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THE RANKIN CONVOLUTIONS AND DIRICHLET SERIES. II

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ABSTRACT

This is the second part of the paper concerning the Riemann hypothesis. The main theorem 2 is proved: every nontrivial zero of the Riemann zeta function not lying on the critical line is simultaneously a zero of all Dirichlet L-series with quadratic characters.

From the main theorems of parts I and II, the Riemann hypothesis is obtained for all Dirichlet L-series with quadratic characters and for the Riemann zeta function.

Dedicated to the memory of the 100th
anniversary of the asymptotic law

1. Introduction

This paper is a continuation of paper [1]. Here we prove the following main theorem.

Theorem 2. *If the Riemann ζ function has a zero $s_0 = \beta + i\gamma$ with $\beta > \frac{1}{2}$, then s_0 is simultaneously a zero of all Dirichlet L-series with quadratic characters $\chi = \chi_q$:*

$$0 = \zeta(s_0) \Longrightarrow L(s_0, \chi_q) = 0, \quad \operatorname{Re} s_0 > \frac{1}{2},$$

where q ranges over all conductor moduli of quadratic charcters χ_q.

It is easily seen that Theorem 2 is the converse to Theorem 1 from [1]. It implies the following consequence.

Corollary 1. *The Riemann hypothesis is valid for the Riemann zeta function: $\zeta(s_0) = 0 \Longrightarrow \operatorname{Re} s_0 = \frac{1}{2}$ if $\operatorname{Im} s_0 \neq 0$.*

Indeed, if we assume that a zero s_0 of the zeta function exists with $\zeta(s_0) = 0$, $s_0 = \beta + i\gamma$, $\beta > \frac{1}{2}$, then Theorem 2 yields a contradiction to the Jutila density theorem from [2], which we cite here as a lemma.

Lemma 1 (M. Jutila [2], Theorem 2)**.**
Let $N(\sigma, T, \chi_q)$ be the number of zeros of the Dirichlet series $L(s, \chi_q)$ with a quadratic character χ_q of conductor modulus q in the region

$$\sigma \leq \operatorname{Re} s \leq 1, \quad |\operatorname{Im} s| \leq T,$$

then for any $\varepsilon > 0$, $T \geq 1$, $\sigma \in \left[\frac{1}{2}, 1\right]$, the following estimate is valid:

$$N_Q(\sigma, T) = \sum_{q \leq Q} N(\sigma, T, \chi_q) \underset{\varepsilon}{\ll} (QT)^{1 - r(\sigma) + \varepsilon}, \tag{0.1}$$

where

$$r(\sigma) = \frac{\left(\sigma - \frac{1}{2}\right)}{1 + 2(1 - \sigma)}.$$

This lemma is a rephrasing of Theorem 2 in [2].

Taking Lemma 1 into account, we assume that there exists a zero of the Riemann zeta function $s_0 = \beta + i\gamma$ with $\beta \geq \frac{1}{2} + 7\varepsilon$, where ε is arbitrary small but fixed from the Jutila estimate (0.1). In this case, by the main Theorem 2, each Dirichlet L-series with quadratic character $\chi = \chi_q$ has s_0 as a zero; therefore, in the region

$$\frac{1}{2} + 6 \cdot \varepsilon \leq \operatorname{Re} s \leq 1, \quad |t| \leq \gamma, \quad \gamma \geq 1 \tag{0.2}$$

the following estimate from below is valid for the Jutila sum $N_Q(\sigma, \gamma)$ from (0.1):

$$N_Q\left(\frac{1}{2} + 6 \cdot \varepsilon; \gamma\right) \geq \sum_{q \leq Q} 1 \geq \frac{1}{4} Q. \tag{0.3}$$

In estimate (0.3), we took into account only square-free odd moduli $q \le Q$, for which we made calculations in [1] and in the present paper.

Moreover, from [18], I it follows that $\gamma \ge 10^8$. (The references from [1] will be accompanied with the Roman numeral I.)

On the other hand, for the Jutila function $r(\sigma)$ we have an estimate from below:

$$r(\sigma) = \frac{(\sigma - \frac{1}{2})}{1 + 2(1-\sigma)} \ge \frac{1}{2}\left(\sigma - \frac{1}{2}\right), \quad \sigma \in \left[\frac{1}{2}, 1\right].$$

From this estimate it follows that

$$r\left(\frac{1}{2} + 6 \cdot \varepsilon\right) \ge 3 \cdot \varepsilon.$$

Therefore, from (0.1) we obtain an estimate from above:

$$N_Q\left(\frac{1}{2} + 6 \cdot \varepsilon; \gamma\right) \underset{\varepsilon}{\ll} (Q\gamma)^{1-2\cdot\varepsilon}. \tag{0.4}$$

But γ is fixed and Q is arbitrary; consequently, we choose Q so large that

$$Q^{\varepsilon} > \gamma.$$

In this case, estimate (0.4) reduces to the form

$$N_Q\left(\frac{1}{2} + 6 \cdot \varepsilon; \gamma\right) \underset{\varepsilon}{\ll} Q^{1-\varepsilon}. \tag{0.5}$$

As $Q \to \infty$, estimate (0.3) contradicts the Jutila estimate (0.5). Therefore, the assumption $\beta \ge \frac{1}{2} + 7\varepsilon$ is not valid, and we have

$$\beta < \frac{1}{2} + 7\varepsilon.$$

But β is fixed for a fixed zero s_0 and $\varepsilon > 0$ is arbitrarily small, therefore $\beta = \frac{1}{2}$.

Thus, the main Theorem 2 implies the Riemann hypothesis for $\zeta(s)$ as was said above.

But in this case the following consequence is valid.

Corollary 2.

The Riemann hypothesis is also valid for all Dirichlet L-series with quadratic characters.

This assertion stems immediately from the main Theorem 1, I and Corollary 1 of the present paper.

Indeed, the main Theorem 1 is equivalent to the statement: if the Riemann hypothesis is valid for $\zeta(s)$, then it is valid for all Dirichlet series with quadratic characters. Together with Corollary 1, this proves Corollary 2.

In order to prove the main Theorem 2, we need one more approximate equation for the Rankin convolution $R_{j,\infty}(s)$ in addition to the two equations that were obtained in [1], Sec. 4.

1. The third approximate equation

In addition to integrals (4.1) and (4.16) from [1], consider the integral

$$J_j^{(3)}(N,s)=\frac{1}{2\pi ki}\int\limits_{\operatorname{Re}w>1} N^w\cdot\Gamma\Big(\frac{w}{k}\Big)\Big(\omega-\frac12\Big)(\omega-1)\cdot q^\omega\zeta_q(2\omega)\cdot R_{j,\infty}(\omega)dw \tag{1.1}$$

with the same parameters and notation as in the two previous cases; in particular, $\omega=w+s$.

In the kernel of integral (1.1), there are two compensating factors: $(\omega-\frac12)$ and $(\omega-1)$. The first factor compensates the pole of $\zeta_q(2\omega)$ at the point $\omega=\frac12$, and the second one compensates the pole of $R_{j,\infty}(\omega)$ at the point $\omega=1$. Therefore, when we move the contour in (1.1) to the left to the line $\operatorname{Re}w=-k$, we meet only one pole of the function $\Gamma(\frac{w}{k})$ at the point $w=0$ and obtain, by the Cauchy theorem, the relation

$$J^{(3)}(N,s)=\Big(s-\frac12\Big)\cdot(s-1)\cdot q^s\cdot\zeta_q(2s)\cdot R_{j,\infty}(s)+\frac{1}{2\pi ki}\int\limits_{\operatorname{Re}w=-k}(\,\cdot\,)dw. \tag{1.2}$$

The integral on the right-hand side in (1.2) is estimated as a similar integral in (4.4) from [1], and its order differs from the quantity of (4.9) in [1] only by $|s|$:

$$\frac{1}{2\pi ki}\int\limits_{\operatorname{Re}w=-k}(\,\cdot\,)dw\ll|s|\cdot\tfrac{ch\pi\varkappa_j}{X^2}\log X. \tag{1.3}$$

On the other hand, using expansion (4.10) from [1], by analogy with (4.12) from [1], we reduce the problem to the calculation of the integral

$$J_{s,k}^{(3)}\Big(\frac{m}{qN}\Big)=\frac{1}{2\pi i}\int\limits_{\operatorname{Re}w>1}\Big(\frac{m}{qN}\Big)^{-w}\cdot\Gamma\Big(\frac{w}{k}\Big)\cdot\frac1k\Big(\omega-\frac12\Big)(\omega-1)dw. \tag{1.4}$$

Further, we use the decomposition

$$\Gamma\Big(\frac{w}{k}\Big)\cdot\frac1k\Big(\omega-\frac12\Big)(\omega-1)=\Big(s-\frac12\Big)(s-1)\cdot\Gamma\Big(\frac{w}{k}\Big)\frac1k+2\Big(s-\frac34\Big)\cdot\Gamma\Big(\frac{w}{k}\Big)\cdot\frac{w}{k}+\Gamma\Big(\frac{w}{k}\Big)\cdot\frac{w^2}{k} \tag{1.5}$$

in the kernel of integral (1.4) (recall that $\omega=w+s$) and the Gauss transformations

$$\Gamma\Big(\frac{w}{k}\Big)\cdot\frac{w}{k}=\Gamma\Big(\frac{w}{k}+1\Big), \tag{1.6}$$

$$\Gamma\Big(\frac{w}{k}\Big)\cdot\frac{w^2}{k}=\Gamma\Big(\frac{w}{k}+1\Big)\cdot w=k\cdot\Big[\Gamma\Big(\frac{w}{k}+2\Big)-\Gamma\Big(\frac{w}{k}+1\Big)\Big]. \tag{1.7}$$

After this, we substitute (1.6) and (1.7) into (1.5) and then into the kernel of integral (1.4). As a result, we reduce the problem to a known integral:

$$\frac{1}{2\pi i}\int\limits_{\operatorname{Re}s>0}x^{-s}\Gamma(s)ds=\exp(-x),\quad x>0.$$

Relation (1.4) transforms to the relation

$$J^{(3)}_{s,k}(x)=\left\{\left(s-\frac{1}{2}\right)(s-1)+k\left[2\left(s-\frac{3}{4}\right)-k\right]x^k+k^2\cdot x^{2k}\right\}\cdot\exp(-x^k), \tag{1.8}$$

where $x=\frac{m}{qN}$.

Comparing relation (1.2) with the representation of integral (1.1) as a sum over m and deviding the two sides of the relation by q^s, we obtain the thitd approximate equation

$$\left(s-\frac{1}{2}\right)\cdot(s-1)\cdot\zeta_q(2s)\cdot R_{j,\infty}(s)=\sum_{m=1}^{\infty}\frac{a_j(m)}{m^s}\cdot J^{(3)}_{s,k}\left(\frac{m}{qN}\right)+O\left(|s|\frac{ch\pi\varkappa_j}{q^{\sigma}\cdot X^2}\log X\right). \tag{1.9}$$

The coefficient $a_j(m)$ from (1.9) is defined in (4.11) from [1].

2. Summation formulas. Continuation

In Sec. 3 of [1], we cited the Kuznetsov theorem from [12], I in Lemma 1.3; moreover, we used only a special case of this theorem for $n=m$. In the present part of the work we need the general formulation of this theorem and give here this theorem.

Lemma 1.2 (N. V. Kuznetsov [12], I).

Under the same assumptions on the function h as in Lemma 1.3 of [1], a more general relation is valid for any integers $n\geq 1$ and $m\geq 1$:

$$\sum_{j=1}^{\infty}\rho_{j,\alpha}(n)\cdot\overline{\rho_{j,\alpha}(m)}\cdot\frac{h(\varkappa_j)}{ch\pi\varkappa_j}+\frac{1}{\pi}\sum_{\beta}\int_{-\infty}^{\infty}\varphi_{\beta,\alpha;n}(s,\chi)\times\overline{\varphi_{\beta,\alpha;m}(s,\chi)}\cdot h(r)dr$$

$$=\frac{\delta_{n,m}}{\pi^2}\int_{-\infty}^{\infty}r\cdot th(\pi r)\cdot h(r)dr+\sum_{c\equiv 0(q)}\frac{S_{\alpha,\alpha}(n,m;c,\chi_q)}{c}\cdot\widehat{h}\left(\frac{4\pi\sqrt{n\cdot m}}{c}\right), \tag{2.1}$$

where $s=\frac{1}{2}+ir$, $\widehat{h}$ is the same function as in Lemma 1.3 from [1].

Now it should be mentioned that in the case of the vertex $\alpha=\infty$, the spectral coefficients $\rho_{j,\infty}(m)$ in Fourier expansion (0.7) from [1] are real for all $m\in\mathbb{Z}-\{0\}$. This assertion was proved in Sec. 7 of Part I of this paper.

Therefore, on the left-hand side in (2.1) for the vertex $\alpha=\infty$ we can omit the bar of complex conjugation over the coefficients $\rho_{j,\infty}(m)$, and we have the following consequence.

Corollary 1.2.

For the vertex ∞ relation (2.1) can be written in the form

$$\sum_{j=1}^{\infty}\rho_{j,\infty}(n)\cdot\rho_{j,\infty}(m)\cdot\frac{h(\varkappa_j)}{ch\pi\varkappa_j}+\frac{1}{\pi}\sum_{\beta}\int_{-\infty}^{\infty}\varphi_{\beta,\infty;n}(s,\chi)\overline{\varphi_{\beta,\infty;m}(s,\chi)}\cdot h(r)dr$$

$$=\frac{\delta_{n,m}}{\pi^2}\int_{-\infty}^{\infty}r\cdot th(\pi r)\cdot h(r)\,dr+\sum_{c\equiv 0(q)}\frac{S(n,m;c,\chi)}{c}\widehat{h}\left(\frac{4\pi\sqrt{n\cdot m}}{c}\right). \tag{2.2}$$

We omitted the double subsript (∞,∞) of the Kloosterman sums s in (2.2), because this case coincides with the traditional sums introduced by Kloosterman in [3].

Moreover, in the kernels of the continuous components in (2.2) we retain the bar of complex conjugation over the coefficient $\varphi_{\beta,\infty;m}$ of the Eisenstein series, because we cannot guarantee the reality of these coefficients under expansion at all vertices β, as was mentioned above.

From relation (2.2) it follows that for $\alpha=\infty$ Kuznetsov asymptotics (3.14) in [1] takes the form

$$\sum_{\varkappa_j\le X}\frac{\rho^2_{j,\infty}(n)}{ch\pi\varkappa_j}=\frac{1}{\pi^2}X^2+\theta\Bigl(X+\sqrt{\frac{n}{q}}\Bigr)\cdot\sigma_0(q)\cdot\sigma_0^2(n)\cdot\log^3(X+n+q). \tag{2.3}$$

For $n\neq m$, the principal term of the asymptotics disappears, because $\delta_{n,m}=0$ for $n\neq m$, and we have an estimate:

$$\sum_{\varkappa_j\le X}\frac{\rho_{j,\infty}(n)\cdot\rho_{j,m}(m)}{ch\pi\varkappa_j}\ll\Bigl(X+\sqrt{\frac{(n,m)}{q}}\Bigr)\cdot\sigma_0(q)\cdot\sigma_0(n)\cdot\sigma_0(m)\cdot\log^3(X+n+m+q). \tag{2.4}$$

Estimate (2.4) can be obtained in the same way as the estimate of the remainder in (3.14) from [1], but for $n\neq m$ the greatest common divisor (n,m) arises in place of the parameter n.

Moreover, the quantity $\sigma_0^2(n)$ from (3.14) in [1] is replaced by the product $\sigma_0(n)\cdot\sigma_0(m)$ in (2.4). This product arises from the estimate of the kernel of the intgral in (2.2) of the continuous component:

$$\Bigl|\varphi_{\beta,\infty;n}(s,\chi)\cdot\overline{\varphi_{\beta,\infty;m}(s,\chi)}\Bigr|\le\frac{\sigma_0(n)\sigma_0(m)}{q\cdot d\cdot|L(2s,\chi)|^2},$$

which, in turn, follows from relation (3.6) in [1] on the line $\mathrm{Re}\,s=\frac{1}{2}$. Here we assume that $\beta=\frac{1}{d}$, $d\mid q$.

In addition, in the remainder of the right-hand side of (2.3) and in estimate (2.4) we omit the quantity $\sqrt{q}/w$, because the condition $\alpha=\infty$ means that $w=q$ and we have $\sqrt{q}/w=1/\sqrt{q}\ll1$. The order 1 is compensated by the quantity $X\ge1$.

3. Divisibility by $\zeta(s)$ and averaging

In [1], Sec. 1, we briefly mentioned that the Rankin convolutions $R_{j,\infty}(s)$ are divisible by $\zeta(s)$ in the right half-plane $\mathrm{Re}\,s>\frac{1}{2}$. This fact will be proved in detail at the end of the present paper, but now use it in order prove the main Theorem 2.

Assume that $s_0=\beta+i\gamma$, $\beta>\frac{1}{2}$, is a zero of $\zeta(s):\zeta(s_0)=0$. In this case, s_0 is also a zero of all Rankin convolutions $R_{j,\infty}(s)$ for any $j\ge1$. Therefore, at the point $s=s_0$ the third approximate equation (1) takes the form

$$0=\sum_{m=1}^{\infty}\frac{a_j(m)}{m^{s_0}}\cdot J^{(3)}_{s_0,k}\Bigl(\frac{m}{q_N}\Bigr)+O\Bigl(|s|\cdot\frac{ch\pi\varkappa_j}{\sqrt{q}\cdot X^2}\log X\Bigr). \tag{3.0}$$

This relation can be averaged over $\varkappa_j$ in the interval $\frac{1}{2}X<\varkappa_j\le X$. Here, on the right-hand side in (3.0) we use the explicit form of the coefficients $a_j(m)$ from [1], Sec. 4, formula (4.11):

$$a_j(m)=\sum_{m=m_1\cdot l^2}|\,\rho_{j,\infty}(m_1)\,|^2. \tag{3.1}$$

In (3.1), we changed the notation of the parameters of summation of the right-hand side as compared with [1].

Now we substitute in (3.1) the relation (0.26) from [1], which we give here for convenience:

$$\mid \rho_{j,\infty}(m_1) \mid^2 = \sum_{\substack{m_1 = m_q \cdot n_1 \cdot d,\\ (n_1 \cdot d, q)=1}} \rho_{j,\infty}^2(m_q) \cdot t_{j,\infty}(n_1^2) \cdot \chi(d). \tag{3.2}$$

The definition of the parameter m_q is given in (0.8) from [1], which we also cite here:

$$m_q = \prod_{p|q} p^{\alpha_p}, \quad \alpha_p \geq 0, \tag{3.3}$$

i.e., m_q consists only of prime divisors of q, and $\alpha_p \geq 0$ is an integer.

When we cited relation (0.26) from [1], we used in (3.2) the fact that the coefficients $\rho_{j,\infty}(\,\cdot\,)$ are real in the form

$$\rho_{j,\infty}^2(m_q) + \overline{\rho_{j,\infty}^2(m_q)} = 2 \cdot \rho_{j,\infty}^2(m_q)$$

and then we canceled the factor 2 on the right and on the left in (3.2).

The condition $(n_1, q) = 1$ and definition (3.3) of the numbers m_q imply the condition $(n_1, m_q) = 1$, which, in turn, enables one to use the multiplicativity of the eigenvalues of Hecke operators and to obtain the relation

$$\rho_{j,\infty}(m_q) \cdot t_{j,\infty}(n_1^2) = \rho_{j,\infty}(m_q \cdot n_1^2). \tag{3.4}$$

Substituting (3.4) into (3.2), we obtain a transformed relation (3.2):

$$\mid \rho_{j,\infty}(m_1) \mid^2 = \sum_{\substack{m_1 = m_q \cdot n_1 \cdot d,\\ (n_1 \cdot d_1, q)=1}} \rho_{j,\infty}(m_q) \cdot \rho_{j,\infty}(m_q \cdot n_1^2) \cdot \chi(d). \tag{3.5}$$

Substituting (3.5) into (3.1), we obtain an explicit form of the coefficient $a_j(m)$ in the new form

$$\mid a_j(m) \mid^2 = \sum_{\substack{m = m_q \cdot n_1 \cdot d \cdot l^2,\\ (n_1 \cdot d_1 \cdot l, q)=1}} \rho_{j,\infty}(m_q) \cdot \rho_{j,\infty}(m_q \cdot n_1^2) \cdot \chi(d). \tag{3.6}$$

From relation (3.6) it follows that the new form of the Kuznetsov asymptotics (2.3) is needed for $n_1 = 1$, and estimate (2.4) is required for $n_1 > 1$. Consequently, substituting (3.6) into (3.0) and averaging the relation obtained over $\varkappa_j$ in the interval $\left(\frac{1}{2}X, X\right)$, we give an averaged version of relation (3.0), normalized by the principal term of asymptotics (2.8), and an estimate of it:

$$\sum_{m=1}^{\infty} \frac{b(m)}{m^{s_0}} \cdot J_{s_0,k}^{(3)}\left(\frac{m}{q_N}\right) \underset{\varepsilon, s_0, q}{\ll} X^{1/2-\beta+\varepsilon}, \tag{3.7}$$

where $\varepsilon > 0$ is arbitrarily small but fixed,

$$b(m) = \sum_{\substack{m = m_q \cdot d \cdot l^2,\\ (dl, q)=1}} \chi(d). \tag{3.8}$$

The constant in Vinogradov's symbol $\ll$ depends on ε, s_0, and q.

On the other hand, using the explicit form of $J^{(3)}_{s_0,k}\left(\frac{m}{q_N}\right)$ from (1.8) and of $b(m)$ from (3.8), it is easy to derive an asymptotics:

$$\sum_{m=1}^{\infty} \frac{b(m)}{m^{s_0}}\, J^{(3)}_{s_0,k}\left(\frac{m}{q_N}\right)$$
$$= \left(s_0 - \frac{1}{2}\right)(s_0 - 1) \prod_{p|q} \left(1 - \frac{1}{p^{s_0}}\right)^{-1} \zeta_q(2s_0) \cdot L(s_0, \chi) + \theta \cdot (q_N)^{1/2-\beta+\varepsilon}, \quad (3.9)$$

where $\theta \ll 1$, $\varepsilon > 0$ is the same as in (3.7).

Substituting (3.9) into (3.7), we obtain an estimate

$$L(s_0, \chi) \ll X^{1/2-\beta+\varepsilon}, \quad (3.10)$$

where the constant in the symbol $\ll$ may depend on ε, s_0, and q.

By condition, $\beta > \frac{1}{2}$ and we choose $\varepsilon > 0$ so that the following condition is satisfied:

$$\beta \geq \frac{1}{2} + 2 \cdot \varepsilon. \quad (3.11)$$

Substituting (3.11) into (3.10) and letting X tend to ∞, we see that

$$L(s_0, \chi) = 0. \quad (3.12)$$

But $\varepsilon > 0$ is arbitrarily small and β is fixed for a fixed zero $s_0 = \beta + i\gamma$. Therefore, relation (3.12) is valid for each zero $s_0 = \beta + i\gamma$, $\beta \neq \frac{1}{2}$, of the Riemann zeta function $\zeta(s)$. This proves the main Theorem 2, from which we have consequences presented in the introduction.

4. The Zagier–Shimura theorem. Continuation

Now we prove that the Rankin convolution $R_{j,\infty}(s)$ is divisible by $\zeta(s)$ in the right half-plane $\operatorname{Re} s > \frac{1}{2}$. In [1], Sec. 1, we proved the divisibility of the Rankin convolution by $L(s, \chi)$ and briefly mentioned about its divisibility by $\zeta(s)$. For convenience of the reader, we repeat the last paragraphs in Sec. 1 from [1], because they are the beginning of the proof.

If we assume that the character χ is even, i.e., $\chi(-1) = +1$, then it is necessary to consider the convolution

$$L^*_j(s, \chi) = \iint\limits_{\Pi} \overline{f_j(z)} \cdot \theta_\chi(z) \cdot y^{\frac{s+1}{2}}\, d\mu(z), \quad (4.1)$$

where

$$\theta_\chi(z) = \sum_{n \in \mathbb{Z}} \chi(n) \cdot \exp(2\pi i z \cdot n^2). \quad (4.2)$$

From (4.1), by analogy with (1.3) in [1], we obtain a representation of the convolution $L^*_j(s, \chi)$ in the region $\operatorname{Re} s > 1$ of absolute convergence:

$$L^*_j(s, \chi) = \Gamma^*_j(s) \cdot L_j(s, \chi) \quad (4.3)$$

with the same Γ-coefficient $\Gamma^*_j(s)$ as in (1.3) from [1].

If χ is odd, i.e., $\chi : \chi(-1) = -1$, we need to consider another convolution:

$$\widetilde{L}_j(s,\chi) = \iint\limits_{\Pi} \overline{f_j(z)} \cdot \widetilde{\theta}_\chi(z) \cdot y^{\frac{s}{2}+1} d\mu(z), \tag{4.4}$$

where

$$\widetilde{\theta}_\chi(z) = \sum_{n\in\mathbb{Z}} n \cdot \chi(n) \cdot \exp(2\pi i z n^2). \tag{4.5}$$

In this case we obtain an analog of relation (4.3) in the region of absolute convergence $\operatorname{Re} s > 1$:

$$\widetilde{L}_j(s,\chi) = \widetilde{\Gamma}_j(s) \cdot L_j(s,\chi). \tag{4.6}$$

An explicit form of the coefficient $\widetilde{\Gamma}_j(s)$ is given in [1], formula (1.23).

First we consider the case of an even character χ, i.e., convolution (4.1). From [1], Sec. 1, it follows that when folding the unit strip Π to the fundamental domain F, we can take the group $\Gamma_0(2q)$ instead of $\Gamma_0(4q)$, because the missing number 2 is contained in the exponent of theta function (4.2). Therefore, relation (4.1) reduces to the form

$$L_j^*(s,\chi) = \iint\limits_{F_{2q}} \overline{f_j(z)} \cdot P_w(z,\theta_\chi) \cdot d\mu(z), \tag{4.7}$$

where

$$P_w(z,\theta_\chi) = \sum_{\sigma\in\Gamma|\Gamma_\infty} \chi(\sigma) \cdot \theta_\chi(\sigma z) \cdot y^w(\sigma z), \tag{4.8}$$

where $w = \frac{s+1}{2}$, $\Gamma = \Gamma_0(2q)/\{\pm E\}$.

Poincare series (4.8) can be expanded into the Fourier series in a neighborhood of each vertex of the fundamental domain F_{2q}, and, decomposing integral (4.7) into the sum of integrals over neighborhoods of the vertices of F_{2q}, we can study each integral saparately.

We consider the case of the vertex $\alpha = \infty$ in detail, because the remaining vertices are considered similarly. One can find the details, for example, in [6, 8], I.

The theta function with character (4.2) is not modular relative to the group $\Gamma_0(2q)$, but it is almost modular relative to this group. The precise meaning of this notion will be clear in the sequel. Following [1], Sec. 1, Lemma 1.1, for $\theta_\chi(z)$ we obtain almost modular relations:

$$\theta_\chi(\sigma z) = \frac{e^{-\frac{\pi}{4}i}}{\sqrt{2}} \cdot \sqrt{z + \frac{d}{c}} \cdot \sum_{n\in\mathbb{Z}} g_{\chi,n}\left(\frac{a}{c}\right) \cdot e^{2\pi i z \cdot n^2}, \tag{4.9}$$

where

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2q), \quad c > 0$$

the Gaussian sum $g_{\chi,n}$ is defined by the relation

$$g_{\chi,n}\left(\frac{a}{c}\right) = \sum_{l=1}^{c} \chi(l) \cdot \exp\left[2\pi i \frac{a}{c}(l + n\cdot d)^2\right]. \tag{4.10}$$

If $\chi \equiv 1$, then the full system of residues $\{l\} \pmod c$ in sum (4.10) takes up the product $n \cdot d$, and we obtain the classical Gaussian sum independent of n, and relation (4.9) transforms to the Jacobi modular relation. For $c < 0$, we need to make the same changes in relation (4.9) as in [1], Lemma 1.1.

Having obtained quasimodular transformation (4.9), we use it in order to transform the Poicaré series (4.8). To this end, in the sum over the group Γ/Γ_∞ we separate the unit element $\sigma = E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$:

$$P_w(z, \theta_\chi) = \theta_\chi(z) \cdot y^w + \sum_{\sigma \neq E} \chi(\sigma) \cdot \theta_\chi(\sigma z) \cdot y^w(\sigma z). \tag{4.11}$$

The first summand of the right-hand side in (4.11) grows as a power of y in a neighborhood of the vertex $\alpha = \infty$, whereas $f_j(z)$ exponentially decays in a neighborhood of each vertex of F_{2q}. Therefore, we have no problems with the continuation the part of integral (4.7) corresponding to this summand.

Using relation (4.9), we transform the sum over $\sigma \neq E$ in (4.11) to the new form

$$\sum_{\sigma \neq E} (\,\cdot\,) = y^w \cdot \frac{e^{-\frac{\pi}{4}i}}{\sqrt{2}} \cdot \sum_{n \in \mathbb{Z}} e^{2\pi i z \cdot n^2} \cdot \sum_{\substack{c \equiv 0(2q), \\ c > 0}} \frac{1}{c^{2w}}$$

$$\times \sum_{\substack{a=1 \\ (a,c)=1}}^{c} \chi(d) \cdot g_{\chi,n}\left(\frac{a}{c}\right) \cdot \sum_{\substack{m \in \mathbb{Z} \\ a \cdot d \equiv 1(c), \\ 1 \leq d < c}} \frac{\sqrt{z + d/c + m}}{|z + d/c + m|^{2w}}. \tag{4.12}$$

In relations (4.9) and (4.12), the symbol $\sqrt{(\,\cdot\,)}$ is meant in the sense of the principal value. Moreover, on the right-hand side in (4.12) we used summation only over $c > 0$, because the initial group Γ in the definition of Poincaré series (4.8) is the congruence group $\Gamma_0(2q)$ factored by the center $\{\pm E\}$. A peculiarity is that any automorphy group G containing the center $\{\pm E\}$ and its factor group by this center $\Gamma = G/\{\pm E\}$ have one and the same fundamental domain F, because the matrices $\pm\sigma \in G$ realize one and the same rational transformation of the fundamental domain F. Therefore, when folding the unit strip Π to the fundamental domain F, we should use the minimal group, i.e., the factor group Γ, which is indicated in the summation of Poincaré series (4.8). One can use the whole group G, but then one should normalize the Poincaré series by the factor 1/2. But often, in this notation, the factor 1/2 is omitted and it is implied that the stabilizer Γ_∞ of the vertex ∞ contains the center $\{\pm E\}$. In particular, such a notation is used in [1]. We note that this situation is a generalization of the relationship between $PSL(2, \mathbb{Z})$ and $SL(2, \mathbb{Z})$.

After this digression, we return to relation (4.12) and recall that $z = x + iy$, $y > 0$. Therefore, expanding the inner sum over $m \in \mathbb{Z}$ into the Fourier series with respect to the point $(x + d/c)$ and making necessary transformations, we reduce (4.12) to the new form

$$\sum_{\sigma \neq E} (\,\cdot\,) = y^{1-w} \cdot \frac{e^{-\frac{\pi}{4}i}}{\sqrt{2}} \cdot \sum_{n \in \mathbb{Z}} e^{2\pi i z \cdot n^2} \cdot \sum_{m \in \mathbb{Z}} e^{2\pi i x m} K_w^{(1/2)}(2\pi m y) \cdot \sum_{\substack{c \equiv 0(2q), \\ c > 0}} \frac{1}{c^{2w}} \cdot G_\chi(c; n, m), \tag{4.13}$$

where

$$G_\chi(c; n, m) = \sum_{(a,c)=1} \chi(d) \cdot g_\chi\left(\frac{a}{c}\right) \cdot e^{2\pi i m \frac{d}{c}}, \tag{4.14}$$

$$K_w^{(1/2)}(2\pi my)=\int\limits_{-\infty}^{\infty}\frac{\sqrt{x+i}}{(x^2+1)^w}\cdot e^{2\pi imyx}\,dx. \tag{4.15}$$

The K-function in (4.15) is the Whittaker function, which is a generalization of the known MacDonald function. Integral (4.15) absolutely converges in the region $\operatorname{Re} w>3/4$ and exponentially converges with respect to m for any $y>0$. For $m=0$, we obtain an explicit value of this integral:

$$K_w^{(1/2)}(0)=e^{\frac{\pi}{4}i}\cdot 2^{\frac{5}{2}-2\cdot w}\cdot\pi\cdot\frac{\Gamma(2w-\frac{3}{2})}{\Gamma(w-\frac{1}{2})\cdot\Gamma(w)}.$$

Using the Gauss duplication formula and putting $w=\frac{s+1}{2}$, we obtain

$$K_{(s+1)/2}^{(1/2)}(0)=e^{\frac{\pi}{4}i}\cdot\sqrt{2\pi}\cdot\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}. \tag{4.16}$$

This relation was used in [1], Sec. 1, when we derived the constant term of the Eisenstein series in formula (1.17) from [1].

We also note that definition (4.15) implies the relation

$$K_w^{(1/2)}(2\pi my)=K_w^{(1/2)}(2\pi|m|y)$$

and the estimate

$$K_w^{(1/2)}(2\pi|m|y)\ll\left(v-\frac{3}{4}\right)^{-1}\cdot e^{-\pi\cdot|m|\cdot y}, \tag{4.17}$$

where $v=\operatorname{Re} w=\frac{\sigma+1}{2}$, $\sigma=\operatorname{Re} s>\frac{1}{2}$.

From (4.17) it follows the exponential convergence of the series over $m\in\mathbb{Z}$ in (4.13) for any $y>0$ and $v>3/4$.

A similar convergence is valid for the series over $n\in\mathbb{Z}$, which follows from the estimate

$$|\exp(2\pi iz\cdot n^2)|\le\exp(-2\pi\cdot y\cdot n^2).$$

Therefore, it remains to study the behavior of the series over $c\equiv 0(\operatorname{mod} 2q)$ in (4.13) as a function of w. To do this, we need to know the behavior and estimates of the Gaussian double sums G_χ in (4.14).

5. Gaussian double sums

Decompose $c\equiv 0(\operatorname{mod} 2q)$ into two factors:

$$c\equiv c_1\cdot c_2, \tag{5.1}$$

where

$$(q,c_1)=1;\quad c_2=\prod_{p|q}q_p,\quad q_p=p^{\alpha_p+1},\quad \alpha_p\ge 0,$$

i.e., c_2 consists only of prime divisors of the conductor modulus q of the character $\chi=\chi_q$.

In accordance with factorization (5.1), we also divide the reduced system of residues $a \pmod c$ into two subsystems:

$$a=a_1\cdot c_2+a_2\cdot c_1, \tag{5.2}$$

where $a_i (\bmod c_i)$ is a reduced system of residues of the modulus c_i, $i = 1, 2$.

We also divide the inverse system $d \pmod c$, where $a \cdot d \equiv 1 \pmod c$, in a similar way:

$$d = d_1 \cdot c_2 + d_2 \cdot c_1, \tag{5.3}$$

where

$$a_1 \cdot c_2^2 \cdot d_1 \equiv 1(c_1), \quad a_2 \cdot c_1^2 \cdot d_2 \equiv 1(c_2).$$

Under such a decomposition, $\chi(d) = \chi(c_1) \cdot \chi(d_2)$ and sum (4.10) splits into factors:

$$g_{\chi,n}\left(\frac{a}{c}\right) = \chi(c_1) \cdot g\left(\frac{a_1 \cdot c_2^2}{c_1}\right) \cdot g_{\chi,n}\left(\frac{a_2 \cdot c_1^2}{c_2}\right). \tag{5.4}$$

The first sum in (5.4) is a Gaussian sum:

$$g\left(\frac{a_1 \cdot c_2^2}{c_1}\right) = g\left(\frac{d_1'}{c_1}\right) = \sum_{l_1=1}^{c_1} \exp\left(2\pi i \frac{d_1'}{c_1} \cdot l_1^2\right), \tag{5.5}$$

where $a_1 \cdot c_2^2 \equiv d_1'(c_1)$ and $d_1 d_1' \equiv 1(c_1)$; this follows from defining relations (5.3).

The absence of the dependence on n in sum (5.5) is explained by the fact that a full system of residues $\{l_1\}(\bmod c_1)$ takes a shifted system $\{l_1 + nd_1\}(\bmod c_1)$ again to a full system. Such an effect does not hold in the second sum on the right-hand side in (5.4), because summation is taken over a reduced system of residues $l_2 (\bmod c_2)$ and the sum has the form

$$g_{\chi,n}\left(\frac{a_2 \cdot c_1^2}{c_2}\right) = g_{\chi,n}\left(\frac{d_2'}{c_2}\right) = \sum_{(l_2,c_2)=1} \chi(l_2) \cdot e^{2\pi i \frac{d_2'}{c_2}(l_2+nd_2)^2}, \tag{5.6}$$

where $a_2 \cdot c_1^2 \equiv d_2'(c_2)$ and $d_2 \cdot d_2' \equiv 1(c_2)$; this follows from defining relations (5.3).

Using relations (5.5) and (5.6), we rewrite (5.4) in a new form:

$$g_{\chi,n}\left(\frac{a}{c}\right) = \chi(c_1) \cdot g\left(\frac{d_1'}{c_1}\right) \cdot g_{\chi,n}\left(\frac{d_2'}{c_2}\right). \tag{5.7}$$

Substituting (5.7) into (4.14), we obtain a factorization of the Gaussian double sum:

$$G_\chi(c; n, m) = G(c_1; m) \cdot G_\chi(c_2; n, m), \tag{5.8}$$

where

$$G(c_1; m) = \sum_{(d_1,c_1)=1} g\left(\frac{d_1'}{c_1}\right) \cdot e^{2\pi i \frac{d_1}{c_1} \cdot m}, \tag{5.9}$$

$$G_\chi(c_2; n, m) = \sum_{(d_2,c_2)=1} g_{\chi,n}\left(\frac{d_2'}{c_2}\right) \cdot e^{2\pi i \frac{d_2}{c_2} \cdot m}. \tag{5.10}$$

Relation (5.8) is an explicit form of the multiplicative laws for the Fourier coefficients of the Eisensten series on the metaplectic group (see [7], I). From relation (5.8) it is seen that these laws hold for all numbers c_1, $(c_1,q)=1$, but for the numbers c_2 these law degenerate. Therefore, for the inner sum over $c\equiv 0(2q)$ in (4.13), for fixed n and m, the following factorization is valid:

$$\sum_{c\equiv 0(2q)} \frac{G_\chi(c;n,m)}{c^{2w}} = L_m(2w)\cdot L_{n,m}(2w,\chi), \tag{5.11}$$

where

$$L_m(2w) = \sum_{\substack{c_1\equiv 0(2)\\ (c_1,q)=1}} \frac{G(c_1;m)}{c_1^{2w}}, \tag{5.12}$$

$$L_{n,m}(2w,\chi) = \sum_{c_2\geq q} \frac{G_\chi(c_2;n,m)}{c_2^{2w}}. \tag{5.13}$$

Therefore, in order to obtain the analytic continuation with respect to w of the initial integral (4.7), it is sufficient to continue analytically of L-series (5.12) and (5.13). The case of L-series (5.12) is standard. From the multiplicativity of the double Gaussian sums $G(c_1;m)$, we find the Euler product

$$L_m(2w) = \Big(\sum_{\alpha=1}^{\infty} \frac{G(2^\alpha;m)}{2^{2\cdot\alpha\cdot w}}\Big)\cdot\prod_{p\nmid 2q}\Big(\sum_{\alpha=0}^{\infty}\frac{G(p^\alpha;m)}{p^{2\cdot\alpha\cdot w}}\Big). \tag{5.14}$$

Next we use the explicit form of the double Gaussian sums:

(1) For $p\neq 2$ and $m>0$,

$$G(p^{2t+1};m) = \begin{cases} p\cdot\chi_4(p)\cdot\left(\dfrac{m}{p}\right), & t=0,\quad p\nmid m\\ \qquad 0, & \text{if } t\geq 1 \ \text{ or } \ p|m; \end{cases}$$

$$G(p^{2t+1};-m) = \begin{cases} p\cdot\left(\dfrac{m}{p}\right), & t=0,\quad p\nmid m,\\ \qquad 0 \ \text{ if } \ t\geq 1 \ \text{ or } \ p|m; \end{cases}$$

$\left(\frac{m}{p}\right)$ is the Legendre symbol;

(2) $p=2$;

$G(2^{2t+1};m)=0$ for all $t\geq 0$ and for any $m\in\mathbb{Z}$.

(3) For $m=0$, for all prime p,

$$G(p^{2t+1};0)=0,\quad t\geq 0,$$

$$G(p^{2t};0)=p^{3t}\left(1-\frac{1}{p}\right),\quad t\geq 1.$$

(4) For $m\geq 1$,

$$G(p^{2t};\pm m)=0,\quad t\geq 1,\quad p\nmid m.$$

(5) If $m=m_0^2\cdot m_1$, where $(m_0,m_1)=1$,

$$m_0=\prod_{p|m_0} p^{t_p(m_0)},\quad m_1=\prod_{p|m_1} p^{2\cdot t_p(m_1)+1},$$

then

(a) for $p|m_0$,

$$G(p^{2t};m)=\begin{cases} p^{3t}\cdot\left(1-\dfrac{1}{p}\right), & 1\le t\le t_p(m_0),\\ 0, & t\ge 1+t_p(m_0);\end{cases}$$

(b) for $p|m_1$,

$$G(p^{2t};m)=\begin{cases} p^{3t}\cdot\left(1-\dfrac{1}{p}\right), & 1\le t\le t_p(m_1),\\ -p^{3t-1}, & t=1+t_p(m_1),\\ 0, & t\ge 1+t_p(m_1).\end{cases}$$

From the above expressions for double Gaussian sums, for $m>0$ and $w=\frac{s+1}{2}$ we obtain an explicit form of relation (5.14):

$$L_m(2w)=\frac{L_{2q}(s,\chi_4\cdot(\frac{m}{\cdot}))}{\zeta_{2q}(2s)}\cdot\sigma_m(2w), \tag{5.15}$$

$$L_{-m}(2w)=\frac{L_{2q}(s,(\frac{m}{\cdot}))}{\zeta_{2q}(2s)}\cdot\sigma_{-m}(2w), \tag{5.16}$$

where L_{2q} and ζ_{2q} are the classical Dirichlet series and ζ is the Riemann function without prime divisors of $2\cdot q$ in its Euler products. An explicit form of the finite products $\sigma_{\pm m}(2w)$ over the prime divisors of m is determined by items (2)–(5) for Gaussian sums; $(\frac{m}{\cdot})$ is the Legendre symbol, which, by the reciprocity law, is a character of modulus m, $4m$, or $8m$. All possible variants of the characters and L-series for fixed m are given in detail in [2], I.

From relations (5.15) and (5.16), one can see that $L_{\pm m}(2w)$ is extendable to the left, to the line $\operatorname{Re} w=\frac{3}{4}$ (i.e., $\operatorname{Re} s=\frac{1}{2}$) in the form of an entire function, except for the case where $m=0$ or the character of the Dirichlet series is principal. In this case, a pole at the point $s=1$ arises.

Moreover, from the theory of Dirichlet series the following estimate is well known:

$$L_{2q}\left(s,\chi_4\cdot\left(\frac{m}{\cdot}\right)\right)\ll\sqrt{m},\quad \operatorname{Re} s\ge\frac{1}{2}, \tag{5.17}$$

and the explicit form of the products $\sigma_{\pm m}(2w)$ implies the estimate

$$\sigma_{\pm m}(2w)\ll\sigma_0(m),\quad \operatorname{Re} w\ge\frac{3}{4}. \tag{5.18}$$

Therefore, series (5.14) grows with respect to t not faster than a power. At the same time, the series over m in (4.13) converges exponentially. Therefore, in order to solve the extendability problem we need to study the behavior of the Gaussian sums in the case of degenerate moduli c_2.

6. Degenerate cases of Gaussian sums

From the multiplicativity of the Gaussian sums, we obtain the factorization

$$G_{\chi_q}(c_2;n,m)=\prod_{p|q}G_{\chi_p}(c_p;n,m), \tag{6.1}$$

where $c_2 = \Pi_{p|q} c_p$ is the factorization from (5.1). Therefore, series (5.13) can be written as a finite product

$$L_{n,m}(2w,\chi_q) = \prod_{p|q}\left(\sum_{\alpha=1}^{\infty}\frac{G_{\chi_p}(p^\alpha;n,m)}{p^{2\alpha\cdot w}}\right), \tag{6.2}$$

$$G_{\chi_p}(c_p;n,m) = \sum_{a=1}^{c_p}\chi_p(a)\cdot g_{\chi_p,n}\left(\frac{a}{c_p}\right)\cdot e^{2\pi i\frac{d}{c_p}m}, \tag{6.3}$$

where $c_p = p^\alpha$, $a\cdot d \equiv 1(c_p)$.

From the definition of $g_{\chi,n}$,

$$g_{\chi,n}\left(\frac{a}{c_p}\right) = \sum_{l=1}^{c_p}\chi_p(l)\cdot e^{2\pi i\frac{a}{c_p}(l+n\cdot d)^2}$$

one can see that, by the change of the reduced system of residues $l\to l\cdot d$ (for fixed d), it can be reduced to the form

$$g_{\chi_p,n}\left(\frac{a}{c_p}\right) = \chi_p(d)\cdot g^*_{\chi_p,n}\left(\frac{d}{c_p}\right), \tag{6.4}$$

where

$$g^*_{\chi_p,n}\left(\frac{d}{c_p}\right) = \sum_{l=1}^{c_p}\chi_p(l)\cdot e^{2\pi i\frac{d}{c_p}(l+n)^2}. \tag{6.5}$$

Substituting (6.4) into (6.3), we obtain

$$G_{\chi_p}(c_p;n,m) = \sum_{\substack{d=1\\(d,p)=1}}^{c_p} g^*_{\chi_p,n}\left(\frac{d}{c_p}\right)e^{2\pi i\frac{d}{c_p}m}. \tag{6.6}$$

Summation over d in (6.6) does not contain a character. This enables us to derive an explicit fprm of sum (6.6) rather simply.

I. Assume that $n\equiv 0(p)$; then $\chi_p(l) = \chi_p(l+n)$ and sum (6.5) does not depend on n:

$$g^*_{\chi_p,n}\left(\frac{d}{c_p}\right) = g_{\chi_p,n}\left(\frac{d}{c_p}\right) = \sum_{l=1}^{c_p}\chi_p(l)\cdot e^{2\pi i\frac{d}{c_p}\cdot l^2}. \tag{6.7}$$

Substituting (6.7) into (6.6) and changing the orders of summation, we obtain the relation

$$G_{\chi_p}(c_p;n,m) = \sum_{l=1}^{c_p}\chi_p(l)\cdot S\left(\frac{m+l^2}{c_p}\right), \tag{6.8}$$

where S is the Ramanujan sum:

$$S\left(\frac{a}{c}\right) = \sum_{(l,c)=1} e^{2\pi i\frac{a}{c}l} = \sum_{d|(a,c)}\mu\left(\frac{c}{d}\right)\cdot d.$$

From the properties of this sum, we have

$\mathrm{I}^{(1)}$ if $m \equiv 0(p)$ and $n \equiv 0(p)$, then for all $\alpha \geq 1$,

$$G_{\chi_p}(c_p; n, m) = 0, \quad c_p = p^{\alpha}. \tag{6.9}$$

$\mathrm{I}^{(2)}$ If $(m,p) = 1$ and $n \equiv 0(p)$, then for all $\alpha \geq 1$,

$$G_{\chi_p}(c_p; n, m) = \frac{1}{2}\big[1 + \chi_p(-m)\big] \cdot \varphi(c_p). \tag{6.10}$$

II. Assume that $(n,p) = 1$. In this case,

$$g^*_{\chi_p,n}\left(\frac{d}{c_p}\right) = \chi_p(n) \cdot \sum_{l=1}^{c_p} \chi_p(l) \cdot e^{2\pi i \frac{d \cdot n^2}{c_p}(l+1)^2}. \tag{6.11}$$

Substituting (6.11) into (6.6) and changing the orders of summation, we obtain an analog of relation (6.8):

$$G_{\chi_p}(c_p; n, m) = \chi_p(n) \sum_{l=1}^{c_p} \chi_p(l) \cdot S\left(\frac{m + n^2 \cdot (l+1)^2}{c_p}\right).$$

$\mathrm{II}^{(1)}$ $m \equiv 0(p)$,

$$G_{\chi_p}(c_p; n, m) = \chi_p(-n) \cdot \varphi(c_p). \tag{6.12}$$

$\mathrm{II}^{(2)}$ Let $(m,p) = 1$.

If there exists a number l from the reduced system of residues $1 \leq l \leq (p-1)$ such that for fixed n and m, the following relation is valid:

$$m + n^2(l+1)^2 \equiv 0(p),$$

then

(a)

$$G_{\chi_p}(c_p; n, m) = \chi_p(-n + \sqrt{-m}) \cdot \varphi(c_p) \tag{6.13}$$

and the symbol $\sqrt{-m}$ is defined by the relation $\sqrt{-m} = m_0$ if $-m \equiv m_0^2(p)$;

(b)

$$G_{\chi_p}(c_p; n, m) = 0 \tag{6.14}$$

if $m + n^2 \equiv 0(p)$ or $\chi_p(-m) = -1$, i.e.,. $-m$ is a quadratic nonresidue $\pmod p$.

Relations (6.9)–(6.14) show that in all cases the following estimate is valid:

$$\mid G_{\chi_p}(c_p; n, m) \mid \leq \varphi(c_p) = p^{\alpha} \cdot \left(1 - \frac{1}{p}\right), \quad \alpha \geq 1. \tag{6.15}$$

Hence, from the multiplicativity we obtain a general estimate

$$\mid G_{\chi_p}(c_p; n, m) \mid \leq \varphi(c_2). \tag{6.16}$$

From estimate (6.16) it follows the absolute convergence of series (5.13) in the region $\mathrm{Re}\, w > \frac{1}{2}$, which means the region $\mathrm{Re}\, s > 0$ for the parameter s ($w = \frac{s+1}{2}$).

The behavior of the Fourier coefficients at any other vertex of the fundamental domain F_{2q} of the Poincaré series (4.8) can be studied in the same way. In principle, it is sufficient to know only the expansion at the vertex $\alpha = \infty$ if one takes into account that the Poincaré series (4.8) grows not faster than y^{-2} as $y \to 0$ (this follows from the estimates obtained in Secs. 4, 5, 6), whereas $f_j(z)$ exponentially decreases at each vertex of F_{2q}. This means that integral (4.7) as an analytic function of s can also be continued to the line $\operatorname{Re} s = \frac{1}{2}$ in the form of an entire function, except for the point $s = 1$, where the kernel of integral (4.7) has a pole of first order. In principle, the residue at this pole must be equal to zero. But it is rather difficult to prove this by the direct evaluation of integral (4.7) at the point $s = 1$. At the same time, upon meromorphic extension of integral (4.7) to the line $\operatorname{Re} s = \frac{1}{2}$, the regularity of $L_j(s, \chi)$ at the point $s = 1$ follows from the factorization of the Rankin convolution:

$$R_{j,\infty}(s) = \zeta_q(s) \cdot L_j(s, \chi) \cdot |r|_{j,q}(s), \tag{6.17}$$

which was obtained in [1], formula (0.15).

As $s \to 1$, the poles of $R_{j,\infty}(s)$ and $\zeta_q(s)$ at this point compensate each other in relation (6.17), and we obtain the regularity of the product

$$L_j(s, \chi) \cdot |r|_{j,q}(s)\Big|_{s=1}. \tag{6.18}$$

Hence it follows the regularity of $L_j(s, \chi)$ at the point $s = 1$ if all prime p, $p|q$, satisfy the condition $p \geq 17$. In this case, the currently available estimate for the eigenvalues of the Hecke operators:

$$|t_{j,\infty}(p)|^2 \leq 4 \cdot \sqrt{p} \tag{6.19}$$

ensures the condition

$$|r|_{j,q}(s)\Big|_{s=1} \neq 0,$$

which implies the regularity of $L_j(s, \chi)$ at the point $s = 1$.

If q is divisible by $p \leq 13$, then for such prime numbers p we can use the conjecture on degenerate estimates for divisors of the congruence modulus:

$$|t_{j,\infty}(p)|^2 \leq 1, \quad p|q. \tag{6.20}$$

These estimates are valid for finite-dimensional spaces (see [4] for details). Conceivably, these estimates are valid for Hilbert spaces of automorphic functions. Note that the presence of the pole of $L_j(s, \chi)$ at the point $s = 1$ does not influence the regularity of $L_j(s, \chi)$ in the remaining part of the critical strip $\frac{1}{2} < \operatorname{Re} s < 1$, which guarantees the divisibility of the Rankin convolution by $\zeta(s)$ in this region.

The case of integral (4.4) with an odd character χ, i.e., $\chi(-1) = -1$, is studied in the same way. A small distinction is in the analog of the modular relation (4.9) for $\widetilde{\theta}_\chi(z)$ from (4.5). In this case, it will be as follows:

$$\widetilde{\theta}_\chi(\sigma z) = \sqrt{2} \cdot e^{-\frac{\pi}{4} i} \cdot \left(z + \frac{d}{c}\right)^{3/2} \cdot \sum_{n \in \mathbb{Z}} n \cdot g_{\chi,n}\left(\frac{a}{c}\right) \cdot e^{2\pi i z \cdot n^2}. \tag{6.21}$$

In comparison with (4.9), on the right-hand side in (6.21) an additional term arises:

$$2 \cdot \left(z + \frac{d}{c}\right) \cdot n,$$

which does not influence the estimates of the same Gaussian sums that were studied in Secs. 5 and 6. Therefore, the analytic continuation of integral (4.4) can be obtained in the same way.

It is obvious from the first that the main idea of the whole paper is the comparison of arithmetic and spectral properties of the Rankin convolutions on two levels.

On the first level of calculation, with the help of the Euler product for the Rankin convolution $R_j(s)$ we prove the divisibility of $R_j(s)$ by $\zeta(s)$ and $L(s,\chi)$ in the region of absolute convergence $\operatorname{Re} s > 1$, and further, by spectral methods, this divisibility is extended to the left half-plane $\operatorname{Re} s \le 1$. (This stage of calculation may be called the Zagier–Shimura stage.) The main result here is the obtaining of the equation $R_j(s_0) = 0$ for a zero s_0 of the Dirichlet series $L(s,\chi)$ in part I and of $\zeta(s)$ in part II. This equation, as distinct from the trivial equation $L(s,\chi) = 0$ in part I (or $\zeta(s) = 0$ in part II), has a free parameter χ_j, which can be used in order to obtain a bound for the zero s_0 in the citical strip.

But the equation $R_j(s_0) = 0$ is implicit in the sense that we do not know the explicit form of the Rankin convolution $R_j(s)$ when we transform it from the region of absolute convergence $\operatorname{Re} s > 1$ to the critical strip $\operatorname{Re} s \in (0,1)$. The necessity of obtaining this explicit form of $R_j(s)$ leads us to the second level of calculation, beginning with Sec. 4 of part I. To do this, we use the fact that the Rankin convolution $R_j(s)$ is simultaneously (with the Euler product) a Dirichlet series in the region of absolute convergence $\operatorname{Re} s > 1$.

In number theory it is well known that any Dirichlet series that has a functional equations can be extended to the critical strip using the technique of approximate equations, which is developed rather well. This is done for $R_j(s)$ in Sec. 4 of part I.

From the approximate equation for $R_j(s)$ in Sec. 4, it is seen that the explicit form of the Rankin convolution in the critical strip has the form of a sum of Laurent residues at poles plus finite parts of the Rankin–Dirichlet series.

Therefore, for $s = s_0$ when $R_j(s) = 0$ we obtain a linear equation connecting the Laurent residues and the finite parts of the Rankin–Dirichlet series.

Upon normalization of this equation by the factor $(\operatorname{ch} \pi \varkappa_j)^{-1}$, we reduce it to a relation between the Laurent residues (which do not depend on the free parameter $\varkappa(j)$ and the finite parts of the Rankin–Dirichlet series whose coefficients depend on this parameter.

This equation is valid for any parameter $\varkappa_j$, therefore it can be averaged with respect to this parameter. Moreover, on the left-hand side of this relation, in which the Laurent residues occur, we must use the Herman Weil asymptotics from Sec. 5 (formula (5.14)). On the right-hand side we use the Kuznetsov asymptotics from Sec. 3, because the coefficients of the Rankin–Dirichlet series depend on the parameter χ_j.

Normalizing the averaged equation obtained, we see that its left-hand side differs from the right-hand side by the quantity $\zeta(s_0)$. Therefore, $\zeta(s_0) = 0$. This was proved in part I.

In part II, this method works, under small changes, when we interchange $\zeta(s)$ and $L(s,\chi)$, i.e., from the relation $\zeta(s_0) = 0$ it follows the relation $L(s_0,\chi) = 0$ if $\operatorname{Re} s > \frac{1}{2}$.

The comparison of the calculation in parts I and II leads to the relation $\operatorname{Re} s_0 = \frac{1}{2}$, i.e., to the proof of the Riemann hypothesis.

It should be mentioned that the Kuznetsov asymptotics in Sec. 3, part I and in Sec. 2, part II has accuracy of degree 1/2 of the principal term. Therefore, in Sec. 5, part I the Weil asymptotics is also obtained with accuracy of degree 1/2 of the principal term, which ensures the proof of the Riemann hypothesis at the final stage.

If we use the Weil asymptotics with accuracy of degree 3/4 of the principal term, which is now well known to specialists, then instead of the Riemann hypothesis, we obtain a quasi-Riemannian shift 3/4.

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References

1. A. I. Vinogradov, *The Rankin convolutions and Dirichlet series, I*, Preprint POMI (1999).
2. M. Jutila, *On mean values of Dirichlet polinomials with real characters*, Acta Arithm., **27** (1975), 191–198.
3. H. D. Kloosterman, *On real representation of number in the form* $ax^2 + by^2 + cz^2 + dt^2$, Acta Math., **49** (1926), 407–464.
4. A. O. Atkin and J. Lehner, *Hecke operator on* $\Gamma_0(m)$, Math. Ann., **185** (1970), 134–160.