambda_j - 1/4} = -\gamma_j, \quad \gamma_j > 0, \tag{2.11}$$

is a pure real negative number. Consequently, the zeroth term in (2.10) does not violate the condition that $f_j(z)$ is square-integrable on F_q if $\rho_{j,\alpha}(0) \neq 0$:

$$\iint_{F_q} |f_j(z)|^2 d\mu(z) < \infty.$$

The Fourier expansion of the Eisenstein series of weight 1/2 at the vertex α has the form

$$E_{\alpha,\frac{1}{2}}(z,\omega) = \delta_{\alpha,\infty} y^{\omega} + \varphi_{\alpha,0}(4\omega-1)\,\frac{\xi(4\omega-2)}{\xi(4\omega-1)}\,y^{1-\omega} +$$

$$+\sqrt{y}\sum_{n\neq 0}\varphi_{\alpha,n}(\omega)a_{\alpha,n}(y,\omega)\,e^{2\pi i x n}, \tag{2.12}$$

where $\varphi_{\alpha,n}(\omega)$ is an arithmetic Fourier coefficient, an analog of $\tau_s(n,\Gamma)$ from (0.2), and $a_{\alpha,n}(y,\omega)$ is the analytic part of it: $(n \neq 0)$

$$a_{\alpha,n}(y,\omega) = \frac{W_{\frac{n}{|n|}\frac{1}{4};\frac{1}{2}-\omega}(4\pi|n|y)}{\Gamma\left(\omega + \frac{n}{|n|}\frac{1}{4}\right)},$$

$\xi(s)$ is the normalized Riemann zeta function from the introduction.

In the general case, the coefficient $\varphi_{\alpha,n}(\omega)$ has a rather cumbersome form (see [3]). But for the vertex $\alpha = \infty$ and $n = 0$, it is rather simple:

$$\varphi_{\infty,0}(\omega) = 2^{1-\omega}\,\frac{\varphi(q)}{\varphi_\omega(q)}, \quad \varphi_\omega(q) = q^{\omega}\prod_{p|q}\left(1 - \frac{1}{p^{\omega}}\right).$$

The symbol $W_{k,m}(z)$ in (2.10) and (2.13) denotes the Whittaker function (a generalization of the MacDonald function from the theory of Bessel functions). In our case of weight 1/2, its explicit form is as follows:

$$W_{k,m}(z) = \frac{e^{-\frac{1}{2}z}z^k}{\Gamma\left(\frac{1}{2}-k+m\right)}\int_0^\infty t^{m-\frac{1}{2}-k}\left(1+\frac{t}{z}\right)^{k-\frac{1}{2}+m} e^{-t}\,dt. \tag{2.13}$$

For details, see [10], Chap. 16, §12.

The only detail that is absent in [10] and that must be taken into account in definition (2.13) is that this definition is valid only for small eigenvalues $\lambda_j < 3/16$ (the strict inequality). If $\lambda_j = 3/16$, then $m = i\varkappa_j = -\sqrt{\frac{1}{4} - \frac{3}{16}} = -\frac{1}{4}$, and for $k = \frac{1}{4}$ we have

$$\frac{1}{2} - k + m = 0, \quad m - \frac{1}{2} - k = -1, \tag{2.13^a}$$

which leads to an indeterminacy of the form $0 \otimes \infty$ in definition (2.13). In order to resolve this indeterminacy, we need to extend this integral analytically to the point (2.13^a). To this end, we put

$$\frac{1}{2} - k + m = \varepsilon,$$

where $\varepsilon > 0$ is arbitrarily small but fixed; then we integrate (2.13) once by parts and let ε tend to $+0$. In the limit, we obtain the value of integral (2.13) at point (2.13^a):

$$W_{\frac{1}{4},-\frac{1}{4}}(z) = e^{-\frac{1}{2}z} z^{\frac{3}{4}} \int_0^\infty \left(1 + \frac{1}{2(t+z)}\right) \frac{e^{-t}dt}{\sqrt{t+z}}. \tag{2.13^b}$$

After this brief digression, we turn again to relation (2.10). Substituting (2.10) into (2.9) and interchanging summation and integration (which is well defined), we obtain an explicit form of the coefficient of the discrete part of the spectral expansion of $P_l(z, w_1)$:

$$a_{j,l}(w_1) = \frac{\rho_j(-l)}{(4\pi l)^{w_1-\frac{1}{2}}} \frac{\Gamma\left(w_1 - \frac{1}{2} - i\varkappa_j\right)\Gamma\left(w_1 - \frac{1}{2} + i\varkappa_j\right)}{\Gamma\left(w_1 + \frac{1}{4}\right)}. \tag{2.14}$$

Coefficient (2.7) of the continuous part of the spectral expansion can be obtained in a similar way:

$$a_{\alpha,l}(\omega) = \frac{e^{-\frac{\pi}{4}i}\pi^{\omega}}{(4\pi l)^{w_1-\frac{1}{2}}} \frac{\Gamma(w_1-\omega)\Gamma(w_1+\omega-1)}{\Gamma\left(\frac{1}{4}+\omega\right)\Gamma\left(\frac{1}{4}+w_1\right)} \varphi_{\alpha,-l}(\omega). \tag{2.15}$$

Having the spectral expansion of the Poincaré series, we can apply it in order to obtain the spectral expansion of the convolution.

3. Spectral expansion of the convolution

In order to obtain the spectral expansion of convolution (1.19), we need to substitute the spectral expansion of Poincaré series (2.6) into the kernel of integral (1.19). But if we do this directly, then subtle questions arise concerning the convergence at the vertices of F_q for the continuous components from (2.6) and for discrete components of the functions $f_j(z)$ which belong to small eigenvalues $\lambda_j < 1/4$, because in the two cases, the Fourier series contain the zeroth Fourier coefficients. These difficulties can be overcome with the help of the Selberg method of disengaging vertices, which we use but in a smooth version, which is more convenient here.

To this end, we add a smooth weight to the initial Eisentein series (0.2) and consider this series in the right half-plane $\operatorname{Re} s > 1$, where it absolutely converges as a series over a group:

$$E^{(\varepsilon)}(z,s) = \sum_{\sigma\in\Gamma_\infty\backslash\Gamma} y^s(\sigma z)\exp[-\varepsilon\, y(\sigma z)], \tag{3.1}$$

where $\varepsilon > 0$ is a fixed number in the interval

$$0 < \varepsilon \le e^{-16}. \tag{3.2}$$

In the sequel we let ε tend to $+0$.

Next, we use the following integral representation:

$$\exp[-\varepsilon\, y(\sigma z)] = \frac{1}{2\pi i}\int_{\operatorname{Re}\Delta=\Delta_0>0} \varepsilon^{-\Delta}\, y^{-\Delta}(\sigma z)\,\Gamma(\Delta)\,d\Delta. \tag{3.3}$$

Substituting this relation into (3.1) and choosing a contour $\mathrm{Re}\,\Delta = \Delta_0$ in such a way that the condition $\mathrm{Re}\,(s-\Delta_0) > 1$ is satisfied, we obtain the right-hand side of (3.1) in the form of an integral mean:

$$E^{(\varepsilon)}(z,s) = \frac{1}{2\pi i}\int_{\substack{\mathrm{Re}\,(s-\Delta_0)>1\\ \mathrm{Re}\,\Delta=\Delta_0>0}} \varepsilon^{-\Delta}\, E(z,s-\Delta)\Gamma(\Delta)\,d\Delta. \tag{3.4}$$

Eisenstein series (0.2) occurs in the kernel of integral (3.4). The condition $\mathrm{Re}\,(s-\Delta_0) > 1$ ensures that its pole at the point $s-\Delta = 1$ lies to the right of the line $\mathrm{Re}\,\Delta = \Delta_0$. Therefore, moving this contour to the left, to the line $\mathrm{Re}\,\Delta = -1$, we meet only one pole of the function $\Gamma(\Delta)$ at the point $\Delta = 0$, by the Cauchy theorem on residues, we transform (3.4) to a new form:

$$E^{(\varepsilon)}(z,s) = E(z,s) + \frac{1}{2\pi i}\int_{\mathrm{Re}\,\Delta=+1} \varepsilon^{\Delta}\, E(z,s+\Delta)\Gamma(-\Delta)d\Delta. \tag{3.5}$$

By the way, we changed the sign of the variable of integration Δ (simultaneously changing the direction of integration), and, thus, we redenoted the contour $\mathrm{Re}\,\Delta = -1$ by $\mathrm{Re}\,\Delta = +1$, bypassing the pole $\Delta = 1$ of the function $\Gamma(-\Delta)$ along a left arc of radius

$$r = \left(\log\frac{1}{\varepsilon}\right)^{-1} \leq \frac{1}{16}. \tag{3.6}$$

This inequality follows from condition (3.2).

We note that the integral over the contour in (3.5) remains regular as a function of s, in the half-plane $\mathrm{Re}\,s > r$, because in this half-plane, the Eisenstein series $E(z,s+\Delta)$ from the kernel in (3.5) lies in the right half-plane $\mathrm{Re}\,(s+\Delta) > 1$, where it converges absolutely as a series over a group.

In order to use (3.5), we rewrite it with respect to the initial Eisenstein series $E(z,s)$:

$$E(z,s) = E^{(\varepsilon)}(z,s) - \frac{1}{2\pi i}\int_{\mathrm{Re}\,\Delta=+1} \varepsilon^{\Delta}\, E(z,s+\Delta)\,\Gamma(-\Delta)d\Delta. \tag{3.7}$$

Substituting this relation into (1.19), we obtain

$$\zeta^*_{q,l}(w,s) = \iint_{F_q} E^{(\varepsilon)}(z,s)\, f(z)\, P_l(z,w_1)d\mu(z) -$$

$$-\frac{1}{2\pi i}\int_{\mathrm{Re}\,\Delta=1} \varepsilon^{\Delta}\, \zeta^*_{q,l}(w,s+\Delta)\,\Gamma(-\Delta)d\Delta. \tag{3.8}$$

In the first integral of the right-hand side in (3.8), we use the spectral expansion of Poincaré series (2.6), because here the vertex ∞ is disengaged with the help of the weight Eisenstein series (3.1) and there are no problems with convergence, which we mentioned above.

In the kernel of the second integral in (3.8) with contour $\mathrm{Re}\,\Delta = 1$, we use representation (1.19) for the convolution $\zeta^*_{q,l}(w,s+\Delta)$, in which the Poincaré series $P_l(z,w_1)$ is written as a series over a group in the half-plane $\mathrm{Re}\,w_1 > 1$. From this writing, the following estimate holds:

$$P_l(\sigma_\alpha z, w_1) \ll y^{1-\mathrm{Re}\,w_1}, \tag{3.9}$$

which is valid for any vertex α of the domain F_q in the half-plane $\mathrm{Re}\, w_1 > 1$. If we sharpen this condition as follows:

$$\mathrm{Re}\, w_1 > \mathrm{Re}\, s + \frac{3}{2} \geq \mathrm{Re}\,(s+\Delta) + \frac{1}{2}, \tag{3.10}$$

where Δ is a variable on the contour (3.8), and take into account the estimate of the Eisenstein series of the vertex ∞ at all vertices:

$$E(\sigma_\alpha^{-1} z, s+\Delta) \ll \delta_{\infty,\alpha}\, y^{\mathrm{Re}\,(s+\Delta)} + y^{1-\mathrm{Re}\,(s+\Delta)},$$

then from (1.19) we obtain an estimate:

$$\zeta^*_{q,l}(w, s+\Delta) \ll 1,$$

which is valid in the half-plane $\mathrm{Re}\, s > r$. Therefore, in this half-plane, the second integral in (3.8) has order of magnitude

$$\varepsilon \log \frac{1}{r}. \tag{3.11}$$

Substituting the spectral expansion of the Poincaré series (2.6) into the first integral in (3.8) and interchanging summation and integration (which is valid in the region $\mathrm{Re}\, s > 1$), we rewrite the first integral in (3.8) as a sum of integrals of spectral functions:

$$\frac{1}{2} \sum_\alpha \frac{1}{2\pi i} \int_{\mathrm{Re}\,\omega = \frac{1}{2}} a_{\alpha,l}(\omega)\, J_\alpha^{(\varepsilon)}(s,\omega) d\omega$$

$$+ \sum_{j=1}^{\infty} a_{j,l}(w_1)\, J_j^{(\varepsilon)}(s), \tag{3.12}$$

where

$$J_\alpha^{(\varepsilon)}(s,\omega) = \iint_{F_q} E^{(\varepsilon)}(z,s)\, f(z) \overline{E_{\alpha,\frac{1}{2}}(z,\omega)}\, d\mu(z) \tag{3.13}$$

and, similarly,

$$J_j^{(\varepsilon)}(s) = \iint_{F_q} E^{(\varepsilon)}(z,s)\, f(z) \overline{f_j(z)}\, d\mu(z). \tag{3.14}$$

Recall that, by definition (1.20), $f(z) = \sqrt[4]{y}\, \theta(z)$, and, consequently, integrals (3.13) and (3.14) are smoothed convolutions of Jacobi theta functions and spectral functions. Therefore, our next problem is to obtain an explicit form of these convolutions and to extend them into the left half-plane $\mathrm{Re}\, s \leq 1$.

4. Continuous components. An explicit form and extendability

In order to obtain an explicit form of component (3.13), it is necessary to "unfold" it from the domain F_q to the unit strip Π, using the smoothed Eisenstein series $E^{(\varepsilon)}(z,s)$ as a protector. As a result, relation (3.13) takes the form

$$J_\alpha^{(\varepsilon)}(s,\omega) = \iint_{\Pi} y^{s+\frac{1}{4}}\, e^{-\varepsilon\, y}\, \theta(z) \overline{E_{\alpha,\frac{1}{2}}(z,\omega)}\, d\mu(z). \tag{4.1}$$

In (4.1), the smooth weight $\exp(-\varepsilon\, y)$ plays the role of the suppressing exponent $\exp(-2\pi l y)$ in (1.1) for $y \to \infty$; without this, integral (4.1) ceases to converge on the product of the zeroth

Fourier coefficients of the series $\theta(z)$ and the series for $\overline{E_{\alpha,\frac{1}{2}}(z,\omega)}$. The zeroth Fourier coefficient of the Jacobi series is equal to 1; therefore, the above product is equal to the zeroth Fourier coefficient of the Eisenstein series $\overline{E_{\alpha,\frac{1}{2}}(z,\omega)}$. By condition, $\mathrm{Re}\,\omega = 1/2$, i.e., $\bar{\omega} = 1-\omega$, therefore it is equal to

$$\delta_{\alpha,\infty}\, y^{1-\omega} + y^{\omega} \frac{\xi(2-4\omega)}{\xi(3-4\omega)}\, \varphi_{\alpha,0}(3-4\omega). \tag{4.2}$$

Consequently, the local part of integral (4.1), where the product $\theta(z)\overline{E_{\alpha,\frac{1}{2}}(z,\omega)}$ in the kernel is replaced by (4.2), is equal to

$$\delta_{\alpha,\infty}\, \varepsilon^{-\left(s-\omega+\frac{1}{4}\right)}\, \Gamma\left(s-\omega+\frac{1}{4}\right)$$

$$+(\cdot)\varepsilon^{-\left(s+\omega-\frac{3}{4}\right)}\, \Gamma\left(s+\omega-\frac{3}{4}\right). \tag{4.3}$$

Recall that $\mathrm{Re}\,s > 1$, therefore the passage from the local part of integral (4.1) to its explicit form (4.3) is well defined, because $\mathrm{Re}\,\omega = \frac{1}{2}$. The coefficient $(\cdot)$ in (4.3) is the same as at the second term in (4.2). Substituting (4.3) into the kernel of the integral of the continuous component in (3.12) for the vertex α and moving the contour $\mathrm{Re}\,\omega = \frac{1}{2}$ for the first term in (4.3) to the right, to the line

$$\mathrm{Re}\left(s-\omega+\frac{1}{4}\right) = -\frac{1}{2}, \tag{4.4}$$

we meet the pole at the point $\omega = s+\frac{1}{4}$ with residue

$$\delta_{\alpha,\infty}\cdot a_{\alpha,l}\left(s+\frac{1}{4}\right). \tag{4.5}$$

Moving the contour $\mathrm{Re}\,\omega = \frac{1}{2}$ of the second term in (4.3) to the left, to the line

$$\mathrm{Re}\left(s+\omega-\frac{3}{4}\right) = -\frac{1}{2}, \tag{4.6}$$

we meet a pole at the point $\omega = \frac{3}{4}-s$ with residue

$$\varphi_{\alpha,0}(4s)\frac{\xi(4s-1)}{\xi(4s)}\, a_{\alpha,l}\left(\frac{3}{4}-s\right). \tag{4.7}$$

Consequently, the principal term of the continuous component (3.12) of the contribution (4.3) is equal to the sum of residues (4.5) and (4.7) over all vertices α:

$$\frac{1}{2}\, a_{\infty,l}\left(s+\frac{1}{4}\right) + \frac{1}{2}\, \frac{(\xi(4s-1)}{\xi(4s)} \sum_{\alpha} \varphi_{\alpha,0}(4s)\, a_{\alpha,l}\left(\frac{3}{4}-s\right). \tag{4.8}$$

The remaining parts over the contours (4.4) and (4.6) vanish as $\varepsilon \to +0$.

The calculation of the explicit form of the remaining part of integral (4.1) creates no problems, because all of the remaining terms of the product $\theta(z)\overline{E_{\alpha,\frac{1}{2}}(z,\omega)}$ converge exponentially as $y \to$

∞, and the role of the suppressing factor $\exp(-\varepsilon\, y)$ is trivial here. Therefore, in the limit as $\varepsilon \to +0$, this part is equal to

$$b_{1-\omega}(s) \sum_{n=1}^{\infty} \frac{\varphi_{\alpha,n^2}(1-\omega)}{n^{2s-\frac{1}{2}}}, \tag{4.9}$$

where φ_{α,n^2} are arithmetic coefficients from (2.12),

$$b_\omega(s) = \frac{\Gamma\left(s+\frac{1}{4}-\omega\right)\Gamma\left(s-\frac{3}{4}+\omega\right)}{\Gamma(s)}. \tag{4.10}$$

The extension of the Rankin–Dirichlet series from (4.9) into the left half-plane $\operatorname{Re} s \le 1$ is accomplished rather simply by using the explicit form of the coefficients φ_{α,n^2}. Moreover, they are much simpler for the squares, because in this case the Gaussian sums are trivial. This is a standard calculation and we omit it. Consequently, the total contribution of the continuous part of the spectrum to the expansion of the convolution $\zeta^*_{q,l}(s,w)$ has the form

$$\frac{1}{2}\, a_{\infty,l}\left(s+\frac{1}{4}\right) + \frac{1}{2}\,\frac{\xi(4s-1)}{\xi(4s)} \sum_{\alpha} \varphi_{\alpha,0}(4s) a_{\alpha,l}\left(\frac{3}{4}-s\right)$$

$$+\frac{1}{2}\sum_{\alpha} \frac{1}{2\pi i}\int_{\operatorname{Re}\omega=\frac{1}{2}} a_{\alpha,l}(\omega) b_{1-\omega}(s) L_{\alpha,1-\omega}\left(2s-\frac{1}{2}\right) d\omega, \tag{4.11}$$

where

$$L_{\alpha,\omega}(s) = \sum_{n=1}^{\infty} \frac{\varphi_{\alpha,n^2}(\omega)}{n^s}. \tag{4.12}$$

Recall that the explicit form of the coefficient $a_{\alpha,l}(\omega)$ from (2.15) depends on the parameter w_1, consequently, the integrals in (4.11) are analytic functions of this parameter, regular in the half-plane $\operatorname{Re} w_1 > \frac{1}{2}$. In order to extend them into the left half-plane $\operatorname{Re} w_1 \le \frac{1}{2}$, we need to use the Sokhotski–Plemejl theorem.

5. Discrete components. An explicit form and extendability

The convergence of integrals (3.14) for large eigenvalues $\lambda_j \ge 1/4$ does not depend on the parameter ε, because in these cases, the eigenfunctions $f_j(z)$ decrease exponentially at all vertices of the domain F_q. Therefore, we consider small eigenvalues $\lambda_j < 1/4$, for which the zeroth coefficient arises in (2.10), which requires careful treatment as in the case of Eisenstein series.

By analogy with the passage from integral (3.13) to (4.1), we pass from (3.14) to an integral over the unit strip:

$$J_j^{(\varepsilon)}(s) = \iint_{\Pi} y^{s+\frac{1}{4}}\, e^{-\varepsilon\, y}\, \theta(z)\overline{f_j(z)}\, d\mu(z). \tag{5.1}$$

When multiplying $\theta(z)$ and $\overline{f_j(z)}$ termwise in the kernel of (5.1), an integral on the product of the zeroth Fourier coefficients $\theta(z)$ and $\overline{f_j(z)}$ stands out:

$$\int_0^{\infty} y^{s-\frac{1}{4}+i\varkappa_j}\, e^{-\varepsilon\, y} \frac{dy}{y} = \varepsilon^{-\left(s-\frac{1}{4}-\gamma_j\right)}\, \Gamma\left(s-\frac{1}{4}-\gamma_j\right), \tag{5.2}$$

where $\gamma_j = -i\varkappa_j \in \left(0, \frac{1}{4}\right]$. Therefore, relation (5.2) is well defined in the half-plane $\operatorname{Re} s > 1$.

The remaining part of integral (5.1) is trivial with respect to the parameter ε and is equal to

$$b_j(s)\sum_{n=1}^{\infty}\overline{\rho_j(n^2)}/n^{2s-\frac{1}{2}}+O(\sqrt{\varepsilon}), \tag{5.3}$$

where

$$b_j(s)=\frac{\Gamma\left(s-\frac{1}{4}-i\varkappa_j\right)\Gamma\left(s-\frac{1}{4}+i\varkappa_j\right)}{\Gamma(s)}. \tag{5.4}$$

In addition, the estimate of the remainder in (5.3) is uniform with respect to ε in the half-plane $\operatorname{Re} s\ge\delta$ with any fixed $\delta>0$.

Hence it follows that for the extendability of the Rankin–Dirichlet series in (5.3) into the left half-plane $\operatorname{Re} s\le 1$, into this half-plane it is necessary to extend integral (3.14) that is equal to integral (5.1) in the domain $\operatorname{Re} s>1$.

To this end, we use the Selberg method of disengaging vertices but in a simplified version (with no functional equation). For this purpose, we write the asymptotic behavior of all functions of the kernel of integral (3.14) at each vertex α of the domain F_q.

Recall that $f(z)=\sqrt[4]{y}\,\theta(z)$; therefore, from the Jacobi transform formula we have

$$f(\sigma_\alpha z)=e_\alpha\sqrt[4]{y}+O(e^{-\pi y})\quad\text{as}\quad y\to\infty,$$

where e_α is a constant dependent on α. Similarly, from (2.10) we obtain

$$\overline{f_j(\sigma_\alpha z)}=\overline{\rho_{j,\alpha}(0)}\,y^{\frac{1}{2}+i\varkappa_j}+O(e^{-\pi y})\quad\text{as}\quad y\to\infty.$$

For $\alpha=\infty$, we have

$$E^{(\varepsilon)}(z,s)=y^s e^{-\varepsilon y}+c_\infty(s)\,y^{1-s}+O(\varepsilon\,y^{-\sigma}),$$

where $\sigma=\operatorname{Re} s$.

For the remaining vertices $\alpha\ne\infty$, we have

$$E_\alpha^{(\varepsilon)}(z,s)=c_\alpha(s)y^{1-s}+O(\varepsilon\,y^{-\sigma})\quad\text{as}\quad y\to\infty.$$

Therefore, integral (3.14) is equal to the sum

$$\iint_{F_q(N)}E^{(\varepsilon)}(z,s)\,f(z)\overline{f_j(z)}\,d\mu(z)+\varepsilon^{-\left(s-\frac{1}{4}-\gamma_j\right)}\,\Gamma\left(s-\frac{1}{4}-\gamma_j,\varepsilon N\right)$$

$$+\frac{N^{\frac{3}{4}-\gamma_i-s}}{s-\frac{3}{4}+\gamma_j}\sum_\alpha e_\alpha\overline{\rho_{j,\alpha}(0)}\,c_\alpha(s)+O(N^{-\sigma-\frac{1}{4}}), \tag{5.5}$$

where $F_q(N)$ is a finite part of the fundamental domain F_q with cut-off vertices at a height $y=N>1$; $\Gamma(s,x)$ is an incomplete Γ-function. The estimate of the remainder in (5.5) is uniform with respect to N in the half-plane $\operatorname{Re} s\ge 0$.

Comparing expansions (5.3) and (5.5) for one and the same integral (3.14), we see that the Rankin–Dirichlet series

$$\overline{L}_j\left(2\overline{s}-\frac{1}{2}\right)=\sum_{n=1}^{\infty}\overline{\rho_j(n^2)}/n^{2s-\frac{1}{2}} \tag{5.6}$$

can be extended meromorphically up to the line $\operatorname{Re} s = 0$ with poles at the points $s_j = \frac{3}{4} - \gamma_j$ with residues

$$\sum_{\alpha} e_\alpha \, \overline{\rho_{j,\alpha}(0)} \, c_\alpha(s_j), \tag{5.7}$$

adding one more pole at the point $s = 1$ of the c-matrix of the Eisenstein series $E^{(\varepsilon)}(z, s)$ with residue

$$\frac{6}{\pi^2} \iint_{F_q} f(z) \, \overline{f_j(z)} \, d\mu(z). \tag{5.8}$$

In fact, these poles are poles of product (5.3), i.e., of series (5.6) by the function $b_j(s)$. Consequently, in order to obtain the residues of series (5.6), it is necessary to divide (5.7) by $b_j(s_j)$ and (5.8) by $b_j(1)$. Therefore, we did not mention the pole $s = \frac{1}{4} + \gamma_j$ from (5.2), because it is a pole of the function $b_j(s)$ but not series (5.6). This also implies that the point $s = \frac{1}{4} - \gamma_j$ is a zero of series (5.6), because it is the second pole of the function $b_j(s)$, at which the whole expression (5.5) is regular.

Therefore, product (5.3) can be extended meromorphically up to the line $\operatorname{Re} s = 0$.

Consequently, in order to solve the problem of analytic extension of the convolution $\zeta^*_{q,l}(w, s)$ with respect to the parameter s into the critical strip $\operatorname{Re} s \in (0, 1]$, we need to extend (3.12) to the left up to the line $\operatorname{Re} s = \frac{1}{4} - \delta$ for any $\delta \in \left(0, \frac{1}{4}\right)$ and then we let ε tend to $+0$. In the limit, on this line we obtain an explicit form of the spectral expansion of the convolution as a sum of the continuous component (4.11) and a discrete part of the form

$$\sum_{j=1}^{\infty} a_{j,l}(w_1) \, b_j(s) \overline{L}_j \left(2\bar{s} - \frac{1}{2}\right). \tag{5.9}$$

Further, we need to extend analytically this representation to the right, to the line $\operatorname{Re} s = \frac{1}{2}$ with condition $s \neq \frac{1}{2}$; i.e., $s = \frac{1}{2} + it$, $t \neq 0$. As a result, this implies the Selberg conjecture, provided that $\rho_j(-l) \neq 0$ and $L_j\left(2s - \frac{1}{2}\right) \not\equiv 0$. These two facts will be established in the next sections.

6. The Hecke operators

In order to solve the two problems stated at the end of the preceding section of the present paper, we need the theory of Hecke operators on the squares of integers, by analogy with paper [5] by Shimura for finite-dimensional spaces. This theory is carried over to infinite-dimensional Hilbert spaces of weight 1/2 in paper [11] in terms of the metaplectic group of A. Weil. A higher weight in [5] ($\geq 5/2$) is related with the guarantee of absolute convergence of Poincaré and Eisenstein series in finite-dimensional classical spaces. In our case, the convergence of these series is ensured by the spectral theory of automorphic functions. The reason of the action of Hecke operators only on the squares of integers is one and the same as in [5] and in [11]: the Gaussian sums are multiplicative only on the moduli that are equal to squares of integers and lose this property in the general case. Therefore, by analogy with [5], the Shimura relation holds in the language of the metaplectic group (see [11]):

$$(T_{p^2} \cdot f_j)(z) = t_j(p) \cdot f_j(z) \tag{6.1}$$

In our case, this relation is guaranteed by the commutative relation for the operators Δ_k and T_{p^2} in our space:

$$\Delta_k \cdot T_{p^2} = T_{p^2} \cdot \Delta_k \tag{6.2}$$

The details of constructing the Hecke operators in spaces of half integral weight are presented in [5, 11]. In our case, the weight factor of the operator T_{p^2} is equal to $1/p$, because $k = 1/2$ and we renormalized the space to the zero weight.

Relations (6.1) and (6.2) are carried over to any integral square by multiplicativity. As a result, we obtain a factorization into the Euler product of the Shimura series $L_j(s)$ from (5.6), which arose there as a Rankin convolution:

$$L_j(s_1) = \rho_j(1) \cdot \prod_p \left[1 - \left(\frac{-1}{p}\right) \cdot \frac{1}{p^{s_1}}\right] \left[1 - \frac{t_j(p)}{p^{s_1}} + \frac{1}{p^{2s_1}}\right] \tag{6.3}$$

We mean that the passage from series (5.6) to the Euler product (6.3) is carried out in the region of absolute convergence $\operatorname{Re} s_1, > 1$ and $s_1 = 2s - 1/2$.

Factorization (6.3) implies the difference from zero of the Rankin–Shimura convolution (5.6), provided that $\rho_j(1) \neq 0$.

Thus, our problem of the residue from Sec. 5 (the difference of it from zero) reduces to the fulfillment of the following two conditions:

$$\rho_j(-l) \neq 0 \quad \text{and} \quad \rho_j(1) \neq 0. \tag{6.4}$$

In the sequel, an eigenfunction $f_j(z)$ is said to be nondegenerate if it has at least one nonzero square-free Fourier coefficient both of negative and positive number:

$$\begin{aligned} &\rho_j(-\nu_1) \neq 0, \quad \nu_1 \geq 1, \quad \mu^2(\nu_1) = 1 \\ &\rho_j(+\nu_2) \neq 0, \quad \nu_2 \geq 1, \quad \mu(\nu_2) = 1 \end{aligned} \tag{6.5}$$

Since the Hecke operators are multiplicative on the squares, it follows that the general coefficient of any number $\pm\nu \cdot n^2$, $\nu \geq 1$, $\mu^2(\nu) = 1$, splits into two factors:

$$\rho_j(\pm\nu \cdot n^2) = \rho_j(\pm\nu) \cdot t_j(n) \tag{6.6}$$

Therefore, if all square-free coefficients $\rho_j(\pm\nu)$, $\nu \geq 1$, $\mu^2(\nu) = 1$, are equal to zero, then we have a complete degeneracy, i.e., $f_j(z) \equiv 0$.

If all negative square-free coefficients are equal to zero:

$$\rho_j(-\nu_1) = 0, \quad \nu_1 \geq 1, \quad \mu^2(\nu_1) = 1,$$

then we call such a function left degenerate. For such a function, at least one positive square-free number $\nu_2 \geq 1$, $\mu^2(\nu_2) = 1$, exists, for which the second condition in (6.5) is satisfied; otherwise, the function is completely degenerate.

If all positive square-free coefficients are equal to zero: $\rho_j(+\nu_2) = 0$, $\nu_2 \geq 1$, $\mu^2(\nu_2) = 0$, then such a function is called right degenerate. For this function, at least one negative square-free coefficient $\rho_j(-\nu_1)$ exists, for which the first condition from (6.5) is satisfied.

Such a complicated gradation of eigenfunctions with respect to the square-free coefficients is explained by the fact that in half integral weights, there is no complete theory of Hecke operators, and $\rho_j(1) \equiv 0$ does not imply that the function $f_j(z)$ is completely degenerate as in the case of an integral weight.

In connection with this gradation, (6.4) is a special case of a more general nondegenerate case (6.5).

7. The general convolution

If $\rho_j(1) = 0$, then we need to change convolution (0.1) a little and to consider a more general case:

$$\zeta_{\nu q,l}(w,s) = \sum_{n=1}^{\infty} \frac{\tau_s(\nu n^2 + l; \Gamma)}{\left(\sqrt{\nu n^2 + l}\right)^w} \tag{7.1}$$

with fixed $\nu \geq 1$, for which the second condition from (6.5) is satisfied. In the language of paper [2], convolution (7.1) can be expanded in the spectrum of a space of weight 0 with automorphy group $\Gamma_0(\nu q)$.

The validity of the Selberg conjecture for the weight 0 implies that a more general convolution (7.1) also has all nontrivial singularities on the line $\operatorname{Re} w = 1/2$ if $\operatorname{Re} s = 1/2$.

An integral analog of (1.1) for convolution (7.1) has in the kernel a more general Jacobi function $\theta(\nu z)$ instead of $\theta(z)$:

$$\zeta^*_{\nu q,l}(w,s) = \iint_{\Pi} E(z,s)\theta(\nu z)\, y^{\frac{w+1}{2}} e^{2\pi i l z} d\mu(z). \tag{7.2}$$

The Jacobi function $\theta(\nu z)$ is modular relative to the group $\Gamma_0(4\nu)$ (see [4]), and the same Eisenstein series $E(z,s)$ as in (0.1) remains automorphic relative to the group $\Gamma_0(q)$, where $q \equiv 0 \pmod 4$. Therefore, the product $E(z,s)\theta(\nu z)$ in the kernel (7.1) is modular relative to the group $\Gamma_0(q\nu)$.

Let

$$\sigma_{q\nu} = \begin{pmatrix} * & * \\ cq\nu & * \end{pmatrix} \in \Gamma_0(q\nu)$$

and

$$\sigma_q = \begin{pmatrix} * & * \\ cq & * \end{pmatrix} \in \Gamma_0(q);$$

then it is easily seen that the following relation is valid:

$$\theta(\nu\sigma_{q\nu}(z)) = \theta(\sigma_q(\nu z)). \tag{7.3}$$

Indeed, let $\sigma_{q\nu} = \begin{pmatrix} a & b \\ cq\nu & d \end{pmatrix}$, then

$$\nu\sigma_{q\nu}(z) = \nu\,\frac{az+b}{cq\nu z + d} = \frac{a(\nu z) + b\nu}{cq(\nu z) + d} = \sigma_q(\nu z),$$

where

$$\sigma_q = \begin{pmatrix} a & b\nu \\ cq & d \end{pmatrix}.$$

This implies that, in this case, an analog of (1.9) has the form

$$\theta(\nu\sigma_{q\nu}(z)) = \varepsilon(\sigma_q)\sqrt{cq\nu z + d}\,\theta(\nu z), \tag{7.4}$$

i.e., the multiplier $\varepsilon(\sigma_{q\nu})$ is trivial with respect to the parameter ν:

$$\varepsilon(\sigma_{q\nu}) = \varepsilon(\sigma_q).$$

Only the analytic part $\sqrt{cq\nu z + d}$ retains the dependence on ν.

This means that the new space of weight $k = 1/2$ of the group $\Gamma_0(q\nu)$ that is constructed with the help of an analog of relation (1.12):

$$\frac{\theta(\nu\sigma_{q\nu}(z))}{\theta(\nu z)} = \varepsilon(\sigma_{q\nu}, z) \cdot \sqrt{|cq\nu z + d|}$$

contains our previous space of the group $\Gamma_0(q)$ as a subspace, because (7.4) implies the relation

$$\varepsilon(\sigma_{q\nu}, z) = \varepsilon(\sigma_q, \nu z).$$

Therefore, all the functions of the discrete spectrum of the first space are contained among the eigenfunctions of the new space as "old forms." Hence it follows that if a certain small eigenvalue $\lambda_j < 1/4$ from the spectrum of the first space has a nondegenerate eigenfunction $f_j(z)$ such that $\rho_j(-l) \neq 0$ and $\rho_j(\nu) \neq 0$ with some fixed square-free integers $l \geq 1$ and $\nu \geq 1$, then we take namely these parameters in convolution (7.1).

In this case, the analog of the Shimura relation (6.3) is a more general relation of the same type:

$$L_j(s_1) = \rho_j(\nu) \prod_p \left[1 - \left(\frac{-\nu}{p}\right) \frac{1}{p^{s_1}}\right] \left[1 - \frac{t_j(p)}{p^{s_1}} + \frac{1}{p^{2s_1}}\right]. \tag{7.5}$$

From the condition $\rho_j(\nu) \neq 0$, it follows that in this case, the Shimura–Rankin series is nontrivial, i.e., it is not equal to zero identically. Therefore, the residues of the two singularities that correspond to a small eigenvalue $\lambda_j < 1/4$ and lie on the real line symmetrically about the point $1/2$ are also different from zero.

Therefore, if the multiplicity of this eigenvalue is equal to one, then, comparing two spectral expansions, we obtain a contradiction; therefore, a small eigenvalue of multiplicity one also does not exist for the weight $k = 1/2$.

If a small eigenvalue has multiplicity greater than one, i.e., two or more eigenfunctions belong to it in the initial space of weight $1/2$ of the group $\Gamma_0(q)$, or several new eigenfunctions are added to it in the new space of the group $\Gamma_0(q\nu)$, then the total residue is equal to the sum of the residues of all these functions. It is much more difficult to prove that such a residue is different from zero. But we can do this by using the pairwise orthogonality of all eigenfunctions on the fundamental domain. To this end, we write the explicit form of the total residue:

$$V_{j,\nu}(l) = \sum_{k=1}^{r_j} \rho_{j,k}(-l)\overline{\rho_{j,k}(\nu)}\, \overline{L^*_{j,k}} \left(2s - \frac{1}{2}\right). \tag{7.6}$$

Here we use the double numeration of eigenfunctions, where the first subscript j means the number of the eigenvalue λ_j to which the function $f_{j,k}(z)$ and its Fourier coefficients belong; the second subscript k means the number of this function among all eigenfunctions belonging to the eigenvalue $\lambda_j < 1/4$; r_j is the quantity of these functions, i.e., the multiplicity of the eigenvalue λ_j. The asterisk of the Rankin–Shimura series $L^*_{j,k}$ means a pure Euler product without Fourier coefficients $\rho_{j,k}(\nu)$:

$$L^*_{j,k}(s_1) = \prod_p \left[1 - \left(\frac{-\nu}{p}\right) \cdot \frac{1}{p^{s_1}}\right] \left[1 - \frac{t_{j,k}(p)}{p^{s_1}} + \frac{1}{p^{2s_1}}\right]. \tag{7.7}$$

It is rather easy to prove that sum (7.6) is not equal to zero identically if two or more Rankin–Shimura series (7.7) differ from one another as analytic functions. But we cannot ensure this. Therefore, in order to prove the condition

$$V_{j,\nu}(l) \neq 0 \tag{7.8}$$

consider the mean square of this residue over the parameter l:

$$W_\varphi(N) = \sum_{l=1}^{\infty} |V_{j,\nu}(l)|^2 \varphi\left(\frac{l}{N}\right), \tag{7.9}$$

where φ is a function guaranteeing the absolute convergence of series (7.9) and we choose its value in the sequel; $N \geq 1$ is an arbitrary fixed parameter.

For simplicity, we omit the subscripts j and ν on the left-hand side in (7.9) because they are fixed.

Substituting (7.6) into (7.9) and interchanging the orders of summation, we arrive at a new form of relation (7.9):

$$W_\varphi(N) = \sum_{k_1=1}^{r_j} \sum_{k_2=1}^{r_j} L_{j,k_1}(s_1)\overline{L_{j,k_2}}(s_1) W_{k_1,k_2}(N), \tag{7.10}$$

where

$$L_{j,k}(s_1) = \rho_{j,k}(\nu) L^{(*)}_{j,k}(s_1)$$

$$W_{k_1,k_2}(N) = \sum_{l=1}^{\infty} \rho_{j,k_1}(-l)\overline{\rho_{j,k_2}(-l)}\,\varphi\left(\frac{l}{N}\right). \tag{7.11}$$

In the sequel, for sum (7.11) we shall obtain an equality of the form

$$W_{k_1,k_2}(N) = \begin{cases} c(\lambda_j)N + \theta N^{1-2\gamma_j+\varepsilon}, & k_1 = k_2 \\ 0 + \theta N^{1-2\gamma_j+\varepsilon}, & k_1 \neq k_2 \end{cases}, \tag{7.12}$$

where

$$c(\lambda_j) > 0, \quad \theta \ll 1, \quad \gamma_j = -i\varkappa_j = \sqrt{\frac{1}{4} - \lambda_j} > 0,$$

$\varepsilon > 0$ is arbitrary small but fixed.

Asymptotics (7.12) is a consequence of the orthonormality of the eigenfunctions $f_{j,k}(z)$ on the fundamental domain $F_{q\nu}$.

Substituting (7.12) into (7.10), we obtain

$$W_\varphi(N) = c(\lambda_j)N \sum_{k=1}^{r_j} |L_{j,k}(s_1)|^2 + \theta N^{1-2\gamma_j+\varepsilon}. \tag{7.13}$$

From (7.12) it follows that for each fixed $\nu \geq 1$ with $\rho_{j,k}(\nu) \neq 0$ (for some k), there are infinitely many $l \geq 1$ such that $\rho_{j,k}(-l) \neq 0$.

The Rankin–Shimura series (7.7) is an analytic function extended to the line $\operatorname{Re} s_1 = \frac{1}{2}$. Therefore, on this straight line, its zeros form a discrete set; consequently, we can always choose a value of $s_1 = \frac{1}{2} + it$ such that $L_{j,k}(s_1) \neq 0$. Therefore, there always exists $l \geq 1$ such that condition (7.8) is satisfied. But this is a contradiction to the spectral expansion of convolution (7.1) in a space of weight 0 (see [2]). Thus, the Selberg conjecture holds for the weight 1/2 if asymptotics (7.12) is valid.

We proceed to the proof of this fact. To this end, we must extend the Rankin convolution of an eigenfunction $f_j(z)$ to the left half-plane.

8. The Rankin convolutions for small eigenvalues

The Rankin integral convolution of the vertex α

$$R_{j,\alpha}(s) = \iint_{\Pi} |f_j(\sigma_\alpha z)|^2\, y^s\, d\mu(z) \tag{8.1}$$

is well defined by this relation in the half-plane $\mathrm{Re}\, s > 1$ only for a large eigenvalue $\lambda_j \geq \frac{1}{4}$ to which the function $f_j(z)$ belongs.

If $f_j(z)$ belongs to a small eigenvalue $\lambda_j < 1/4$, then the definition of convolution (8.1) ceases to have meaning for the same reason as for the Eisenstein series; the zeroth Fourier coefficient of its Fourier series

$$f_j(\sigma_\alpha z) = \rho_{j,\alpha}(0) y^{\frac{1}{2}+i\varkappa_j} + \sum_{n\neq 0} (\cdot) e^{2\pi i n x} \tag{8.2}$$

is an obstacle to this definition.

The kernel of (8.2) is the same as in (2.10). For convenience, we cite it here again:

$$(\cdot) = \rho_{j,\alpha}(n)\sqrt{y}\, W_{\frac{1}{4}\frac{n}{|n|}, i\varkappa_j}(4\pi|n|y). \tag{8.3}$$

Therefore, we change the definition of convolution (8.1) for small eigenvalues, introducing a cut unit strip Π_N:

$$\Pi_N = \{z = x + iy,\ |x| \leq \frac{1}{2},\ 0 < y \leq N\} \tag{8.4}$$

$N \geq 1$ is an arbitrary fixed parameter. The convolution

$$R^{(N)}_{j,\alpha}(s) = \iint_{\Pi_N} |f_j(\sigma_\alpha z)|^2 y^s d\mu(z), \tag{8.5}$$

is well defined for a small eigenvalue $\lambda_j < 1/4$.

Sunstituting (8.2) into (8.5) and assuming that $\mathrm{Re}\, s > 1$, we obtain an explicit form of this convolution:

$$R^{(N)}_{j,\alpha}(s) = \frac{N^{s-2\gamma_j}}{s-2\gamma_j} |\rho_{j,\alpha}(0)|^2 + \sum_{n\neq 0} J^{(N)}_{j,n}(s) \frac{|\rho_{j,\alpha}(n)|^2}{(4\pi|n|)^s}, \tag{8.6}$$

where

$$J^{(N)}_{j,n}(s) = \int_0^{4\pi|n|N} W^2_{\frac{1}{4}\frac{n}{|n|}, -\gamma_j}(y)\, y^{s-1} dy, \tag{8.7}$$

$\gamma_j = -i\varkappa_j = \sqrt{1/4 - \lambda_j}$.

If all the parameters of the W-function are real, then the W-function itself is real; therefore, in the kernel of integral (8.7) we replaced the square of its modulus merely by the square. From the definition of integral (8.7), we have the following asymptotic relation:

$$J^{(N)}_{j,n}(s) = J_{j,\frac{n}{|n|}}(s) + \theta \exp[-2\pi|n|N], \tag{8.8}$$

where

$$J_{j,\frac{n}{|n|}}(s) = \int_0^\infty W^2_{\frac{1}{4}\frac{n}{|n|}, i\varkappa_j}(y)\, y^{s-1} dy. \tag{8.9}$$

Therefore, relation (8.6) can be written as an asymptotic relation with respect to the parameter N:

$$R_{j,\alpha}^{(N)}(s) = \frac{N^{s-2\gamma_j}}{s-2\gamma_j}\,|\rho_{j,\alpha}(0)|^2 + R_{j,\alpha}(s) + O(e^{-\pi N}), \tag{8.10}$$

where

$$R_{j,\alpha}(s) = J_{j,+1}(s)\,R_{j,\alpha}^{(+1)}(s) + J_{j,-1}(s)R_{j,\alpha}^{-1}(s) \tag{8.11}$$

$$R_{j,\alpha}^{(\pm1)}(s) = \sum_{n=1}^{\infty} \frac{|\rho_{j,\alpha}(\pm n)|^2}{(4\pi n)^s}. \tag{8.12}$$

The sign (±1) of the superscript on the left in (8.12) is made consistent with the sign $(\pm n)$ on the right in this relation.

The constant in the O-symbol of the right-hand side in (8.12) is uniformly bounded in any compact set of the s-plane $\mathbb{C}$. In order to extend convolution (8.5) to the critical strip, we need to observe that the difference between the unit strip Π and the cut strip Π_N (if they are regarded as sets) is simply a neighborhood of the vertex ∞:

$$\Pi - \Pi_N = V_\infty(N). \tag{8.13}$$

If $N \geq 1$, then the remaining $V_\alpha(N)$ obtained from (8.13) by the transformation $\sigma_\alpha(z)$ do not intersect. Representing (8.13) in the form

$$\Pi_N = \Pi - V_\infty(N),$$

we may observe that the folding of the cut strip Π_N to the fundamental domain F_q reduces to the folding of the whole strip Π without $V_\infty(N)$. Thus, relation (8.5) takes the form

$$R_{j,\alpha}^{(N)}(s) = \iint_{F_q} |f_j(z)|^2\, E_\alpha(z,s)d\mu(z) - \iint_{V_\infty(N)} |f_j(\sigma_\alpha z)|^2\, y^s\, d\mu(z). \tag{8.14}$$

The integral over the fundamental domain F_q in (8.14) is meant as a sum:

$$\iint_{F_q}(\cdot) = \iint_{F_q(N)}(\cdot) + \sum_{\beta\neq\alpha}\ \iint_{V_\beta(N)}(\cdot) + \iint_{V_\alpha(N)}(\cdot), \tag{8.15}$$

where $F_q(N)$ is the fundamental domain without vertices.

The integral over $V_\alpha(N)$ is transformed again to the vertex ∞ and is written as follows:

$$\iint_{V_\alpha(N)} = \iint_{V_\infty(N)} |f_j(\sigma_\alpha z)|^2\, E_\alpha(\sigma_\alpha z,s)d\mu(z). \tag{8.16}$$

For the Eisenstein series $E_\alpha(\sigma_\alpha z,s)$, we use the Fourier expansion:

$$E_\alpha(\sigma_\alpha z,s) = y^s + C_{\alpha,\alpha}(s)y^{1-s} + \sum_{n\neq0} \tau_{\alpha,n}^{(\alpha)}\sqrt{y}\,K_{s-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx}, \tag{8.17}$$

where $\tau_{\beta,n}^{(\alpha)}(s)$ is the arithmetic Fourier coefficient of number n, of the vertex α of the expansion of the Eisenstein series at the vertex β.

Comparing integral (8.16) with the integral over the neighborhood $V_\infty(N)$ on the right-hand side in (8.14), we see that it annihilates the corresponding part of integral (8.16) with the first zeroth Fourier coefficient in expansion (8.17).

Therefore, when rigorosly defining the right-hand side of (8.14), it is understood as the right-hand side of (8.15) in which the singularity is removed from the integral over the neighborhood $V_\alpha(N)$:

$$R^{(N)}_{j,\alpha}(s) = \iint_{F_q(N)} |f_j(z)|^2 E_\alpha(z,s)d\mu(z) + \sum_\beta \iint_{V_\beta(N)} |f_j(z)|^2[E_\alpha(z,s) - \delta_{\alpha,\beta}\, y^s(\sigma_\beta^{-1}z)]d\mu(z), \tag{8.18}$$

where $\delta_{\alpha,\beta}$ is the Kronecker delta.

We note that the sum over all vertices β on the right in (8.18) can be transformed to the form

$$\sum_\beta \iint_{V_\infty(N)} |f_j(\sigma_\beta z)|^2[E_\alpha(\sigma_\beta z,s) - \delta_{\alpha,\beta}\, y^s]d\mu(z), \tag{8.19}$$

then we use the Fourier series for the Eisenstein series E_α expanded with respect to the vertex β:

$$E_\alpha(\sigma_\beta z,s) = \delta_{\alpha,\beta}\, y^s + C_{\alpha,\beta}(s)y^{1-s} + \sum_{n\neq 0} \tau^{(\alpha)}_{\beta,n}(s)\sqrt{y}\, K_{s-\frac{1}{2}}(2\pi|n|y)e^{2\pi inx}.$$

As a result, we obtain an asymptotic behavior of sum (8.19) with respect to the parameter N:

$$\frac{N^{1-s-2\gamma_j}}{s+2\gamma_j-1} \sum_\beta |\rho_{j,\beta}(0)|^2 C_{\alpha,\beta}(s) + \theta\, e^{-\pi N}, \tag{8.20}$$

where $\theta \ll 1$.

Substituting (8.10) and (8.20) into (8.18), we rewrite this relation in the form

$$\iint_{F_q(N)} |f_j(z)|^2 E_\alpha(z,s)d\mu(z) = R_{j,\alpha}(s) + \frac{N^{s-2\gamma_j}}{s-2\gamma_j}\,|\rho_{j,\alpha}(0)|^2$$

$$-\frac{N^{1-s-2\gamma_j}}{s+2\gamma_j-1} \sum_\beta |\rho_{j,\beta}(0)|^2 C_{\alpha,\beta}(s) + \theta_\alpha\, e^{-\pi N}, \tag{8.21}$$

where $\theta_\alpha \ll 1$.

Combining relations (8.21) for all vertices α, we obtain a vector-matrix relation:

$$\iint_{F_q(N)} |f_j(z)|^2 \tilde{E}(z,s)d\mu(z) = \tilde{R}_j(s) + \frac{N^{s-2\gamma_j}}{s-2\gamma_j}\,\tilde{\rho}_j - \frac{N^{1-s-2\gamma_j}}{s+2\gamma_j-1}\,\Phi(s)\tilde{\rho}_j + \tilde{\theta}_1\, e^{-\pi N}, \tag{8.22}$$

where

$$\begin{aligned} \tilde{E} &= {}^t(E_{\alpha_1},\dots,E_{\alpha_\tau}),\\ \tilde{R}_j &= {}^t(R_{j,\alpha_1},\dots,R_{j,\alpha_\tau}),\\ \tilde{\rho}_j &= {}^t(|\rho_{j,\alpha_1}(0)|^2,\dots,|\rho_{j,\alpha_\tau}(0)|^2),\\ \tilde{\theta}_1 &= {}^t(\theta_{\alpha_1},\dots,\theta_{\alpha_\tau}); \end{aligned}$$

$\Phi(s)$ is the Selberg matrix of the zeroth coefficients $C_{\alpha,\beta}(s)$, τ is the number of vertices of the fundamental domain F_q. This relation is extended meromorphically to the left half-plane $\mathrm{Re}\, s \le 1$.

Carrying it over to the line $\mathrm{Re}\, s = \frac{1}{2}$, we use on this line the Selberg functional equation for a vector Eisenstein series:

$$\tilde{E}(z,s) = \Phi(s)\,\tilde{E}(z,1-s).$$

As a result, relation (8.22) takes the form

$$\Phi(s) \iint_{F_q(N)} |f_j(z)|^2\, \tilde{E}(z,1-s) d\mu(z) = \tilde{R}_j(s)$$

$$+\frac{N^{s-2\gamma_j}}{s-2\gamma_j}\,\tilde{\rho}_j - \frac{N^{1-s-2\gamma_j}}{s+2\gamma_j-1}\,\Phi(s)\,\tilde{\rho}_j + \tilde{\theta}_1\, e^{-\pi N}. \tag{8.23}$$

Now, we substitute (8.22) into the left-hand side of (8.23) changing s by $(1-s)$ and cancel similar terms on the right and on the left in the relation obtained. As a result, we obtain a new relation:

$$\Phi(s)\tilde{R}_j(1-s) = \tilde{R}_j(s) + \tilde{\theta}_1 e^{-\pi N}. \tag{8.24}$$

Now we let N tend to ∞, which yields the following functional equation for the vector Rankin convolution:

$$\tilde{R}_j(s) = \Phi(s)\tilde{R}_j(1-s). \tag{8.25}$$

When passing from (8.23) to (8.24), we used the fact that the Selberg matrix is unitary:

$$\Phi(s)\Phi(1-s) = 1$$

where 1 is the identity matrix.

One can define and study a more general convolution of two eigenfunctions in precisely the same way:

$$R^{(N)}_{j_1 j_2;\alpha}(s) = \iint_{\Pi_N} f_{j_1}(\sigma_\alpha z)\,\overline{f_{j_2}(\sigma_\alpha z)}\, y^s\, d\mu(z). \tag{8.5^a}$$

Convolution (8.5) is a special case of convolution (8.5^a) with $j_1 = j_2 = j$.

As $N \to \infty$, from (8.5^a) we obtain a generalization of convolution (8.11):

$$R_{j_1 j_2;\alpha}(s) = J_{j_1 j_2;+1}(s)\, R^{(+1)}_{j_1 j_2;\alpha}(s) + J_{j_1 j_2;-1}(s)\, R^{(-1)}_{j_1 j_2;\alpha}(s), \tag{8.11^a}$$

where

$$R^{(\pm 1)}_{j_1 j_2;\alpha}(s) = \sum_{n=1}^{\infty} \rho_{j_1,\alpha}(\pm n)\overline{\rho_{j_2,\alpha}(\pm n)}/(4\pi n)^s. \tag{8.12^a}$$

The signs (± 1) on the right and on the left are made consistent,

$$J_{j_1 j_2;\pm 1}(s) = \int_0^\infty W_{\pm\frac{1}{4}, i\varkappa_{j_1}}(y)\overline{W_{\pm\frac{1}{4}, i\varkappa_{j_2}}(y)}\, y^{s-1}\, dy. \tag{8.9^a}$$

This is a generalization of integral (8.9). In this relation the signs (± 1) are also made consistent.

If in relations (8.5^a)–(8.12^a) we have two function of one and the same eigenvalue $\lambda_j < \frac{1}{4}$, then, passing to the double numeration

$$\begin{aligned} f_{j_1}(z) &= f_{j,k_1}(z) \\ f_{j_2}(z) &= f_{j,k_2}(z), \end{aligned} \tag{8.26}$$

where the second subscript "k" means the number of the function $f_{j,k}(z)$ in the set of all functions belonging to λ_j, we can simplify integral (8.9^a) and write it again as integral (8.9), because it depends on the eigenvalue λ_j and does not depend on the subscript k:

$$J_{j_1 j_2;\pm 1}(s) \equiv J_{j,\pm 1}(s) = \int_0^\infty W^2_{\pm\frac{1}{4};-\gamma_j}(y) y^{s-1}\, dy. \tag{8.9^b}$$

For the general vector-convolution

$$\tilde{R}_{j_1 j_2}(s) = {}^t(R_{j_1 j_2,\alpha_1}, \dots, R_{j_1 j_2,\alpha_\tau})(s),$$

we also obtain the functional equation

$$\tilde{R}_{j_1 j_2}(s) = \Phi(s)\, \tilde{R}_{j_1 j_2}(1-s). \tag{8.25^a}$$

Equation (8.25) is a special case of this equation for $j_1 = j_2 = j$.

The absence of a pole at the point $s = 1$ is a unique essential distinction of convolution (8.11^a) with $j_1 \neq j_2$ from convolution (8.11) with $j_1 = j_2 = j$. This fact is a consequence of the orthonormality of the basis of eigenfunctions on F_q:

$$\iint_{F_q} f_{j_1}(z)\, \overline{f_{j_2}(z)}\, d\mu(z) = \begin{cases} 1, & j_1 = j_2 \\ 0, & j_1 \neq j_2. \end{cases} \tag{8.27}$$

In the case of two functions from (8.26), these conditions are transformed either to the equality $k_1 = k_2$, or to the condition $k_1 \neq k_2$.

9. Nonhomogeneous Rankin convolutions

In the preceding section of this paper, we continued the Rankin convolution (8.11) to the critical strip and singled out a pole of it at the point $s = 1$. Hence it follows the existence of infinitely many coefficients of it that are different from zero. But in order to prove that $f_j(z)$ is left and right nondegenerate, we must prove this fact separately for coefficients with positive numbers $\rho_j(n)$ and negative numbers $\rho_j(-n)$, $n \geq 1$. Moreover, from the concluding part of Sec. 7 it is seen that this fact is needed in the version of the asymptotics for the squares of the moduli of these coefficients.

In order to obtain these results, we need to extend the convolution with the derivative of the function $f_j(z)$

$$R^*_{j(s)} = \frac{1}{2\pi i} \iint_\Pi \overline{f_j(z)}\, \frac{\partial}{\partial x}\, f_j(z)\, y^{s+1} d\mu(z) \tag{9.1}$$

analytically to the critical strip. Substituting into the right-hand side of (9.1) the Fourier series (2.10) of the function $f_j(z)$ and its derivative:

$$\frac{\partial}{\partial x}\, f_j(x) = 2\pi i \sum_{n\neq 0} n \cdot \rho_j(n)\, \sqrt{y}\, W_{\frac{1}{4}\frac{n}{|n|}, i\varkappa_j}(4\pi|n|y) e^{2\pi i n x}, \tag{9.2}$$

we obtain an explicit form of convolution (9.1) in the region of absolute convergence $\mathrm{Re}\, s > 1$:

$$R_j^*(s) = J_{j,+1}(s+1)\, R_j^{(+1)}(s) - J_{j,-1}(s+1) R_j^{(-1)}(s). \tag{9.3}$$

Observe that the Fourier series of the derivative (9.1) converges exponentially at the vertex $\alpha = \infty$, therefore in (9.1) we took the whole unit strip Π. In order to fold it to the fundamental domain F_q in (9.1), we need to know automorphic properties of derivative (9.2). They are more complicated than similar properties of the function $f_j(z)$ itself. First we introduce two auxiliary functions:

$$\begin{aligned} F_{-j}(z) &= \frac{\partial}{\partial x} f_j(z) - i \frac{\partial}{\partial y} f_j(z) \\ F_{+j}(z) &= \frac{\partial}{\partial x} f_j(z) + i \frac{\partial}{\partial y} f_j(z) \end{aligned} \tag{9.4}$$

(see [1], vol. 228).

The definition of functions (9.4) implies an obvious relation:

$$\frac{\partial}{\partial x} f_j(z) = \frac{1}{2}[F_{-j}(z) + F_{+j}(z)]. \tag{9.5}$$

It is easy to verify that for function (9.4), the following automorphic relations are valid for any matrix $\sigma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_\infty \setminus \Gamma$:

$$\begin{aligned} \bar{\xi}(\sigma, z)\, F_{-j}(\sigma z) &= (cz+d)^2\, F_{-j}(z) + \frac{1}{2} c(cz+d) f_j(z) \\ \bar{\xi}(\sigma, z)\, F_{+j}(\sigma z) &= (c\bar{z}+d)^2\, F_{+j}(z) - \frac{1}{2} c(c\bar{z}+d) f_j(z). \end{aligned} \tag{9.6}$$

Further, we substitute relation (9.5) into the kernel of integral (9.1), subdivide it into two parts according to the two terms on the right-hand side in (9.5), and use the automorphic relation

$$\overline{f_j}(\sigma z) = \bar{\xi}(\sigma, z)\, \overline{f_j}(z)$$

and relations (9.6) in each integral when folding the strip Π to the fundamental domain F_q.

But here a difficulty arises that should be taken into account. Adding and subtracting the derivative $\frac{\partial}{\partial y} f_j(z)$ on the right in (9.4) and (9.5), we constructed the functions $F_{\pm j}(z)$ whose Fourier series have zeroth coefficients, because

$$\frac{\partial}{\partial y} f_j(z) = \left(\frac{1}{2} + i\varkappa_j\right) y^{-\frac{1}{2}+i\varkappa_j}\, \rho_j(0) + \sum_{n\neq 0} (\cdot)'_y\, e^{2\pi i n x}.$$

Therefore, strictly speaking, we cannot decompose integral (9.1) into two integrals according to relation (9.5), because each of them is divergent at the vertex ∞.

In order to overcome this difficulty, we preliminarily split the unit strip in integral (9.1) into two parts:

$$2\pi i R_j^*(s) = \left(\iint_{\Pi_N} + \iint_{V_\infty(N)}\right) (\cdot) d\mu(z) \tag{9.7}$$

with the same notation for Π_N and $V_\infty(N)$ as in Sec. 8 and with the same kernel as in (9.1). We leave unchanged the kernel of the integral over $V_\infty(N)$ in (9.7), and we continue to study the integral over the cut strip:

$$R^*_{j,N}(s) = \frac{1}{2\pi i} \iint_{\Pi_N} \overline{f_j}(z)\, \frac{\partial}{\partial x}\, f_j(z)\, y^{s+1} d\mu(z). \tag{9.8}$$

It is obvious that this is an analog of integral (8.5) for the vertex $\alpha = \infty$. Therefore, folding integral (9.8) to the finite fundamental domain $F_q(N)$ in the same way as in (8.18), we obtain

$$2\pi i R^*_{j,N}(s) = \iint_{F_q(N)} \overline{f_j}(z)\, E_j(z,s) d\mu(z) + \sum_\beta \iint_{V_\beta(N)} \overline{f_j}(z)$$

$$\times \left[E_j(z,s) - \delta_{\infty,\beta}\, y^{s+1} \frac{\partial}{\partial x}\, f_j(z) \right] d\mu(z), \tag{9.9}$$

where

$$E_j(z,s) = \sum_{\sigma \in \Gamma_\infty \backslash \Gamma} \bar{\xi}(\sigma, z)\, \frac{\partial}{\partial x_\sigma}\, f_j(\sigma z) y^{s+1}(\sigma z),$$

$$x_\sigma = x(\sigma z). \tag{9.10}$$

Substituting (9.5) and (9.6) into (9.10), we find an explicit form of this Eisenstein series:

$$E_j(z,s) = \frac{1}{2} F_{-j}(z)\, E^{(2)}(z,s) + \frac{1}{2}\, F_{+j}(z) E^{(2)}(\bar{z},s) + \frac{1}{4}\, f_j(z)\, [E^{(1)}(z,s) - E^{(1)}(\bar{z},s)], \tag{9.11}$$

where

$$E^{(2)}(z,s) = \sum_{\sigma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma} |y(\sigma z)|^{s+1} (cz+d)^2 \tag{9.12}$$

$$E^{(1)}(z,s) = \sum_{\sigma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma} |y(\sigma z)|^{s+1} c(cz+d). \tag{9.13}$$

The Eisenstein series $E^{(2)}(\bar{z},s)$ and $E^{(1)}(\bar{z},s)$ are defined by series (9.12) and (9.13) with $(cz+d)$ replaced by $(c\bar{z}+d)$; therefore, we used the symbol of modulus $|y(\sigma z)|$, because if $z \in H$ (the upper half-plane), then a transitive relation is valid:

$$|y(\sigma \bar{z})| = |y(\sigma z)| = y(\sigma z).$$

From definition (9.13) it follows that for the difference of Eisestein series with supercript (1) from (9.11), the following relation is valid:

$$\Delta E^{(1)}(z,s) = E^{(1)}(z,s) - E^{(1)}(\bar{z},s) = 2i\, y \sum_{\sigma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma} y^{s+1}(\sigma z) c^2. \tag{9.14}$$

From the definition of the right-hand side in (9.14), it follows that in the sum, the identity matrix $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is absent, because in this case $c = 0$. Therefore, the Fourier series of difference (9.14) in a neighborhood of the vertex $\alpha = \infty$ has the form

$$\Delta E^{(1)}(z,s) = i\, C_1(s)\, y^{1-s} + \frac{2iy}{s\xi(2s)} \sum_{n\neq 0} \tau_s(n,\Gamma)\sqrt{y}\, K_{s+\frac{1}{2}}(2\pi|n|y)e^{2\pi inx}, \tag{9.15}$$

where

$$C_1(s) = \left(2 - \frac{1}{s}\right) \frac{\varphi_1(q)}{\varphi_{2s}(q)} \frac{\xi(2s-1)}{\xi(2s)} \tag{9.16}$$

$$\varphi_s(q) = q^s \prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

We observe that the Fourier expansion (9.15) can be use at any other vertex α. If $z = x + iy \in V_\alpha(N) \subset F_q$ and $\alpha \neq \infty$, then for $y \to +0$ and $\operatorname{Re} s \in [0,1]$, the following estimate is valid:

$$y \cdot \sum_{n\neq 0} (\cdot) \ll 1, \tag{9.17}$$

where the kernel $(\cdot)$ in sum (9.17) is the same as in the sum in (9.15).

Estimate (9.17) can be obtained rather simply if $\operatorname{Re} s \neq \frac{1}{2}$. If $\operatorname{Re} s = \frac{1}{2}$, then we need to calculate the asymptotic behavior of sum (9.17) at the point $s = \frac{1}{2} + it$ for $t \neq 0$. The expression for this asymptotics consists of two parts. The first part is a function of the parameter $\left(s - \frac{1}{2}\right) = it$, and the second part is the same function of the parameter $\left(\frac{1}{2} - s\right) = -it$. Each of them has a pole at the point $t = 0$ with one and the same residue, but with different signs. Therefore, the singularity at the point $t = 0$ is solvable and we obtain regularity and an estimate of sum (9.17) at the point $s = \frac{1}{2}$. At the remaining points of the critical line (and the whole critical strip), regularity and an estimate of sum (9.17) can be obtained significantly simpler.

10. The Eisenstein series of weight 2

The Eisenstein series (9.12) can be normalized and written in the form

$$E^{(2)}(z,s) = y\, E(z,s;\xi_2), \tag{10.1}$$

where

$$E(z,s;\xi_2) = \sum_{\sigma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma} y^s(\sigma z)\xi_2(\sigma, z), \tag{10.2}$$

$$\xi_2(\sigma, z) = (cz+d)^2/|cz+d|^2. \tag{10.3}$$

In this notation, it is clear that series (10.2) is an Eisenstein series of weight 2 in terms of paper [3] with trivial arithmetic multiplier equal to 1.

Therefore, for series (9.12) with the conjugate parameter $\bar{z}$, relation (10.1) takes the form

$$E^{(2)}(\bar{z},s) = y\, E(z,s;\bar{\xi}_2), \tag{10.4}$$

where $\bar{\xi}_2$ is the complex conjugate of the weight function (10.3);

$$E(z,s;\bar{\xi}_2) = \sum_{\sigma=\begin{pmatrix} * & * \\ c & d\end{pmatrix}\in\Gamma_\infty\backslash\Gamma} y^s(\sigma z)\bar{\xi}_2(\sigma,z). \tag{10.5}$$

From definitions (10.2) and (10.5), it is clear that these Eisenstein series belong to complex conjugate differential operators (2.2) for $k=2$.

At the vertex $\alpha=\infty$, the Fourier expansion of Eisenstein series (10.2) is of the form

$$E(z,s;\xi_2) = y^s + \sigma_{1-2s}(0,\Gamma)\frac{\zeta(2s-1)}{\zeta(2s)}\,J_0^{(-1)}(s)y^{1-s} + \frac{y^{1-s}}{\zeta(2s)}\sum_{n\neq 0}\sigma_{1-2s}(n,\Gamma)\,J_n^{(-1)}(s)e^{2\pi i n x}. \tag{10.6}$$

The function $\sigma_{1-2s}(n,\Gamma)$ is defined in the introduction by relation (0.4) for $n\neq 0$ and by relation (0.5) for $n=0$:

$$J_n^{(-1)}(s) = \int_{-\infty}^{\infty}\frac{(t-i)^2}{(t^2+1)^{s+1}}\,e^{-2\pi i n y\,t}\,dt. \tag{10.7}$$

For $n=0$, from definition (10.7) we obtain an explicit form of this integral:

$$J_0^{(-1)}(s) = \frac{(1-s)}{s}\,\frac{\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}. \tag{10.8}$$

If $n\neq 0$, then the relation

$$\pi^{-s}\Gamma(s+1)J_n^{(-1)}(s)y^{1-s} = \frac{1}{2}|n|^{s-\frac{1}{2}}\sqrt{y}\,I_n^{(-1)}(s) \tag{10.9}$$

is valid, where

$$I_n^{(-1)}(s) = \int_0^{\infty}\left[1-2\pi|n|y\,t\left(1-\frac{\mathrm{sgn\ n}}{t}\right)\right]e^{-\pi|n|y\left(t+\frac{1}{t}\right)}t^{s-\frac{1}{2}}\,\frac{dt}{t}. \tag{10.10}$$

We observe that integral (10.10) is a modified Whittaker function for the weight $k=2$, because its explicit form from Sec. 2 is unsuitable in this case.

Therefore, the Fourier series (10.6) can be rewritten in the form

$$E(z,s;\xi_2) = y^s + C_2(s)y^{1-s} + \frac{1}{2s\xi(2s)}\sum_{n\neq 0}\tau_s(n,\Gamma)\sqrt{y}\,I_n^{(-1)}(s)e^{2\pi i n x}, \tag{10.11}$$

where

$$C_2(s) = \frac{\varphi_1(q)}{\varphi_{2s}(q)}\,\frac{\xi(2s-1)}{\xi(2s)}\,\frac{(1-s)}{s}; \tag{10.12}$$

$$\tau_s(n,\Gamma) = |n|^{s-\frac{1}{2}}\sigma_{1-2s}(n,\Gamma);$$

$$\varphi_s(q) = q^s\prod_{p|q}\left(1-\frac{1}{p^s}\right),\quad \xi(s) = \pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

From the explicit form of C_2-matrix (10.12), it follows that it is regular at the point $s=1$ because

$$\lim_{s\to 1}(1-s)\xi(2s-1) = -\frac{1}{2}.$$

From the finiteness of this limit and the explicit form (10.10) for the function $I_n^{(-1)}(s)$, we obtain an analog of estimate (9.17) as $y\to +0$ for $z\in F_q$:

$$E^{(2)}(z,s) = y\,E(z,s;\xi_2) \ll 1. \tag{10.13}$$

The validity of this estimate follows from the Fourier expansion (10.11) for the whole critical strip $\operatorname{Re} s\in[0,1]$.

The analog of the Fourier series for the Eisenstein series (10.5) differs from (10.6) only in that the integral

$$J_n^{(+1)}(s) = \int_{-\infty}^{\infty} \frac{(t+i)^2}{t^2+1)^{s+1}}\, e^{-2\pi inyt}\,dt$$

occurs there in place of the integral $J_n^{(-1)}(s)$ from (10.7). If $n=0$, then it is easy to see that

$$J_0^{(+1)}(s) = J_0^{(-1)}(s). \tag{10.14}$$

Therefore, the C_2-matrix of the Eisenstein series (10.5) is the same as in (10.6) and also does not have a pole at the point $s=1$.

In the analog of relation (10.9) for $n\neq 0$, the integral $I_n^{(+1)}(s)$ occurs, for which a generalization of (10.14) is valid:

$$I_n^{(+1)}(s) = I_{-n}^{(-1)}(s).$$

Therefore, from the Fourier series for the Eisenstein series (10.5) the relation

$$E(z,s;\bar{\xi}_2) = E(-\bar{z},s;\xi_2) \tag{10.15}$$

follows. Consequently, estimate (10.13) remains valid also for series (10.4):

$$E^{(2)}(\bar{z},s) = y\,E(z,s;\bar{\xi}_2) \ll 1, \tag{10.16}$$

which is valid for any $z\in F_q$ as $y\to +0$.

Having obtained an explicit form of the Fourier expansion at the vertex ∞ for the series $E^{(2)}(z,s)$ and $E^{(2)}(\bar{z},s)$, we can extend convolution (9.9) analytically to the critical strip up to the line $\operatorname{Re} s=\frac{1}{2}$. To do this, we make use of relation (9.5) and transform the difference in square brackets in the kernel of the integral from (9.9) for the vertex $V_\infty(N)$:

$$E_j(z,s) - y^{s+1}\frac{\partial}{\partial x}\, f_j(z) = E_j(z,s) - y^{s+1}\frac{1}{2}\,F_{-j}(z) - y^{s+1}\,\frac{1}{2}\,F_{+j}(z). \tag{10.17}$$

Substituting (9.11) into the right-hand side of (10.17), we find an explicit form of the left-hand side of (10.17):

$$E_j(z,s) - y^{s+1}\frac{\partial}{\partial x}\, f_j(z) = \frac{1}{2}F_{-j}(z)[E^{(2)}(z,s)-y^{s+1}]$$
$$+\frac{1}{2}F_{+j}(z)[E^{(2)}(\bar{z},s)-y^{s+1}] + \frac{1}{4}\,f_j(z)\Delta\,E^{(1)}(z,s). \tag{10.18}$$

But we know that the C_2-matrix in the Fourier expansions of the series $E^{(2)}(z,s)$ and $E^{(2)}(\bar{z},s)$ is one and the same (from (10.12)), therefore we can transform the right-hand side in (10.18) once more and reduce it to the form

$$\frac{1}{2}\,F_{-j}(z)[E^{(2)}(z,s)-y^{s+1}-C_2(s)y^{2-s}]+\frac{1}{2}\,F_{+j}(z)[E^{(2)}(\bar{z},s)-y^{s+1}-C_2(s)y^{2-s}]$$

$$+\frac{\partial}{\partial x}\,f_j(z)\,C_2(s)y^{2-s}+\frac{1}{4}\,f_j(z)\,\Delta E^{(1)}(z,s). \tag{10.19}$$

In the third term, we used the inverse of relation (9.5):

$$\frac{1}{2}\,F_{-j}(z)+\frac{1}{2}\,F_{+j}(z)=\frac{\partial}{\partial x}\,f_j(z).$$

From the Fourier expansion (2.10) for the vertex ∞, it follows that, in a neighborhood $V_\infty(N)$ $\frac{\partial}{\partial x}\,f_j(z)$ decays exponentially as $y\to\infty$. The same is true for both differences in the first two brackets in (10.19). This immediately follows from Fourier expansion (10.11) if its zeroth coefficient is rerranged to the left-hand side with an inverse sign. Therefore, the first three terms in (10.19) decay exponentially as $y\to\infty$.

Now we transform the last term in (10.19) to the form

$$\frac{1}{4}\,f_j(z)\Delta\,E^{(1)}(z,s)=\frac{i}{4}\,f_j(z)C_1(s)y^{1-s}+\frac{1}{4}\,f_j(z)[\Delta\,E^{(1)}(z,s)-i\,C_1(s)y^{1-s}]. \tag{10.20}$$

The difference in the brackets in (10.20) also decays exponentially as $y\to\infty$, as follows from the Fourier expansion (9.15). Therefore, the first term of the right-hand side of (10.20) is the only nontrivial summand in the whole expression (10.19). Consequently, for the left-hand side of (10.18) the following asymptotic behavior is valid as $y\to\infty$:

$$E_j(z,s)-y^{s+1}\,\frac{\partial}{\partial x}\,f_j(z)=\frac{i}{4}\,f_j(z)\,C_1(s)y^{1-s}+O(e^{-\frac{3}{2}\,\pi y}).$$

Using this relation and the Fourier expansion (2.10) for $f_j(z)$ at the vertex $\alpha=\infty$, for $\operatorname{Re}s>1$ we obtain an explicit form of the integral over the neighborhood $V_\infty(N)$ from (9.9):

$$\iint_{V_\infty(N)}\overline{f_j(z)}\,[E_j(z,s)-y^{s+1}\,\frac{\partial}{\partial x}\,f_j(z)]d\mu(z)=\frac{i}{4}\,C_1(s)|\rho_{j,\infty}(0)|^2\,\frac{N^{1-s-2\gamma_j}}{s+2\gamma_j-1}+O(e^{-\pi N}). \tag{10.21}$$

We recall that $\gamma_j=-i\varkappa_j=\sqrt{\frac{1}{4}-\lambda_j}\le\frac{1}{4}$. From relation (10.21), it follows an analytic continuation of convolution (9.8) into the critical strip $\operatorname{Re}s\le 1$ with extracting the poles at the points $s=1$ and $s=1-2\gamma_j$. In addition, the residue of convolution (9.8) is equal to

$$\frac{1}{8\pi}\,C_1(1-2\gamma_j)\,|\rho_{j,\infty}(0)|^2 \tag{10.22}$$

at the pole $s=1-2\gamma_j$ and to

$$\frac{1}{16\pi}\,\frac{\sigma_1(0,\Gamma)}{\xi(2)} \tag{10.23}$$

at the pole $s = 1$. If we remind that the residue of the homogeneous convolution from Sec. 8 at the pole $s = 1$ is equal to

$$\frac{1}{2}\,\frac{\sigma_1(0,\Gamma)}{\xi(2)} = \frac{3}{\pi}\,\frac{1}{q\prod\limits_{p|q}\left(1+\frac{1}{p}\right)}, \tag{10.24}$$

then it is clear that the residue (10.23) for the nonhomogeneous convolution (9.8) is 8π times less than the residue (10.24). Simultaneously, we also obtain an explicit form of the nonhomogeneous convolution (9.8) in the critical strip, which can be used for estimates up to the line $\operatorname{Re} s = \frac{1}{2}$. This restriction is caused by the fact that the Fourier series of all Eisenstein series contain the Riemann function $\zeta(2s)$ in the denominator, which yields meromorphicity in the half-plane $\operatorname{Re} s < \frac{1}{2}$. But this meromorphicity is artificial, and we can easily get rid of it if we multiply our convolutions by $\zeta(2s)$. For such a product, all the estimates can automatically be continued to the left part of the critical strip. The singularity at the point $s = \frac{1}{2}$ that arises in this case is solvable. This fact is well known in the theory of Eisenstein series, therefore we shall not dwell on it.

In concluding this section, we note that everything we have said about nonhomogeneous convolution (9.1) remains valid for its generalization to two eigenfunctions:

$$R^{*}_{j_1 j_2}(s) = \frac{1}{2\pi i}\iint_{\Pi} \overline{f_{j_1}(z)}\,\frac{\partial}{\partial x}\,f_{j_2}(z)\,y^{s+1}\,d\mu(z), \tag{9.1^a}$$

which can be written as a difference similar to (9.3):

$$R^{*}_{j_1 j_2}(s) = J_{j_1 j_2+1}(s+1)\,R^{(+1)}_{j_1 j_2}(s) - J_{j_1 j_2-1}(s+1)\,R^{(-1)}_{j_1 j_2}(s). \tag{9.3^a}$$

The terms of this difference are integrals and series in (8.12^a) from the concluding part of Sec. 8 for the vertex ∞. In addition, for simplicity we omit the symbol of the vertex ∞ in the notation of the convolution $R^{(\pm1)}_{j_1 j_2 \infty}(s)$, because we do not consider the other vertices.

The absence of a pole of convolution (9.1^a) at the point $s = 1$ is the only essential distinction of convolution (9.1^a) from (9.1) for $j_1 \neq j_2$. This fact is a consequence of the orthogonality of the functions $f_{j_1}(z)$ and $f_{j_2}(z)$ on the fundamental domain F_q (see the second relation in (8.27)).

11. The final section

Reading relations (8.11) and (9.3) from right to left, for the convolutions $R^{(\pm1)}_j(s)$ we obtain a system:

$$\begin{aligned} J_{+j}(s+1)R^{(+1)}_j(s) - J_{-j}(s+1)R^{(-1)}_j(s) &= R^{*}_j(s) \\ J_{+j}(s)R^{(+1)}_j(s) + J_{-j}(s)R^{(-1)}_j(s) &= R_j(s), \end{aligned} \tag{11.1}$$

where, for simplicity, we put

$$\begin{aligned} J_{+j}(s) &\equiv J_{j,+1}(s) \\ J_{-j}(s) &\equiv J_{j,-1}(s). \end{aligned} \tag{11.2}$$

Solving this system relative to the convolutions $R^{(\pm1)}_j(s)$, we obtain two relations:

$$\begin{aligned} D_j(s)\,R^{(+1)}_j(s) &= J_{-j}(s+1)R_j(s) + J_{-j}(s)R^{*}_j(s) \\ D_j(s)\,R^{(-1)}_j(s) &= J_{+j}(s+1)R_j(s) - J_{+j}(s)R^{*}_j(s), \end{aligned} \tag{11.3}$$

where

$$D_j(s) = J_{+j}(s)J_{-j}(s+1) + J_{+j}(s+1)J_{-j}(s). \tag{11.4}$$

The discriminant of system (11.4) is an analytic function that is not equal to zero identically. This immediately follows from the fact that the $J_{\pm j}(s)$ are real and positive functions on the real line. Therefore, system (11.3) enables us to extend the convolutions $R_j^{(\pm 1)}(s)$ analytically to the critical strip, provided that this has already been done for the initial convolutions $R_j(s)$ and $R_j^*(s)$.

In order to obtain the first asymptotic relation in (7.12), we consider the integral

$$\frac{1}{2\pi i}\int_{\mathrm{Re}\, s>1} N^s D_j(s)\, R_j^{(-1)}(s)ds. \tag{11.5}$$

The contour of integration in (11.5) is a fixed straight line with $\mathrm{Re}\, s > 1$ in the region of the absolute convergence of the convolution $R_j^{(-1)}(s)$. Therefore, we may interchange summation and integration and write integral (11.5) in the form of an absolutely convergent series:

$$\sum_{l=1}^{\infty} |\rho_j(-l)|^2\, \varphi\left(\frac{l}{N}\right), \tag{11.6}$$

where

$$\varphi(x) = \frac{1}{2\pi i}\int_{\mathrm{Re}\, s>1} x^{-s}\, D_j(s)ds. \tag{11.7}$$

From the explicit form of the function $\varphi(x)$ in (11.7), we have an estimate:

$$\varphi(x) \ll \exp(-\sqrt{x}), \quad x \to \infty. \tag{11.8}$$

Although this estimate is rather nontrivial, the computations nevertheless are quite routine, in the spirit of estimations by the saddle-point method of the product of two Bessel functions (see [12]); therefore, we omit the details. We shall keep in mind namely this function when obtaining asymptotics (7.12). On the other hand, substituting the right-hand side of the second relation from (11.3) into the kernel of integral (11.5), we obtain another representation for this integral:

$$\frac{1}{2\pi i}\int_{\mathrm{Re}\, s>1} N^S [J_{+j}(s+1)\, R_j(s) - J_{+j}(s)R_j^*(s)]ds. \tag{11.9}$$

We calculate integral (11.9) by using the Cauchy theorem on residues, moving the contour to the left to the line $\mathrm{Re}\, s = 1/2$. As a result, we derive the following asymptotics for this integral:

$$\frac{N}{\|F_{q\nu}\|}\left[J_{+j}(2) - J_{+j}(1)\frac{1}{8\pi}\right] + \theta N^{1-2\gamma_j+\varepsilon}, \tag{11.10}$$

where $\|\cdot\|$ is the volume of the fundamental domain:

$$\|F_{q\nu}\| = q\nu \prod_{p|q\nu}\left(1+\frac{1}{p}\right)\frac{\pi}{3}.$$

This is the asymptotics from (7.12). It remains to prove that the difference in the principal term in (11.10) is different from zero. Although this problem is also rather nontrivial, as in estimate

(11.8), it belongs to the same class of problems with standard calculations in the spirit of [12]; therefore, we also omit these calculations and only indicate that this difference is truly different from zero.

If the coefficient $\rho_{j,\infty}(0) \neq 0$ of the residue (10.22) at the pole $s = 1 - 2\gamma_j$, then in (11.10) we can extract the second term of order $N^{1-2\gamma_j}$ of the asymptotics, and the remainder becomes of order $N^{\frac{1}{2}+\varepsilon}$.

The asymptotics for the first relation in (11.3) is computed in precisely the same way and is of the form

$$\sum_{l=1}^{\infty} |\rho_j(l)|^2 \, \varphi\left(\frac{l}{N}\right) = \frac{N}{\|F_{q\nu}\|} \left[J_{-j}(2) + J_{-j}(1) \, \frac{1}{8\pi} \right] + \theta \, N^{1-2\gamma+\varepsilon}. \tag{11.11}$$

The bracket in the principal term in (11.11) is also different from zero. Moreover, here we also can extract the second term of the asymptotics as in (11.10). Asymptotics (11.10) and (11.11) show that each eigenfunction $f_j(z)$ for small $\lambda_j < 1/4$ is nondegenerate both from the left and from the right.

The analog of system (11.1) that consists of relations (9.3^a) and (8.11^a) may be restricted to the functions that belong to one and the same eigenvalue λ_j, because the other cases are not required to us.

In this case, the system of two functions can be simplified, because the integral coefficients depend only on an eigenvalue, and, thus, they will be the same as in the case of coinciding functions that we have already considered.

When an eigenvalue $\lambda_j < 1/4$ is fixed, we denote by $R^{(\pm 1)}_{k_1 k_2}(s)$ the convolutions of the functions $f_{j,k_1}(z)$ and $f_{j,k_2}(z)$ and rewrite the system for them in the form

$$\begin{aligned} J_{+j}(s+1) \, R^{(+1)}_{k_1 k_2}(s) - J_{-j}(s+1) \, R^{(-1)}_{k_1 k_2}(s) &= R^{*}_{k_1 k_2}(s) \\ J_{+j}(s) \, R^{(+1)}_{k_1 k_2}(s) + J_{-j}(s) \, R^{(-1)}_{k_1 k_2}(s) &= R_{k_1 k_2}(s). \end{aligned} \tag{11.12}$$

By analogy with (11.1) and (11.3), from system (11.12) we pass to the inverse of it:

$$\begin{aligned} D_j(s) R^{(+1)}_{k_1 k_2}(s) &= J_{-j}(s+1) R_{k_1 k_2}(s) + J_{-j}(s) R^{*}_{k_1 k_2}(s) \\ D_j(s) R^{(-1)}_{k_1 k_2}(s) &= J_{+j}(s+1) R_{k_1 k_2}(s) - J_{+j}(s) R^{*}_{k_1 k_2}(s). \end{aligned} \tag{11.13}$$

In this case, the analog of integral (11.5) is of the form

$$\frac{1}{2\pi i} \int_{\operatorname{Re} s > 1} N^s \, D_j(s) R^{(-1)}_{k_1 k_2}(s) ds = \sum_{l=1}^{\infty} \varphi\left(\frac{l}{N}\right) \, \rho_{j,k_1}(-l) \, \overline{\rho_{j,k_2}(-l)}. \tag{11.14}$$

If $k_1 \neq k_2$, then from the second relation in (11.13) we derive that sum (11.14) is of order of the second estimate in (7.12), because the convolutions $R_{k_1 k_2}(s)$ and $R^{*}_{k_1 k_2}(s)$ do not have a pole at the point $s = 1$ in this case. Together with asymptotics (11.10), this solves the problem of a nonzero value of residue (7.8) and, thus, proves the Selberg conjecture for the weight 1/2.

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