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RELATIONSHIP BETWEEN THE SPECTRAL THEORIES OF AUTOMORPHIC FUNCTIONS OF WEIGHT 0 AND 1/2

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ABSTRACT

In the present paper, it is established a spectral expansion of the convolution of an Eisenstein series of weight zero and the classical Jacobi theta function with a nontrivial shift in eigenfunctions of the space of automorphic functions of weight zero, invariant relative to the group $SL(2,\mathbb{Z})$.

Notation

Introduce the following notation:

$\mathbb{N}$ is the set of positive integers;

$\mathbb{Z}$ is the set of rational integers (for brevity, we shall call them merely integers);

$\mathbb{R}$ is the set of real numbers;

$\mathbb{C}$ is the set of complex numbers;

$\Gamma(s)$ is the gamma function;

$\zeta(s)$ is the Riemann zeta function.

For $n \in \mathbb{N}$ and $s \in \mathbb{C}$, we set

$$\sigma_s(n) = \sum_{d \backslash n} d^s, \quad \tau(n) = \sigma_0(n) = \sum_{d \backslash n} 1.$$

For $q \in \mathbb{N}$ and $a \in \mathbb{Z}$, we set

$$\delta_q(a) = \begin{cases} 1 & \text{if} \quad a \equiv 0(q); \\ 0 & \text{if} \quad a \not\equiv 0(q). \end{cases}$$

Introduction

The question of the asymptotic behavior of the sum

$$\sum_{\sqrt{|D|}<|n|\le P} \tau(n^2+D) \tag{0.1}$$

($D \in \mathbb{Z}$, $D \ne -k^2$) was originally considered in [21], where the following asymptotic formula was proved:

$$\sum_{\sqrt{|D|}<|n|\le P} \tau(n^2+D) = c_1(D) P \ln P + O_D(P \ln\ln P). \tag{0.2}$$

Attempts to extract one more principal term in formula (0.2) and, thus, to diminish the degree of the remainder term have led to the necessity of estimating sums of the form

$$\sum_{1\le q\le P} \sum_{1\le t\le q} \delta_q(t^2+D) e^{2\pi i \frac{mt}{q}} \tag{0.3}$$

for $m \ne 0$ (the definition of $\delta_q(a)$ is given in the notation). If we set

$$\rho_{q,D}(m) = \sum_{1\le t\le q} \delta_q(t^2+D) e^{2\pi i \frac{mt}{q}}, \tag{0.4}$$

then (0.3) can be written in the form

$$\sum_{1\le q\le P} \rho_{q,D}(m). \tag{0.5}$$

Yu. V. Linnik, in his report at the International Mathematical Congress in Stockholm in 1962 (see [10]), has pointed to the importance of the estimates of sums of the form (0.5) in connection with the Petersson conjecture. In 1963, Hooley in [17] proved the inequality

$$\sum_{P_1<q\le P} \rho_{q,D}(m) \underset{D,\varepsilon}{\ll} P^{\varepsilon} \left(P^{\frac{3}{4}} + |m|^{\frac{4}{5}} \right) \tag{0.6}$$

(here, $D \in \mathbb{Z}$, $D \neq -k^2$, m is an integer different from zero, ε is an arbitrarily small positive number). In that paper, with the help of (0.6), he derived an asymptotic formula:

$$\sum_{\sqrt{|D|}<|n|\leq P} \tau(n^2+D) = c_1(D)P\ln P + c_0(D)P + O_{D,\varepsilon}\left(P^{\frac{8}{9}+\varepsilon}\right) \tag{0.7}$$

(see also [6], where a logarithmic refinement of (0.7) is obtained). Using the methods of [17], one can estimate the remainder term in (0.7) as

$$O_{D,\varepsilon}\left(P^{\frac{5}{6}+\varepsilon}\right),$$

where ε is any positive number. This was carried out in [4]. In [14, 17, 18], applications of inequality (0.6) to some other problems were found.

In 1964, Selberg (see [23]) suggested a new approach to the solution of some number-theoretic results, which applies the spectral theory of automorphic function originated by him. Developing the ideas of Selberg, N. V. Kuznetsov (see [8]) obtained radically new results in the course of proving the Linnik conjecture concerning the behavior of a sum of Kloosterman's sums. In [9, 16], with the help of the spectral theory of automorphic functions, a meromorphic continuation to the whole plane of complex variable was obtained for the series

$$\sum_{n=-\infty}^{\infty} \frac{\tau(n^2+D)}{(n^2+D)^{\frac{s}{2}}} \quad (D \neq -k^2), \tag{0.8}$$

$$\sum_{q=1}^{\infty} \rho_{q,D}(m)\frac{1}{q^s} \quad (D<0,\, D \neq -k^2). \tag{0.9}$$

In the present paper, the spectral theory of automorphic functions is applied to the study of sums of the form (0.1) and (0.5); moreover, the analytic continuation of series (0.8) and (0.9) for $D>0$ is considered in a different way. It should be noted that the techniques from [8, 17] is used essentially.

The present paper consists of three chapters. The first chapter is of an auxiliary nature. It contains results related to the theory of positive definite quadratic forms and the spectral theory of automorphic forms. This chapter contains nothing new except for Lemmas 6 and 7. These two lemmas, despite the simplicity of the proofs, play an important role in the present paper, because they establish a relationship between number-theoretic expressions of the form (0.1) or (0.5) and the automorphic functions.

In Sec. 1 of Chap. II, series (0.9) is extended meromorphically to the whole plane of complex variable for $D>0$. In Sec. 2 of Chap. II, estimate (0.6) is sharpened for $2\pi|m|\sqrt{D} \leq P_1 < P$. First we prove an auxiliary theorem in which sums of the form

$$\sum_{q=1}^{\infty} f\left(\frac{2\pi|m|\sqrt{D}}{q}\right)\rho_{q,D}(m)$$

(f is an arbitrary function with sufficiently good analytic properties) are expressed in terms of the Fourier coefficients of eigenfunctions of the Laplace–Beltrami operator. Further, choosing the function f in a special way, we obtain an estimate

$$\sum_{P_1<q\leq P} \rho_{q,D}(m) \underset{D,\varepsilon}{\ll} P^{\frac{2}{3}+\varepsilon} + |m|^{\frac{1}{4}+\varepsilon}P^{\frac{1}{2}}, \tag{0.10}$$

where ε is an arbitrary positive number, $2\pi|m|\sqrt{D} \le P_1 < P$. This inequality sharpens estimate (0.6). It should be noted that, in (0.10), summation over q begins not with $P_1 \ge 1$ as in (0.6), but with some number not less than $2\pi|m|\sqrt{D}$. But this is not essential for applications.

In Sec. 1 of Chap. III, a meromorphic continuation of series (0.8) to the whole plane of complex variable is proved. Finally, in Sec. 2 of Chap. III, the asymptotic formula

$$\sum_{|n|\le P} \tau(n^2+D) = c_1(D)P\ln P + c_0(D)P + O_{D,\varepsilon}\left(P^{\frac{2}{3}+\varepsilon}\right) \tag{0.11}$$

is proved (here, $D > 0$ and ε is any positive number) This result improves all previously-known results.

The main results of the present paper were published in [3, 4].

Chapter I

§1. $PSL_2(\mathbb{Z})$ as a discrete subgroup of the group of motions on the Lobachevskii plane and the integral points on the surface $\alpha\gamma - \beta^2 = D$

1. In this subsection, we recall some facts and notions concerning the theme of this section and introduce the notation that will be used in the sequel.

Denote by H the upper half-plane in $\mathbb{C}$, i.e., the set of complex numbers z whose imaginary part is greater than zero. As usial, $SL_2(\mathbb{R})$ is the group of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with real entries such that $ad - bc = 1$. We define an action of $SL_2(\mathbb{R})$ on $\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ as follows:

$$\text{if} \quad \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an element of $SL_2(\mathbb{R})$ and $z \in \mathbb{C}$, then we set

$$\sigma(z) = \sigma z = \frac{az+b}{cz+d}.$$

The fact that this is an action of $SL_2(\mathbb{R})$ on $\tilde{\mathbb{C}}$ (i.e., the composition of transformations and σ_1 : and $\sigma_2 : \tilde{\mathbb{C}} \to \tilde{\mathbb{C}}$ corresponds to the product of the matrices $\sigma_1, \sigma_2 \in SL_2(\mathbb{R})$ can be verified immediately. The formula

$$\operatorname{Im} \sigma z = \frac{\operatorname{Im} z}{|cz+d|^2}$$

is easily checked. From this it follows that H is invariant relative to $SL_2(\mathbb{R})$. Therefore, the elements of $SL_2(\mathbb{R})$ define analytic isomorphisms of H onto irself. Note that the element

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

from $SL_2(\mathbb{R})$ acts trivially on H; consequently, one may think that, in fact, the group

$$G = PSL_2(\mathbb{R}) = SL_2(\mathbb{R}) / \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

acts (and the action of this group is faithful; one can show that it is the group of all analytic automorphisms of $\tilde{\mathbb{C}}$).

It is immediately verified that the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \qquad (z = x + iy)$$

on H is G-invariant. Poincaré proved that the triple (H, G, ds^2) is a model of the Lobachevskii plane. Here, G is a group of motions, $\sqrt{ds^2}$ is an element of length. The role of "straight lines" on this plane is played by the arcs of semicircles with centers at the line $\operatorname{Im} z = 0$ and by the vertical straight lines. A G-invariant measure dz on the Lobachevskii plane is given by the relation

$$dz = \frac{dx\, dy}{y^2}.$$

For the Lobachevskii plane, the discrete subgroups of G are analogs of the lattices on the Euclidean plane. More precisely, $\Gamma \subset G$ is a discrete subgroup if the identical transformation is isolated from the remaining ones. As is known, a lattice on the Euclidean plane is characterized by a fundamental parallelogram of periods. For the Lobachevskii plane, a fundamental domain of the group Γ is an analog of this notion.

A fundamental domain of a discrete group Γ is a domain $\Omega \subset H$ such that for any elements $\sigma_1 \neq \sigma_2$ from Γ, the images of Ω under the action of σ_1 and σ_2 do not intersect and the closure of the set

$$\bigcup_{\sigma \in \Gamma} \sigma(\Omega)$$

coincides with H ($\sigma(\Omega)$ is the image of Ω under the action of σ). Obviously, Ω can be given in different ways (for example, if Ω is a fundamental domain of Γ, then for any $\sigma \in \Gamma$, $\sigma(\Omega)$ is also a fundamental domain). Usually, a fundamental domain of a group is denoted by $\Gamma \setminus H$. The invariant (not Euclidean) measure of a fundamental domain of the group Γ is not always finite, and if the measure of $\Gamma \setminus H$ is finite, then Γ is called a discrete group of the first kind. Siegel proved that if Γ is a discrete group of the first kind, then as $\Gamma \setminus H$ one can choose a non-Euclidean polygon with finite number of sides (see [7], pp. 27–34). The full modular group

$$\Gamma = PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z}) / \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is an example of a discrete subgroup in G of the first kind. Here, $SL_2(\mathbb{Z})$ is a subgroup in $SL_2(\mathbb{R})$ that consists of the matrices with integer entries. We deal with Γ in the sequel. Let

$$F = \left\{ z \in H \mid -\frac{1}{2} < \operatorname{Re} z < 0,\ |z| > 1 \right\} \bigcup \left\{ z \in H \mid 0 \leq \operatorname{Re} z \leq \frac{1}{2},\ |z| \geq 1 \right\} \tag{1.1}$$

and let

$$\Gamma_z = \{ \sigma \in \Gamma \mid \sigma z = z \}$$

for $z \in H$.

Lemma 1. *The interior of F is a fundamental domain of Γ.*

Lemma 2. *For any $z \in H$, there exist elements $z_0 \in F$ and $\sigma_0 \in \Gamma/\Gamma_{z_0}$ such that*

$$\sigma_0 z_0 = z;$$

moreover, σ_0 and z_0 are determined uniquely and

$$\Gamma_{z_0} = \begin{cases} \left\{\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\} & \textit{for } z_0 = i; \\ \left\{\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\right\} & \textit{for } z_0 = e^{\frac{\pi i}{3}}; \\ \left\{\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} & \textit{for the remaining numbers } z_0 \in F. \end{cases}$$

For the proofs of these lemmas, see [7] and [11].

Let D be a positive integer. By K_D we denote the set of positive definite quadratic forms of discriminant $-4D$, of the form

$$Q(u, v) = \alpha u^2 + 2\beta u\mathrm{v} + \gamma\mathrm{v}^2.$$

In other words, K_D is a subset in $\mathbb{R}^3$ given by the equation

$$\alpha\gamma - \beta^2 = D \quad (\alpha > 0).$$

The action of $PSL_2(\mathbb{R})$ in K_D is defined by the rule

$$Q(u, \mathrm{v}) \xrightarrow{\sigma} Q(au + c\mathrm{v}, bu + d\mathrm{v}) = (\sigma Q)(u, \mathrm{v}),$$

where

$$\sigma = \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The same action can be written in the form

$$(\alpha, \beta, \gamma) \xrightarrow{\sigma} (\alpha(\sigma), \beta(\sigma), \gamma(\sigma)),$$

where

$$(\alpha, \beta, \gamma) \in K_D, \quad (\alpha(\sigma), \beta(\sigma), \gamma(\sigma)) \in K_D,$$

$$\begin{cases} \alpha(\sigma) = Q(a, b) = \alpha a^2 + 2\beta ab + \gamma b^2; \\ \beta(\sigma) = \alpha ac + \beta(ad + bc) + \gamma bd; \\ \gamma(\sigma) = Q(c, d) = \alpha c^2 + 2\beta cd + \gamma d^2; \end{cases} \tag{1.2}$$

The objects H and K_D are identified with the help of the isomorphism

$$H \xrightarrow{\Pi} K_D,$$

given by the relation

$$\Pi(z) = \left(\sqrt{D}\frac{|z|^2}{\operatorname{Im} z}, \sqrt{D}\frac{\operatorname{Re} z}{\operatorname{Im} z}, \sqrt{D} \cdot \frac{1}{\operatorname{Im} z}\right). \tag{1.3}$$

The inverse mapping Π^{-1} is given by the relation

$$\Pi^{-1}((\alpha, \beta, \gamma)) = \frac{\beta}{\gamma} + \frac{\sqrt{D}}{\gamma} i. \tag{1.4}$$

The following remarkable property of Π is due to Dirichlet (it enables us to identify H and K_D).

Lemma 3. *For any $\sigma \in PSL_2(\mathbb{R})$, the diagram*

$$\begin{array}{ccc} H & \xrightarrow{\sigma} & H \\ {\scriptstyle \Pi}\Big\downarrow & & {\scriptstyle \Pi}\Big\downarrow \\ K_D & \xrightarrow{\sigma} & K_D \end{array}$$

is commutative; i.e., if $z_1 \in H$ is taken to $z_2 \in H$ under the action of σ, then $\Pi(z_1) \in K_D$ is taken to $\Pi(z_2)$ under the action of σ.

This lemma is proved by direct calculations.

It is easily seen that the Π-image of F (see (1.1)) has the form

$$\left\{(\alpha,\beta,\gamma) \in K_D \mid -\frac{1}{2}\gamma < \beta < 0,\ \gamma < \alpha\right\} \bigcup \left\{(\alpha,\beta,\gamma) \in K_D \mid 0 \le \beta \le \frac{1}{2}\gamma,\ \gamma \le \alpha\right\} \tag{1.5}$$

and is a domain of reduction of positive definite quadratic forms.

Let Γ_∞ be the subgroup in $\Gamma = PSL_2(\mathbb{Z})$ that consists of the matrices

$$\pm\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \quad (n \in \mathbb{Z})$$

(the stabilizer of ∞).

It is easily checked that a right coset in $\Gamma_\infty \setminus \Gamma$, different from Γ_∞, is uniquely determined by a representative

$$\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where c is a positive integer, α is an integer, a and b are determined by the relations

$$ad - bc = 1, \quad -\frac{1}{2}c < a \le \frac{1}{2}c.$$

2. Let $\mathbb{Z}K_D$ be the subset of K_D that consists of the forms with integer α, β, and γ. In other words,

$$\mathbb{Z}K_D = \{(\alpha,\beta,\gamma) \mid \alpha,\gamma \in \mathbb{N},\ \beta \in \mathbb{Z},\ \alpha\gamma - \beta^2 = D\}.$$

Obviously, for any $Q \in \mathbb{Z}K_D$ and any $\sigma \in \Gamma$, we have

$$\sigma Q \in \mathbb{Z}K_D.$$

It is well known that in the reduction domain (1.5), there exist only finitely many elements from $\mathbb{Z}K_D$. By $h(-4D)$ we denote the number of such forms (the number of classes).

Lemma 4. *There exists a collection*

$$(\alpha_k, \beta_k, \gamma_k) \in \mathbb{Z}K_D \quad (k = 1, \dots, h(-4D))$$

such that

$$z_k = \frac{\beta_k}{\gamma_k} + \frac{\sqrt{D}}{\gamma_k} i \in F \quad (k = 1, \dots, h(-4D))$$

and all the solutions of the equation

$$\alpha\gamma = \beta^2 + D \quad (\alpha, \gamma \in \mathbb{N}; \beta \in \mathbb{Z})$$

without repitition can be written in the form

$$(\alpha_k(\sigma), \beta_k(\sigma), \gamma_k(\sigma)) \quad (\sigma \in \Gamma/\Gamma_{z_k};\ k = 1, \dots, h(-4D)).$$

This lemma is an immediate consequence of Lemma 1 (taking into account (1.4)).

In the sequel, by $Q_k(u, \mathrm{v})$ we shall denote the quadratic form

$$\alpha_k u^2 + 2\beta_k u\mathrm{v} + \gamma_k \mathrm{v}^2;$$

naturally, we use the notation from Lemma 4. Define a Γ-invariant (automorphic) function $\lambda(z)$ by the equality

$$\lambda(z) = \begin{cases} \frac{1}{2} & \text{for} \quad z = i; \\ \frac{1}{3} & \text{for} \quad z = e^{\frac{\pi i}{3}}; \\ 1 & \text{for the remaining} \quad z \in F. \end{cases}$$

Lemma 5. *Let m be an integer, q a positive integer. Then, under the notation of Lemma 4, the following relation holds:*

$$\rho_{q,D}(m) = \sum_{k=1}^{h(-4D)} \lambda(z_k) \sum_{\sigma \in \Gamma_\infty \setminus \Gamma; Q_k(c,d) = q} e^{2\pi i m \mathrm{Re}\, \sigma z_k}. \tag{1.6}$$

Here we assume that

$$\sigma = \pm \begin{pmatrix} * & * \\ c & d \end{pmatrix}.$$

(The definition of $\rho_{q,D}(m)$ is in (0.4).)

Proof. From the definition of the symbol ($\equiv$) (congruence modulo an integer), we obtain

$$\rho_{q,D}(m) = \sum_{\beta=1}^{q} \delta_q(\beta^2 + D) e^{2\pi i \frac{m\beta}{q}} = \sum_{\gamma = q; \alpha\gamma = \beta^2 + D; -\frac{1}{2}\gamma < \beta \le \frac{1}{2}\gamma} e^{2\pi i \frac{m\beta}{\gamma}}.$$

Applying Lemma 4, we find

$$\rho_{q,D}(m) = \sum_{k=1}^{h(-4D)} \lambda(z_k) \sum_{\sigma \in \Gamma; Q_k(c,d) = q; -\frac{1}{2} < \mathrm{Re}\, \sigma z_k \le \frac{1}{2}} e^{2\pi i m \mathrm{Re}\, \sigma z_k}.$$

The condition $-\frac{1}{2} < \mathrm{Re}\, \sigma z_k \le \frac{1}{2}$ can be replaced by the condition $\sigma \in \Gamma_\infty \setminus \Gamma$, i.e.,

$$\rho_{q,D}(m) = \sum_{k=1}^{h(-4D)} \lambda(z_k) \sum_{\sigma \in \Gamma_\infty \setminus \Gamma; Q_k(c,d) = q} e^{2\pi i m \mathrm{Re}\, \sigma z_k}.$$

The lemma is proved.

In the sequel, we shall use the following fact: the relation $Q_k(c, d) = q$ is equivalent to the relation

$$\mathrm{Im}\, \sigma z_k = \frac{\sqrt{D}}{q}. \tag{1.7}$$

This is easily seen from the relations

$$\mathrm{Im}\, \sigma z_k = \frac{\sqrt{D}}{\gamma_k(\sigma)} = \frac{\sqrt{D}}{Q_k(c, d)}$$

(see (1.2) and (1.4)).

Lemma 6. *Let $f(t)$ be a complex-valued function defined on $(0,\infty)$ and such that the series*

$$\sum_{q=1}^{\infty} \rho_{q,D}(0) \left| f\left(\frac{\sqrt{D}}{q}\right) \right|$$

converges. Then the following relation holds (see the notation in Lemma 4):

$$\sum_{q=1}^{\infty} \rho_{q,D}(m) f\left(\frac{\sqrt{D}}{q}\right) = \sum_{k=1}^{h(-4D)} \lambda(z_k) \sum_{\sigma\in\Gamma_\infty\backslash\Gamma} f(\mathrm{Im}\,\sigma z_k) e^{2\pi i m \mathrm{Re}\,\sigma z_k}.$$

Proof. From (1.6) and (1.7), we have

$$\sum_{q=1}^{\infty} \rho_{q,D}(m) f\left(\frac{\sqrt{D}}{q}\right)$$

$$= \sum_{q=1}^{\infty} \left(\sum_{k=1}^{h(-4D)} \lambda(z_k) \sum_{\sigma\in\Gamma_\infty\backslash\Gamma; Q_k(c,d)=q} e^{2\pi i m \mathrm{Re}\,\sigma z_k} \right) f\left(\frac{\sqrt{D}}{q}\right)$$

$$= \sum_{k=1}^{h(-4D)} \lambda(z_k) \sum_{q=1}^{\infty} \sum_{\sigma\in\Gamma_\infty\backslash\Gamma; Q_k(c,d)=q} f(\mathrm{Im}\,\sigma z_k) e^{2\pi i m \mathrm{Re}\,\sigma z_k}$$

$$= \sum_{k=1}^{h(-4D)} \lambda(z_k) \sum_{\sigma\in\Gamma_\infty\backslash\Gamma} f(\mathrm{Im}\,\sigma z_k) e^{2\pi i m \mathrm{Re}\,\sigma z_k}.$$

The fact that the interchange of the orders of summation is valid follows from the obvious inequality

$$|\rho_{q,D}(m)| \le \rho_{q,D}(0)$$

and the convergence of the series

$$\sum_{q=1}^{\infty} \rho_{q,D}(0) \left| f\left(\frac{\sqrt{D}}{q}\right) \right|.$$

The lemma is proved.

Lemma 7. *Let D be a positive integer, f a complex-valued function defined on $(0,\infty)$ and such that the series*

$$\sum_{n=-\infty}^{\infty} \tau(n^2+D) f\left(\frac{1}{D}(n^2+D)\right)$$

converges absolutely (recall that $\tau(m)$ is the number of divisors of the positive integer m). Then the following relation holds:

$$\sum_{n=-\infty}^{\infty} \tau(n^2+D) f\left(\frac{1}{D}(n^2+D)\right) = \sum_{k=1}^{h(-4D)} \lambda(z_k) \sum_{\sigma\in\Gamma} f\left(\frac{|\sigma z_k|^2}{(\mathrm{Im}\,\sigma z_k)^2}\right)$$

(see the notation in Lemma 4).

Proof. From the definition of the function τ, we have

$$\sum_{n=-\infty}^{\infty} \tau(n^2+D) f\left(\frac{1}{D}(n^2+D)\right) = \sum_{\beta=-\infty}^{\infty} \sum_{\substack{\alpha,\gamma=1 \\ \alpha\gamma=\beta^2+D}}^{\infty} f\left(\frac{\alpha\gamma}{D}\right).$$

Now we make use of Lemma 4 and relation (1.3). Then

$$\sum_{\beta=-\infty}^{\infty} \sum_{\substack{\alpha,\gamma=1 \\ \alpha\gamma=\beta^2+D}}^{\infty} f\left(\frac{\alpha\gamma}{D}\right)$$

$$= \sum_{k=1}^{h(-4D)} \lambda(z_k) \sum_{\sigma\in\Gamma} f\left(\frac{1}{D}\,\frac{\sqrt{D}|\sigma z_k|^2}{\operatorname{Im}\sigma z_k}\cdot\frac{\sqrt{D}}{\operatorname{Im}\sigma z_k}\right)$$

$$= \sum_{k=1}^{h(-4D)} \lambda(z_k) \sum_{\sigma\in\Gamma} f\left(\frac{|\sigma z_k|^2}{(\operatorname{Im}\sigma z_k)^2}\right).$$

Lemma 7 is proved.

§2. Some information on special functions

This section contains several known facts concerning special functions. The proofs of the results given below can be found in [1] and [2].

As is known, the modified Bessel functions $I_s(z)$ and $K_s(z)$ are defined by the relations

$$I_s(z) = \sum_{m=0}^{\infty} \frac{1}{\Gamma(m+1)\Gamma(m+1+s)} \left(\frac{z}{2}\right)^{2m+s}, \tag{1.8}$$

$$K_s(z) = \frac{\pi}{2}\cdot\frac{I_s(z)-I_{-s}(z)}{\sin\pi s}. \tag{1.9}$$

For fixed z, the functions $I_s(z)$ and $K_s(z)$ are holomorphic with respect to s in $\mathbb{C}\setminus\{0\}$.

We give the following three identities that are required in the sequel: for complex numbers s with $\operatorname{Re} s > -\frac{1}{2}$,

$$\int_{-\infty}^{\infty} \frac{e^{itx}dt}{(t^2+1)^{s+\frac{1}{2}}} = \frac{2\sqrt{\pi}}{\Gamma\left(s+\frac{1}{2}\right)} \left(\frac{x}{2}\right)^{s} \cdot K_s(x); \tag{1.10}$$

for complex numbers s and ν with $\operatorname{Re} s > |\operatorname{Re}\nu| - \frac{1}{2}$,

$$\int_0^{\infty} x^{s-\frac{1}{2}} K_\nu(x)dx = 2^{s-\frac{3}{2}}\Gamma\left(\frac{s}{2}+\frac{\nu}{2}+\frac{1}{4}\right)\Gamma\left(\frac{s}{2}-\frac{\nu}{2}+\frac{1}{4}\right); \tag{1.11}$$

for complex numbers s, μ, ν with $\operatorname{Re}(s\pm\nu\pm\nu+1)>0$,

$$\int_0^{\infty} K_\mu(x)K_\nu(x)x^s dx = \frac{2^{s-2}}{\Gamma(s+1)}\Gamma\left(\frac{1+\mu+\nu+s}{2}\right)\Gamma\left(\frac{1+\mu-\nu+s}{2}\right)$$

$$\times\Gamma\left(\frac{1-\mu+\nu+s}{2}\right)\Gamma\left(\frac{1-\mu-\nu+s}{2}\right). \tag{1.12}$$

The behavior of the function $K_s(y)$ can be inferred by the following two estimates. For $0 < c \le x$, an asymptotic formula is valid:

$$K_s(x) = \frac{\sqrt{\pi}}{\sqrt{2x}} \cdot e^{-x}\left(1 + O_c\left(\frac{1+|s|^2}{x}\right)\right). \tag{1.13}$$

For real numbers r and x such that $0 < c \le r^2 + x^2$, the following estimate holds:

$$\frac{1}{x}\int_x^{2x} |K_{it}(y)|dy \underset{c}{\ll} \frac{e^{-\frac{\pi|t|}{2}}}{(t^2+x^2)^{\frac{1}{4}}}. \tag{1.14}$$

Finally, we give the Stirling asymptotic formula: for $|\arg z| < \pi - \delta$ ($\delta > 0$)

$$\Gamma(z) = e^{-z} z^{z-\frac{1}{2}}\left(\sqrt{2\pi} + O_\delta\left(\frac{1}{|z|}\right)\right). \tag{1.15}$$

From (1.15), the estimate

$$|t|^{\sigma-\frac{1}{2}} e^{-\frac{\pi|t|}{2}} \underset{\sigma,c}{\ll} |\Gamma(\sigma+it)| \underset{\sigma,c}{\ll} |t|^{\sigma-\frac{1}{2}} e^{-\frac{\pi|t|}{2}} \tag{1.16}$$

is valid for real numbers σ and t with $0 < c \le |t|$.

§3. Eigenfunctions of the Laplace-Beltrami operator and their properties

In this section, as in the previous one, we recall known results. For details and proofs, see [5, 8, 12, 19].

By $\mathcal{L}^2(\Gamma \setminus H)$ ($\Gamma = PSL_2(\mathbb{Z})$) we denote the Hilbert space of Γ-invariant (automorphic) complex-valued functions defined on H (the half-plane) square integrable over a fundamental domain of the group Γ. The inner product in $\mathcal{L}^2(\Gamma \setminus H)$ is given by the relation

$$(f_1, f_2) = \iint_{\Gamma\setminus H} f_1(z)\overline{f_2(z)}\, dz.$$

The invariant Laplace operator

$$\mathcal{L} = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

can be defined on an everywhere dense subset in $\mathcal{L}^2(\Gamma \setminus H)$ as a selfadjoint operator. It is known that $\mathcal{L}$ has a discrete spectrum $0 = \lambda_0 < \lambda_1 \le \lambda_2 \dots$ with $\lambda_1 > 2$, and $\lambda_j = 12j + o(j)$ as $j \to \infty$. The operator $\mathcal{L}$ has also a continuous spectrum of multiplicity 1, which covers the half-line $\left(\frac{1}{4}, \infty\right)$; moreover, the eigenfunctions that correspond to the continuous spectrum are regarded as generalized functions.

The eigenfunctions of the discrete spectrum $u_j(z)$, corresponding to eigenvalues λ_j, are called parabolic forms. It is known that the parabolic forms are real-analytic functions in the variables x and y. Moreover, the following estimate holds uniformly in χ:

$$u_j(x+iy) \underset{c,j}{\ll} e^{-\pi y} \quad (0 < c \le y). \tag{1.17}$$

The eigenfunctions corresponding to the continuous spectrum are the Eisenstein series

$$E(z,s)=\sum_{\sigma\in\Gamma_\infty\backslash\Gamma}(\operatorname{Im}\sigma z)^s,$$

extended to the straight line $\operatorname{Re}s=\frac{1}{2}$ with respect to s in a holomorphic way. The function $E(z,s)$ satisfies the functional equation

$$\pi^{-s}\Gamma(s)\zeta(2s)E(z,s)=\pi^{-(1-s)}\Gamma(1-s)\zeta(2(1-s))E(z,1-s). \tag{1.18}$$

In the sequel, we assume that $u_j(z)$ are normalized (i.e., $(u_j,u_j)=1$) and are eigenfunctions of the reflection operator

$$T_{-1}:\mathcal{L}^2(\Gamma\setminus H)\to\mathcal{L}^2(\Gamma\setminus H),$$

given by the relation

$$(T_{-1}f)(z)=f(-\bar z).$$

Since $T_{-1}(T_{-1}f)=f$, we have

$$T_{-1}(u_j)=\psi(j)u_j, \tag{1.19}$$

where $\psi(j)=\pm1$.

The following Fourier-series expansions are valid:

$$u_j(x+iy)=\sum_{n=-\infty}^{\infty}\rho_j(n)\sqrt{y}\,K_{i\varkappa_j}(2\pi|n|y)e^{2\pi inx}\quad(j=1,2,\dots) \tag{1.20}$$

(here $\varkappa_j=\sqrt{\lambda_j-\frac{1}{2}}$),

$$E(z,s)=\sum_{n=-\infty}^{\infty}a_n(y)e^{2\pi inx}, \tag{1.21}$$

where

$$a_0(y)=y^s+\sqrt{\pi}\cdot\frac{\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}\,y^{1-s}, \tag{1.22}$$

and for $n\neq0$

$$a_n(y)=\frac{\sqrt{y}\,\pi^s|n|^{s-\frac{1}{2}}}{\Gamma(s)\zeta(2s)}\,\sigma_{1-2s}(|n|)K_{s-\frac{1}{2}}(2\pi|n|y). \tag{1.23}$$

From (1.19), for $\rho_j(n)$ that occur in the right-hand side of (1.20) we have

$$\rho_j(-n)=\psi(j)\rho_j(n). \tag{1.24}$$

4. Auxiliary lemmas

For $z,\ z'\in H$ we denote by $\rho(z,z')$ the geodesic distance corresponding to the invariant Poincaré metric. Instead of the distance $\rho(z,z')$, we make use of a function of it:

$$u(z,z')=\frac{1}{2}(ch\rho(z,z')-1)$$

the fundamental invariant of a pair of points. The following relation is valid:

$$u(z,z')=\frac{|z-z'|^2}{yy'},$$

where $z=x+iy$, $z'=x'+iy'$.

Lemma 8. *Let K be an arbitrary compact set in H. Then for any $\varepsilon > 0$, the series*

$$\sum_{\sigma\in\Gamma} \frac{1}{(1+u(\sigma z, z))^{1+\varepsilon}}$$

converges uniformly with respect to $z \in K$.

For the proof, see [12].

Lemma 9. *Let a function $h : \mathbb{C} \to \mathbb{C}$ possess the following properties:*
(a) *$h(s)$ is analytic in s in the strip $-\varepsilon <$ Re $s < 1+\varepsilon$ for some number $\varepsilon > 0$;*
(b) *satisfies the estimate*

$$h(s) \underset{\delta_l,\varepsilon}{\ll} (1+|s|)^{-2-\delta} \quad (-\varepsilon < \operatorname{Re} s < 1+\varepsilon)$$

for some number $\delta > 0$. Then the following equality is valid:

$$\sum_{j=0}^{\infty} h(\varkappa_j) \mid u_j(z) \mid^2 + \frac{1}{4\pi} \int\limits_{-\infty}^{\infty} h(t) \left| E\left(z, \frac{1}{2} + it\right)\right|^2 dt = \sum_{\sigma\in\Gamma} k(u(\sigma z, z)),$$

where the function $k : [0, +\infty) \to \mathbb{C}$ is related with the function h by the following transformation (the inversion formula due to Selberg and Harish Chandra):

$$\int\limits_{\omega}^{\infty} \frac{k(r)dr}{\sqrt{r-\omega}} = Q(\omega), \quad k(r) = -\frac{1}{\pi} \int\limits_{r}^{\infty} \frac{dQ(\omega)}{\sqrt{\omega - r}},$$

$$Q\left(e^u + e^{-u} - 2\right) = g(u) \quad (u \geq 0),$$

$$h(t) = \int\limits_{-\infty}^{\infty} e^{itu}\, g(u)du, \quad g(u) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{-itu}\, h(t)dt.$$

This lemma is a well-known result due to Selberg (see [22]). The proof of it can be found in [5] and [19].

Lemma 10. *Let K be an arbitrary compact set in H. Then for positive P, the following inequality holds:*

$$\sum_{0\leq\varkappa_j\leq P^{1/2}} \mid u_j(z) \mid^2 \underset{K}{\ll} P.$$

Proof. Obviously, this lemma is valid for $0 \leq P < 1$. Therefore, we assume that $P \geq 1$. We put

$$h(s) = e^{-s^2/P}.$$

The function h satisfies the assumptions of Lemma 9. Since for $\varkappa_j \leq P^{1/2}$ we have

$$e^{-\varkappa_j^2/P} \geq \frac{1}{e},$$

it follows that

$$\sum_{0\le\varkappa_j\le P^{1/2}} \mid u_j(z)\mid^2 \le e\Bigg(\sum_{j=0}^{\infty} h(\varkappa_j)\mid u_j(z)\mid^2 + \frac{1}{4\pi}\int\limits_{-\infty}^{\infty} h(t)\Big|E\Big(z,\frac{1}{2}+it\Big)\Big|^2\Bigg)dt.$$

From Lemma 9 we conclude that

$$\sum_{0\le\varkappa_j\le P^{1/2}} \mid u_j(z)\mid^2 \le e\Bigg(\sum_{\sigma\in\Gamma} k(u(\sigma z, z))\Bigg). \tag{1.25}$$

Since

$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-itu}\,h(t)dt = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-itu}\,e^{-t^2/P}dt = \frac{\sqrt{P}}{2\sqrt{\pi}}\,e^{-1/4Pu^2},$$

we have

$$g(u) = \frac{\sqrt{P}}{2\sqrt{\pi}}\,e^{-1/4Pu^2}.$$

Solving the equation

$$e^u + e^{-u} - 2 = \omega \quad (\omega\ge 0,\quad u\ge 0),$$

we obtain

$$u = \ln\Bigg(1+\frac{1}{2}\omega+\sqrt{\frac{1}{4}\omega^2+\omega}\Bigg).$$

Therefore,

$$Q(\omega) = \frac{\sqrt{P}}{2\sqrt{\pi}}\,e^{-1/4Pu^2}\,e^{-\frac{P}{4}\ln(1+\frac{1}{2}\omega+\sqrt{\frac{1}{4}\omega^2+\omega})}.$$

This implies that

$$k(r) = -\frac{1}{\pi}\int\limits_{r}^{\infty}\frac{P^{1/2}}{2\sqrt{\pi}}\Big(-\frac{P}{4}\Big)\,e^{-\frac{P}{4}\ln^2(1+\frac{1}{2}\omega+\sqrt{\frac{1}{4}\omega^2+\omega})}$$

$$\times 2\cdot\ln e^{(1+\frac{1}{2}\omega+\sqrt{\frac{1}{4}\omega^2+\omega})}\,\frac{\frac{1}{2}+\frac{1}{2}\frac{1+\frac{1}{2}\omega}{\sqrt{\frac{1}{4}\omega^2+\omega}}}{1+\frac{1}{2}\omega+\sqrt{\frac{1}{4}\omega^2+\omega}}\cdot\frac{d\omega}{\sqrt{\omega-2}} \tag{1.26}$$

$$= \frac{P^{3/2}}{8\pi^{3/2}}\int\limits_{r}^{\infty}\Big(e^{-\frac{P}{4}\ln^2(1+\frac{1}{2}\omega+\sqrt{\frac{1}{4}\omega^2+\omega})}\Big)\,\frac{\ln(1+\frac{1}{2}\omega+\sqrt{\frac{1}{4}\omega^2+\omega}}{\sqrt{\frac{1}{4}\omega^2+\omega}\sqrt{\omega-r}}\,d\omega.$$

From this relation, we find

$$k(r)\ll P^{3/2}\int\limits_{r}^{\infty} e^{-\frac{P}{4}\ln^2(1+\frac{1}{2}\omega+\sqrt{\frac{1}{4}\omega^2+\omega})}\,\frac{d\omega}{\sqrt{\omega-r}}\ll P^{3/2}\int\limits_{0}^{\infty} e^{-\frac{P}{4}\ln^2(1+\frac{1}{2}t+\sqrt{\frac{1}{4}t^2+t})}\,\frac{dt}{\sqrt{t}}\ll P.$$

Thus, for any $r\ge 0$,

$$k(r)\ll P. \tag{1.27}$$

With the help of (1.26), for $r \geq 1$ we obtain

$$k(r) \ll P^{3/2} \int\limits_r^\infty e^{-\frac{P}{4}\ln^2(1+r)} \frac{d\omega}{\omega\sqrt{\omega - r}} \ll \frac{P}{r^2}.$$

Combining the latter estimate with (1.27), we conclude that

$$k(r) \ll \frac{P}{(1+r)^2}.$$

This inequality, together with (1.25), leads to an estimate:

$$\sum_{0 \leq \varkappa_j \leq P^{1/2}} | u_j(z) |^2 \ll P\Big(\sum_{\sigma \in \Gamma} \frac{1}{(1 + u(\sigma z, z))^2} \Big).$$

Applying Lemma 8, we obtain

$$\sum_{0 \leq \varkappa_j \leq P^{1/2}} | u_j(z) |^2 \ll_K P.$$

Lemma 10 is proved.

Lemma 11. *Let P be a real number greater than 2, let n and m be positive integers. Then, for any $\varepsilon > 0$,*

$$\sum_{0 \leq \varkappa_j \leq P} \frac{1}{ch\pi\varkappa_j} \rho_j(n)\, \overline{\rho_j(m)} = \frac{\delta_{n,m}}{\pi^2} P^2 + O\varepsilon \left(P^{1+\varepsilon} + P^{1/2+\varepsilon} \left(n^{1/4} + m^{1/4} \right) + (mn)^{1/4+\varepsilon} \right).$$

Here, $\delta_{n,m} = 1$ for $n = m$ and $\delta_{n,m} = 0$ otherwise (for the definition of $\rho_j(n)$, see (1.20)*).*

The proof of this lemma for $n = m$ in contained in [8]. The general case can be obtained by repeating the arguments from [8] word for word.

Lemma 12 (expansion theorem)**.** *Let $f(z)$ be a continuous function from $\mathcal{L}^2(\Gamma \backslash H)$ that satisfies the following conditions:*

(a) *the integral*

$$\iint\limits_{\Gamma \backslash H} f(z)\, y^{1/2}\, dz$$

converges absolutely;

(b) *for the quantities $\mathcal{E}_j(f)$ and $\mathcal{E}(t, f)$ defined by the relations*

$$\mathcal{E}_j(f) = (f, u_j) = \iint\limits_{\Gamma \backslash H} f(z)\, \overline{u_j(z)}\, dz,$$

$$\mathcal{E}(t, f) = \Big(f, E\Big(z, \frac{1}{2} + it\Big)\Big) = \iint\limits_{\Gamma \backslash H} f(z)\, \overline{E\Big(z, \frac{1}{2} + it\Big)}\, dz,$$

the series

$$\sum_{j=0}^{\infty} \mathcal{E}_j(f)\, u_j(z)$$

and the integral

$$\int_{-\infty}^{\infty} \mathcal{E}(t,f)\, E\Big(z, \frac{1}{2}+it\Big)\, dt$$

converge absolutely and uniformly on any compact set from H. Then the following relation holds:

$$f(z) = \sum_{j=0}^{\infty} \mathcal{E}_j(f)\, u_j(z) + \frac{1}{4\pi} \int_{-\infty}^{\infty} \mathcal{E}(t,f)\, E\Big(z, \frac{1}{2}+it\Big)\, dt.$$

For the proofs, see [5, 12, 19].

Lemma 13. *Let $0 < c \le 4|r| \le y$. Then*

$$K_{ir}(y) \ll_c \frac{1}{y^2\, ch \frac{\pi|r|}{2}}.$$

Proof. Since

$$K_{-s}(y) = K_s(y),$$

is suffices to prove the lemma for $r \ge 0$. With the help of (1.10) and (1.16), we find

$$K_{ir}(y) \ll_c \frac{1}{ch \frac{\pi|r|}{2}} \cdot \left| \int_{-\infty}^{\infty} \frac{e^{ity} dt}{(t^2+1)^{1/2+ir}} \right|. \tag{1.28}$$

Further,

$$\int_{-\infty}^{\infty} \frac{e^{ity} dt}{(t^2+1)^{1/2+ir}} = \int_{-\infty}^{\infty} \frac{e^{i(ty - r\ln(t^2+1))}}{(t^2+1)^{1/2}} dt = -i \int_{-\infty}^{\infty} \frac{\sqrt{t^2+1}}{y(t^2+1)-2rt} de^{i(ty-r\ln(t^2+1))}$$

$$= i \int_{-\infty}^{\infty} \frac{(2r - t^3y - ty)e^{i(ty-r\ln(t^2+1))}}{(y(t^2+1)-2rt)^2\sqrt{t^2+1}} dt = \int_{-\infty}^{\infty} \frac{(2r - t^3y - ty)\sqrt{t^2+1}}{(yt^2+y-2rt)^3} de^{i(ty-r\ln(t^2+1))}.$$

Applying the second theorem on the mean value, we obtain

$$\int_{-\infty}^{\infty} \frac{e^{ity}\, dt}{(t^2+1)^{1/2+ir}} \ll \frac{1}{y^2}.$$

This estimate, together with (1.28) yields Lemma 13.

Lemma 14. *For any $\varepsilon > 0$, the following estimate holds:*

$$\frac{1}{\zeta(1+it)} \underset{\varepsilon}{\ll} (1+|t|)^{\varepsilon}.$$

For the proofs, see [20].

Lemma 15. *Let K be a compact set in H. Then for any $\varepsilon > 0$,*

$$E\left(z, \frac{1}{2}+it\right) \underset{\varepsilon,K}{\ll} (1+|t|)^{1/2+\varepsilon}.$$

Proof. Relation (1.22), the functional equation for the Riemann zeta function, and the estimate in Lemma 14 imply that

$$E\left(z, \frac{1}{2}+it\right) \underset{\varepsilon,K}{\ll} (1+|t|)^{\varepsilon/2} \sum_{n=1}^{\infty} \tau(n) \frac{|K_{it}(2\pi ny)|}{|\Gamma(1/2+it)|}.$$

On applying (1.14), (1.16), and Lemma 13, we conclude that

$$E\left(z, \frac{1}{2}+it\right) \underset{\varepsilon,K}{\ll} 1+(1+|t|)^{\varepsilon}\left(\sum_{0<2\pi ny\leq 4|t|} \frac{\tau(n)}{\sqrt{|t|}}\right)+(1+|t|)^{\varepsilon/2}\left(\sum_{2\pi ny>4|t|} \frac{\tau(n)}{n^2}\right) \underset{\varepsilon,K}{\ll} (1+|t|)^{1/2+\varepsilon}.$$

The lemma is proved.

Lemma 16. *Let $P \geq 2\varkappa_j$. Then*

$$\frac{1}{ch\pi\varkappa_j} \sum_{1\leq n\leq P} |\rho_j(n)|^2 \ll P \ln \frac{P}{\varkappa_j}.$$

For the proof, see [15].

Define $\mathcal{L}_j(s)$ (for $j = 1, 2, \ldots$) by the relation

$$\mathcal{L}_j(s) = \sum_{n=-\infty}^{\infty} \frac{\rho_j(n)}{|n|^s} \tag{1.29}$$

(if $\psi(j) = -1$, then (1.24) implies the identity $\mathcal{L}_j(s) \equiv 0$). From the estimate of Lemma 16, it follows that the series defining $\mathcal{L}_j(s)$ converges absolutely for $\operatorname{Re} s > 1$ and defines a holomorphic function in this part of the plane; moreover,

$$\mathcal{L}_j(s) \underset{s}{\ll} \sqrt{\varkappa_j}\, ch\frac{\pi\varkappa_j}{2} \quad (\operatorname{Re} s > 1). \tag{1.30}$$

We construct an analytic extension and a functional equation for $\mathcal{L}_j(s)$, using the ideas due to Hecke.

Lemma 17. *Let $u_j(z)$ be the jth eigenfunction of the discrete spectrum of the Laplace–Beltrami operator. Then the integral*

$$\int\limits_{\infty}^{\infty} y^{s-3/2}\, u_j(iy)\, dy$$

converges absolutely for any $s \in \mathbb{C}$ and defines a holomorphic function on the whole plane of complex variable.

Proof. The integral

$$\int\limits_{1}^{\infty} y^{s-3/2}\, u_j(iy)\, dy$$

converges absolutely for any $s \in \mathbb{C}$ and defines a holomorphic function on the whole plane (see estimate (1.17)). Since

$$u_j(iy) = u_j\left(\frac{-1}{iy}\right) = u_j\left(\frac{i}{y}\right) \tag{1.31}$$

(u_j is an automorphic function), the following estimate holds for $0 < y \le 1$:

$$u_j(iy) \underset{j}{\ll} e^{-\pi/y}.$$

Therefore, the integral

$$\int\limits_{0}^{1} y^{s-3/2} u_j(iy)\, dy$$

converges absolutely for any $s \in \mathbb{C}$ and defines a holomorphic function on $\mathbb{C}$. Now, it remains to use the relation

$$\int\limits_{0}^{\infty} y^{s-3/2} u_j(iy)\, dy = \int\limits_{0}^{1} y^{s-3/2} u_j(iy)\, dy + \int\limits_{1}^{\infty} y^{s-3/2} u_j(iy)\, dy.$$

Lemma 17 is proved.

Lemma 18. *The function $\mathcal{L}_j(s)$ is extended holomorphically to the whole plane of complex variable and satisfies the relation*

$$\pi^{-s}\Gamma\left(\frac{s}{2}+\frac{i\varkappa_j}{2}\right)\Gamma\left(\frac{s}{2}-\frac{i\varkappa_j}{2}\right)\mathcal{L}_j(s) = \pi^{-(1-s)}\Gamma\left(\frac{1-s}{2}+\frac{i\varkappa_j}{2}\right)\Gamma\left(\frac{1-s}{2}-\frac{i\varkappa_j}{2}\right)\mathcal{L}_j(1-s).$$

Proof. From (1.20), for $\operatorname{Re} s > 1$ we find

$$\int\limits_{0}^{\infty} y^{s-3/2} u_j(iy)\, dy = \int\limits_{0}^{\infty} y^{s-3/2}\left(\sum_{n=-\infty}^{\infty}{}' \rho_j(n)\sqrt{y}\, K_{i\varkappa_j}(2\pi|n|y)\right) dy$$

$$= \sum_{n=-\infty}^{\infty}{}' \rho_j(n) \int\limits_{0}^{\infty} y^{s-1}\, K_{i\varkappa_j}(2\pi|n|y) dy = (2\pi)^{-s}\left(\sum_{n=-\infty}^{\infty}{}' \frac{\rho_j(n)}{|n|^s}\right)\int\limits_{0}^{\infty} y^{s-1}\, K_{i\varkappa_j}(y)\, dy;$$

i.e.,

$$\int_0^\infty y^{s-3/2} u_j(iy)\,dy = \frac{1}{4}\pi^{-s}\Gamma\Big(\frac{s}{2}+\frac{i\varkappa_j}{2}\Big)\Gamma\Big(\frac{s}{2}-\frac{i\varkappa_j}{2}\Big)\mathcal{L}_j(s) \tag{1.32}$$

(see formula (1.11)). Integration term by term is ensured by the estimate

$$K_s(y) \underset{c,s}{\ll} \frac{1}{\sqrt{y}}\,e^{-y} \quad (0 < c \le y),$$

which follows from (1.13), and also by the estimate from Lemma 16. Formula (1.32) and Lemma 17 enable us to extend $\mathcal{L}_j(s)$ holomorphically to the whole plane. From (1.31), we obtain

$$\int_0^\infty y^{s-3/2} u_j(iy)\,dy = \int_0^\infty y^{s-3/2} u_j\Big(\frac{i}{y}\Big)\,dy = \int_0^\infty y^{-s-1/2} u_j(iy)\,dy.$$

These relations and formula (1.32) yield

$$\frac{1}{4}\pi^{-s}\Gamma\Big(\frac{s}{2}+\frac{i\varkappa_j}{2}\Big)\Gamma\Big(\frac{s}{2}-\frac{i\varkappa_j}{2}\Big)\mathcal{L}_j(s) = \frac{1}{4}^{-(1-s)}\Gamma\Big(\frac{1-s}{2}+\frac{i\varkappa_j}{2}\Big)\Gamma\Big(\frac{1-s}{2}-\frac{i\varkappa_j}{2}\Big)\mathcal{L}_j(1-s)$$

This is the required functional equation. The lemma is proved.

Lemma 19. *For any $\varepsilon > 0$ and $\operatorname{Re} s \ge 0$, the following estimate holds:*

$$\mathcal{L}_j(s) \underset{s,\varepsilon}{\ll} \varkappa_j^{3/2+\varepsilon} ch\frac{\pi\varkappa_j}{2}.$$

Proof. Apply the Lindelöf principle. For $-1 \le \mu \le 2$, formulas (1.16) and (1.32) imply the estimate

$$\mathcal{L}_j(\mu+it) \underset{s,\varepsilon}{\ll} (1+|t|)^2\, e^{\pi|t|/2}.$$

This estimate enables us to make use of the Lindelöf principle in the strip $-\varepsilon < \operatorname{Re} s < 1+\varepsilon$, where $0<\varepsilon<\frac{1}{2}$. From the functional equation of Lemma 18,

$$\mathcal{L}_j(-\varepsilon+it) = \frac{\pi^{-(1+\varepsilon-it)}\Gamma(\frac{1+\varepsilon-it}{2}+\frac{i\varkappa_j}{2})\Gamma(\frac{1+\varepsilon-it}{2}-\frac{i\varkappa_j}{2})}{\Gamma(\frac{-\varepsilon+it}{2}+\frac{i\varkappa_j}{2})\Gamma(\frac{-\varepsilon+it}{2}-\frac{i\varkappa_j}{2})}\mathcal{L}_j(1+\varepsilon-it)$$

and from the estimate of Lemma 16, as well as from (1.16), we conclude that

$$\mathcal{L}_j(-\varepsilon_i t) \underset{\varepsilon}{\ll} \sqrt{\varkappa_j}(1+\varkappa_j+|t|)^{1+2\varepsilon} ch\frac{\pi\varkappa_j}{2}.$$

With the help of the Lindelöf principle, the latter estimate implies the required estimate. Lemma 19 is proved.

CHAPTER II

§1. Meromorphic continuation of the series $\sum\limits_{q=1}^{\infty}\rho_{q,\mathcal{D}}(m)\frac{1}{q^s}$

Recall that the Poincaré–Selberg series of index m (m is a positive integer) are defined by the relation

$$U_m(z,s) = \sum_{\sigma\in\Gamma_\infty\backslash\Gamma} (\operatorname{Im}\sigma z)^s\, e^{2\pi i m\sigma z} \quad (z\in H).$$

The series $U_m(z,s)$ converges absolutely for $\operatorname{Re} s > 1$. In [23], Selberg extended these series meromorphically in s to the whole plane of complex variable and found the position of their poles. We state his result in the next lemma.

Lemma 20. *For any natural m, the series $U_m(z,s)$ can be extended meromorphically to the whole plane of complex variable, and the poles of $U_m(z,s)$ belong to the set*

$$\left\{ -k+\frac{1}{2}\pm i\varkappa_j \middle| j=1,2,\dots,;k=0,1,\dots\right\}\bigcup\{-k+s\in\mathbb{C}|\zeta(2s)=0;k=0,1,\dots\}. \tag{2.1}$$

A detailed proof of this lemma is contained in [8].

Proposition 1. *Let $\mathcal{D}$ be a positive integer, m an integer different from zero. Then the series*

$$A_{\mathcal{D}}(m,s) = \sum_{q=1}^{\infty}\frac{\rho_{q,\mathcal{D}}(m)}{q^s}$$

converges absolutely for $\operatorname{Re} s > 1$, is extended meromorphically to the whole plane of complex variable, and the poles of $A_{\mathcal{D}}(m,s)$ belong to the set (2.1).

Proof. Since

$$\rho_{q,\mathcal{D}}(m) = \sum_{\beta=1}^{q}\delta_q(\beta^2+\mathcal{D})\,e^{2\pi i\frac{m\beta}{q}} = \sum_{\beta=1}^{q}\delta_q(\beta^2+\mathcal{D})\,e^{-2\pi i\frac{m\beta}{q}} = \rho_{q,\mathcal{D}}(-m),$$

we have

$$A_{\mathcal{D}}(m,s) = A_{\mathcal{D}}(-m,s).$$

Therefore, it suffices to prove the proposition for positive integers m. For $\operatorname{Re} s > 1$, with the help of Lemma 6 we derive

$$\sum_{q=1}^{\infty}\rho_{q,\mathcal{D}}(m)\left(\frac{\sqrt{\mathcal{D}}}{q}\right)^s = \sum_{k=1}^{h(-4\mathcal{D})}\lambda(z_k)\sum_{\sigma\in\Gamma_\infty\backslash\Gamma}(\operatorname{Im}\sigma z_k)^s\,e^{2\pi i m\operatorname{Re}\sigma z_k}.$$

Consequently, it is sufficient to prove the proposition for the series

$$\Phi_m(z,s) = \sum_{\sigma\in\Gamma_\infty\backslash\Gamma}(\operatorname{Im}\sigma z)^s\,e^{2\pi i m\operatorname{Re}\sigma z}.$$

Write $U_m(z,s)$ in the form

$$\sum_{\sigma\in\Gamma_\infty\backslash\Gamma}(\operatorname{Im}\sigma z)^s\,e^{2\pi i m\operatorname{Re}\sigma z_k}\,e^{2\pi m\operatorname{Im}\sigma z}.$$

Since

$$e^{-2\pi m \operatorname{Im} \sigma z} = 1 + O_{z,m}(\operatorname{Im} \sigma z),$$

it follows that for $\operatorname{Re} s > 1$,

$$U_m(z,s) = \Phi_m(z,s) + O_{m,z}\Bigg(\sum_{\sigma \in \Gamma_\infty \backslash \Gamma} (\operatorname{Im} \sigma z_k)^{\operatorname{Re} s + 1} \Bigg).$$

The series

$$\sum_{\sigma \in \Gamma_\infty \backslash \Gamma} (\operatorname{Im} \sigma z_k)^{\operatorname{Re} s + 1}$$

converges absolutely for $\operatorname{Re} s > 0$. Therefore (see Lemma 20), $\Phi_m(z,s)$ is meromorphically extended to the strip $\operatorname{Re} s > 0$, and the poles of $\Phi_m(z,s)$ belong to the set (2.1). Using the relation

$$e^{-2\pi m \operatorname{Im} \sigma z} = 1 - 2\pi m \operatorname{Im} \sigma z + O_{z,m}\big((\operatorname{Im} \sigma z)^2\big),$$

we obtain

$$U_m(z,s) = \Phi_m(z,s) - 2\pi m \cdot \Phi_m(z,s+1) + \Bigg(\sum_{\sigma \in \Gamma_\infty \backslash \Gamma} (\operatorname{Im} \sigma z_k)^{\operatorname{Re} s + 2} \Bigg).$$

This implies the meromorphic extendability of $\Phi_m(z,s)$ to the strip $\operatorname{Re} s > -1$. Repeating these arguments a countable number of times (by induction), we conclude that $\Phi_m(z,s)$ is meromorphically extended to the whole plane of complex variable and the poles of this function belong to (2.1). The proposition is proved.

§2. Estimation of the sum $\sum\limits_{P_1 < q \le P} \rho_{q,\mathcal{D}}(m)$

Lemma 21. *Let $f(t)$ be a continuous complex-valued function on $(0, +\infty)$ with compact support set and such that for the function*

$$\widehat{f}(r) = \int_0^\infty f(y)\, K_{ir} \frac{dy}{y^{3/2}}$$

and for integers $m \neq 0$ the series

$$\sum_{j=1}^\infty \widehat{f}(\varkappa_j)\, \overline{\rho_j(m)}\, u_j(z)$$

and the integral

$$\int_0^\infty \frac{(\pi|m|)^{-ir} \sigma_{2ir}(|m|)}{\Gamma(1/2 - ir)\zeta(1 - 2ir} E\Big(z, \frac{1}{2} + ir\Big) \widehat{f}(r) dr$$

converge absolutely and uniformly on any compact subset of H. Then the following relation holds:

$$\sum_{\sigma \in \Gamma_\infty \backslash \Gamma} f(2\pi|m| \operatorname{Im} \sigma z)\, e^{\pi i m \operatorname{Re} \sigma z} = \sqrt{2\pi|m|} \sum_{j=1}^\infty \widehat{f}(\varkappa_j)\, \overline{\rho_j(m)}\, u_j(z)$$

$$+\sqrt{\frac{|m|}{2}}\int\limits_0^\infty \frac{(\pi|m|)^{-ir}\sigma_{2ir}(|m|)}{\Gamma(1/2-ir)\zeta(1-2ir}E\Big(z,\frac{1}{2}+ir\Big)\widehat{f}(r)dr$$

Proof. This lemma is an immediate consequence of Lemma 12. Only the calculation of $\mathcal{E}_j(\psi)$ and $\mathcal{E}(t,\psi)$ deserves attention, where

$$\psi(z)=\sum_{\sigma\in\Gamma_\infty\backslash\Gamma}(2\pi|m|\operatorname{Im}\sigma z)\,e^{2\pi im\operatorname{Re}\sigma z}.$$

We have

$$\mathcal{E}_j(\psi)=\iint\limits_{\Gamma\backslash H}\psi(z)\,\overline{u_j(z)}\,dz=\iint\limits_{\Gamma\backslash H}\Big(\sum_{\sigma\in\Gamma_\infty\backslash\Gamma}f(2\pi|m|\operatorname{Im}\sigma z)\,e^{2\pi im\operatorname{Re}\sigma z}\Big)\overline{u_j(z)}\,dz$$

$$=\sum_{\sigma\in\Gamma_\infty\backslash\Gamma}\iint\limits_{\sigma(\Gamma\backslash H)}f(2\pi|m|\operatorname{Im}\sigma z)\,e^{2\pi im\operatorname{Re}\sigma z}\,\overline{u_j(z)}\,dz=\iint\limits_{\Gamma_\infty\backslash H}f(2\pi|m|y)\,e^{2\pi imx}\,\overline{u_j(z)}\frac{dxdy}{y^2}$$

$$=\int\limits_0^\infty\Big(\int\limits_0^1 e^{2\pi imx}\,\overline{u_j(z)}\,dx\Big)f(2\pi|m|y)\frac{dy}{y^2}$$

$$=\sqrt{2\pi|m|}\,\overline{\rho_j(m)}\int\limits_0^\infty f(y)\,K_{i\varkappa_j}(y)\frac{dy}{y^{3/2}}=\sqrt{2\pi|m|}\cdot\overline{\rho_j(m)}\,\widehat{f}(\varkappa_j).$$

Here we made use of formula (1.20). Similarly, with the help of (1.21), we obtain

$$\mathcal{E}(t,\psi)=\iint\limits_{\Gamma\backslash H}\psi(z)\overline{E\Big(z,\frac{1}{2}+it\Big)}\,dz=\iint\limits_{\Gamma_\infty\backslash H}f(2\pi|m|y)\,e^{2\pi imx}\,\overline{E\Big(z,\frac{1}{2}+it\Big)}\,dz$$

$$=\sqrt{2\pi|m|}\cdot\frac{2\pi^{1/2-it}}{\Gamma(1/2-it)\zeta(1-2it)}\cdot|m|^{-it}\,\sigma_{2it}(|m|)\widehat{f}(t).$$

Substituting the above expressions for $\mathcal{E}_j(\psi)$ and $\mathcal{E}(t,\psi)$ into the formula

$$\psi(z)=\sum_{j=0}^\infty\mathcal{E}_j(\psi)u_j(z)+\frac{1}{4\pi}\int\limits_{-\infty}^\infty\mathcal{E}(t,\psi)E\Big(z,\frac{1}{2}+it\Big)\,dt,$$

we arrive at the required relation of the lemma.

Theorem 1. *Let m be an integer different from zero, let $\mathcal{D}$ be a positive integer, and let $f(t)$ satisfy the assumptions of Lemma 21. Then following relation holds:*

$$\sum_{q=1}^\infty f\Big(\frac{2\pi|m|\sqrt{\mathcal{D}}}{q}\Big)\rho_{q,\mathcal{D}}(m)=\sqrt{2\pi|m|}\sum_{j=1}^\infty\widehat{f}(\varkappa_j)\,\overline{\rho_j(m)}\Big(\sum_{k=1}^{h(-4\delta_l)}\lambda(z_k)u_j(z_k)\Big)$$

$$+\sqrt{\frac{|m|}{2}}\int\limits_0^\infty \frac{(\pi|m|)^{-ir}\sigma_{2ir}(|m|)}{\Gamma(1/2-ir)\zeta(1-2ir}\widehat{f}(r)\Big(\sum_{k=1}^{h(-4\delta_l)}\lambda(z_k)E\Big(z,\frac{1}{2}+ir\Big)\Big)\,dr.$$

(See the notation in Lemma 4).

Proof. From Lemma 6 we find

$$\sum_{q=1}^\infty f\Big(\frac{2\pi|m|\sqrt{\mathcal{D}}}{q}\Big)\rho_{q,\mathcal{D}}(m)=\sum_{k=1}^{h(-4\delta_l)}\lambda(z_k)\sum_{\sigma\in\Gamma_\infty\backslash\Gamma}f(2\pi|m|\operatorname{Im}\sigma z_k)\,e^{2\pi i m\operatorname{Re}\sigma z_k}.$$

Using Lemma 2.1 on the right-hand side in this relation, we obtain the required equality. The theorem is proved.

Theorem 2. *Let m be an integer different from zero, $\mathcal{D}$ a positive integer with $2\pi|m|\sqrt{\mathcal{D}}\le P_1<P$. Then for any $\varepsilon>0$,*

$$\sum_{P_1<q\le P}\rho_{q,\mathcal{D}}(m)\underset{\varepsilon,\mathcal{D}}{\ll}P^{2/3+\varepsilon}+P^{1/2}|m|^{1/4+\varepsilon}.$$

Proof. If $P-P_1\le P^{2/3}+P^{1/2}|m|^{1/4}$, then the theorem is obvious. Therefore, it is sufficient to consider the case $P-P_1>P^{2/3}+^{1/2}|m|^{1/4}$. Let $0<\Delta_1<\frac{1}{2}P_1$, $0<\Delta<\frac{1}{2}P$, $P_1+\Delta_1<P-\Delta$. We set

$$f_1(t)=\frac{1}{2\Delta}\int\limits_{-\Delta}^{\Delta}\chi_{(-\infty,P]}(t+\theta)d\theta,$$

$$f_2(t)=\frac{1}{2\Delta_1}\int\limits_{-\Delta_1}^{\Delta}\chi_{[P_1,\infty)}(t+\theta)d\theta.$$

Here, $\chi_{(-\infty,P]}$ and $\chi_{[P_1,\infty)}$ are the characteristic functions of the half-intervals $(-\infty,P]$ and $[P_1,\infty)$, respectively. Let

$$f(t)=f_1(t)f_2(t)$$

(a function smoothing a peak of a function, Vinogradov's "glass"). Since $\rho_{q,\mathcal{D}}(m)\underset{\varepsilon,\mathcal{D}}{\ll}q^\varepsilon$, $0\le f(t)\le 1$ for all t and

$$f(t)=\begin{cases}1 & \text{for } t\in[P_1+\Delta_1,P-\Delta];\\ 0 & \text{for } t\in\mathbb{R}\setminus[P_1-\Delta_1,P+\Delta],\end{cases}$$

we have

$$\sum_{P_1<q\le P}\rho_{q,\mathcal{D}}(m)=\sum_{q=1}^\infty f(q)\rho_{q,\mathcal{D}}(m)+O_{\varepsilon,\mathcal{D}}\big((1+\Delta_1+\Delta)P^\varepsilon\big).$$

Transforming the right-hand side with the help of Theorem 2, we obtain

$$\sum_{P_1<q\le P}\rho_{q,\mathcal{D}}(m)=\sum_{j=1}^\infty f^0(\varkappa_j)\,\overline{\rho_j(m)}\Big(\sum_{k=1}^{h(-4\delta_l)}\lambda(z_k)u_j(z_k)\Big)$$

$$+\sqrt{\frac{|m|}{2}}\int\limits_0^\infty \frac{(\pi|m|)^{-ir}\sigma_{2ir}(|m|)}{\Gamma(1/2-ir)\zeta(1-2ir}f^0(r)\Big(\sum_{k=1}^{h(-4\delta_l)}\lambda(z_k)E\Big(z,\frac{1}{2}+ir\Big)\Big)\,dr+O_{\varepsilon,\mathcal{D}}\big((1+\Delta_1+\Delta)P^{\varepsilon}\big),$$

where

$$f^0(r)=\int\limits_0^\infty f\left(\frac{2\pi|m|\sqrt{\mathcal{D}}}{y}\right)K_{ir}(y)\frac{dy}{y^{3/2}}.$$

(We did not verify the conditions imposed on $f(t)$ in Theorem 2. The estimates obtained below will guarantee the fulfillment of these conditions.)

Make the change of variable

$$y=\frac{2\pi|m|\sqrt{\mathcal{D}}}{y_1},$$

and obtain

$$f^0(r)=\frac{1}{\sqrt{2\pi|m|\sqrt{\mathcal{D}}}}\cdot\int\limits_0^\infty f(y_1)\,K_{ir}\left(\frac{2\pi|m|\sqrt{\mathcal{D}}}{y_1}\right)\frac{dy_1}{\sqrt{y_1}}.$$

From the formula

$$K_{ir}(y)=\frac{\pi}{2}\cdot\frac{1}{sh\pi r}\left(\sum_{n=0}^\infty\left(\frac{1}{n!\Gamma(n+ir+1)}\left(\frac{y}{2}\right)^{2n+ir}+\frac{1}{n!\Gamma(n-ir+1)}\left(\frac{y}{2}\right)^{2n-ir}\right)\right)$$

(see (1.8) and (1.9)), we find

$$f^0(r)=\frac{\pi}{2}\cdot\frac{1}{sh\pi r}\left(\sum_{n=0}^\infty\left(\frac{1}{n!\Gamma(n+ir+1)}f^0_{n+}(r)+\frac{1}{n!\Gamma(n-ir+1)}f^0_{n-}(r)\right)\right),\tag{2.3}$$

where

$$f^0_{n\pm}(r)=\int\limits_0^\infty f(y_1)\left(\frac{\pi|m|\sqrt{\mathcal{D}}}{y_1}\right)^{2n\pm ir}\frac{dy_1}{\sqrt{y_1}}.$$

With the help of the second theorem on the mean value, we have

$$f^0_{n\pm}(r)=(\pi|m|\sqrt{\mathcal{D}})^{2n\pm ir}\int\limits_0^\infty f(y_1)dy_1^{-2n\mp ir+1/2}$$

$$\ll\frac{(\pi|m|\sqrt{\mathcal{D}})^{2n}}{(|1+|n|+|r|)P_1^{2n-1/2}}+\frac{(\pi|m|\sqrt{\mathcal{D}})^{2n}}{(|1+|n|+|r|)P^{2n-1/2}}.$$

Substituting the estimate from (2.3), we conclude that

$$f^0(r)\ll\left(\frac{P}{|m|}\right)^{1/2}\frac{1}{(1+|r|)^{3/2}ch\frac{\pi r}{2}}.\tag{2.4}$$

Integration by parts yields:

$$f^0_{n\pm}(r)=(\pi|m|\sqrt{\mathcal{D}})^{2n\pm ir}\left(\frac{1}{2\Delta_1}\int\limits_{P_1-\Delta_1}^{P_1+\Delta_1}\frac{-y_1^{-2n\mp ir+1/2}}{-2n\mp ir+1/2}dy_1\right.$$

$$+\frac{1}{2\Delta_1}\int\limits_{P-\Delta}^{P+\Delta}\frac{y_1^{-2n\mp ir+1/2}}{-2n\mp ir+1/2}dy_1\Bigg)$$

$$\ll\frac{1}{(1+|n|+|r|)^2}\bigg(\frac{P^{3/2}}{\Delta_1}+\frac{P^{3/2}}{\Delta}\bigg).$$

From this, we arrive at the estimate

$$f^0(r)\ll\frac{1}{(1+|r|)^{5/2}\,ch\frac{\pi r}{2}}\bigg(\frac{P^{3/2}}{\Delta_1}+\frac{P^{3/2}}{\Delta}\bigg). \tag{2.5}$$

For any $A\geq 1$, from (2.2), (2.4), and (2.5) we have

$$\sum_{P_1<\leq P}\rho_{q,\mathcal{D}}(m)\underset{\mathcal{D},\varepsilon}{\ll}\sum_{\varkappa_j\leq A}\frac{P^{1/2}}{\varkappa_j^{3/2}}\cdot\frac{|\rho_j(m)|}{ch\frac{\pi\varkappa_j}{2}}\Bigg(\sum_{k=1}^{h(-4\mathcal{D})}|u_j(z_k)|\Bigg)$$

$$+\tau(|m|)P^{1/2}\int\limits_0^P\frac{1}{|\zeta(1+2ir)|(1+|r|)^{3/2}}\Bigg(\sum_{k=1}^{h(-4\mathcal{D})}\bigg|E\Big(z_k,\frac{1}{2}+ir\Big)\bigg|dr$$

$$\sum_{\varkappa_j>A}\bigg(\frac{P^{3/2}}{\Delta_1}+\frac{P^{3/2}}{\Delta}\bigg)\frac{|\rho_j(m)|}{\varkappa^{5/2}ch\frac{\pi\varkappa_j}{2}}\Bigg(\sum_{k=1}^{h(-4\mathcal{D})}|u_j(z_k)|\Bigg)$$

$$+\tau(|m|)\bigg(\frac{P_1^{3/2}}{\Delta_1}+\frac{P^{3/2}}{\Delta}\bigg)\int\limits_P^\infty\frac{1}{|\zeta(1+2ir)|(1+|r|)^{5/2}}\Bigg(\sum_{k=1}^{h(-4\mathcal{D})}\bigg|E\Big(z_k,\frac{1}{2}+ir\Big)\bigg|\Bigg)dr+(1+\Delta_1+\Delta)P^{\varepsilon}.$$

Applying the estimates of Lemmas 10, 11, 14, and 15, we obtain

$$\sum_{P_1<q\leq P}\rho_{q,\mathcal{D}}(m)\underset{\mathcal{D},\varepsilon}{\ll}P^{1/2}\left(A^{1/2}+|m|^{1/4+\varepsilon}\right)+\bigg(\frac{P^{3/2}}{\Delta_1}+\frac{P^{3/2}}{\Delta}\bigg)\left(A^{-1/2}+|m|^{1/4+\varepsilon}A^{-1}\right)$$

$$+\tau(|m|)P^{1/2+\varepsilon}+\tau(|m|)\bigg(\frac{P^{3/2}}{\Delta_1}+\frac{P^{3/2}}{\Delta}\bigg)\cdot\frac{P^{\varepsilon}}{P}+(1+\Delta_1+\Delta)P^{\varepsilon}.$$

For $A=P^{1/3}$, the latter inequality is rewritten in the form

$$\sum_{P_1<q\leq P}\rho_{q,\mathcal{D}}(m)\underset{\mathcal{D},\varepsilon}{\ll}P^{2/3}+P^{1/2}|m|^{1/4+\varepsilon}+\frac{P^{4/3}}{\Delta_1}+\frac{P^{4/3}}{\Delta}+\bigg(\frac{1}{\Delta_1}+\frac{1}{\Delta}\bigg)P^{1/2}|m|^{1/4+\varepsilon}+(\Delta_1+\Delta)P^{\varepsilon}.$$

Choosing Δ_1 of order $P_1^{3/2}$ and Δ of order $P^{3/2}$, we arrive at the inequality

$$\sum_{P_1<q\leq P}\rho_{q,\mathcal{D}}(m)\underset{\mathcal{D},\varepsilon}{\ll}P^{2/3+\varepsilon}+P^{1/2}|m|^{1/4+\varepsilon}.$$

The theorem is proved.

CHAPTER III

Lemma 22. *Let K be a compact set from H. Then for any $\varepsilon > 0$, the series*

$$\sum_{\sigma\in\Gamma} \left(\frac{\operatorname{Im}\sigma z}{|\sigma z|}\right)^{1+\varepsilon}$$

converges uniformly in $z \in K$.

Proof. Since for

$$\sigma = \pm\begin{pmatrix} a & b \\ c & d\end{pmatrix}$$

the equality

$$\left(\frac{\operatorname{Im}\sigma z}{|\sigma z|}\right)^{2} = \frac{y^2}{|az+b|^2|cz+d|^2}$$

is valid, we have

$$\sum_{\sigma\in\Gamma} \left(\frac{\operatorname{Im}\sigma z}{|\sigma z|}\right)^{1+\varepsilon} \ll_{\varepsilon,K} \sum_{\sigma\in\Gamma} \frac{1}{(|az+b||cz+d|)^{1+\varepsilon}} = 2\cdot \sum_{\substack{a,b,c,d=-\infty \\ ac-bd=1}}^{\infty} \frac{1}{(|az+b||cz+d|)^{1+\varepsilon}}$$

$$\leq 4\cdot \sum_{\substack{a,b,c,d=-\infty \\ ac-bd=1; \\ |az+b|\leq|cz+d|}}^{\infty} \frac{1}{(|az+b||cz+d|)^{1+\varepsilon}} = 4\cdot \sum_{\substack{a,b=-\infty \\ (a,b)=1}}^{\infty} \sum_{\substack{t=-\infty \\ |az+b|\leq \\ \leq|c_0z+d_0+t(az+b)|}}^{\infty} \frac{1}{(|az+b||cz+d|)^{1+\varepsilon}}$$

$$= 4\cdot \sum_{\substack{a,b=-\infty \\ (a,b)=1}}^{\infty} \frac{1}{|az+b|^{2+\varepsilon}} \sum_{\substack{t=-\infty \\ 1\leq|\frac{c_0z+d_0}{az+b}+t|}}^{\infty} \frac{1}{|t+\frac{c_0z+d_0}{az+b}|^{1+\varepsilon}} \ll_{\varepsilon,K} 1.$$

The lemma is proved.

Lemma 23. *Let $y > 0$. Then for any $x \in \mathbb{R}$ and any $\operatorname{Re} s > 1$, the following relation is valid:*

$$\sum_{n=-\infty}^{\infty} \frac{1}{(y^2+(x+n)^2)^{s/2}} = \sqrt{\pi}\frac{\Gamma(\frac{s-1}{2})}{\Gamma(\frac{s}{2})}\, y^{1-s} + \frac{2\pi^{s/2}}{\Gamma(\frac{s}{2})}\, y^{\frac{1-s}{2}} \sum_{n=-\infty}^{\infty}{}' \, |m|^{\frac{s-1}{2}}\, K_{\frac{s-1}{2}}(2\pi|m|y)\, e^{2\pi imx}.$$

Proof. Obviously, the series

$$\sum_{n=-\infty}^{\infty} \frac{1}{(y^2+(x+n)^2)^{s/2}}$$

converges absolutely, and, for fixed y, it is a periodic continuous function in x with period 1. Now we calculate $\widehat{G}(m)$, the Fourier coefficients of the function

$$G(s;x,y) = \sum_{n=-\infty}^{\infty} \frac{1}{(y^2+(x+n)^2)^{s/2}}$$

($\operatorname{Re} s > 1$). For $m = 0$ we have

$$\widehat{G}(0) = \int_0^1 G(s;x,y)dx = \int_0^1 \left(\sum_{n=-\infty}^{\infty} \frac{1}{(y^2+(x+n)^2)^{s/2}} \right) dx$$

$$= \sum_{n=-\infty}^{\infty} \int_0^1 \frac{dx}{(y^2+(x+n)^2)^{s/2}} = \sum_{n=-\infty}^{\infty} \int_{-n}^{-n+1} \frac{dx}{(y^2+(x+n)^2)^{s/2}} = \int_{-\infty}^{\infty} \frac{dx}{(y^2+x^2)^{s/2}}.$$

After the change $x \to yx$, we obtain

$$\widehat{G}(0) = y^{1-s} \int_0^{\infty} \frac{dx}{(y^2+(x+n)^2)^{s/2}} = y^{1-s} \int_0^{\infty} (x_1+1)^{-s/2}\, x_1^{-1/2} dx_1 = y^{1-s} \int_1^{\infty} x_2^{-s/2} (x_2-1)^{-1/2} dx_2$$

$$= y^{1-s} \int_1^0 x_3^{s/2} \left(\frac{1}{x_3} - 1 \right)^{-1/2} \left(-\frac{1}{x_3^2} \right) dx_3 = y^{1-s} \int_0^1 x_3^{\frac{s-1}{2}-1} (1-x_3)^{1/2-1} dx_3$$

$$= y^{1-s} B\left(\frac{s-1}{2}, \frac{1}{2} \right) = y^{1-s} \frac{\Gamma(\frac{s-1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{s}{2})} = \sqrt{\pi}\, y^{1-s} \frac{\Gamma(\frac{s-1}{2})}{\Gamma(\frac{s}{2})};$$

i.e.,

$$\widehat{G}(0) = \sqrt{\pi}\, y^{1-s} \frac{\Gamma(\frac{s-1}{2})}{\Gamma(\frac{s}{2})}.$$

For $m \neq 0$,

$$\widehat{G}(m) = \int_0^1 G(s;y,x) e^{-2\pi i m x}\, dx = \int_0^1 \left(\sum_{n=-\infty}^{\infty} \frac{1}{(y^2+(x+n)^2)^{s/2}} \right) e^{-2\pi i m x}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{-2\pi i m x}}{(y^2+x^2)^{s/2}} dx = y^{1-s} \cdot \int_{-\infty}^{\infty} \frac{e^{-2\pi i m x}}{(1+x^2)^{s/2}} dx.$$

With the help of (1.10), we conclude that

$$\widehat{G}(m) = y^{1-s} \frac{2\sqrt{\pi}}{\Gamma(\frac{s}{2})} (\pi |m| y)^{\frac{s-1}{2}}\, K_{\frac{s-1}{2}}(2\pi |m| y) = \frac{2\pi^{s/2}}{\Gamma(\frac{s}{2})}\, y^{\frac{1-s}{2}}\, |m|^{\frac{s-1}{2}}\, K_{\frac{s-1}{2}}(2\pi |m| y).$$

Substituting the expressions for $\widehat{G}(m)$ into the formula

$$G(s;x,y) = \sum_{m=-\infty}^{\infty} \widehat{G}(m)\, e^{2\pi i m x}$$

(the Fourier expansion of $G(s;x,y)$), we arrive at the relation of the lemma.

Lemma 24. *Let m be an integer different from zero, $\operatorname{Re} s > 1$ and $\operatorname{Re} s_0 > \frac{1}{2}$. Then the following expansion holds:*

$$\P_m(z; s, s_0) = \sum_{\sigma \in \Gamma_\infty \backslash \Gamma} (2\pi|m| \operatorname{Im} \sigma z)^{\frac{s+1}{2}+s_0} K_{\frac{s-1}{2}}(2\pi|m| \operatorname{Im} \sigma z)\, e^{2\pi i m \operatorname{Re} \sigma z}$$

$$= \frac{\sqrt{2\pi}\, 2^{s/2+s_0}}{8\Gamma(\frac{s}{2}+s_0)} \sum_{j=1}^{\infty} \Gamma\Big(\frac{s}{2} - \frac{1}{4} + \frac{i\varkappa_j}{2} + \frac{s_0}{2}\Big) \Gamma\Big(\frac{s}{2} - \frac{1}{4} - \frac{i\varkappa_j}{2} + \frac{s_0}{2}\Big) \Gamma\Big(\frac{1}{4} + \frac{i\varkappa_j}{2} + \frac{s_0}{2}\Big)$$

$$\times \Gamma\Big(\frac{1}{4} - \frac{i\varkappa_j}{2} + \frac{s_0}{2}\Big) \sqrt{|m|}\, \overline{\rho_j(m)}\, u_j(z) + \frac{\sqrt{2}\; 2^{s/2+s_0}}{16\Gamma(\frac{s}{2}+s_0)} \int_{-\infty}^{\infty} \Gamma\Big(\frac{s}{2} - \frac{1}{4} + \frac{it}{2} + \frac{s_0}{2}\Big)$$

$$\times \Gamma\Big(\frac{s}{2} - \frac{1}{4} - \frac{it}{2} + \frac{s_0}{2}\Big) \Gamma\Big(\frac{1}{4} + \frac{it}{2} + \frac{s_0}{2}\Big) \Gamma\Big(\frac{1}{4} - \frac{it}{2} + \frac{s_0}{2}\Big) \frac{\pi^{-it} |m|^{1/2-it} \sigma_{2it}(|m|)}{\Gamma(\frac{1}{2} - it)\zeta(1-2it)}\, E\Big(z, \frac{1}{2} + it\Big)\, dt.$$

Proof. From the expansion of $K_s(y)$ in powers of y (see (1.8)), the following estimate holds:

$$K_{\frac{s-1}{2}}(y) \ll_s y^{\operatorname{Re}(\frac{1-s}{2})} + y^{\operatorname{Re}(\frac{s-1}{2})} \quad (y > 0). \tag{3.1}$$

Therefore,

$$\P_m(z; s, s_0) \ll_{m,s,s_0} y^{\operatorname{Re}(\frac{s+1}{2}+s_0)} \Big|K_{\frac{s-1}{2}}(2\pi|m|y)\Big|$$

$$+ \sum_{\substack{\sigma + \Gamma_\infty; \\ \sigma \neq \Gamma_\infty}} (\operatorname{Im} \sigma z)^{\operatorname{Re}(\frac{s+1}{2}+s_0)} \Big((\operatorname{Im} \sigma z)^{\operatorname{Re}(\frac{1-s}{2})} + (\operatorname{Im} \sigma z)^{\operatorname{Re}(\frac{s-1}{2})}\Big)$$

$$= y^{\operatorname{Re}(\frac{s+1}{2}+s_0)} \Big|K_{\frac{s-1}{2}}(2\pi|m|y)\Big| + E(z, \operatorname{Re}(1+s_0)) - y^{\operatorname{Re}(1+s_0)} + E(z, \operatorname{Re}(s+s_0)) - y^{\operatorname{Re}(s+s_0)}.$$

Expansion (1.21) and estimate (1.13) yield

$$\P_m(z; s, s_0) \ll_{m,s,s_0} y^{\operatorname{Re}(1-s_0)}.$$

Therefore, for $\operatorname{Re} s > 1$ and $\operatorname{Re} s_0 > \frac{1}{2}$,

$$\P_m(z; s, s_0) \in \mathcal{L}^2(\Gamma \setminus H)$$

and the integral

$$\iint_{\Gamma\backslash H} \P_m(z; s, s_0) y^{1/2}\, dz$$

converges absolutely; i.e., $\P_m(z; s, s_0)$ satisfies condition (a) of Lemma 12. Calculate the inner products:

$$(\P_m(z; s, s_0), u_j(z)), \quad (\P_m(z; s, s_0), E\Big(z, \frac{1}{2} + it\Big)).$$

We have

$$(\P_m(z; s, s_0), u_j(z)) = \iint_{\Gamma\backslash H} \P_m(z; s, s_0),\ \overline{u_j(z)}\, dz$$

$$= \iint\limits_{\Gamma\backslash H} \Big(\sum_{\sigma\in\Gamma_\infty\backslash\Gamma} (2\pi|m|\operatorname{Im}\sigma z)^{\frac{s+1}{2}+s_0} K_{\frac{s-1}{2}}(2\pi|m|\operatorname{Im}\sigma z)\, e^{2\pi i m \operatorname{Re}\sigma z}\Big) \overline{u_j(z)}\, dz$$

$$= \sum_{\sigma\in\Gamma_\infty\backslash\Gamma} \iint\limits_{\sigma(\Gamma\backslash H)} 2\pi|m|y)^{\frac{s+1}{2}+s_0} K_{\frac{s-1}{2}}(2\pi|m|y)\, e^{2\pi i m \operatorname{Re}\sigma z}\Big) \overline{u_j(z)}\, \frac{dx\,dy}{y^2}$$

$$= \iint\limits_{\Gamma_\infty\backslash\Gamma} (2\pi|m|y)^{\frac{s+1}{2}+s_0} K_{\frac{s-1}{2}}(2\pi|m|y)\, e^{2\pi i m \operatorname{Re}\sigma z}\Big) \overline{u_j(z)}\, \frac{dx\,dy}{y^2}.$$

With the help of (1.20), this implies that

$$(\P_m(z;s,s_0),u_j(z)) = \int_0^\infty \Big(\int_0^1 e^{2\pi i m x}\, \overline{u_j(z)}\, dx\Big) (2\pi|m|y)^{\frac{s+1}{2}+s_0} K_{\frac{s-1}{2}}(2\pi|m|y)\, \frac{dy}{y^2}$$

$$= \overline{\rho_j(m)} \int_0^\infty K_{i\varkappa_j}(2\pi|m|y)\, K_{\frac{s-1}{2}}(2\pi|m|y)\,(2\pi|m|y)(2\pi|m|y)^{\frac{s+1}{2}+s_0}\, \frac{dy}{y^{3/2}}$$

$$= \sqrt{2\pi|m|}\,\overline{\rho_j(m)} \int_0^\infty K_{i\varkappa_j}(y)\, K_{\frac{s-1}{2}}(y)\, y^{\frac{s}{2}+s_0-1}\, dy.$$

Using formula (1.12), we find

$$(\P_m(z;s,s_0),u_j(z)) = \frac{\sqrt{2\pi}\, 2^{s/2+s_0}}{8\Gamma(\frac{s}{2}+s_0)} \sum_{j=1}^\infty \Gamma\Big(\frac{s}{2}-\frac14+\frac{i\varkappa_j}{2}+\frac{s_0}{2}\Big)\Gamma\Big(\frac{s}{2}-\frac14-\frac{i\varkappa_j}{2}+\frac{s_0}{2}\Big)\Gamma\Big(\frac14+\frac{i\varkappa_j}{2}+\frac{s_0}{2}\Big)$$

$$\times\Gamma\Big(\frac14-\frac{i\varkappa_j}{2}+\frac{s_0}{2}\Big)\sqrt{|m|}\,\overline{\rho_j(m)}.$$

Now we calculate the corresponding inner product for the continuous spectrum. Using the previous arguments, we obtain

$$\Big(\P_m(z;s,s_0), E\Big(z,\frac12+it\Big)\Big) = \iint\limits_{\Gamma\backslash H} \P_m(z;s,s_0)\, \overline{E\Big(z,\frac12+it\Big)}\, dz$$

$$= \int_0^\infty \Big(\int_0^1 e^{2\pi i m x}\, \overline{E\Big(z,\frac12+it\Big)}\, dx\Big) (2\pi|m|y)^{\frac{s+1}{2}+s_0} K_{\frac{s-1}{2}}(2\pi|m|y)\, \frac{dy}{y^2}.$$

From (1.2), we see that

$$\Big(\P_m(z;s,s_0), E\Big(z,\frac12+it\Big)\Big) = \frac{2\pi^{1/2-it}|m|^{-it}\sigma_{2it}(|m|)}{\Gamma(\frac12-it)\zeta(1-2it)}$$

$$\times \int_0^\infty K_{it}(2\pi|m|y)\, K_{\frac{s-1}{2}}(2\pi|m|y)\,(2\pi|m|y)(2\pi|m|y)^{\frac{s+1}{2}+s_0}\, \frac{dy}{y^{3/2}}$$

$$\frac{\sqrt{2\pi}\,2^{s/2+s_0}}{4\Gamma(\frac{s}{2}+s_0)}\Gamma\Big(\frac{s}{2}-\frac{1}{4}+\frac{it}{2}+\frac{s_0}{2}\Big)\Gamma\Big(\frac{s}{2}-\frac{1}{4}-\frac{it}{2}+\frac{s_0}{2}\Big)\Gamma\Big(\frac{1}{4}+\frac{it}{2}+\frac{s_0}{2}\Big)$$
$$\times\Gamma\Big(\frac{1}{4}-\frac{it}{2}+\frac{s_0}{2}\Big)\frac{\pi^{-it}|m|^{-it}\sigma_{2it}(|m|)}{\Gamma(\frac{1}{2}-it)\zeta(1-2it)}\sqrt{|m|}.$$

With the help of (1.16) and the estimate of Lemma 14, we arrive at

$$(\P_m(z;s,s_0),u_j(z))\ll_{s,s_0}\varkappa_j^{c(s,s_0)}\,e^{-\pi\varkappa_j}\sqrt{|m|}\,|\rho_j(m)|$$

and

$$\Big(\P_m(z;s,s_0),E\Big(z,\frac{1}{2}+it\Big)\Big)\ll_{s,s_0}(1+|t|)^{c(s,s_0)}\,e^{-\pi|t|}\sqrt{|m|}\,\tau|(m)|\,e^{\frac{\pi|t|}{2}}.$$

These estimates and Lemmas 10, 11, and 15 imply that $\P_m(z;s,s_0)$ satisfies condition (b) of Lemma 12. Therefore,

$$(\P_m(z;s,s_0))=\sum_{j=1}^{\infty}(\P_m(z;s,s_0),u_j(z))u_j(z)+\frac{1}{4\pi}\int_{-\infty}^{\infty}\Big(\P_m(z;s,s_0),E\Big(z,\frac{1}{2}+it\Big)\Big)E\Big(z,\frac{1}{2}+it\Big)\,dt.$$

Substituting the above expressions for the inner products into this formula, we obtain the required relation. The lemma is proved.

For $z\in H$ and $s,\ s_0\in\mathbb{C}$, we set

$$\Pi(z;s,s_0)=\sum_{\sigma\in\Gamma}\frac{(\operatorname{Im}\sigma z)^{s+s_0}}{|\sigma z|^s}. \tag{3.2}$$

Lemma 25. *Let* $\operatorname{Re}s>1$, $\operatorname{Re}s_0>\frac{1}{2}$, $z\in H$. *Then the function* $\Pi(z;s,s_0)$ *defined in* (3.2) *is expressed as*

$$\Pi(z;s,s_0)=\sqrt{\pi}\frac{\Gamma(\frac{s-1}{2})}{\Gamma\frac{s}{2}}E(z,1+s_0)+\frac{\pi^{-s_0}}{4\Gamma(\frac{s}{2})\Gamma(\frac{s}{2}+s_0)}$$
$$\sum_{j=1}^{\infty}\Big(\Gamma\Big(\frac{s}{2}-\frac{1}{4}+\frac{i\varkappa_j}{2}+\frac{s_0}{2}\Big)\Gamma\Big(\frac{s}{2}-\frac{1}{4}-\frac{i\varkappa_j}{2}+\frac{s_0}{2}\Big)\Gamma\Big(\frac{1}{4}+\frac{i\varkappa_j}{2}+\frac{s_0}{2}\Big)$$
$$\times\Gamma\Big(\frac{1}{4}-\frac{i\varkappa_j}{2}+\frac{s_0}{2}\Big)\overline{\mathcal{L}_j\Big(\frac{1}{2}+\overline{s}_0\Big)}u_j(z)\Big)+\frac{\pi^{-s_0}}{4\Gamma(\frac{s}{2})\Gamma(\frac{s}{2}+s_0)}$$
$$\times\int_{-\infty}^{\infty}\Gamma\Big(\frac{s}{2}-\frac{1}{4}+\frac{it}{2}+\frac{s_0}{2}\Big)\Gamma\Big(\frac{s}{2}-\frac{1}{4}-\frac{it}{2}+\frac{s_0}{2}\Big)\Gamma\Big(\frac{1}{4}+\frac{it}{2}+\frac{s_0}{2}\Big)$$
$$\Gamma\Big(\frac{1}{4}-\frac{it}{2}+\frac{s_0}{2}\Big)\pi^{-it}\frac{\zeta(\frac{1}{2}+it+s_0)\zeta(\frac{1}{2}-it+s_0)}{\Gamma(\frac{1}{2}-it)\zeta(1-2it)}E\Big(z,\frac{1}{2}+it\Big)\,dt.$$

Proof. By definition,

$$\Pi(z;s,s_0)=\sum_{\sigma\in\Gamma}(\operatorname{Im}\sigma z)^{s+s_0}\frac{1}{|\sigma z|^s}=\sum_{\sigma\in\Gamma_\infty\backslash\Gamma}(\operatorname{Im}\sigma z)^{s+s_0}\sum_{n=-\infty}^{\infty}\frac{1}{((\operatorname{Im}\sigma z)^2+(\operatorname{Re}\sigma z+n)^2)^{s/2}}.$$

With the help of Lemma 23, we obtain

$$\Pi(z;s,s_0)=\sum_{\sigma\in\Gamma}(\operatorname{Im}\sigma z)^{s+s_0}\Bigg(\sqrt{\pi}\frac{\Gamma(\frac{s-1}{2})}{\Gamma(\frac{s}{2})}(\operatorname{Im}\sigma z)^{1-s}$$

$$+\frac{2\pi^{s/2}}{\Gamma(\frac{s}{2})}\sum_{n=-\infty}^{\infty}{}'\,|m|^{\frac{s-1}{2}}\,(\operatorname{Im}\sigma z)^{\frac{1-s}{2}}\,K_{\frac{s-1}{2}}(2\pi|m|\operatorname{Im}\sigma z)\,e^{2\pi im\operatorname{Re}\sigma z}\Bigg)$$

$$=\sqrt{\pi}\frac{\Gamma(\frac{s-1}{2})}{\Gamma(\frac{s}{2})}E(z,1+s_0)+\sqrt{\pi}\frac{\Gamma(\frac{s-1}{2})}{\Gamma(\frac{s}{2})}E(z,1+s_0)+\frac{(2\pi)^{-s_0}2^{-s/2+1/2}}{\sqrt{\pi}\Gamma(\frac{s}{2})}\sum_{\sigma\in\Gamma_\infty\backslash\Gamma}\sum_{n=-\infty}^{\infty}{}'\,|m|^{-1-s_0}$$

$$(2\pi|m|\operatorname{Im}\sigma z)^{\frac{s+1}{2}+s_0}\,K_{\frac{s-1}{2}}\,(2\pi|m|\operatorname{Im}\sigma z)\,e^{2\pi im\Re\sigma z}.$$

Interchanging summation over σ and m (the validity of such a change is ensured by estimates (1.13) and (3.1)), we find

$$\Pi(z;s,s_0)=\sqrt{\pi}\frac{\Gamma(\frac{s-1}{2})}{\Gamma\frac{s}{2}}E(z,1+s_0)$$

$$+\frac{(2\pi)^{-s_0}2^{-s/2+1/2}}{\sqrt{\pi}\Gamma(\frac{s}{2})}\sum_{n=-\infty}^{\infty}{}'\,|m|^{-1-s_0}\Bigg(\sum_{\sigma\in\Gamma_\infty\backslash\Gamma}(2\pi|m|\operatorname{Im}\sigma z\Bigg)^{\frac{s+1}{2}+s_0}$$

$$\times K_{\frac{s-1}{2}}(2\pi|m|\operatorname{Im}\sigma z)\,e^{2\pi im\operatorname{Re}\sigma z}\Bigg)=\sqrt{\pi}\frac{\Gamma(\frac{s-1}{2})}{\Gamma(\frac{s}{2})}E(z,1+s_0)$$

$$+\frac{(2\pi)^{-s_0}2^{-s/2+1/2}}{\sqrt{\pi}\Gamma(\frac{s}{2})}\sum_{n=-\infty}^{\infty}{}'\,|m|^{-1-s_0}\,\P_m(z;s,s_0).$$

From Lemma 24 it follows that

$$\Pi(z;s,s_0)=\sqrt{\pi}\frac{\Gamma(\frac{s-1}{2})}{\Gamma(\frac{s}{2})}E(z,1+s_0)+\frac{\pi^{-s_0}}{4\Gamma(\frac{s}{2})\Gamma(\frac{s}{2}+s_0)}$$

$$\times\sum_{j=1}^{\infty}\Gamma\left(\frac{s}{2}-\frac{1}{4}+\frac{i\varkappa_j}{2}+\frac{s_0}{2}\right)\Gamma\left(\frac{s}{2}-\frac{1}{4}-\frac{i\varkappa_j}{2}+\frac{s_0}{2}\right)\Gamma\left(\frac{1}{4}+\frac{i\varkappa_j}{2}+\frac{s_0}{2}\right)$$

$$\times\Gamma\left(\frac{1}{4}-\frac{i\varkappa_j}{2}+\frac{s_0}{2}\right)\sqrt{|m|}\;\overline{\rho_j(m)}u_j(z)+\frac{\pi^{-s_0}}{8\sqrt{\pi}\Gamma(\frac{s}{2})\Gamma(\frac{s}{2}+s_0)}$$

$$\times\sum_{n=-\infty}^{\infty}{}'\,|m|^{-1-s_0}\int_{-\infty}^{\infty}\Gamma\left(\frac{s}{2}-\frac{1}{4}+\frac{it}{2}+\frac{s_0}{2}\right)\times\Gamma\left(\frac{s}{2}-\frac{1}{4}-\frac{it}{2}+\frac{s_0}{2}\right)$$

$$\times\Gamma\left(\frac{1}{4}+\frac{it}{2}+\frac{s_0}{2}\right)\Gamma\left(\frac{1}{4}-\frac{it}{2}+\frac{s_0}{2}\right)\frac{\pi^{-it}|m|^{it}\sigma_{2it}(|m|)}{\Gamma(\frac{1}{2}-it)\zeta(1-2it)}\,E\left(z,\frac{1}{2}+it\right)dt.$$

Since

$$\sum_{n=-\infty}^{\infty}{}'\,|m|^{-1/2-s_0}\,\overline{\rho_j(m)}=\overline{\mathcal{L}_j\left(\frac{1}{2}+\overline{s}_0\right)}$$

and

$$\sum_{n=-\infty}^{\infty}{}' |m|^{-1/2-s_0-it}\,\sigma_{2it}(|m|) = 2\cdot\sum_{m=1}^{\infty}|m|^{-1/2-s_0-it}\sum_{m_1\backslash m} m_1^{2it}$$

$$= 2\cdot\sum_{m_1,m_2=1}^{\infty} m_1^{-1/2-s_0-it}\,m_2^{-1/2-s_0-it} = 2\zeta\left(\frac{1}{2}-it+s_0\right)\zeta\left(\frac{1}{2}+it+s_0\right),$$

we have

$$\Pi(z;s,s_0) = \sqrt{\pi}\frac{\Gamma(\frac{s-1}{2})}{\Gamma(\frac{s}{2})}E(z,1+s_0) + \frac{\pi^{-s_0}}{4\Gamma(\frac{s}{2})\Gamma(\frac{s}{2}+s_0)}$$

$$\times\sum_{j=1}^{\infty}\Gamma\left(\frac{s}{2}-\frac{1}{4}+\frac{i\varkappa_j}{2}+\frac{s_0}{2}\right)\Gamma\left(\frac{s}{2}-\frac{1}{4}-\frac{i\varkappa_j}{2}+\frac{s_0}{2}\right)\Gamma\left(\frac{1}{4}+\frac{i\varkappa_j}{2}+\frac{s_0}{2}\right)$$

$$\times\Gamma\left(\frac{1}{4}-\frac{i\varkappa_j}{2}+\frac{s_0}{2}\right)\overline{\mathcal{L}_j\left(\frac{1}{2}+\overline{s}_0\right)}\,u_j(z) + \frac{\pi^{-s_0-1/2}}{4\sqrt{\pi}\Gamma(\frac{s}{2})\Gamma(\frac{s}{2}+s_0)}$$

$$\times\int_{-\infty}^{\infty}\Gamma\left(\frac{s}{2}-\frac{1}{4}+\frac{it}{2}+\frac{s_0}{2}\right)\times\Gamma\left(\frac{s}{2}-\frac{1}{4}-\frac{it}{2}+\frac{s_0}{2}\right)$$

$$\times\Gamma\left(\frac{1}{4}+\frac{it}{2}+\frac{s_0}{2}\right)\Gamma\left(\frac{1}{4}-\frac{it}{2}+\frac{s_0}{2}\right)\frac{\pi^{-it}}{\Gamma(\frac{1}{2}-it)\zeta(1-2it)}$$

$$\times\zeta\left(\frac{1}{2}-it+s_0\right)\zeta\left(\frac{1}{2}+it+s_0\right)E\left(z,\frac{1}{2}+it\right)dt.$$

The lemma is proved.

As usual, we denote by $\eta(z)$ the Dedekind η-function defined by the relation

$$\eta(z) = e^{\frac{\pi i z}{12}}\prod_{n=1}^{\infty}\left(1-e^{2\pi i n z}\right)\quad (z\in H).$$

In the sequel, the following well-known first limit Kronecker's formula will be required:

$$E(z,s) = \frac{3}{\pi(s-1)} + \frac{3}{\pi^2}C - \frac{3}{4\pi^2}\ln\left(\left|y^{1/4}\eta(z)\right|\right) + c_1(z)(s-1)+\ldots \tag{3.3}$$

(Laurent-series expansion of $E(z,s)$ at the point $s=1$. Here, C is the Euler constant).

Lemma 26.

$$\Pi(z;s) = \Pi(z;s,s_0) = \frac{-3}{\sqrt{\pi}}\cdot\frac{\Gamma'(\frac{s-1}{2})}{\Gamma(\frac{s}{2})} + \frac{6}{\pi\sqrt{\pi}}\frac{\Gamma(\frac{s}{2}-\frac{1}{4})}{\Gamma(\frac{s}{2})}$$

$$\times\left(2c-\ln\left(\left|y^{1/4}\eta(z)\right|\right)\right) + \frac{3}{\sqrt{\pi}}\frac{\Gamma(\frac{s}{2}-\frac{1}{4})\Gamma'(\frac{s}{2})}{\Gamma^2(\frac{s}{2})}$$

$$+\frac{1}{4\Gamma^2(\frac{s}{3})}\sum_{j=1}^{\infty}\left(\Gamma\left(\frac{s}{2}-\frac{1}{4}+\frac{i\varkappa_j}{2}\right)\Gamma\left(\frac{s}{2}-\frac{1}{4}-\frac{i\varkappa_j}{2}\right)\Gamma\left(\frac{1}{4}+\frac{i\varkappa_j}{2}\right)\right.$$

$$\times\Gamma\left(\frac{1}{4}-\frac{i\varkappa_j}{2}\right)\overline{\mathcal{L}_j\left(\frac{1}{2}\right)}\,u_j(z)\Bigg)+\frac{1}{4\sqrt{\pi}\,\Gamma^2(\frac{s}{2})}\times\int\limits_{-\infty}^{\infty}\Gamma\left(\frac{s}{2}-\frac{1}{4}+\frac{it}{2}\right)\times\Gamma\left(\frac{s}{2}-\frac{1}{4}-\frac{it}{2}\right)$$

$$\times\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)\frac{\pi^{-it}|\zeta(\frac{1}{2}+it)|^2}{\Gamma(\frac{1}{2}-it)\zeta(1-2it)}E\left(z,\frac{1}{2}+it\right)dt.$$

Proof. From the estimate of Lemma 22, it follows that for $\operatorname{Re}s>1$ and $\operatorname{Re}s_0>0$, the function $\Pi(z;s,s_0)$ is holomorphic in s_0; moreover, for $\operatorname{Re}s>\frac{1}{2}$, the expansion in eigenfunctions of the Laplace–Beltrami operator, indicated in Lemma 25, holds. From the estimates of Lemmas 10 and 19, as well as from (1.16), it follows that the function

$$\frac{\pi^{-s_0}}{4\Gamma(\frac{s}{2})\Gamma(\frac{s}{2}+s_0)}\sum_{j=1}^{\infty}\Bigg(\Gamma\left(\frac{s}{2}-\frac{1}{4}+\frac{i\varkappa_j}{2}+\frac{s_0}{2}\right)\Gamma\left(\frac{s}{2}-\frac{1}{4}-\frac{i\varkappa_j}{2}+\frac{s_0}{2}\right)$$

$$\times\Gamma\left(\frac{1}{4}+\frac{i\varkappa_j}{2}+\frac{s_0}{2}\right)\times\Gamma\left(\frac{1}{4}-\frac{i\varkappa_j}{2}+\frac{s_0}{2}\right)\overline{\mathcal{L}_j\left(\frac{1}{2}+\overline{s}_0\right)}\,u_j(z)\Bigg) \tag{3.4}$$

is extended holomorphically in s_0 to the strip $\operatorname{Re}s_0>0$. Difficulties emerge when we extend in s_0 the integral

$$\int\limits_{-\infty}^{\infty}\Gamma\left(\frac{s}{2}-\frac{1}{4}+\frac{it}{2}+\frac{s_0}{2}\right)\Gamma\left(\frac{s}{2}-\frac{1}{4}-\frac{it}{2}+\frac{s_0}{2}\right)\Gamma\left(\frac{1}{4}+\frac{it}{2}+\frac{s_0}{2}\right)$$

$$\times\Gamma\left(\frac{1}{4}-\frac{it}{2}+\frac{s_0}{2}\right)\pi^{-it}\frac{\zeta(\frac{1}{2}+it+s_0)\zeta(\frac{1}{2}-it+s_0)}{\Gamma(\frac{1}{2}-it)\zeta(1-2it)} \tag{3.5}$$

$$E\left(z,\frac{1}{2}+it\right)dt,$$

to the same strip, because for $\operatorname{Re}s_0=1/2$, the expression under the integral sign has singularities at the points $t=\pm\operatorname{Im}s_0$ (at the poles of the zeta function). At the same time, for $0<\operatorname{Re}s_0<\frac{1}{2}$, integral (3.5) is well defined and is a holomorphic function in s_0 in this strip. Moreover, as was mentioned above, $\Pi(z;s,s_0)$ is holomorphic in s_0 in the strip $\operatorname{Re}s_0<0$. Consequently, in order to write a formula for $\Pi(z;s,s_0)$ (in the strip $0<\operatorname{Re}s_0<\frac{1}{2}$) similar to the formula from Lemma 25, we need to find a relation (in the sense of holomorphic extension) connecting integral (3.5) for the values of s_0 lying to the right of the straight line $\operatorname{Re}s_0=\frac{1}{2}$ and integral (3.5) for the values of s_0 in the strip bounded by the lines $\operatorname{Re}s_0=0$ and $\operatorname{Re}s_0=\frac{1}{2}$. By $W^+(z;s,s_0)$ we denote the function (holomorphic in s_0) defined by integral (3.5) for $\operatorname{Re}s_0>\frac{1}{2}$, and let $W^-(z;s,s_0)$ denote the function (holomorphic in s_0) defined by the same integral for $0<\operatorname{Re}s_0<\frac{1}{2}$. We write the integrand in (3.5) in the form

$$\frac{\psi_1(z;s,s_0}{-\frac{1}{2}+it+s_0}+\frac{\psi_2(z;s,s_0}{-\frac{1}{2}-it+s_0}+\psi(z;s,s_0,t),$$

where

$$\psi_1(z;s,s_0),\quad \psi_2(z;s,s_0),\quad\text{and}\quad \psi(z;s,s_0,t)$$

are holomorphic in s_0 in the strip $\operatorname{Re} s_0 > 1$ (we extract the poles of the zeta function that occur in the integrand). Now we can write

$$W^+(z;s,s_0) = \psi_1(z;s,s_0)\int\limits_{-\infty}^{\infty}\frac{dt}{-\frac{1}{2}+it+s_0} + \psi_2(z;s,s_0)\int\limits_{-\infty}^{\infty}\frac{dt}{-\frac{1}{2}-it+s_0} + \int\limits_{-\infty}^{\infty}\psi(z;s,s_0,t)\,dt$$

for $\operatorname{Re} s_0 > \frac{1}{2}$. Similarly,

$$W^-(z;s,s_0) = \psi_1(z;s,s_0)\int\limits_{-\infty}^{\infty}\frac{dt}{-\frac{1}{2}+it+s_0} + \psi_2(z;s,s_0)\int\limits_{-\infty}^{\infty}\frac{dt}{-\frac{1}{2}+it+s_0} + \int\limits_{-\infty}^{\infty}\psi(z;s,s_0,t)\,dt$$

for $\operatorname{Re} s_0 < \frac{1}{2}$. From the definition of the function $\psi(z;s,s_0,t)$, it follows that the integral

$$\int\limits_{-\infty}^{\infty}\psi(z;s,s_0,t)\,dt$$

defines a function holomorphic in s_0 in the strip $\operatorname{Re} s_0 > 0$. Therefore, it is sufficient to compare the holomorphic extension of the function

$$\psi_1(z;s,s_0)\int\limits_{-\infty}^{\infty}\frac{dt}{-\frac{1}{2}+it+s_0} + \psi_2(z;s,s_0)\int\limits_{-\infty}^{\infty}\frac{dt}{-\frac{1}{2}-it+s_0}$$

$$= \big(\psi_1(z;s,s_0)+\psi_2(z;s,s_0)\big)\int\limits_{-\infty}^{\infty}\frac{dt}{-\frac{1}{2}-it+s_0}$$

to the strip $0 < \operatorname{Re} s_0 < 1$ (this follows from the fact that $\Pi(z;s,s_0)$ is a function holomorphic in s_0) with the expression

$$\big(\psi_1(z;s,s_0)+\psi_2(z;s,s_0)\big)\int\limits_{-\infty}^{\infty}\frac{dt}{-\frac{1}{2}-it+s_0},$$

defining a function holomorphic in s_0 in the strip $0 < \operatorname{Re} s_0 < \frac{1}{2}$. For $r > 0$,

$$\int\limits_{-\infty}^{\infty}\frac{dt}{-\frac{1}{2}-it+(\frac{1}{2}+r+it_0)} = \int\limits_{-\infty}^{\infty}\frac{dt}{r+i(t+t_0)} = \int\limits_{-\infty}^{\infty}\frac{2r\,dt}{r^2+t^2} = \pi.$$

Similarly, for $r < 0$ we have

$$\int\limits_{-\infty}^{\infty}\frac{dt}{-\frac{1}{2}-it+(\frac{1}{2}+r+it_0)} = -\pi.$$

Therefore,

$$W^+(z;s,s_0) = \pi\big(\psi_1(z;s,s_0) + \psi_2(z;s,s_0)\big) + \int\limits_{-\infty}^{\infty} \psi(z;s,s_0,t)\,dt$$

and

$$W^-(z;s,s_0) = -\pi\big(\psi_1(z;s,s_0) + \psi_2(z;s,s_0)\big) + \int\limits_{-\infty}^{\infty} \psi(z;s,s_0,t)\,dt$$

This implies that for the holomorphic extension in s_0 of the function $W^+(z;s,s_0)$ to the strip $0 < \operatorname{Re} s_0 < \frac{1}{2}$, the following relation is valid:

$$W^+(z;s,s_0) = 2\pi\big(\psi_1(z;s,s_0) + \psi_2(z;s,s_0)\big) + W^-(z;s,s_0).$$

From the definition of the quantities $\psi_1(z;s,s_0)$ and $\psi_2(z;s,s_0)$, we deduce that

$$\psi_1(z;s,s_0) + \psi_2(z;s,s_0) = \Gamma\left(\frac{s}{2} - \frac{1}{2} + \frac{1}{2}(1-s_0) + \frac{s_0}{2}\right)\Gamma\left(\frac{s}{2} - \frac{1}{2}(1-s_0) + \frac{s_0}{2}\right)$$

$$\times\Gamma\left(\frac{1}{2}(1-s_0) + \frac{s_0}{2}\right)\Gamma\left(\frac{s}{2} - \frac{1}{2}(1-s_0) + \frac{s_0}{2}\right)\pi^{1/2}\,\pi^{-(-s_0)}$$

$$\times\frac{\zeta(1-(1-s_0)+s_0)}{\Gamma(1-(1-s_0))\zeta(2-2(1-s_0))}\,E(z,1-s_0) + \Gamma\left(\frac{s}{2} - \frac{1}{2} + s_0\right)$$

$$\times\Gamma\left(\frac{s}{2}\right)\Gamma(s_0)\Gamma\left(\frac{1}{2}\right)\pi^{1/2}\,\pi^{-s_0}\,\frac{\zeta(2s_0)}{\Gamma(1-s_0)\zeta(2-2s_0)}\,E(z,s_0)$$

$$= \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2} - \frac{1}{2} + s_0\right)\pi^{s_0}\,E(z,1-s_0) + \frac{\Gamma(\frac{s}{2})\Gamma(\frac{s}{2} - \frac{1}{2} + s_0)\Gamma(s_0)\zeta(2s_0)\,\pi^{1-s_0}}{\Gamma(1-s_0)\,\zeta(2-2s_0)}\,E(z,s_0).$$

With the help of the functional equation for $E(z,s_0)$ (see (1.8)), we find

$$\psi_1(z;s,s_0) + \psi_2(z;s,s_0) = \pi^{s_0}\cdot\Gamma\left(\frac{s}{2} - \frac{1}{2} + s_0\right)\Gamma\left(\frac{s}{2}\right)\,E(z,1-s_0)$$

$$+\pi^{s_0}\Gamma\left(\frac{s}{2} - \frac{1}{2} + s_0\right)\Gamma\left(\frac{s}{2}\right)\,E(z,1-s_0)$$

$$= 2\pi^{s_0}\Gamma\left(\frac{s}{2} - \frac{1}{2} + s_0\right)\Gamma\left(\frac{s}{2}\right)\,E(z,1-s_0).$$

Combining all these formulas for $0 < \operatorname{Re} s_0 < \frac{1}{2}$ we obtain

$$\Pi(z;s,s_0) = \sqrt{\pi}\frac{\Gamma(\frac{s-1}{2})}{\Gamma(\frac{s}{2})}E(z,1+s_0) - \sqrt{\pi} + \frac{\Gamma(\frac{s-1}{2} + s_0)}{\Gamma(\frac{s}{2} + s_0)}E(z,1+s_0)$$

$$+\frac{\pi^{-s_0}}{4\Gamma(\frac{s}{2})\Gamma(\frac{s}{2}+s_0)}\sum_{j=1}^{\infty}\left(\Gamma\left(\frac{s}{2} - \frac{1}{4} + \frac{i\varkappa_j}{2} + \frac{s_0}{2}\right)\Gamma\left(\frac{s}{2} - \frac{1}{4} - \frac{i\varkappa_j}{2} + \frac{s_0}{2}\right)\right.$$

$$\times\Gamma\Big(\frac14+\frac{i\varkappa_j}{2}+\frac{s_0}{2}\Big)\Gamma\Big(\frac14-\frac{i\varkappa_j}{2}+\frac{s_0}{2}\Big)\,\overline{\mathcal{L}_j\Big(\frac12+\overline{s}_0\Big)\,u_j(z)\Big)}$$

$$+\frac{\pi^{-s_0-1/2}}{4\Gamma(\frac{s}{2})\Gamma(\frac{s}{2}+s_0)}\int\limits_{-\infty}^{\infty}\Gamma\Big(\frac{s}{2}-\frac14+\frac{it}{2}+\frac{s_0}{2}\Big)\Gamma\Big(\frac{s}{2}-\frac14-\frac{it}{2}+\frac{s_0}{2}\Big)$$

$$\times\Gamma\Big(\frac14+\frac{it}{2}+\frac{s_0}{2}\Big)\Gamma\Big(\frac14-\frac{it}{2}+\frac{s_0}{2}\Big)\,\pi^{-it}\frac{\zeta(\frac12+it+s_0)\,\zeta(\frac12-it+s_0)}{\Gamma(\frac12-it)\,\zeta(1-2it)}E\Big(z,\frac12+it\Big)\,dt.$$

Turning s_0 to zero in this relation and using (3.3) we arrive at the required relation. The lemma is proved.

Proposition 2. *The series*

$$\sum_{n=-\infty}^{\infty}\frac{\tau(n^2+\mathcal{D})}{(n^2+\mathcal{D})^{s/2}}$$

(which converges for $\operatorname{Re}s>1$*) is extended in* s *meromorphically to the whole plane of complex variable, and the poles of this series belong to the set*

$$\{1-2k\big|k=0,1,\dots\}\bigcup\Big\{\frac12\pm i\varkappa_j-2k\big|j=1,2,\dots;k=0,1,\dots\Big\}$$

$$\bigcup\Big\{\frac12\rho-2k\big|\zeta(\rho)=0;k=0,1,\dots\Big\}. \qquad (3.6)$$

Proof. Consider the expression

$$-\frac{3}{\sqrt{\pi}}\frac{\Gamma'(\frac{s-1}{2})}{\Gamma(\frac{s}{2})}+\frac{6}{\pi\sqrt{\pi}}\frac{\Gamma(\frac{s-1}{2})}{\Gamma(\frac{s}{2})}\Big(2c-\ln\Big(|\,y^{1/4}\,\eta(z)\,|\Big)\Big)+\frac{3}{\sqrt{\pi}}\frac{\Gamma(\frac{s-1}{2})\Gamma'(\frac{s}{2})}{\Gamma^2(\frac{s}{2})}$$

$$+\frac{1}{4\Gamma^2(\frac{s}{2})}\sum_{j=1}^{\infty}\Big(\Gamma\Big(\frac{s}{2}-\frac14+\frac{i\varkappa_j}{2}\Big)\Gamma\Big(\frac{s}{2}-\frac14-\frac{i\varkappa_j}{2}\Big)\Gamma\Big(\frac14+\frac{i\varkappa_j}{2}\Big)\times\Gamma\Big(\frac14-\frac{i\varkappa_j}{2}\Big)\,\overline{\mathcal{L}_j\Big(\frac12+\overline{s}_0\Big)}\,u_j(z).$$

From the estimates if Lemmas 10 and 19, it follows that this expression defines a function meromorphic in s, whose poles belong to the set

$$\{1-2k\big|k=0,1,\dots\}\bigcup\Big\{\frac12\pm i\varkappa_j-2k\big|j=1,2,\dots;k=0,1,\dots\Big\}$$

Now consider the integral

$$\frac{1}{2\sqrt{\pi}\Gamma^2(\frac{s}{2})}\int\limits_{-\infty}^{\infty}\Gamma\Big(\frac{s}{2}-\frac14+\frac{it}{2}\Big)\Gamma\Big(\frac{s}{2}-\frac14-\frac{it}{2}\Big)\Gamma\Big(\frac14+\frac{it}{2}\Big)$$

$$\times\Gamma\Big(\frac14-\frac{it}{2}\Big)\frac{\pi^{-it}|\zeta(\frac12+it)|^2}{\Gamma(\frac12-it)\zeta(1-2it)}E\Big(z,\frac12+it\Big)\,dt$$

$$= \frac{\sqrt{\pi}}{2\Gamma^2(\frac{s}{2})} \cdot \frac{1}{2\pi i} \int\limits_{1/2-i\infty}^{1/2+i\infty} \Gamma\Big(\frac{s}{2} - \frac{1}{2} + \frac{1}{2}\omega\Big)\Gamma\Big(\frac{s}{2} - \frac{1}{2}\omega\Big)\Gamma\Big(\frac{1}{2}\omega\Big)$$

$$\Gamma\Big(\frac{1}{2} - \frac{1}{2}\omega\Big)\pi^{1/2-\omega}\frac{\zeta(\omega)\,\zeta(1-\omega)}{\Gamma(1-\omega)\,\zeta(2-2\omega)}\,E(z,\omega)\,d\omega.$$

Obviously, for any $k = 0, 1, 2, \ldots$ this integral defines a function holomorphic in s in the strip $\frac{1}{2}-2k < \operatorname{Re} s < \frac{3}{2}-2k$. Following the same line of arguments as in the proof of Lemma 26, we can show that the meromorphic extension of the above integral from the strip $\frac{1}{2}-2k < \operatorname{Re} s < \frac{3}{2}-2k$ to the strip $\frac{1}{2}-2(k+1) < \operatorname{Re} s < \frac{3}{2}-2(k+1)$ differs from the respective integral by the quantity

$$\frac{\sqrt{\pi}\Gamma(s+k-\frac{1}{2})\Gamma(\frac{s}{2}+k)\Gamma(\frac{1}{2}-\sqrt{2}-k)\pi^{1/2-s-2k}\zeta(s+2k)\zeta(1-s-2k)}{\Gamma^2(\frac{s}{2})\Gamma(1-s-2k)\zeta(2-2s-4k)}$$

$$\times E(z, s+2k),$$

whose poles (regarded as a function of the complex variables s) belong to the set

$$\Big\{\frac{1}{2}\rho - 2k \,\big|\, \zeta(\rho) = 0\Big\}.$$

With the help of Lemma 26, from the above discussion we conclude that $\Pi(z;s)$ is extended meromorphically to the whole plane of complex variable and the poles of $\Pi(z;s)$ belong to set (3.6). Now the theorem follows from the relations

$$\mathcal{D}^{s/2}\sum_{n=-\infty}^{\infty}\frac{\tau(n^2+\mathcal{D})}{(n^2+\mathcal{D})^{s/2}} = \sum_{k=1}^{h(-4\mathcal{D})}\lambda(z_k)\sum_{\sigma\in\Gamma}\left(\frac{\operatorname{Im}\sigma z_k}{|\sigma z_k|}\right)^s = \sum_{k=1}^{h(-4\mathcal{D})}\lambda(z_k)\Pi(z_k;s),$$

which stem from Lemma 7. The proof of the proposition is complete.

§2. Behavior of the function of the number of divisors of a quadratic polynomial

Lemma 27. *Let $1 \le P_1 \le P_2$, $1 \le \Delta = P_2 - P_1$, and let $f(x)$ be a function satisfying the following conditions:*
(a) *$f(x) = 1$ on the interval $[-P_1, P_1]$;*
(b) *$f(x) = 0$ outside the interval $[-P_2, P_2]$;*
(c) *$|f(x)| \le 1$ for the remaining numbers x.*
Then for any $\varepsilon > 0$ and $P_1 \le P \le P_2$,

$$\sum_{|n|\le P}\tau(n^2+\mathcal{D}) = 2\sum_{1\le q\le P}\ \sum_{n=-\infty}^{\infty} f(n)\,\delta_q(n^2+\mathcal{D}) - 4\sum_{1\le q\le P}\rho_{q,\mathcal{D}}(o) + \mathcal{O}_{\mathcal{D},\varepsilon}(\Delta P_2^{\varepsilon}).$$

Proof. From the definition of the function $\tau(n)$, it follows that

$$\tau(n) = \sum_{d_1 d_2 = n} 1 = 2\sum_{\substack{d_1 d_2 = n\\ d_1 < d_2}} 1 + \delta(n) = 2\sum_{\substack{d\backslash n;\\ d<\sqrt{n}}} 1 + \delta(n),$$

where $\delta(n) = 1$ if n is the square of a positive integer and $\delta(n) = 0$ in the remaining cases. Therefore, taking into account the estimate $\tau(n) \underset{\varepsilon}{\ll} n^{\varepsilon}$, we obtain

$$\sum_{|n|\le P} \tau(n^2+\mathcal{D}) = 2 \sum_{|n|\le P} \sum_{\substack{q\backslash(n^2+\mathcal{D}) \\ q<\sqrt{n^2+\mathcal{D}}}} 1 + \sum_{n^2+\mathcal{D}=k^2} 1$$

$$= 2 \sum_{|n|\le P} \sum_{q^2\le n^2} \delta_q(n^2+\mathcal{D}) + 2 \sum_{|n|\le P} \sum_{n^2<q^2\le n^2+\mathcal{D}} \delta_q(n^2+\mathcal{D}) + \mathcal{O}_{\mathcal{D}}(1).$$

Since

$$\sum_{|n|\le P} \sum_{n^2<q^2\le n^2+\mathcal{D}} \delta_q(n^2+\mathcal{D}) \le \sum_{k=1}^{\mathcal{D}} \sum_{\substack{|n|\le P; \\ n^2+k=q}} 1 \le \sum_{k=1}^{\mathcal{D}} \sum_{(q-|n|)(q+|n|)=k} 1 \le 2\sum_{k=1}^{\mathcal{D}} \tau(k) \underset{\mathcal{D}}{\ll} 1,$$

we have

$$\sum_{|n|\le P} \tau(n^2+\mathcal{D}) = 2 \sum_{|n|\le P} \sum_{q\le|n|} \delta_q(n^2+\mathcal{D}) + \mathcal{O}_{\mathcal{D}}(1)$$

$$= 2 \sum_{1\le q\le P} \sum_{q\le|n|\le P} \delta_q(n^2+\mathcal{D}) + \mathcal{O}_{\mathcal{D}}(1)$$

$$= 2 \sum_{1\le q\le P} \sum_{|n|\le P} \delta_q(n^2+\mathcal{D}) - 2 \sum_{1\le q\le P} \sum_{|n|<q} \delta_q(n^2+\mathcal{D}) + \mathcal{O}_{\mathcal{D}}(1).$$

But

$$\sum_{1\le q\le P} \sum_{|n|<q} \delta_q(n^2+\mathcal{D}) = 2 \sum_{1\le q\le P} \rho_{q,\mathcal{D}}(0) - \sum_{\substack{q\backslash\mathcal{D}; \\ 1\le q\le P}} 1 = 2 \sum_{1\le q\le P} \rho_{q,\mathcal{D}}(0) + \mathcal{O}_{\mathcal{D}}(1).$$

Therefore

$$\sum_{|n|\le P} \tau(n^2+\mathcal{D}) = 2 \sum_{1\le q\le P} \sum_{|n|\le P} \delta_q(n^2+\mathcal{D}) - 4 \sum_{1\le q\le P} \rho_{q,\mathcal{D}}(0) + \mathcal{O}_{\mathcal{D}}(1).$$

Further,

$$\left| \sum_{1\le q\le P} \sum_{|n|\le P} \delta_q(n^2+\mathcal{D}) - \sum_{1\le q\le P} \sum_{|n|\le P} f(n)\,\delta_q(n^2+\mathcal{D}) \right|$$

$$\le \sum_{1\le q\le P} \sum_{P_1\le|n|\le P_2} \delta_q(n^2+\mathcal{D}) \le \sum_{P_1\le|n|\le P_2} \tau(n^2+\mathcal{D}) \underset{\varepsilon,\mathcal{D}}{\ll} \Delta P_2^{\varepsilon}.$$

Therefore,

$$\sum_{|n|\le P} \tau(n^2+\mathcal{D}) = 2 \sum_{1\le q\le P} \sum_{|n|\le P} f(n)\,\delta_q(n^2+\mathcal{D}) - 4 \sum_{1\le q\le P} \rho_{q,\mathcal{D}}(0) + \mathcal{O}_{\mathcal{D},\varepsilon}(\Delta P_2^{\varepsilon}),$$

which is required to be proved.

Lemma 28. *Let $f(x)$ be a continuous, piecewise differentiable function on $(-\infty,\infty)$ with compact support set. Then for any natural q, the following relation holds:*

$$\sum_{n=-\infty}^{\infty} f(n)\delta_q(n^2+\mathcal{D}) = \sum_{m=-\infty}^{\infty} \frac{\rho_{q,\mathcal{D}}(m)}{q} \int\limits_{-\infty}^{\infty} f(x)\, e^{-2\pi i \frac{mx}{q}}\, dx.$$

Proof. From the definition of $\delta_n(n)$, we have

$$\sum_{n=-\infty}^{\infty} f(n)\delta_q(n^2+\mathcal{D}) = \sum_{\beta=1}^{q} \sum_{n_1=-\infty}^{\infty} f(qn_1+\beta)\delta_q\big((qn_1+\beta)^2+\mathcal{D}\big)$$

$$= \sum_{\beta=1}^{q} \delta_q(\beta^2+\mathcal{D}) \sum_{n_1=-\infty}^{\infty} f(qn_1+\beta).$$

With the help of the Poisson summation formula, we find

$$\sum_{n=-\infty}^{\infty} f(n)\delta_q(n^2+\mathcal{D}) = \sum_{\beta=1}^{q} \delta_q(n^2+\mathcal{D})\Bigg(\sum_{m=-\infty}^{\infty} \int\limits_{-\infty}^{\infty} f(qy+\beta)\, e^{-2\pi i m y}\, dy\Bigg)$$

$$= \sum_{\beta=1}^{q} \delta_q(\beta^2+\mathcal{D})\Bigg(\sum_{m=-\infty}^{\infty} \frac{1}{q} \int\limits_{-\infty}^{\infty} f(y+\beta)\, e^{-2\pi i \frac{my}{q}}\, dy\Bigg)$$

$$= \sum_{\beta=1}^{q} \delta_q(\beta^2+\mathcal{D})\Bigg(\sum_{m=-\infty}^{\infty} \frac{1}{q}\, e^{-2\pi i \frac{my}{q}} \int\limits_{-\infty}^{\infty} f(y)\, e^{-2\pi i \frac{my}{q}} = \sum_{m=-\infty}^{\infty} \frac{\rho_{q,\mathcal{D}}(m)}{q}\Bigg(\int\limits_{-\infty}^{\infty} f(y)\, e^{-2\pi i \frac{my}{q}}\, dy\Bigg).$$

The lemma is proved.

Let t be a fixed positive integer, $P \geq 1$,

$$B = P^{+\frac{1}{3}+\frac{1}{t}}, \quad \Delta = \frac{1}{2t}\, P^{2/3}, \quad N = \left[P^{1/2-1/2t}\right]. \tag{3.8}$$

Lemma 29. *Let $\mathcal{D}$ be a positive integer. Then*

$$\sum_{|n|\leq P} \tau(n^2+\mathcal{D}) = 4 \sum_{1\leq q\leq P} \left(\frac{P}{q}-1\right) \rho_{q,\mathcal{D}}(0)$$

$$+\frac{4}{\pi}\, \frac{1}{(2\Delta)^t} \int\limits_{-\Delta}^{\Delta} \cdots \int\limits_{-\Delta}^{\Delta} H(P+\theta_1+\cdots+\theta_t)d\theta_1 \ldots d\theta_t + \mathcal{O}_{\mathcal{D},t}\left(P^{2/3+1/t}\right),$$

where

$$H(P_1) = \sum_{1\leq m\leq B} \frac{1}{m}\Bigg(\sum_{P_1/N\leq q\leq P_1} \rho_{q,\mathcal{D}}(m) \sin \frac{2\pi m P_1}{q}\Bigg).$$

Proof. Let
$$f_0(x) = f_0(P, x) = \chi_{[-P,P]}(x)$$
be the characteristic function of the interval $[-P, P]$. For $1 \le \ell \le t$, we put

$$f_l(P) = f_l(P, x) = \frac{1}{2\Delta} \int\limits_{-\Delta}^{\Delta} f_{l-1}(P+\theta, x)\, d\theta$$

(Vinogradov's "glass"). For $l = t$, the following relation holds:

$$f_t(x) = \frac{1}{(2\Delta)^t} \int\limits_{-\Delta}^{\Delta} \dots \int\limits_{-\Delta}^{\Delta} \chi_{[-(P+\theta_1+\dots+\theta_t),(P+\theta_1+\dots+\theta_t)]}(x)\, d\theta_1 \dots d\theta_t. \tag{3.9}$$

It is easily verified that

(a) $f_t(x) = 1$ for x lying in the interval $[-(P - t\Delta), (P - t\Delta)]$;

(b) $f_t(x) = 0$ for x lying outside the interval $[-(P + t\Delta), (P + t\Delta)]$;

(c) $0 \le f_t(x) \le 1$ for the remaining numbers x.

By Lemma 27, we have

$$\sum_{|n| \le P} \tau(n^2+\mathcal{D}) = 2 \sum_{1 \le q \le P} \sum_{n=-\infty}^{\infty} f_t(n)\, \delta_q(n^2+\mathcal{D}) - 4 \sum_{1 \le q \le P} \rho_{q,\mathcal{D}}(0) + \mathcal{O}_{\mathcal{D},t}\left(P^{2/3+1/t}\right). \tag{3.10}$$

Next,
$$\sum_{1 \le q \le P} \sum_{n-\infty}^{\infty} f_t(n)\, \delta_q(n^2+\mathcal{D}) =$$

$$\sum_{1 \le q \le \frac{P}{N}} \sum_{n-\infty}^{\infty} f_t(n)\, \delta_q(n^2+\mathcal{D}) + \sum_{\frac{P}{N} \le q \le P} \sum_{n-\infty}^{\infty} f_t(n)\, \delta_q(n^2+\mathcal{D}) = w_1 + w_2. \tag{3.11}$$

With the help of (3.9), for w_1 we obtain

$$w_1 = \sum_{1 \le q \le \frac{P}{N}} \sum_{n-\infty}^{\infty} f_t(n)\, \delta_q(n^2+\mathcal{D}) = \frac{1}{(2\Delta)^t} = \int\limits_{-\Delta}^{\Delta} \dots \int\limits_{-\Delta}^{\Delta}$$

$$\times \Bigg(\sum_{1 \le q < \frac{P}{N}} 1 \sum_{|n| \le P+\theta_1+\dots+\theta_t} \delta_q(n^2+\mathcal{D}) \Bigg) d\theta_1 \dots d\theta_t = \frac{1}{(2\Delta)^t}$$

$$\times \int\limits_{-\Delta}^{\Delta} \dots \int\limits_{-\Delta}^{\Delta} \Bigg(\sum_{1 \le q < \frac{P}{N}} 2\, \frac{P+\theta_1+\dots+\theta_t}{q}\, \rho_{q,\mathcal{D}}(m) + \mathcal{O}\left(\rho_{q,\mathcal{D}}(0)\right) \Bigg)$$

$$\times d\theta_1 \dots d\theta_t = 2 \sum_{1 \le q < \frac{P}{N}} \rho_{q,\mathcal{D}}(0) \frac{P}{q} + \mathcal{O}\Bigg(\sum_{1 \le q < \frac{P}{N}} \rho_{q,\mathcal{D}}(0) \Bigg) = 2P \sum_{1 \le q < \frac{P}{N}} \frac{\rho_{q,\mathcal{D}}(0)}{q} + \mathcal{O}_{\mathcal{D},t}\left(P^{2/3+1/t}\right).$$

Now consider

$$w_2 = \sum_{\frac{P}{N} \le q \le P} \sum_{n=-\infty}^{\infty} f_t(n)\, \delta_q(n^2 + \mathcal{D}).$$

Applying Lemma 28 to the inner sum, we deduce that

$$w_2 = \sum_{\frac{P}{N} \le q \le P} \sum_{m=-\infty}^{\infty} \frac{\rho_{q,\mathcal{D}}(m)}{q} \int\limits_{-\infty}^{\infty} f_t(x)\, e^{-2\pi i \frac{mx}{q}}\, dx.$$

Next,

$$\int\limits_{-\infty}^{\infty} f_t(x)\, e^{-2\pi i \frac{mx}{q}}\, dx = \frac{1}{(2\Delta)^t} \int\limits_{-\Delta}^{\Delta} \cdots \int\limits_{-\Delta}^{\Delta}$$

$$\times \Bigg(\int\limits_{-\infty}^{\infty} \chi_{[-(P+\theta_1+\cdots+\theta_t),(P+\theta_1+\cdots+\theta_t)]}(x)\ e^{-2\pi i \frac{mx}{q}}\, dx \Bigg) d\theta_1 \ldots d\theta_t$$

$$= \frac{1}{(2\Delta)^t} \int\limits_{-\Delta}^{\Delta} \cdots \int\limits_{-\Delta}^{\Delta} \Bigg(\int\limits_{-(P+\theta_1+\cdots+\theta_t)}^{(P+\theta_1+\cdots+\theta_t)} e^{-2\pi i \frac{mx}{q}}\, dx \Bigg) d\theta_1 \ldots d\theta_t \ll_t \frac{1}{\Delta^t} \Bigg(\frac{q}{|m|} \Bigg)^{t+1}.$$

Consequently,

$$w_2 = \sum_{\frac{P}{N} \le q \le P} \sum_{|m| \le B} \frac{\rho_{q,\mathcal{D}}(m)}{q} \int\limits_{-\infty}^{\infty} f_t(x)\, e^{-2\pi i \frac{mx}{q}}\, dx + R,$$

where

$$R \ll_t \sum_{\frac{P}{N} \le q \le P} \sum_{|m| \le B} \frac{\rho_{q,\mathcal{D}}(m)}{q} \frac{1}{\Delta^t} \Bigg(\frac{q}{|m|} \Bigg)^{t+1}$$

$$\le \Bigg(\sum_{1 \le q \le P} \frac{\rho_{q,\mathcal{D}}(0)}{q} \Bigg) \Bigg(\sum_{|m| > B} \frac{P^{t+1}}{P^{\frac{2}{3}t}} \frac{1}{|m|^{t+1}} \Bigg) \ll_{\mathcal{D},t} P^{1/t} \frac{P^{t+1}}{P^{\frac{2}{3}t} B^t} = \frac{P^{t+1+\frac{1}{t}}}{P^{\frac{2}{3}t+\frac{1}{3}t+\frac{1}{3}}} = P^{\frac{2}{3}+\frac{1}{t}}.$$

Thus,

$$w_2 = \sum_{\frac{P}{N} \le q \le P} \sum_{|m| \le B} \frac{\rho_{q,\mathcal{D}}(m)}{q} \int\limits_{-\infty}^{\infty} f_t(x)\, e^{-2\pi i \frac{mx}{q}}\, dx + \mathcal{O}_{\mathcal{D},t}\Big(P^{\frac{2}{3}+\frac{1}{t}} \Big).$$

The above expressions for w_1 and w_2, together with relations (3.10) and (3.11), lead to the formula

$$\sum_{|n| \le P} \tau(n^2 + \mathcal{D}) = 4P \sum_{1 \le q < \frac{P}{N}} \frac{\rho_{q,\mathcal{D}}(0)}{q} + 2 \sum_{\frac{P}{N} \le q \le P}$$

$$\times \sum_{|m| \le B} \frac{\rho_{q,\mathcal{D}}(m)}{q} \int\limits_{-\infty}^{\infty} f_t(x)\, e^{-2\pi i \frac{mx}{q}}\, dx - 4 \sum_{1 \le q \le P} \rho_{q,\mathcal{D}}(0) + \mathcal{O}_{\mathcal{D},t}\Big(P^{\frac{2}{3}+\frac{1}{t}} \Big).$$

Since

$$\int_{-\infty}^{\infty} f_t(x)\,dx = 2P,$$

it follows that the latter formula can be rewritten in the form

$$\sum_{|n|\le P}\tau(n^2+\mathcal{D}) = 4\sum_{1\le q\le P}\left(\frac{P}{q}-1\right)\rho_{q,\mathcal{D}}(0)$$

$$+2\sum_{\frac{P}{N}\le q\le P}\ \sum_{|m|\le B}\frac{\rho_{q,\mathcal{D}}(m)}{q}\int_{-\infty}^{\infty} f_t(x)\,e^{-2\pi i\,\frac{mx}{q}}\,dx + \mathcal{O}_{\mathcal{D},t}\Big(P^{\frac{2}{3}+\frac{1}{t}}\Big).$$

Since for $|\theta_1|\le\Delta,\ldots,|\theta_t|\le\Delta$ we have

$$\sum_{1\le m\le B}\frac{1}{m}\Bigg(\sum_{\substack{\frac{P+\theta_1+\cdots+\theta_t}{N}\le\\ \le q\le P+\theta_1+\cdots+\theta_t}}\rho_{q,\mathcal{D}}(m)\,\sin\frac{2\pi m(P+\theta_1+\cdots+\theta_t)}{q}\Bigg)$$

$$=\sum_{1\le m\le B}\frac{1}{m}\Bigg(\sum_{\frac{P}{N}\le q\le P}\rho_{q,\mathcal{D}}(m)\,\sin\frac{2\pi m(P+\theta_1+\ldots+\theta_t)}{q}\Bigg)+\mathcal{O}_{\mathcal{D},t}\Big(P^{\frac{2}{3}+\frac{1}{t}}\Big),$$

it follows that

$$\sum_{\frac{P}{N}\le q\le P}\ \sum_{|m|\le B}\frac{\rho_{q,\mathcal{D}}(m)}{q}\int_{-\infty}^{\infty} f_t(x)\,e^{-2\pi i\,\frac{mx}{q}}\,dx$$

$$=\frac{1}{(2\Delta)^t}\int_{-\Delta}^{\Delta}\cdots\int_{-\Delta}^{\Delta}\Bigg(\sum_{|m|\le B}\ \sum_{\frac{P}{N}\le q\le P}\frac{\rho_{q,\mathcal{D}}(m)}{q}\int_{-(P+\theta_1+\cdots+\theta_t)}^{(P+\theta_1+\cdots+\theta_t)} e^{-2\pi i\frac{mx}{q}}\,dx\Bigg)d\theta_1\ldots d\theta_t$$

$$=\frac{2}{\pi}\,\frac{1}{(2\Delta)^t}\int_{-\Delta}^{\Delta}\cdots\int_{-\Delta}^{\Delta}\Bigg(\sum_{1\le m\le B}\frac{1}{m}\Bigg(\sum_{\frac{P}{N}\le q\le P}\rho_{q,\mathcal{D}}(m)\times\sin\left(2\pi m\frac{P+\theta_1+\cdots+\theta_t}{q}\right)\Bigg)\Bigg)d\theta_1\ldots d\theta_t$$

$$=\frac{2}{\pi}\,\frac{1}{(2\Delta)^t}\int_{-\Delta}^{\Delta}\cdots\int_{-\Delta}^{\Delta} H(P+\theta_1+\cdots+\theta_t)d\theta_1\ldots d\theta_t+\mathcal{O}_{\mathcal{D},t}\Big(P^{\frac{2}{3}+\frac{1}{t}}\Big).$$

The lemma is proved.

Lemma 30. *Let $f(x)$ and $g(x)$ be real-valued functions, twice continuously differentiable on $[A,B]$, let $g(x)\ne 0$, and let $(f'/g)'$ have no more than q distinct zeros on the interval $[A,B]$, and let*

$$\left|\frac{f'(x)}{g(x)}\right|\le c>0$$

on $[A,B]$. Then

$$\left|\int_A^B g(x)\,e^{if(x)}\,dx\right|\le\frac{4(q+1)}{c}.$$

Lemma 31. *Let $f(x)$ be twice continuously differentiable on $[A,B]$ and let $0<c\le|f''(x)|$ on $[A,B]$. Let $g(x)$ be twice continuously differentiable on $[A,B]$, $|g(x)|\le M$ for $A\le x\le B$; let $g(x)\ne 0$ and let $(f'/g)'$ have no more than q distinct zeros on $[A,B]$. Then*

$$\Big|\int\limits_A^B g(x)\,e^{2\pi i f(x)}\,dx\Big| \le \frac{8M(q+1)}{\sqrt{c}}.$$

For the proof of these lemmas, see [13].

Theorem 3. *Let $\mathcal{D}$ be a positive integer. Then for any $\varepsilon>0$, the following relation holds:*

$$\sum_{|n|\le P}\tau(n^2+\mathcal{D}) = c_1(\mathcal{D})P\ln P + c_0(\mathcal{D})P + \mathcal{O}_{\mathcal{D},\varepsilon}\left(P^{\frac{2}{3}+\varepsilon}\right).$$

Proof. Choose t, B, Δ, and N such as in (3.8), $\frac{1}{2}P\le P_1\le 2P$. Putting

$$f_m(x)=\begin{cases}\sin\dfrac{P_1 x}{\sqrt{\mathcal{D}}} & \text{for } \dfrac{2\pi|m|\sqrt{\mathcal{D}}}{P_1}\le x\le \dfrac{2\pi|m|\sqrt{\mathcal{D}}\,N}{P_1};\\ 0 & \text{for the remaining numbers } \ x\ge 0,\end{cases}$$

with the help of Theorem 1 we obtain

$$H(P_1)=\sum_{1\le m\le B}\frac{1}{m}\Bigg(\sum_{\frac{P_1}{N}\le q\le P_1}\rho_{q,\mathcal{D}}(m)\sin\frac{2\pi m P_1}{q}\Bigg) = \sum_{1\le m\le B}\frac{1}{m}\sum_{q=1}^{\infty} f_m\left(\frac{2\pi m\sqrt{\mathcal{D}}}{q}\right)\rho_{q,\mathcal{D}}(m)$$

$$=\sqrt{2\pi}\sum_{1\le m\le B}\frac{1}{m}\sum_{j=1}^{\infty}\widehat{f}_m(\varkappa_j)\,\overline{\rho_j(m)}\Bigg(\sum_{k=1}^{h(-4\mathcal{D})}\lambda(z_k)\,u_j(z_k)\Bigg)$$

$$+\frac{1}{\sqrt{2}}\sum_{1\le m\le B}\frac{1}{m}\int\limits_{-\infty}^{\infty}\frac{(\pi m)^{-ir}\,\sigma_{2ir}(m)}{\Gamma(\frac{1}{2}-ir)\zeta(1-2ir)}\widehat{f}_m(r)\Bigg(\sum_{k=1}^{h(-4\mathcal{D})}\lambda(z_k)E\left(z_k,\frac{1}{2}+ir\right)\Bigg)dr,$$

where

$$\widehat{f}_m(r)=\int\limits_{\frac{2\pi m\sqrt{\mathcal{D}}}{P_1}}^{\frac{2\pi m\sqrt{\mathcal{D}}\,N}{P_1}}\sin\frac{P_1 y}{\sqrt{\mathcal{D}}}\,K_{ir}(y)\frac{dy}{y^{3/2}}.$$

From (1.8), we have

$$f_m(r)=\frac{\pi}{2}\,\frac{1}{sh\pi r}\sum_{n=0}^{\infty}\left(\widehat{f}_{m,n}(r)+\widehat{f}_{m,n}(-r)\right), \tag{3.12}$$

where

$$\widehat{f}_{m,n}(r)=\frac{1}{n!\Gamma(n+1+ir)}\int\limits_{\frac{2\pi m\sqrt{\mathcal{D}}}{P_1}}^{\frac{2\pi m\sqrt{\mathcal{D}}\,N}{P_1}}\sin\frac{P_1 y}{\sqrt{\mathcal{D}}}\left(\frac{y}{2}\right)^{2n+ir}\frac{dy}{y^{3/2}}.$$

For $n>1$, a simple estimation by modulus yields the bound

$$\widehat{f}_{m,n}(r) \ll_{\mathcal{D}} \frac{e^{\frac{\pi|r|}{2}}}{n!(1+|r|)^{5/2}}. \tag{3.13}$$

Now let $n=1$. Then

$$\widehat{f}_{m,n}(r) = \frac{1}{\Gamma(2+ir)} \int\limits_{\frac{2\pi m\sqrt{\mathcal{D}}}{P_1}}^{\frac{2\pi m\sqrt{\mathcal{D}}\,N}{P_1}} \sin\frac{P_1 y}{\sqrt{\mathcal{D}}}\left(\frac{y}{2}\right)^{2+ir}\frac{dy}{y^{3/2}} \ll_{\mathcal{D}} \frac{e^{\frac{\pi|r|}{2}}}{P^{3/2}(1+|r|)^{3/2}}\left|\int\limits_{2\pi m}^{2\pi mN} y^{\frac{1}{2}+ir}\sin y\,dy\right|.$$

Integration by parts leads to estimates:

$$\widehat{f}_{m,1}(r) \ll_{\mathcal{D}} \frac{e^{\frac{\pi|r|}{2}}}{P^{3/2}(1+|r|)^{3/2}}\left|\int\limits_{2\pi m}^{2\pi mN}\frac{y^{\frac{3}{2}+ir}}{\frac{3}{2}+ir}\cos y\,dy\right|$$

$$\ll_t \frac{e^{\frac{\pi|r|}{2}}}{P^{3/2}(1+|r|)^{3/2}}\,\frac{(P^{\frac{2}{3}+\frac{1}{2t}})^{5/2}}{1+|r|} \le \frac{P^{\frac{1}{6}+\frac{2}{t}}\,e^{\frac{\pi|r|}{2}}}{(1+|r|)^{5/2}},$$

i.e.,

$$\widehat{f}_{m,1}(r) \ll_{\mathcal{D},t} \frac{P^{\frac{1}{6}+\frac{2}{t}}\,e^{\frac{\pi|r|}{2}}}{(1+|r|)^{5/2}}. \tag{3.14}$$

The estimates in Lemmas 10, 11, 14, 15, and 16 and estimates (3.11)–(3.14) imply that

$$\begin{aligned} H(P_1) = \frac{\pi\sqrt{\pi}}{2}\sum_{1\le m\le B}\frac{1}{\sqrt{m}}\sum_{j=1}^{\infty}\frac{\overline{\rho_j(m)}}{sh\pi\varkappa_j}\Bigg(\left(\frac{2P_1}{\sqrt{\mathcal{D}}}\right)^{\frac{1}{2}-i\varkappa_j}\frac{\varphi_m(\varkappa_j)}{\Gamma(1+i\varkappa_j)} \\ +\left(\frac{2P_1}{\sqrt{\mathcal{D}}}\right)^{\frac{1}{2}+i\varkappa_j}\frac{\varphi_m(-\varkappa_j)}{\Gamma(1-i\varkappa_j)}\Bigg)\left(\sum_{k=1}^{h(-4\mathcal{D})}\lambda(z_k)\,u_j(z_k)\right) \\ +\frac{\pi}{4}\sum_{1\le m\le B}\frac{1}{m}\times\int\limits_{-\infty}^{\infty}\frac{(\pi m)^{-ir}\,\sigma_{2ir}(m)}{\Gamma(\frac{1}{2}-ir)\zeta(1-2ir)}\,\frac{1}{sh\pi r}\Bigg(\left(\frac{2P_1}{\sqrt{\mathcal{D}}}\right)^{\frac{1}{2}-ir}\frac{\varphi_m(r)}{\Gamma(1+ir)} \\ +\left(\frac{2P_1}{\sqrt{\mathcal{D}}}\right)^{\frac{1}{2}+ir}\frac{\varphi_m(-r)}{\Gamma(1-ir)}\Bigg)\left(\sum_{k=1}^{h(-4\mathcal{D})}\lambda(z_k)\,E\left(z_k,\frac{1}{2}+ir\right)\right)dr+\mathcal{O}_{\mathcal{D},t}\left(P^{\frac{1}{2}+\frac{2}{t}}\right). \end{aligned} \tag{3.15}$$

Here, the $\varphi_m(r)$ are determined from the relations

$$\widehat{f}_{m,0}(r) = \frac{1}{\sqrt{2}}\left(\frac{2P_1}{\sqrt{\mathcal{D}}}\right)^{\frac{1}{2}-ir}\frac{1}{\Gamma(1+ir)}\int\limits_{2\pi m}^{2\pi mN}\sin y\,y^{-\frac{3}{2}+ir}\,dy$$

$$= \frac{1}{2\sqrt{2}}\left(\frac{2P_1}{\sqrt{\mathcal{D}}}\right)^{\frac{1}{2}-ir}\frac{1}{(1-2ir)\Gamma(1+ir)}\times\left(\int\limits_{2\pi m}^{2\pi mN}\frac{e^{i(r\ln y+y)}-e^{i(r\ln y-y)}}{\sqrt{y}}\,dy\right)$$

$$= \frac{1}{2\sqrt{2}}\left(\frac{2P_1}{\sqrt{\mathcal{D}}}\right)^{\frac{1}{2}-ir}\ \frac{\varphi_m(r)}{\Gamma(1+ir)}. \tag{3.16}$$

For $|r| \geq 4\pi mN$, from Lemma 30 we conclude that

$$\varphi_m(r) \ll \frac{(mN)^{1/2}}{(1+|r|)^2}. \tag{3.17}$$

For any r, from Lemmas 30 and 31 we have

$$\varphi_m(r) \ll \frac{1}{1+|r|}. \tag{3.18}$$

By Lemma 15, estimates (3.17) and (3.18) yield

$$\sum_{1\leq m\leq B} \frac{1}{m}\int_{-\infty}^{\infty} \frac{(\pi m)^{-ir}\,\sigma_{2ir}(m)}{\Gamma(\frac{1}{2}-ir)\zeta(1-2ir)}\,\frac{1}{sh\pi r}\left(\left(\frac{2P_1}{\sqrt{\mathcal{D}}}\right)^{\frac{1}{2}-ir}\frac{\varphi_m(r)}{\Gamma(1+ir)}\right.$$

$$\left.+\left(\frac{2P_1}{\sqrt{\mathcal{D}}}\right)^{\frac{1}{2}+ir}\frac{\varphi_m(-r)}{\Gamma(1-ir)}\right)\left(\sum_{k=1}^{h(-4\mathcal{D})}\lambda(z_k)E\left(z_k,\frac{1}{2}+ir\right)\right)dr$$

$$\ll_{\mathcal{D},t} \sum_{1\leq m\leq B}\frac{\tau(m)}{\sqrt{m}}\left(\int_0^{4\pi mN}\frac{P_1^{\frac{1}{2}}(1+r)^{\frac{1}{2}+\frac{1}{4t}}}{(1+r)^{3/2}}dr+\int_{4\pi mN}^{\infty}\frac{P_1^{\frac{1}{2}}(mN)^{\frac{1}{2}}(1+r)^{\frac{1}{2}+\frac{1}{4t}}}{(1+r)^{5/2}}\,dr\right) \ll_t P^{\frac{2}{3}+\frac{1}{t}}.$$

Moreover, (3.17) implies the estimate (see also Lemmas 10 and 11)

$$\sum_{1\leq m\leq B}\frac{1}{\sqrt{m}}\sum_{\varkappa_j\geq 4\pi mN}\frac{\overline{\rho_j(m)}}{sh\pi\varkappa_j}\left(\left(\frac{2P_1}{\sqrt{\mathcal{D}}}\right)^{\frac{1}{2}-i\varkappa_j}\frac{\varphi_m(\varkappa_j)}{\Gamma(1+i\varkappa_j)}\right.$$

$$\left.+\left(\frac{2P_1}{\sqrt{\mathcal{D}}}\right)^{\frac{1}{2}+i\varkappa_j}\frac{\varphi_m(-\varkappa_j)}{\Gamma(1-i\varkappa_j)}\right)\left(\sum_{k=1}^{h(-4\mathcal{D})}\lambda(z_k)\,u_j(z_k)\right)\ll_{\mathcal{D},t}\sum_{1\leq m\leq B}\frac{1}{\sqrt{mN}}\ll_t P^{\frac{2}{3}+\frac{1}{t}}.$$

Consequently, (3.15) can be rewritten in the form

$$H(P_1)=\frac{\pi\sqrt{\pi}}{4}\sum_{1\leq m\leq B}\frac{1}{\sqrt{m}}\sum_{\varkappa_j<4\pi mN}\frac{\overline{\rho_j(m)}}{sh\pi\varkappa_j}\left(\left(\frac{2P_1}{\sqrt{\mathcal{D}}}\right)^{\frac{1}{2}-i\varkappa_j}\frac{\varphi_m(\varkappa_j)}{\Gamma(1+i\varkappa_j)}\right.$$

$$\left.+\left(\frac{2P_1}{\sqrt{\mathcal{D}}}\right)^{\frac{1}{2}+i\varkappa_j}\frac{\varphi_m(-\varkappa_j)}{\Gamma(1-i\varkappa_j)}\right)\left(\sum_{k=1}^{h(-4\mathcal{D})}\lambda(z_k)\,u_j(z_k)\right)+\mathcal{O}_{\mathcal{D},t}\left(P^{\frac{2}{3}+\frac{1}{t}}\right). \tag{3.19}$$

From relation (3.16), Lemmas 30 and 31, we find

(a) for $0\leq r\leq 2\pi m+1$,

$$\varphi_m(r)\ll\frac{\sqrt{m}}{(1+r)(r-2\pi m+2)};$$

(b) for $2\pi m+1 \le r \le 2\pi mN-1$,

$$\varphi_m(r) = \frac{1}{1-2ir}\int\limits_{\frac{1}{2}r}^{2r} \frac{e^{i(r\ln y - y}}{\sqrt{y}}dy + O\biggl(\frac{1}{1+r}\biggl(\frac{\sqrt{m}}{r-2\pi m} + \frac{\sqrt{mN}}{2\pi mN - r}\biggr)\biggr);$$

(c) for $2\pi mN-1 \le r \le 4\pi mN$,

$$\varphi_m(r) \ll \frac{\sqrt{mN}}{(1+r)(r-2\pi m+2)};$$

(d) for any $r>0$

$$\int\limits_{\frac{1}{2}r}^{2r} \frac{e^{i(r\ln y - y}}{\sqrt{y}}dy \ll 1. \tag{3.20}$$

From this (see also Lemmas 10 and 11) it follows that (3.19) can be written on the form

$$H(P_1) = \frac{\pi\sqrt{\pi}}{4}\sum_{1\le m\le B}\frac{1}{\sqrt{m}}\sum_{2\pi m\le \varkappa_j\le 2\pi mN}\frac{\overline{\rho_j(m)}}{sh\pi\varkappa_j}\biggl(2\,\mathrm{Re}\biggl(\biggl(\frac{2P_1}{\sqrt{\mathcal{D}}}\biggr)^{\frac{1}{2}-i\varkappa_j}$$

$$\frac{u(\varkappa_j)}{(1-2i\varkappa_j)\Gamma(1+i\varkappa_j)}\biggr)\biggr)\biggl(\sum_{k=1}^{h(-4\mathcal{D})}\lambda(z_k)\,u_j(z_k)\biggr) + \mathcal{O}_{\mathcal{D},t}\Bigl(P^{\frac{2}{3}+\frac{1}{t}}\Bigr),$$

where

$$u(r) = \int\limits_{\frac{1}{2}r}^{2r}\frac{e^{i(r\ln y - y)}}{\sqrt{y}}dy.$$

Putting

$$M(r) = \biggl(\frac{2}{\sqrt{\mathcal{D}}}\biggr)^{\frac{1}{2}-ir}\frac{u(r)}{(1-2ir)\Gamma(1+ir)},$$

we arrive at the relation

$$H(P_1) = \pi\sqrt{\pi}\sum_{\varkappa_j\le 2\pi BN}\frac{\mathrm{Re}\,(P^{\frac{1}{2}-i\varkappa_j}\,M(\varkappa_j))}{2sh\pi\varkappa_j}T(\varkappa_j)$$

$$\times\biggl(\sum_{k=1}^{h(-4\mathcal{D})}\lambda(z_k)\,u_j(z_k)\biggr) + \mathcal{O}_{\mathcal{D},t}\Bigl(P^{\frac{2}{3}+\frac{1}{t}}\Bigr), \tag{3.21}$$

where

$$T(\varkappa_j) = \sum_{\substack{\frac{\varkappa_j}{2\pi N}\le m\le \\ \le\min(\frac{\varkappa_j}{2\pi},B)}}\frac{\overline{\rho_j(m)}}{\sqrt{m}}.$$

Taking into account Lemma 29, this yields

$$\sum_{|n|\le P}\tau(n^2+\mathcal{D})=4\sum_{1\le q\le P}\left(\frac{P}{q}-1\right)\rho_{q,\mathcal{D}}(0)+R+\mathcal{O}_{\mathcal{D},t}\left(P^{\frac{2}{3}+\frac{1}{t}}\right).$$

Here,

$$R=\frac{4}{\pi}\,\frac{1}{(2\Delta)^t}\int\limits_{-\Delta}^{\Delta}\dots\int\limits_{-\Delta}^{\Delta}H(P+\theta_1+\dots+\theta_t)d\theta_1\dots d\theta_t$$

$$\underset{\mathcal{D},t}{\ll}\sum_{\varkappa_j\le P^{\frac{1}{3}}}\frac{P^{\frac{1}{2}}}{\varkappa_j^{\frac{3}{2}}}\frac{|T(\varkappa_j)|}{ch\frac{\pi\varkappa_j}{2}}\left(\sum_{k=1}^{h(-4\mathcal{D})}|\,u_j(z_k)\,|\right)+\frac{P^{\frac{3}{2}}}{\Delta}\sum_{P^{\frac{1}{3}}<\varkappa_j\le 2\pi BN}\frac{1}{\varkappa_j^{\frac{5}{2}}}\left(\sum_{k=1}^{h(-4\mathcal{D})}|\,u_j(z_k)\,|\right)+P^{\frac{2}{3}+\frac{2}{t}}.$$

Since

$$\sum_{\varkappa_j\le A}\frac{1}{ch\pi\varkappa_j}\,|\,T(\varkappa_j)\,|^2=\sum_{\varkappa_j\le A}\frac{1}{ch\pi\varkappa_j}\sum_{\substack{\frac{\varkappa_j}{2\pi N}\le m_1,\\ m_2\le\min(\frac{\varkappa_j}{2\pi},B)}}\frac{\rho_j(m_1)\,\overline{\rho_j(m_2)}}{\sqrt{m_1m_2}}$$

$$=\sum_{\substack{1\le m_1\\ m_2\le B}}\frac{1}{\sqrt{m_1m_2}}\sum_{\substack{\mathcal{L}_1(A,m_1,m_2,B)\le\varkappa_j\le\\ \le\mathcal{L}_2(A,m_1,m_2,B)\le A}}\frac{\rho_j(m_1)\,\overline{\rho_j(m_2)}}{ch\pi\varkappa_j}\ll\sum_{\substack{1\le m_1\\ m_2\le B}}$$

$$\times\frac{1}{\sqrt{m_1m_2}}\;\left(\delta_{m_1,m_2}\,A^2+A^{1+\varepsilon}+A^{\frac{1}{2}+\varepsilon}\left(m^{\frac{1}{4}}{}_1+m_2^{\frac{1}{4}}\right)+(m_1,m_2)^{\frac{1}{4}+\varepsilon}\right)$$

$$\underset{\varepsilon}{\ll}A^2\,B^{\varepsilon}+A^2+A^{1+\varepsilon}\,B+A^{\frac{1}{2}+\varepsilon}\,B^{\frac{5}{4}}+B^{\frac{3}{2}+\varepsilon},$$

(we applied Lemma 11), we obtain

$$R\underset{\mathcal{D},t}{\ll}P^{\frac{1}{6}}\,P^{\frac{1}{6}+\frac{1}{t}}+P^{\frac{5}{6}}\,P^{-\frac{1}{6}+\frac{1}{t}}+P^{\frac{2}{3}+\frac{2}{t}}\ll P^{\frac{2}{3}+\frac{2}{t}}.$$

Thus,

$$\sum_{|n|\le P}\tau(n^2+\mathcal{D})=4\sum_{1\le q\le P}\left(\frac{P}{q}-1\right)\rho_{q,\mathcal{D}}(0)+\mathcal{O}_{\mathcal{D},t}\left(P^{\frac{2}{3}+\frac{1}{t}}\right).$$

Next, we find (see Lemma 6)

$$\sum_{1\le q\le P}\rho_{q,\mathcal{D}}(0)=\frac{1}{2}\sum_{k=1}^{h(-4\mathcal{D})}\lambda(z_k)\sum_{\substack{Q_k(c,d)\le P\\ (c,d)=1}}1$$

$$=\frac{1}{2}\sum_{k=1}^{h(-4\mathcal{D})}\lambda(z_k)\sum_{1\le a\le\sqrt{P}}\mu(a)\left(-1+\sum_{Q_k(c,d)\le\frac{P}{a^2}}\right)=c(\mathcal{D})P+\mathcal{O}_{\mathcal{D}}\left(P^{\frac{1}{2}}\ln P\right).$$

With the help of the Abel transformation, these relations imply:

$$4\sum_{1\le q\le P}\left(\frac{P}{q}-1\right)\rho_{q,\mathcal{D}}(0)=c_1(\mathcal{D})P\ln P+c_0(\mathcal{D})P+\mathcal{O}_{\mathcal{D}}\left(P^{\frac{1}{2}}\ln P\right).$$

Consequently,

$$\sum_{|n|\le P}\tau(n^2+\mathcal{D})=c_1(\mathcal{D})P\ln P+c_0(\mathcal{D})P+\mathcal{O}_{\mathcal{D},t}\left(P^{\frac{2}{3}+\frac{2}{t}}\right).$$

Putting $t>\frac{1}{2\varepsilon}$, we obtain the required relation. The theorem is proved.

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