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THE RANKIN CONVOLUTIONS AND DIRICHLET SERIES. I

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ABSTRACT

In the present paper, the Riemann hypothesis is proved for classical Dirichlet series with quadratic characters, including the Riemann zeta function, which occurs in this set as a Dirichlet series with trivial quadratic character modulo 1.

The paper consists of two parts. In part I, a spectral-number technique is elaborated and it is shown how this technique works with an example of effectivization of the Siegel theorem, i.e., the Riemann hypothesis is proved for the Siegel zero. In part II, this proof is carried over to all of the remaining nontrivial zeros.

Dedicated to the 100th birthday
of C. L. Siegel and to the
60th anniversary of his paper [1]

Introduction

In [2, 3], it is proved that the Rankin convolution is divisible by the Dedekind zeta function of the quadratic field $k = Q\left(\sqrt{\chi_q(-1)q}\right)$ in finite-dimensional spaces of classical modular forms.

If we carry over this result to infinite-dimensional Hilbert spaces of automorphic functions of weight zero, invariant relative to the congruence group $\Gamma_0(q)$ with multiplier $\chi = \chi_q$ (the quadratic character of conductor modulus q) in the invariance relation, then the Zagier–Shimura theorem is a background for the proof of the main theorem of the present paper.

Theorem 1. *Let $s_0 = \beta + i\gamma$ be a zero of the Dirichlet series $L(s, \chi)$: $L(s_0, \chi) = 0$, $\operatorname{Re} s_0 = \beta \geq \frac{1}{2}$, let χ be a quadratic character of conductor modulus q. If $\beta \neq \frac{1}{2}$, then this zero is also a zero of the Riemann zeta function: $\zeta(s_0) = 0$.*

This implies several consequences.

Corollary 1. *If $s_0 = \beta$ is a real Siegel zero of $L(s, \chi)$ (see* [1]*), then $\beta = \frac{1}{2}$.*

This consequence can be immediately obtained from the main theorem if one takes into account a well-known fact that $\zeta(s)$ does not have nontrivial zeros on the real axis. Corollary 1 solves the problem of effectivization of the Siegel theorem (see [1]).

Corollary 2. *The table Riemann hypothesis, obtained in* [18] *for $\zeta(s)$ for a height $|t| \leq 10^8$ of the critical strip, is valid for the same height for all Dirichlet L-series with quadratic characters, but, possibly, their zeros are not of multiplicity one, contrary to the zeros of $\zeta(s)$, which is proved in* [18].

Corollary 3. *The Ingham–Montgomery density theorem for $\zeta(s)$ (see* [19]*, Chap.* 12*, Theorem* 12.1 *for $q = 1$) is valid for all Dirichlet L-series with quadratic characters $\chi = \chi_q$ if it is multiplied by $\log(qT)$, the maximal multiplicity of a zero of $L(s, \chi_q)$ for a height $|t| \leq T \geq 1$.*

Corollary 4. *Vinogradov's bound for the zeros of $\zeta(s)$ (see* [5]*,* VI*, §*5*) is valid for all Dirichlet L-series with quadratic characters.*

We restricted ourselves to these four consequences, which are most important in the applications.

The present paper is devoted to the implementation of this program. We observe that paper [17] was the first nontrivial step on the road to the effectivization of the Siegel theorem [1].

In the sequel, we shall use the term "the conductor modulus q of a quadratic character $\chi = \chi_q$" instead of "the fundamental dicriminant q of a character χ" in order not to take into account of the sign of q in numerous estimates in which q occurs. The conductor modulus is always positive, whereas the fundamental discriminant is obtained from it by multiplying by $\chi_q(-1) = \pm 1$. This expression of the dicriminant was used above, in the first paragraph, in the notation of quadratic fields k.

Our first problem is to prove the Zagier–Shimura theorem for the Hilbert spaces mentioned above. Here, of primary interest are the functions $f_j(z)$, $j \geq 1$, of the discrete spectrum of the Laplace operator

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right),$$

$$\Delta f_j(z) = \lambda_j f_j(z), \tag{0.1}$$

automorphic relative to the group $\Gamma_0(q)$ with multiplier $\chi(\sigma)$ in the invariance relation

$$f_j(\sigma z) = \chi(\sigma) f_j(z), \quad \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q),$$

where the multiplier $\chi(\sigma)$ is defined by the relation

$$\chi(\sigma) = \chi_q(d) \tag{0.2}$$

χ_q is the quadratic character of conductor modulus q.

In the sequel, we shall consider the groups $\Gamma_0(q)$, where q coincides with the conductor modulus of the quadratic character $\chi = \chi_q$. This means that either q is odd and square-free, or $q = 4q_1$, or $q = 8q_1$, where q_1 is odd and square-free. We perform the calculations for odd and square-free q in order to simplify the technical part of the paper. The presence of a power of 2 in the modulus: 2^ν, $\nu = 2, 3$, does not influence the method of proving but requires additional calculations as always occurs with powers of 2 (see, for example, [6]). For the same purpose, we assume that the Selberg spectral conjecture (see [2]) on the absence of small eigenvalues in the spectrum of the congruence group $\Gamma_0(q)$ holds in the version with a nontrivial multiplier. Note that this conjecture is now proved in the usual version and in the version with a nontrivial multiplier (see [22]). But these papers have been published recently, and not very many specialists are familiar with them. For this reason, we prove the main theorem (Theorem 1) in such a way that it is valid independently of the Selberg conjecture.

After this short digression, we return to the functions $f_j(z)$ in (0.1). We choose these functions, which constitute an orthonormal basis for the Hilbert space, in such a way that they are also eigenfunctions of all Hecke operators T_m:

$$T_m f_j(z) = t_j(m) f_j(z),$$

where $t_j(m)$ is the eigenvalue of the Hecke operator T_m for the function $f_j(z)$. Moreover, we define the operators T_m by the relation

$$T_m f_j(z) = \frac{1}{\sqrt{m}} \sum_{\substack{ad=m \\ 1 \le b \le d}} \chi(a) f_j\left(\frac{az+b}{d}\right). \tag{0.3}$$

Note that, in the sequel, we shall often use one and the same symbol χ to denote the multiplier χ and the character χ_q. This does not lead to a confusion, because there are no other characters in the text.

To the system of Hecke operators we also assign the reflection operator T_{-1}:

$$T_{-1} f_j(z) = f_j(-\bar{z}) = \xi_j f_j(z). \tag{0.4}$$

The eigenvalue $\xi_j = \pm 1$ of this operator is determined by the eveness or oddness of an eigenfunction $f_j(z)$.

If m and n are relatively prime with q, i.e., $(mn, q) = 1$, then for the operators T_m and T_n, the following multiplicative relation is valid:

$$T_m T_n = \sum_{d|(m,n)} \chi(d) T_{\frac{mn}{d^2}}.$$

From this, a similar relation holds for their eigenvalues:

$$t_j(m)t_j(n) = \sum_{d|(m,n)} \chi(d)t_j\left(\frac{mn}{d^2}\right). \tag{0.5^a}$$

From this general relation, a special important case follows:

$$t_j^2(n) = \sum_{d|n} \chi(d)t_j\left(\left(\frac{n}{d}\right)^2\right). \tag{0.5}$$

We observe that all these assertions for Hecke operators are classical and date back to the papers of Hecke himself concerning finite-dimensional spaces of functions of even weight. The proof of his results for the Hilbert spaces of functions of weight zero may be found, for example, in [6, 12, 16].

Formulas (0.5^a) and (0.5) are well known in the classical theory of numbers for the function of the number of divisors with a character:

$$\tau_s(m,\chi) = m^{s-\frac{1}{2}}\sigma_{1-2s}(m,\chi), \tag{0.6}$$

where

$$\sigma_s(m,\chi) = \sum_{d|m} \chi(d)d^s,$$

s is an arbitrary complex parameter. Such a relationship is rather natural, because for $s = \frac{1}{2}+it$, the function τ_s from (0.6) is an eigenvalue of the Hecke operator T_m for the Eisenstein series of the group $\Gamma_0(q)$ with multiplier χ.

In order to find a relationship between the above results and the Rankin convolutions for $f_j(z)$, consider the Fourier series for this function:

$$f_j(z) = \sum_{n\neq 0} \rho_j(n)\sqrt{y}K_{i\varkappa_j}(2\pi|n|y)e^{2\pi inx}, \tag{0.7}$$

where $z = x+iy$, $y > 0$, $\varkappa_j = \sqrt{\lambda_j - 1/4}$; K is the MacDonald function from the theory of Bessel functions (see [20]):

$$K_s(x) = \frac{1}{2}\int_0^\infty y^{s-1}e^{-\frac{1}{2}x(y+y^{-1})}dy.$$

The arithmetic Fourier coefficients $\rho_j(n)$ in (0.7) for $n = \pm m$, $m \geq 1$, are connected by the relation

$$\rho_j(-m) = \xi_j\overline{\rho_j(m)},$$

where $\xi_j = \pm 1$ is an eigenvalue of the reflection operator T_{-1} from (0.4); the bar means complex conjugation.

If we rearrange the terms of series (0.7) and rewrite (0.7) in the form

$$f_j(z) = \sum_{m=1}^{\infty} \left[\rho_j(m)e^{2\pi imx} + \xi_j\overline{\rho_j(m)}e^{-2\pi imx}\right]\sqrt{y}\,K_{i\varkappa_j}(2\pi my),$$

then $f_j(z)$ is a real-valued function if it is even ($\xi_j = +1$) and pure imaginary if it is odd ($\xi_j = -1$).

This is an analog of the classical Euler decomposition

$$e^{2\pi imx} = \cos 2\pi mx + i\sin 2\pi imx$$

in the theory of Fourier series into the pure real function $\cos 2\pi mx$ and the pure imaginary odd part $i\sin 2\pi mx$.

The MacDonald function $K_{i\varkappa_j}$ is a real-valued function if $\varkappa_j$ is either pure real or pure imaginary, which corresponds to large eigenvalues $\lambda_j \geq 1/4$ in the first case and to small eigenvalues $\lambda_j < 1/4$ in the second.

If we use the factorization

$$m = m_q n, \tag{0.8}$$

where $(n,q) = 1$, $m_q = \prod_{p|q} p^{\alpha_p}$ (i.e., m_q consists only of prime divisors of q), then the following factorization of the Fourier coefficients holds:

$$\rho_j(m) = \rho_j(m_q) t_j(n).$$

Next we note that the Hecke operators from (0.3) are also well defined for the numbers m_q from (0.8). Moreover, the multiplicative properties for them and their eigenvalues remain valid. The details may be found in [16]; here we underline the difference in the definitions of Hecke operators in (0.3) and in [16]. The definition of T_n in [16] corresponds to $\chi(n)T_n$ in the sense of (0.3). This explains the coincidence of definition (0.3) with the operator U_p from [16] for $p|q$.

Therefore, a complete factorization of the Fourier coefficients $\rho_j(m)$ is of the form

$$\rho_j(m) = \rho_j(1) t_j(m_q) t_j(n), \tag{0.9}$$

where $\rho_j(1)$ is a normalizing coefficient whose value is determined by the relation

$$\iint_{F_q} |f_j(z)|^2 d\mu(z) = 1.$$

From (0.9) and (0.7) it follows that $\rho_j(1) \neq 0$, except for a trivial case $f_j(z) \equiv 0$, which we shall not consider.

Returning to the Hecke operators T_n and T_{m_q}, we note that they are essentially different from one another despite the common definition (0.3).

The Hecke operators T_n are Hermitian operators with a character:

$$\langle T_n f, g\rangle = \chi(n)\langle f, T_n g\rangle, \quad (n,q) = 1$$

whereas the T_{m_q} do not posses this property.

The property of being Hermitian with a character means that T_n is purely Hermitian for quadratic residues $n \pmod q$ and anti-Hermitian for quadratic nonresidues $n \pmod q$, which corresponds to the cases $\chi(n) = \pm 1$.

Therefore, the eigenvalues $t_j(n)$ are real in the first case, $t_j(n) \in \mathbb{R}$, and are pure imaginary in the second case, $t_j(n) \in i\mathbb{R}$.

The reader can find the proof of these facts in [15] for finite-dimensional spaces and in [16] for Hilbert spaces of functions of weight zero.

Therefore, the following relation holds:

$$|t_j(n)|^2 = \chi(n)t^2(n). \tag{0.10}$$

Squaring the moduli of the two sides of (0.9):

$$|\rho_j(m)|^2 = |\rho_j(m_q)|^2 \, |t_j(n)|^2$$

and using relation (0.10), we obtain an important consequence

$$|\rho_j(m)|^2 = |\rho_j(m_q)|^2 \chi(n) t_j^2(n), \tag{0.11}$$

where

$$|\rho_j(m_q)|^2 = |\rho_j(1)|^2 |t_j(m_q)|^2.$$

I express my deep gratitude to Professor N. V. Proskurin for discussions about the Hecke operators in finite-dimensional spaces and to Professor O. M. Fomenko for similar discussions concerning Hilbert spaces, and also for references [15] and [16] suggested by them.

The normed Rankin convolution $R_j^*(s)$ for the function $f_j(z)$ is defined by the relation

$$R_j^*(s) = \iint_{\Pi} |f_j(z)|^2 y^s \, d\mu(z), \tag{0.12}$$

where Π is the unit strip of the upper half-plane H:

$$\Pi = \left\{ z = x + iy, \ |x| \le \frac{1}{2}, \ y > 0 \right\}.$$

$d\mu(z)$ is the invariant differential relative to $SL_2(\mathbb{R})$:

$$d\mu(z) = \frac{dx \, dy}{y^2}.$$

Substituting (0.7) into (0.12) and interchanging summation and integration (which is valid in the region of absolute convergence $\operatorname{Re} s > 1$), we obtain an explicit form of the convolution $R_j^*(s)$ in this field:

$$R_j^*(s) = 2\Gamma_j(s) R_j(s), \tag{0.13}$$

where

$$\Gamma_j(s) = 2^{s-3} \frac{(2\pi)^{-s}}{\Gamma(s)} \Gamma^2\left(\frac{s}{2}\right) \Gamma\left(\frac{s}{2} + i\varkappa_j\right) \Gamma\left(\frac{s}{2} - i\varkappa_j\right),$$

$$R_j(s) = \sum_{m=1}^{\infty} |\rho_j(m)|^2 / m^s. \tag{0.14}$$

Substituting (0.11) and (0.5) into (0.14) and interchanging the orders of summation, we find

$$R_j(s) = \zeta_q(s) L_j(s, \chi) |r|_{j,q}(s), \tag{0.15}$$

where

$$\zeta_q(s) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right),$$

$$L_j(s,\chi) = \sum_{(n,q)=1} \chi(n) t_j(n^2)/n^s, \tag{0.16}$$

$$|r|_{j,q}(s) = \sum_{m_q \geq 1} |\rho_j(m_q)|^2/m_q^s, \tag{0.17}$$

$\zeta(s)$ is the classical Riemann zeta function.

Relation (0.15) implies that the Rankin convolution $R_j(s)$ is divisible by $\zeta(s)$ in the region of absolute convergence $\operatorname{Re} s > 1$. Henceforth, the phrase "a function is divisible by another function" means that the ratio of them is a regular function in the region under consideration."

On the other hand, if $f_j(z)$ is even, then it is real-valued and we have

$$|f_j(z)|^2 = f_j^2(z). \tag{0.18}$$

If $f_j(z)$ is odd, then it is pure imaginary and we have

$$|f_j(z)|^2 = -f_j^2(z). \tag{0.19}$$

These two relations are combined into a single one:

$$|f_j(z)|^2 = (T_{-1}f_j)(z) f_j(z) = f_j(-\bar{z}) f_j(z). \tag{0.20}$$

Substituting (0.20) into (0.12), we obtain an analog of relation (0.13):

$$R_j^*(s) = \Gamma_j(s)\big[V_j(s) + \overline{V_j(\bar{s})}\big], \tag{0.21}$$

where

$$V_j(s) = \sum_{m=1}^{\infty} \rho_j^2(m)/m^s. \tag{0.22}$$

Comparing relations (0.21) and (0.13), we obtain a new representation for the Rankin convolution:

$$2R_j(s) = V_j(s) + \overline{V_j(\bar{s})}. \tag{0.23}$$

Substituting (0.9), (0.10), and (0.5) into (0.22) and interchanging the orders of summation (which is valid in the region of absolute convergence $\operatorname{Re} s > 1$), we obtain a factorization for $V_j(s)$:

$$V_j(s) = L(s,\chi) L_j(s) r_{j,q}(s), \tag{0.24}$$

where

$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n)/n^s$ is the classical Dirichlet series of a character χ.

$L_j(s) = \sum_{(n,q)=1} t_j(n^2)/n^s$ is an analog of Hecke series (0.16) without a character;

$r_{j,q}(s) = \sum_{m_q \geq 1} \rho_j^2(m_q)/m_q^s$ is an analog of series (0.17).

Substituting (0.24) into (0.23) and taking into account that for a real character χ and real eigenvalues $t_j(n^2)$, the relations

$$\overline{L(\bar{s},\chi)} = L(s,\chi), \quad \overline{L_j(\bar{s})} = L_j(s)$$

hold, we obtain the second factorization of the Rankin convolution:

$$2R_j(s) = L(s,\chi)L_j(s)\bigl[r_{j,q}(s) + \overline{r_{j,q}(\bar{s})}\bigr]. \tag{0.25}$$

This implies that the Rankin convolution is divisible by $L(s,\chi)$ in the region of absolute convergence.

Equality (0.25) of Dirichlet series implies equality of their coefficients:

$$2|\rho_j(m)|^2 = \sum_{m=m_q nd} \chi(d)t_j(n^2)\bigl[\rho_j^2(m_q) + \overline{\rho_j^2(m_q)}\bigr], \tag{0.26}$$

where m_q is defined in (0.8), $(nd,q)=1$.

Combining relations (0.15) and (0.25), we conclude that, in the region $\operatorname{Re} s > 1$, the Rankin convolution $R_j(s)$ is divisible by the function

$$[\zeta(s), L(s,\chi)] = \frac{\zeta(s)L(s,\chi)}{(\zeta(s), L(s,\chi))}, \tag{0.27}$$

where $[\cdot,\cdot]$ is the least common multiple, $(\cdot,\cdot)$ is the greatest common divisor in the sense of the Weierstrass products for $\zeta(s)$ and $L(s,\chi)$.

This is a generalization of the Zagier–Shimura theorem to Hilbert spaces in the region of absolute convergence. In the next section we continue this divisibility to the left half-plane $\operatorname{Re} s \le 1$.

In concluding this section, I would like to express my sincere gratitude to Professor M. M. Skriganov for numerous discussions about the theory of operators in Hilbert spaces and, in particular, about the structure of the ring of Hermitian operators and their eigenvalues.

1. The Zagier–Shimura theorem

The finite Rankin convolution $|r|_{j,q}(s)$ from (0.17) and $r_{j,q}(s)$ from (0.24) converge absolutely in the half-plane $\operatorname{Re} s > \frac{1}{2}$. This follows immediately from a well-known estimate:

$$|\rho_j(m_q)|^2 \ll m_q^{\frac{1}{2}+\varepsilon} \cosh \pi\varkappa_j, \tag{1.1}$$

where $\varepsilon > 0$ is arbitrarily small but fixed. Estimate (1.1) reduces the problem to the summation either of a geometric progression for prime q, or of a product of such progressions for composite q. In Sec. 3, we make estimate (1.1) more precise. Therefore, the problem reduces to the extension of the series $L_j(s,\chi)$ from (0.16) and $L_j(s)$ from (0.24).

We begin with the series $L_j(s)$ from (0.24) as the simplest and most important series in this paper. To this end, we use the convolution

$$L_j^*(s) = \iint_\Pi \overline{f_j(z)}\, \theta_q(z) y^{\frac{s+1}{2}}\, d\mu(z), \tag{1.2}$$

where

$$\theta_q(z) = \sum_{(m,q)=1,\ m\in\mathbb{Z}} \exp(2\pi i z m^2).$$

Substituting series (0.7) and $\theta_q(z)$ into the kernel of integral (1.2) and interchanging integration and summation (which can be done in the region of absolute convergence $\operatorname{Re} s > 1$), we obtain an explicit form for L_j^*:

$$L_j^*(s) = \overline{\rho_j(1)}\, \frac{\pi^{\frac{1-s}{2}}}{2^{s-1}}\, \frac{\Gamma\left(\frac{s}{2} + i\varkappa_j\right)\Gamma\left(\frac{s}{2} - i\varkappa_j\right)}{\Gamma\left(\frac{s+1}{2}\right)}\, L_j(s). \tag{1.3}$$

From (1.3) it is seen that convolution (1.2) is a normalized series $L_j(s)$ with trivial (in the sense of extension) normalizing functions. Therefore, it is sufficient to extend convolution (1.2) in order to solve the problem of extension for $L_j(s)$.

To this end, we get rid of the condition $(m,q)=1$ in the definition of the theta function $\theta_q(z)$ in (1.2). This may be done with the help of the discontinuous sum

$$\sum_{\delta|(m,q)} \mu(\delta) = \begin{cases} 1, & (m,q)=1, \\ 0, & (m,q)>1, \end{cases}$$

where $\mu(\delta)$ is the Möbius function.

Substituting this sum into the definition of $\theta_q(z)$ and interchanging the orders of summation, we obtain the relation

$$\theta_q(z) = \sum_{\delta|q} \mu(\delta)\theta(2\delta^2 z). \tag{1.4}$$

The ordinary Jacobi theta functions occur in the right-hand side of (1.4):

$$\theta(2\delta^2 z) = \sum_{m\in\mathbb{Z}} \exp(2\pi i \delta^2 z m^2).$$

Substituting (1.4) into (1.2), we obtain the relation

$$L_j^*(s) = \sum_{\delta|q} \mu(\delta) J_{j,\delta}(s), \tag{1.5}$$

where

$$J_{j,\delta}(s) = \iint_{\Pi} \overline{f_j(z)}\, \theta(2\delta^2 z) y^{\frac{s+1}{2}}\, d\mu(z). \tag{1.6}$$

For simplicity, we assume that q is odd and square-free; then, for any divisor $\delta|q$, the product $\bar{f}_j\,\theta$ is invariant relative to the group $\Gamma = \Gamma_0(4\delta q)$. Moreover, $f_j(z)$ is automorphic for $\Gamma_0(q)$ with multiplier χ, and the theta function $\theta(2\delta^2 z)$ is modular relative to the group $\Gamma_0(4\delta^2)$. An explicit form of its modular factor can be found in [4, 6]. Here we obtain another form of the modular factor, which is more convenient from the point of view of summation over these transformations.

Lemma 1.1. *For $\theta(2\delta^2 z)$ the following modular relation is valid:*

$$\theta(2\delta^2\sigma z) = \frac{\delta}{\sqrt{2}}\, e^{-\frac{\pi}{4}i} g\left(\frac{a\delta^2}{c}\right) \sqrt{z + \frac{d}{c}}\, \theta(2\delta^2 z), \tag{1.7}$$

where

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4\delta q), \quad c > 0,$$

$$g\left(\frac{a\delta^2}{c}\right) = \sum_{1 \le x \le c|\delta^2} \exp\left(2\pi i \frac{a\delta^2}{c} x^2\right).$$

Proof. Write the relation

$$2\delta^2 \sigma z = 2\delta^2 \frac{az+b}{cz+d} = 2\frac{a(\delta^2 z) + \delta^2 b}{c^*(\delta^2 z) + d} \tag{1.8}$$

with $c^* = c/\delta^2$.

Using the unimodularity of the matrix σ, we transform relation (1.8) to the form

$$2\delta^2 \sigma z = \frac{2a}{c^*} - \alpha, \quad \text{where} \quad \alpha = \frac{2}{c^*(c^* z^* + d)} \quad \text{and} \quad z^* = \delta^2 z.$$

Therefore, the following relation is valid:

$$\theta(2\delta^2 \sigma z) = \sum_{m \in \mathbb{Z}} \exp\left[\pi i \left(\frac{2a}{c^*} - \alpha\right) m^2\right]. \tag{1.10}$$

By condition, $c^* = c/\delta^2 \equiv 0 \left(4\frac{q}{\delta}\right)$, therefore c^* is even, and, thus, $c_1 = \frac{1}{2}c^*$ is an integer; we subdivide all $m \in \mathbb{Z}$ into progressions:

$$m = l_1 + c_1 n, \quad n \in \mathbb{Z}, \quad 1 \le l_1 \le c_1. \tag{1.11}$$

As a result, relation (1.10) takes the form

$$\theta(2\delta^2 \sigma z) = \sum_{l_1=1}^{c_1} e^{2\pi i l_1^2 \frac{a}{c^*}} \sum_{n \in \mathbb{Z}} \exp\left[-\pi i \alpha c_1^2 \left(n + \frac{l_1}{c_1}\right)^2\right]; \tag{1.12}$$

expanding the inner series over $n \in \mathbb{Z}$ into the Fourier series and making necessary transformations, we reduce (1.12) to the form

$$\theta(2\delta^2 \sigma z) = \sqrt{2}\delta e^{-\frac{\pi}{4}i} \sqrt{z + \frac{d}{c}} \sum_{m \in \mathbb{Z}} g_m^{(1)}\left(\frac{a}{c^*}\right) e^{2\pi i \delta^2 z m^2}, \tag{1.13}$$

where

$$g_m^{(1)}\left(\frac{a}{c^*}\right) = \sum_{l_1=1}^{c_1} \exp\left(2\pi i \frac{a}{c^*}(dm + l_1)^2\right).$$

Taking another system of residues $\pmod{c_1}$ in the system of progressions (1.11):

$$m = l_2 + c_1 n, \quad n \in \mathbb{Z}, \quad l_2 = c_1 + l_1; \quad 1 \le l_1 \le c_1,$$

and making the same transformations, we obtain an analog of relation (1.13) with another system of residues:

$$\theta(2\delta^2 \sigma z) = \sqrt{2}\,\delta e^{-\frac{\pi}{4}i} \sqrt{z + \frac{d}{c}} \sum_{m \in \mathbb{Z}} g_m^{(2)}\left(\frac{a}{c^*}\right) e^{2\pi i \delta^2 z m^2}, \tag{1.14}$$

where

$$g_m^{(2)}\left(\frac{a}{c^*}\right) = \sum_{l_2=c_1+1}^{2c_1} \exp\left[2\pi i \frac{a}{c^*}(dm+l_2)^2\right].$$

Adding relations (1.13) and (1.14) term by term, we obtain relation (1.7), because $2c_1 = c^*$ and

$$(g_m^{(1)} + g_m^{(2)})\left(\frac{a}{c^*}\right) = g\left(\frac{a}{c^*}\right) = g\left(\frac{a\delta^2}{c}\right),$$

where

$$g\left(\frac{a}{c^*}\right) = \sum_{x=1}^{c^*} \exp\left(2\pi i \frac{a}{c^*}\, x^2\right)$$

is the full Gaussian sum $\pmod{c^*}$.

The quantity $\sqrt{2}$ in the denominator of the right-hand side of (1.7) is compensated by the presence of 2 at the degree of the Gaussian sum $g\left(\frac{a}{c^*}\right)$, because in the classical modular Jacobi transformation (see [14]) this number 2 is absent.

In the case of $c < 0$, relation (1.7) takes the form

$$\theta(2\delta^2\sigma z) = \frac{\delta}{\sqrt{2}}\, e^{-\frac{\pi}{4}i} g\left(\frac{-a\delta^2}{|c|}\right)\sqrt{z - \frac{d}{|c|}}\,\theta(2\delta^2 z). \tag{1.15}$$

In the sequel, by the ratio

$$\theta(2\delta^2\sigma z)/\theta(2\delta^2 z)$$

we mean the modular coefficient in (1.7) or in (1.15) in accordance with the sign of c.

Having obtained the explicit form of modular transformation (1.7) and (1.15), we now can fold the unit strip Π in integral (1.6) to the fundamental domain $F_{4\delta q}$ of the group $\Gamma_0(4\delta q)$ and transform relation (1.6) to the form

$$J_{j,\delta}(s) = \iint_{F_{4\delta q}} \overline{f_j(z)}\,\theta(2\delta^2 z)\, E_{\chi,\delta}\left(\frac{s+1}{2}, z, \theta\right) d\mu(z), \tag{1.16}$$

where

$$E_{\chi,\delta}(w,z;\theta) = \sum_{\sigma\in\Gamma|\Gamma_\infty} \chi(\sigma) y^w(\sigma z)\frac{\theta(2\delta^2\sigma z)}{\theta(2\delta^2 z)}.$$

This is the Eisenstein series on the metaplectic group generated by the group $\Gamma = \Gamma(4\delta q)$ with multiplier χ. Now the theory of these series is well developed (see [7]). In particular, in [7], an analytic extension of these series to the entire complex plane and a functional equation of the vector-matrix type are obtained. For our purposes, it is sufficient that the Eisenstein series in (1.16) is regular in the half-plane $\mathrm{Re}\, w \geq 3/4$, except for the point $w = 1$ where it has a pole of the first order. For the parameter s ($w = \frac{s+1}{2}$), this corresponds to the half-plane $\mathrm{Re}\, s \geq 1/2$ and the pole at the point $s = 1$. These facts (except for the functional equation) can easily be obtained from the zeroth Fourier coefficient of the Eisenstein series that is equal to

$$y^{\frac{s+1}{2}} + \frac{\varphi(q)}{\varphi_{2s}(2q)}\,\frac{\xi(2s-1)}{\xi(2s)}\, y^{\frac{(1-s)+1}{2}}, \tag{1.17}$$

$$\varphi_s(n) = n^s \prod_{p|n} \left(1 - \frac{1}{p^s}\right),$$

where

$$\xi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

The coefficient dependent on s in (1.17), without normalizing multiplier $\Gamma\left(\frac{s+1}{2}\right)$, is a coefficient of the Selberg matrix, and it determines the region of regularity of the Eisenstein series and its points of singularity.

The question of the absolute convergence of integral (1.16) at the vertices of the fundamental domain $F_{4\delta q}$ is solved rather simply, because $f_j(z)$ decays exponentially at the vertices of F_q, which is a part of the vertices of $F_{4\delta q}$; in the remaining vertices of $F_{4\delta q}$, $f_j(z)$ also decays exponentially because all these vertices are equivalent to the vertices of F_q from the point of view of the space of functions $f_j(z)$. The Eisenstein series and the theta functions grow at these vertices no faster than a power of y (see, for example, [6]); therefore, the whole integral (1.16) as an analytic function in s is also regular in the half-plane $\operatorname{Re} s \geq \frac{1}{2}$, except for the point $s = 1$, where it has a pole of multiplicity one, as is seen from the zeroth Fourier coefficient (1.17). A finite sum of such integrals in (1.5) also possesses all of the above properties; therefore, the series $L_j(s)$ also enjoys these properties, because it differs from such a sum equal to $L_j^*(s)$ only by a normalizing factor.

In this part of the paper we need not use the fact of the divisibility of $R(s)$ by $\zeta(s)$; therefore, we restrict ourselves only to brief comments concerning the possibility of analytically extending relation (0.15) to the left half-plane $\operatorname{Re} s \leq 1$.

To this end, in the case of an even character $\chi : \chi(-1) = +1$, it is necessary to take an analog of convolution (1.1) with a character:

$$L_j^*(s, \chi) = \iint_\Pi \overline{f_j(z)}\, \theta(z, \chi) y^{\frac{s+1}{2}}\, d\mu(z), \tag{1.18}$$

where

$$\theta(z, \chi) = \sum_{n \in \mathbb{Z}} \chi(n) \exp(2\pi i z n^2). \tag{1.19}$$

By analogy with (1.3), from (1.18) we obtain the following representation in the region of absolute convergence $\operatorname{Re} s > 1$:

$$L_j^*(s, \chi) = \Gamma_j^*(s) L_j(s, \chi) \tag{1.20}$$

with the same Γ^*-coefficient as in (1.3).

If χ is an odd character: $\chi(-1) = -1$, it is necessary to take another convolution:

$$\tilde{L}_j(s, \chi) = \iint_\Pi \overline{f_j(z)}\, \tilde{\theta}(z, \chi) y^{\frac{s}{2}+1}\, d\mu(z), \tag{1.21}$$

where

$$\tilde{\theta}(z, \chi) = \sum_{n \in \mathbb{Z}} n\, \chi(n) \exp(2\pi i z n^2). \tag{1.22}$$

In this case, in the region $\operatorname{Re} s > 1$ we obtain an analog of relation (1.20) but with another Γ-factor:

$$\tilde{L}_j(s, \chi) = \tilde{\Gamma}_j(s)\, L_j(s, \chi), \tag{1.23}$$

where

$$\tilde{\Gamma}_j(s) = \overline{\rho_j(1)}\,\frac{(4\pi)^{-\frac{s}{2}}}{2}\,\frac{\Gamma\left(\frac{s+1}{2}+i\varkappa_j\right)\Gamma\left(\frac{s+1}{2}-i\varkappa_j\right)}{\Gamma\left(\frac{s}{2}+1\right)}.$$

Modular composition laws for the Jacobi theta functions with a character (1.19) and (1.22) are more complicated than the classical Jacobi laws for $\theta(z)$; nevertheless, this distinction is not principal, and with the help of them one can extend integrals (1.18) and (1.21), as analytic functions in s, to the left half-plane $\operatorname{Re} s \le 1$ by analogy with $L_j^*(s)$.

We refer the reader who would like to study this question for himself to the excellent book by Mumford [21] concerning the doubly periodic theta functions to which the two theta functions with character (1.19) and (1.22) reduce by translating the multiplicative character into the language of the additive one:

$$\chi(n) = \frac{1}{\tau_\chi}\sum_{l=1}^{q}\chi(l)\exp\left(2\pi i\frac{l}{q}n\right),$$

where τ_χ is the Gaussian sum.

2. A functional equation

Return to relation (0.14). In this relation, we tacitly mean that the Rankin convolution $R_j(s)$ is considered relative to the vertex $\alpha=\infty$, because we used the Fourier expansion (0.7) with respect to this vertex.

In fact, for any eigenfunction $f_j(z)$ there exist as many different Rankin convolutions as there are vertices α in the fundamental domain F_q of the group $\Gamma_0(q)$. In each of these cases, we use the Fourier expansion of $f_j(z)$ with respect to a vertex α:

$$f_j(\sigma_\alpha z) = \sum_{n\ne 0}\rho_{j,\alpha}(n)\sqrt{y}\,K_{i\varkappa_j}(2\pi|n|y)e^{2\pi i n x}, \tag{2.1}$$

where σ_α is the normalizing matrix that takes the vertex ∞ to α:

$$\sigma_\alpha(\infty)=\alpha.$$

Details concerning the structure of the matrices σ_α may be found, for example, in [8].

We note that in relation (2.1) and in the special case (0.7) of it, we used the Selberg conjecture [22] in the version with a nontrivial multiplier, because relation (2.1) is valid only for large eigenvalues $\lambda_j \ge 1/4$.

For small eigenvalues $\lambda_j < 1/4$, the zeroth Fourier coefficient

$$+\rho_{j,\alpha}(0)y^{\frac{1}{2}+i\varkappa_j} \tag{2.1^a}$$

arises on the right-hand side in (2.1). Details of this phenomenon may be found, for example, in [24, 25].

Because of the zeroth Fourier coefficient (2.1^a), the properties of the function $f_j(z)$ at the vertices of the fundamental domain F_q change abruptly. It ceases to decay exponentially at the vertices (which was essentially used in Sec. 1) and begin to grow as a power at each vertex, i.e., it behaves in the same way as Eisenstein series. But at the same time, it is finite on F_q:

$$\iint_{F_q}|f_j(z)|^2 d\mu(z) < \infty,$$

which is not typical of the Eisenstein series.

This phenomenon is explained by the fact that if $\lambda_j < 1/4$, then $i\varkappa_j = -\sqrt{\frac{1}{4} - \lambda_j} = -\delta_j$, where $0 < \delta_j \le 1/4$. Here, the upper bound for δ_j is a consequence of the lower bound for $\lambda_j \ge 3/16$, which is valid for the congruence groups $\Gamma_0(q)$ by a known estimate due to André Weil for Kloosterman sums.

Consequently, these functions are intermediate between the Eisenstein series and the classical discrete spectrum for large eigenvalues $\lambda_j \ge 1/4$.

For functions $f_j(z)$ with small eigenvalues $\lambda_j < 1/4$, one can obtain the extension of the Shimura convolution from Sec. 1, but this is significantly more difficult than for $\lambda_j \ge 1/4$. To this end, one should use Selberg's technique of cutting-off vertices, because the $f_j(z)$ behave as Eisenstein series at the vertices.

But we shall prove Theorem 1 in such a way that we shall not make use of small eigenvalues. For this reason, we assume that the Selberg conjecture is valid in order not to perform unnecessary calculations.

There are finitely many functions $f_j(s)$ with small eigenvalues $\lambda_j < 1/4$; the number of them does not exceed the volume of the fundamental domain $\|F_q\| \ll q^{1+\varepsilon}$ (see [26]). This circumstance enables us to disregard these functions and to work only with large eigenvalues $\lambda_j \ge 1/4$, for which expansions (0.7) and (2.1) are valid. This will be used in Secs. 5 and 6, therefore the assumption on the validity of the Selberg conjecture simplifies the technical part but is not principal.

After this digression, we return to the general Rankin convolution.

The normalized Rankin convolution $R^*_{j,\alpha}(s)$ with respect to the vertex α is defined by the relation

$$R^*_{j,\alpha}(s) = \iint_{\Pi} |f_j(\sigma_\alpha z)|^2 y^s \, d\mu(z). \tag{2.3}$$

Relation (0.12) is a special case of relation (2.3) for $\alpha = \infty$. In this case, $\sigma_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, therefore $\sigma_\infty(z) = z$ and relation (2.3) transforms to (0.12).

Substituting Fourier expansion (2.1) into (2.3), we obtain the general form of relation (0.13):

$$R^*_{j,\alpha}(s) = 2\Gamma_j(s) R_{j,\alpha}(s), \tag{2.4}$$

where $\Gamma_j(s)$ is the same as in (0.13),

$$R_{j,\alpha}(s) = \sum_{m=1}^{\infty} |\rho_{j,\alpha}(m)|^2 / m^s. \tag{2.5}$$

For a generic vertex α, we cannot obtain an analog of relation (0.15), because this relation is valid only for $\alpha = \infty$ and $\alpha = 0$.

The reason for this phenomenon is the following. The ring of Hecke operators can be constructed with respect to any vertex α of the fundamental domain F_q of the group $\Gamma_0(q)$. One can choose a basis of eigenfunctions $f_j(z)$ in such a way that the Fourier coefficients $\rho_{j,\alpha}(m)$ of this basis and the eigenvalues $t_{j,\alpha}(m)$ of the Hecke operators of the vertex α are connected by an analog of relation (0.9):

$$\rho_{j,\alpha}(m) = \rho_{j,\alpha}(1) t_{j,\alpha}(m_q) t_{j,\alpha}(n).$$

But this basis may lose multiplicative properties when we construct the Fourier expansion with respect to another vertex β. The multiplicative properties of the coefficients hold only under passage $\alpha \to \beta = \alpha^{-1}$, where the inverse vertex $\beta = \alpha^{-1}$ is defined by the relation

$$\alpha\beta = \frac{1}{q}.$$

In the special case where $\alpha = \frac{1}{q} \sim \infty$, the inverse vertex $\beta = 1 \sim 0$.

The phenomenon of multiplicative properties of Fourier coefficients at an inverse vertex may be explained from the spectral point of view as follows. If we write the set of all eigenfunctions belonging to an eigenvalue λ_j:

$$\{f_{j,1}(z), \dots, f_{j,k}(z)\}, \quad k \geq 1 \text{ is the multiplicity of } \lambda_j, \tag{2.6}$$

and make the matrix $\sigma_0 = \begin{pmatrix} 0 & 1/\sqrt{q} \\ -\sqrt{q} & 0 \end{pmatrix}$ of the vertex $\alpha = 0$, $\sigma_0(\infty) = 0$, to act on the set (2.6):

$$\{f_{j,1}(\sigma_0 z), \dots, f_{j,k}(\sigma_0 z)\}, \tag{2.7}$$

then we obtain the same set of functions (2.6) up to a permutation. This means that σ_0 is an automorphism of set (2.6).

Hence it follows the phenomenon of the pairwise multiplicativity of the coefficients of all vertices α of F_q. If q is prime, then F_q has two nonequivalent vertices $\alpha = \infty$ and $\alpha = 0$. Therefore, if we choose a multiplicative basis for one of them, the other basis will be multiplicative automatically.

Professor V. Gritsenko informed me about the rings of Hecke operators for all vertices. Professor A. Venkov informed me about the special interpretation of the pairwise multiplicativity of bases of all vertices. I express them much gratitude for this important information.

For our purposes, we choose a multiplicative basis of orthonormal functions $f_j(z)$ with the ring of Hecke operators relative to the vertex $\alpha = \infty$. Although Rankin convolutions (2.5) with respect to other vertices α (except for $\alpha = 0 \sim 1$) may fail to have Euler products, this does not influence the general theory of analytic extension of them and the existence of a vector-matrix functional equation, which are based on the additive notation of the Rankin convolution (2.5).

After this brief digression, we return again to relation (2.3). Acting on the argument of the kernel of (2.3) by the set of elements $\sigma_\alpha^{-1}\sigma$, where $\sigma \in \Gamma|\Gamma_\infty$, $\Gamma = \Gamma_0(q)$, we reduce (2.3) to the form

$$R^*_{j,\alpha}(s) = \iint_{F_q} |f_j(z)|^2 E_\alpha(z,s)\, d\mu(z), \tag{2.8}$$

where E_α is the Eisenstein series of the vertex α:

$$E_\alpha(z,s) = \sum_{\sigma \in \Gamma|\Gamma_\infty} y^s(\sigma_\alpha^{-1}\sigma z). \tag{2.9}$$

Using the theory of Eisenstein series (for details, see, for example, [8, 9]), with the help of relation (2.8) we obtain the meromorphic extension of the convolution $R^*_{j,\alpha}(s)$ to the left half-plane $\operatorname{Re} s \leq 1$ for all vertices α. In particular, the residue of the Eisenstein series (2.9) at the pole $s = 1$ does not depend on the vertex α and determines (together with normalizing factor (2.4) and relation (2.8)) the residue of the Rankin convolution:

$$\operatorname*{res}_{s=1} R^*_{j,\alpha}(s) = \operatorname*{res}_{s=1} E_\alpha(z,s) = q^{-1} \prod_{p|q} \left(1 + \frac{1}{p}\right)^{-1} \zeta^{-1}(2)\frac{\pi}{2}. \tag{2.10}$$

Detailed calculations may be found in [8]. In addition to the extension to the left half-plane $\operatorname{Re} s \le 1$, in the sequel we need estimates of the Rankin convolutions $R_{j,\alpha}(s)$ in the left half-plane. In order to obtain them, it is necessary to have a functional equation for $R_{j,\alpha}(s)$ that enables us to use for the Rankin convolutions analogs of the classical technique of approximate equations and estimates using convexity (see [10, 11]). But these techniques require that the series under study should be entire functions on the whole complex plane except for finitely many poles on the real axis. In order to include the Rankin convolutions into this class of functions, it is necessary to note that the meromorphy of the Rankin convolution is related on the whole with the classical Riemann zeta function of the variable $2s$:

$$\zeta_q(2s) = \zeta(2s) \prod_{p|q} \left(1 - \frac{1}{p^{2s}}\right) \tag{2.11}$$

without prime divisors of q in the Euler product (it is encountered in (0.15) for the variable s). The reason is that $\zeta_q(2s)$ occurs in the denominator of the C-matrix of the Eisenstein series E_α in (2.9). Therefore, multiplying (2.8) by $\xi_q(2s)$, we transform it to the form

$$\xi_q(2s) R^*_{j,\alpha}(s) = \iint_{F_q} |f_j(z)|^2 \xi_q(2s) E_\alpha(z,s) d\mu(z), \tag{2.12}$$

where

$$\xi_q(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta_q(s).$$

The Fourier expansion of the Eisenstein series E_α relative to the vertex $\beta = \infty$ (for details, see [8]) shows that the product $\xi_q(2s)E_\alpha(z,s)$ is an entire function in s on the whole complex plane except for the points $s = 1$ and $s = \frac{1}{2}$, where it has simple poles.

From relation (2.12) it follows that the left-hand side of this relation possesses the same properties. Moreover, for the residues of the left-hand side of (2.12) at the above points, the following relations are valid:

$$\begin{aligned} \operatorname*{res}_{s=1} \zeta_q(2s) R_{j,\alpha}(s) &= V_q \cosh \pi\varkappa_j \\ \operatorname*{res}_{s=\frac{1}{2}} \zeta_q(2s) R_{j,\alpha}(s) &= \frac{1}{2} \frac{\varphi(q)}{q} R_{j,\alpha}\left(\frac{1}{2}\right), \end{aligned} \tag{2.13}$$

where

$$V_q = \frac{2\varphi(q)}{q^2}.$$

The first relation in (2.13) is a consequence of (2.10), (2.4), and the definition of $\Gamma_j(s)$ in (0.13), because

$$2\Gamma_j(s)\big|_{s=1} = \frac{\pi}{\cosh \pi\varkappa_j}.$$

The second relation in (2.13) is a consequence of the entirety of $R_{j,\alpha}(s)$ at the point $s = \frac{1}{2}$ and of the relation

$$\operatorname*{res}_{s=\frac{1}{2}} \zeta_q(2s) = \frac{1}{2} \frac{\varphi(q)}{q}.$$

In order to obtain functional equations for the Rankin convolutions $R^*_{j,\alpha}(s)$, we collect relations (2.8) for all vertices α and write them in vector form:

$$\tilde{R}^*_j(s) = \iint_{F_q} |f_j(z)|^2 \tilde{E}(z,s) d\mu(z), \tag{2.14}$$

where

$$\tilde{R}_j^*(s) = {}^t[R_{j,\alpha_1}^*(s), \dots, R_{j,\alpha_\tau}^*(s)],$$

$$\tilde{E}(z,s) = {}^t[E_\alpha(z,s), \dots, E_{\alpha_\tau}(z,s)],$$

$\tau = \tau(q)$ is the number of divisors of q that is equal to the number of vertices in F_q. Note that such a coincidence is valid for conductor moduli q of quadratic characters, because, in the general case, the number of vertices is equal to

$$\sum_{d|q} \varphi\left(\left(d, \frac{q}{d}\right)\right),$$

and if q is a conductor modulus, then $\left(d, \frac{q}{d}\right) = 1$ or 2, and therefore $\varphi\left(\left(d, \frac{q}{d}\right)\right) = 1$ for all $d|q$.

Using the functional equation

$$\tilde{E}(z,s) = \Phi(s)\tilde{E}(z, 1-s)$$

(see [9]) for the vector $\tilde{E}(z,s)$, where

$$\Phi(s) = (c_{\alpha,\beta}(s))$$

and c is the Selberg matrix of constant Fourier coefficients for E_α relative to the vertex β, from (2.14) we obtain a functional equation for the Rankin convolutions:

$$\tilde{R}_j^*(s) = \Phi(s)\tilde{R}_j^*(1-s). \tag{2.15}$$

Such an equation can be obtained for small $\lambda_j < 1/4$ with the help of Selberg's technique of cutting-off vertices. But we shall not need it in the sequel, as was mentioned at the beginning of this section.

From vector-matrix equation (2.15), we obtain a functional equation for Rankin convolutions for each vertex α:

$$R_{j,\alpha}^*(s) = \sum_\beta c_{\alpha,\beta}(s) R_{j,\beta}^*(1-s). \tag{2.16}$$

Since the Selberg matrix Φ is symmetric, i.e., $\Phi = {}^t\Phi$, Eq. (2.16) can be rewritten in the form

$$R_{j,\alpha}^*(s) = \sum_\beta c_{\beta,\alpha}(s) R_{j,\beta}^*(1-s). \tag{2.17}$$

Hence, as a special case, we obtain a functional equation for the vertex $\alpha = \infty$:

$$R_{j,\infty}^*(s) = \sum_\beta c_{\beta,\infty}(s) R_{j,\beta}^*(1-s). \tag{2.18}$$

We have already noted above that the number of vertices in F_q for a conductor modulus q coincides with the number of divisors of q. This enables us to give all the vertices β by the condition

$$\beta = \frac{1}{w}, \quad w|q. \tag{2.19}$$

We observe that here we use the notation of paper [8] in order to simplify the comparison of the relations obtained in the present paper and the formulas from [8]. When we represent the vertices by relations (2.19), it should be remembered that for $w = 1$, the vertex $\beta = 1$ is equaivalent to the vertex 0, and for $w = q$, the vertex $\beta = 1/q$ is equivalent to the vertex ∞. The proof of the first assertion can be found in [6] (the second assertion is trivial).

Substituting (2.19) into (2.18), we obtain

$$R^*_{j,\infty}(s) = \sum_{w|q} c_{\frac{1}{w},\infty}(s) R^*_{j,\frac{1}{w}}(1-s). \tag{2.20}$$

An explicit form of the coefficients $c_{\frac{1}{w},\infty}(s)$ is given in [8]:

$$c_{\frac{1}{w},\infty}(s) = \frac{\varphi(w)}{\varphi\left((w,\frac{q}{w})\right)} \left(\frac{(w,\frac{q}{w})}{wq}\right)^s \frac{\xi_{\frac{q}{w}}(2s-1)}{\xi_q(2s)}, \tag{2.21}$$

where $\xi_{\frac{q}{w}}(s)$ is the normed Riemann zeta function from (2.12). The following relation is valid for it:

$$\xi_{\frac{q}{w}}(s) = \xi_1(s) \prod_{p\left|\frac{q}{w}\right.} \left(1 - \frac{1}{p^s}\right);$$

therefore, the invariance relation $\xi_1(s) = \xi_1(1-s)$ yields the functional equation

$$\xi_{\frac{q}{w}}(2s-1) = \xi_1(2(1-s)) \prod_{p\left|\frac{q}{w}\right.} \left(1 - \frac{1}{p^{2s-1}}\right), \tag{2.22}$$

where $\xi_1(s) \equiv \xi(s)$ is the classical Riemann normalizing factor for $\zeta(s)$.

We can simplify formulas (2.21) and calculations if we assume that q is odd and square-free. In this case, (2.21) takes the form

$$c_{\frac{1}{w},\infty}(s) = \frac{\varphi(w)}{(qw)^s} \frac{\xi_{\frac{q}{w}}(2s-1)}{\xi_q(2s)}. \tag{2.23}$$

In the sequel, when studying functional equation (2.22) in order to obtain necessary estimates in the left half-plane $\operatorname{Re} s < \frac{1}{2}$, it should be remembered that the finite product

$$\prod_{p\left|\frac{q}{w}\right.} \left(1 - \frac{1}{p^{2s-1}}\right)$$

does not occur in the Riemann equation ($s \to 1-s$), and, thus, if we move the contours to the left half-plane, this product begins to grow as a quantity of order

$$\left(\frac{q}{w}\right)^{-2s+1}, \quad \operatorname{Re} s < 0.$$

Using the explicit form of coefficients (2.23), we multiply the two parts of Eq. (2.20) by the quantity

$$q^s \xi_q(2s).$$

On the right-hand side of the equation, this multiplier cancels with the same multiplier in the denominator of (2.23). As a result, from (2.20) we obtain an equation of the form

$$q^s \xi_q(2s) R^*_{j,\infty}(s) = \sum_{w|q} \frac{\varphi(w)}{w^s}\, \xi_{\frac{q}{w}}(2s-1) R^*_{j,\frac{1}{w}}(1-s). \tag{2.24}$$

In Eq. (2.24), we have entire functions on the left and on the right, except for the points $s = 1$ and $s = \frac{1}{2}$ where they have simple poles. Thus, we have obtained the standard form of a functional equation [10, 11], which may be applied in order to derive approximate equations and estimates in the critical strip, using the convexity.

3. Kuznetsov's summation formulas

In the sequel, we need to prove the results of paper [12] (in particular, Theorem 6) for the group $\Gamma_0(q)$ with a multiplier $\chi = \chi_q$.

Without a multiplier, these results are proved in detail in paper [8] for any q. The condition that q is square-free significantly simplifies the proofs in [8], and the addition of the character χ_q does not complicate the proofs (see, also, [6, 16]). In particular, for any $w|q$ relation (2.4) from [8] takes the form

$$S_{\frac{1}{w},\frac{1}{w}}(m,n;q\gamma;\chi_q) = e\left(\frac{m-n}{q}\right) \sum_{\delta_1 \pmod{q\gamma}} \chi_w(\delta_1)\chi_{\frac{q}{w}}(\gamma-\delta_1) \exp\left[2\pi i\left(m\frac{\alpha_1}{q\gamma} + n\frac{\delta_1}{q\gamma}\right)\right] \tag{3.1}$$

if we consider the case with a character χ_q in the notation that is used in Lemma 2.5 from [8].

For sum (3.1), Lemma 2.6 from [8] remains valid:

$$\left|S_{\frac{1}{w},\frac{1}{w}}(m,n;q\gamma;\chi_q)\right| \le (m,n,q\gamma)^{\frac{1}{2}}\sqrt{q\gamma}\,\tau(q\gamma). \tag{3.2}$$

One can write relation (3.1) and estimate (3.2) for a generic pair of vertices (α,β), but we need not it in the sequel. A characteristic feature of the Kloosterman sums with character χ_q for a pair of vertices $\left(\frac{1}{w},\frac{1}{w}\right)$ is that the character splits into two parts: $\chi_q \to \chi_{\frac{q}{w}}\chi_w$. Moreover, the first of them $\chi_{\frac{q}{w}}(\gamma)$ occurs in the outer sum over the moduli γ, whereas $\chi_w(\delta_1)$ is in the Kloosterman sum. Taking all the vertices $\frac{1}{w}$ beginning with $w = q$ (i.e., $\frac{1}{q} \sim \infty$) and ending with $w = 1$ (i.e., $1 \sim 0$), we place the character χ_q to the outer sum, and the Kloosterman sum will be without a character.

We note that Kloosterman sums (3.1) arise when we expand into a Fourier series the Poincaré series of the vertex $\alpha = \frac{1}{w}$ with multiplier χ_q, of number m relative to the same vertex $\alpha = \frac{1}{w}$.

In the sequel, we also need the Fourier coefficients $\varphi_{\alpha,\beta;n}(s,\chi_q)$, $n \neq 0$, of Eisenstein series $E_\alpha(\sigma_\beta z, s, \chi_q)$ of the vertex α with multiplier χ_q, expanded relative to the vertex β:

$$E_\alpha(\sigma_\beta z, s;\chi_q) = \delta_{\alpha,\beta} y^s + \sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\varphi_{\alpha,\beta;0}(s,\chi_q) y^{1-s}$$

$$+\frac{2\pi^s\sqrt{y}}{\Gamma(s)}\sum_{n\neq 0}|n|^{s-\frac{1}{2}}\varphi_{\alpha,\beta;n}(s,\chi_q)K_{s-\frac{1}{2}}(2\pi|n|y)e^{2\pi i n x}. \tag{3.5}$$

In the special case of two vertices $\left(\frac{1}{w}, \infty\right)$, these coefficients are calculated relatively simply (for details, see, for example, [6, 8]) and are determined by the relation

$$\varphi_{\frac{1}{w},\infty;n}(s,\chi_q) = \chi_q(n)\frac{\bar{\varepsilon}_w\sqrt{w}}{(qw)^s}\,\varphi_n(s,\chi_q), \tag{3.6}$$

where

$$\varphi_n(s,\chi_q) = \frac{\sigma_{1-2s}(n,\chi_q)}{L(2s,\chi_q)}, \tag{3.7}$$

ε_w is the sign of the Gaussian sum of modulus w; $L(2s,\chi_q)$ is the classical Dirichlet series of the character χ_q; for $n>0$ the function $\sigma_s(n,\chi)$ is defined in (0.6), and for $n \in \mathbb{Z}-\{0\}$ it is defined by the relation

$$\sigma_s(n,\chi) = \sum_{d|n,d>0} \chi(d)d^s. \tag{3.8}$$

For two generic vertices $\left(\frac{1}{w_1}, \frac{1}{w_2}\right)$, the coefficient $\varphi_{\frac{1}{w_1},\frac{1}{w_2};n}(s,\chi_q)$ has a complicated arithmetic structure because of common divisors of n, q, w_1, and w_2; here we do not give an explicit form of it because it is very cumbersome. In addition, in the sequel we need only an estimate of its modulus on the line $\operatorname{Re} s = \frac{1}{2}$, and such an estimate is rather simple:

$$\left|\varphi_{\frac{1}{w_1},\frac{1}{w_2};n}(s,\chi_q)\right| \le \frac{2}{\sqrt{q_1 w_2}}\,|\varphi_n(s,\chi_q)|, \tag{3.9}$$

where

$$q_1 = q\Big/\frac{w_1 w_2}{\Delta}, \qquad \Delta = (w_1, w_2). \tag{3.10}$$

The fact that this estimate is rather sharp can easily be obtained if we check it for the coefficient $\varphi_{\frac{1}{w},\infty;n}(s,\chi_q)$, $\operatorname{Re} s = \frac{1}{2}$, and compare it with the explicit form in (3.6). In this case, from (3.9) we have $w_2 = q$, $\Delta = (w_1,w_2) = (w_1,q) = w_1$; therefore, $q_1 = 1$ and, consequently, from (3.9) we have

$$\left|\varphi_{\frac{1}{w},\infty;n}(s,\chi_q)\right| \le \frac{2}{\sqrt{q}}|\varphi_n(s,\chi_q)|, \quad \operatorname{Re} s = \frac{1}{2}.$$

This estimate differs from the precise value from (3.6) only by the coefficient equal to 2.

From relations (3.7) and (3.8), the following estimate holds:

$$|\varphi_n(s,\chi_q)| \le \frac{\sigma_0(n)}{|L(2s,\chi_q)|}, \quad \operatorname{Re} s = \frac{1}{2}. \tag{3.11}$$

Substituting (3.11) into (3.9) and squaring the two sides of the inequality, we obtain an estimate of the square of the modulus of the Fourier coefficient on the line $\operatorname{Re} s = \frac{1}{2}$:

$$\left|\varphi_{\frac{1}{w_1},\frac{1}{w_2};n}(s,\chi_q)\right|^2 \le \frac{4}{q_1 w_2}\,\frac{\sigma_0^2(n)}{|L(2s,\chi_q)|^2}. \tag{3.12}$$

Using these results, we can carry over Kuznetsov's summation formulas from [12] to the group $\Gamma_0(q)$ with multiplier χ_q. Kuznetsov's inverse theorems remain the same as they stated in [8], only the multiplier $\chi = \chi_q$ occurs in the notation used in these theorems (we should remember that χ_q splits into two parts). In the sequel, we need a direct theorem (Theorem 1 from [12]), which is not contained in [8] and which we give here in the form of a lemma.

Lemma 1.3 (N. V. Kuznetsov). *Let an even function $h(r)$ of a complex variable r be regular in the strip $|\operatorname{Im} r| \leq \Delta$ for some $\Delta > \frac{1}{2}$, and for some fixed $\delta > 0$, as $|r| \to \infty$, $|\operatorname{Im} r| \leq \Delta$, let the following estimate be valid:*

$$h(r) \ll |r|^{-2-\delta}.$$

Then for any integer $n \geq 1$ and any vertex $\alpha = \frac{1}{w}$, $w|q$, the following relation holds:

$$\sum_{j=1}^{\infty} \frac{|\rho_{j,\alpha}(n)|^2}{\cosh \pi \varkappa_j} h(\varkappa_j) + \frac{1}{\pi} \sum_{\beta} \int_{-\infty}^{\infty} |\varphi_{\beta,\alpha;n}(s,\chi)|^2 h(r) dr =$$

$$\frac{1}{\pi^2} \int_{-\infty}^{\infty} r \tanh(\pi r) h(r) dr + \sum_{c \equiv 0(q)} \frac{s_{\alpha,\alpha}(n,n;c,\chi)}{c} \hat{h}\left(\frac{4\pi n}{c}\right), \tag{3.13}$$

where in the kernel of the integral from the left-hand side $s = \frac{1}{2} + ir$,

$$\hat{h}(x) = \frac{2i}{\pi} \int_{-\infty}^{\infty} J_{2ir}(x) \frac{r h(r)}{\cosh \pi r} dr.$$

As a consequence, Lemma 1.3 implies an analog of Theorem 6 from [12] for $\Gamma_0(q)$ with multiplier $\chi = \chi_q$.

Lemma 2.3 (N. V. Kuznetsov). *For any $x \geq 2$ and any integer $n \geq 1$, for any vertex $\alpha = \frac{1}{w}$ of the fundamental domain F_q, the following asymptotic relation is valid ($\theta \ll 1$):*

$$\sum_{\varkappa_j \leq X} \frac{|\rho_{j,\alpha}(n)|^2}{\cosh \pi \varkappa_j} = \frac{1}{\pi^2} X^2 + \theta \left[X + \frac{\sqrt{q}}{w} + \sqrt{\frac{n}{q}} \right] \times \sigma_0(q) \sigma_0^2(n) \log^3(X + q + n). \tag{3.14}$$

The passage from (3.13) to (3.14) is the same as in §5 of [12] in the proof of Theorem 6. The principal term of the right-hand side in (3.14) does not depend on the volume of the fundamental domain F_q, because the square of the normalizing coefficient $|\rho_{j,\alpha}(n)|^2$ from (0.9) contains this volume, as was mentioned in the introduction. Here, the estimates of the sum of Kloosterman sums yield the quantity $\sqrt{\frac{n}{q}}$ instead of $\sqrt{n}$ in [12], because summation is taken over the moduli $c \equiv 0 \pmod{q}$ (see estimate (3.2)).

The quantity n^{ε} from [12] is replaced here by the quantity $\sigma_0(n)$. Moreover, the continuous part of the spectrum yields the quantity

$$\frac{\sqrt{q}}{w} \sigma_0(q) \sigma_0^2(n) \log q, \tag{3.15}$$

which is absent in [12]. It can be obtained as follows. From (3.13) and estimate (3.12), it follows that the contribution of the continuous component to the remainder of Theorem 6 from [12] for $\Gamma_0(q)$ with multiplier $\chi = \chi_q$ is a quantity of order

$$\frac{\sigma_0(q) \sigma_0^2(n)}{w} \int_0^{2X} |L(1+it,\chi)|^{-2} dt. \tag{3.16}$$

A known bound for the zeros of Dirichlet L-series (see [13]) ensures the relation

$$\int_1^{2X} |L(1+it,\chi)|^{-2}dt = 2\frac{\zeta(2)}{\zeta(4)} \prod_{p|q} \left(1+\frac{1}{p^2}\right) X + O\left(\frac{X}{\log X} + \log^3 q\right). \tag{3.17}$$

This is a generalization of relation (5.21) from [12] to Dirichlet L-series.

For X lying in the interval $2 \le X \le \log^3 q$, the remainder in (3.17) exceeds the principal term, and asymptotics (3.17) estimates the integral by a quantity of order of the remainder.

In order to estimate the remaining part of integral (3.17) over the unit interval $0 \le t \le 1$, one can use an analog of Theorem 1 from [14] for Dirichlet L-series, replacing the pole of $\zeta(s)$ at the point $s = 1$ by the Siegel zero β. This analog takes the reverse form as compared with Theorem 1 from [14], but all the functions occurring in Theorem 1 from [14] remain the same. Theorem 1 in [14] for $L(s,\chi)$ can be obtained by using the Deuring–Heilbronn effect (see [3]) and is a multiplicative analog of the approximate equation for $L(s,\chi)$. If $(1-\beta) \le 1/\sqrt[6]{q}$, then this theorem ensures an estimate:

$$|L(1+it,\chi)|^2 \ll \frac{(\log\log q)^2}{(1-\beta)^2+t^2}, \qquad 0 \le t \le 1,$$

and an estimate of the derivative:

$$L'(\beta,\chi) \ll (\log\log q)^2,$$

which, in turn, guaranties an estimate from below for the Siegel zero:

$$(1-\beta)^{-1} \ll \sqrt{q}(\log\log q)^2.$$

Combining these results, we obtain an estimate

$$\int_0^1 |L(1+it,\chi)|^{-2}dt \ll \sqrt{q}\log q. \tag{3.18}$$

If the reverse inequality $(1-\beta) > 1/\sqrt[6]{q}$ is valid, then estimate (3.18) is trivial.

Together with the coefficient at the integral (3.16), estimate (3.18) generates quantity (3.15), which was absent in Theorem 6 from [12].

Owing to estimate (3.2), the sum of Kloosterman sums in (3.13) is estimated in the same way as in §5 of [12] and yields a quantity of order

$$\sqrt{\frac{n}{q}}\,\sigma_0(q)\sigma_0(n)\log(n+q). \tag{3.19}$$

Combining estimates (3.19), (3.17), and (3.15) together with the quantity $X\log X$, we obtain the remainder of the right-hand side in (3.14). Moreover, $\log^3(X+n+q)$ majorizes all of the logarithmic factors that arose in the estimates of different components. If in Lemma 2.3 we replace the condition $X \ge 1$ by the condition $X \ge q$, then asymptotics (3.14) is valid independent of the Selberg conjecture, because the addition of small eigenvalues does not change (3.14).

Relation (3.14) implies an individual estimate for each vertex.

Corollary 1.3. *For any vertex $\alpha = \frac{1}{w}$ and any $n \geq 1$, the following estimate holds for the group $\Gamma_0(q)$ with multiplier $\chi = \chi_q$:*

$$|\rho_{j,\alpha}(n)|^2/\cosh \pi\varkappa_j \ll \left(\varkappa_j + \frac{\sqrt{q}}{w} + \sqrt{\frac{n}{q}}\right) \sigma_0(q)\sigma_0^2(n) \log^3(\varkappa_j + n + q). \tag{3.20}$$

This estimate is uniform with respect to all parameters with an absolute constant in the sign $\ll$.

Corollary 2.3. *For $n \geq q^2\varkappa_j^2$ from (3.20), a simpler estimate is valid:*

$$|\rho_{j,\alpha}(n)|^2/\cosh \pi\varkappa_j \ll \sqrt{\frac{n}{q}}\sigma_0(q)\sigma_0^2(n) \log^3 n \tag{3.21}$$

with an absolute constant in the sign $\ll$.

Although estimates (3.20) and (3.21) are rather rough in comparison with the Peterson conjecture, nevertheless they work very well in many applications. In particular, they ensure the absolute convergence of the finite Rankin convolutions $|r|_{j,q}$ and $r_{j,q}$ from the introduction in the region $\operatorname{Re} s > \frac{1}{2}$.

Moreover, in order to justify the absolute convergence of the Rankin series $R_{j,\alpha}(s)$ in the right half-plane $\operatorname{Re} s > 1$, we indicate a consequence of functional equation (2.15) and the estimates, using the convexity on the line $\operatorname{Re} s = \frac{1}{2}$.

Lemma 3.3. *The following asymptotic behavior holds as $X \to \infty$:*

$$\sum_{n=1}^{\infty} |\rho_{j,\alpha}(n)|^2 \exp\left(-\frac{n}{X}\right) = c_j(q)X + O(X^{\frac{1}{2}+\varepsilon}), \tag{3.22}$$

where

$$c_j(q) = \frac{3}{\pi^2} \frac{\cosh \pi\varkappa_j}{q\prod_{p|q}\left(1 + \frac{1}{p}\right)},$$

$\varepsilon > 0$ is arbitrarily small but fixed.

Relation (3.22) yields the following estimate.

Corollary 3.3. *For all $X \geq X_0(q, \varkappa_j)$, the estimate*

$$\sum_{n \leq X} |\rho_{j,\alpha}(n)|^2 \ll c_j(q)X \tag{3.23}$$

holds with an absolute constant in the symbol $\ll$.

The above results are sufficient to obtain an approximate equation for the Rankin convolution $R_{j,\infty}(s)$ and the asymptotics of Hermann Weyl for the number of eigenvalues in a given interval.

4. An approximate equation

We need functional equation (2.24) in order to obtain an approximate functional equation in the critical strip for the explicit form of the Rankin convolution $R_{j,\infty}(s)$. At present, the technique of deriving such equations is well elaborated (see [10, 11]), and all that we need is contained implicitely in [11]. But it is rather difficult to translate purely linear statements in

[11] into the vector-matrix language of Eq. (2.15). Therefore, for convenience of the reader, we give here the necessary calculations for Eq. (2.24). To this end, we consider an integral:

$$J_j^{(1)}(N,s) = \frac{1}{2\pi ki} \int\limits_{\operatorname{Re} W>1} N^w \Gamma\left(\frac{w}{k}\right)\left(\omega - \frac{1}{2}\right) q^\omega \zeta_q(2\omega) R_{j,\infty}(\omega)dw, \quad \omega = w+s, \tag{4.1}$$

s is a fixed parameter in the critical strip:

$$s = \sigma + it \in \left\{\frac{1}{2} < \sigma < 1,\ |t| \le T\right\}. \tag{4.2}$$

The parameter $k \ge 2$ is arbitrarily large but fixed. In the sequel, we put

$$\frac{1}{k} = \varepsilon, \tag{4.3}$$

where $\varepsilon > 0$ is an arbitrarily small but fixed number in the interval $0 < \varepsilon \le \frac{1}{2}$.

The linear factor $\left(\omega - \frac{1}{2}\right)$ is inserted into the kernel of integral (4.1) in order to suppress the pole of $\zeta_q(2\omega)$ at the point $\omega = \frac{1}{2}$ (i.e., $w = \frac{1}{2} - s$); therefore, the kernel in (4.1) has only two poles in the half-plane $\operatorname{Re} w > -k$. One of these poles at the point $w = 0$ belongs to the function $\Gamma\left(\frac{w}{k}\right)$, whereas the other locates at the point $\omega = 1$ (i.e., $w = 1 - s$) and belongs to the Rankin convolution $R_{j,\infty}(\omega)$. Consequently, upon moving the contour in (4.1) to the left, to the line $\operatorname{Re} w = -k$, relation (4.1), by the Cauchy theorem, transforms to the relation

$$J_j^{(1)}(N,s) = qv_q \cosh \pi\varkappa_j N^{1-s} \frac{1}{2k}\Gamma\left(\frac{1-s}{k}\right) + q^s\left(s - \frac{1}{2}\right)\zeta_q(2s)R_{j,\infty}(s) + \frac{1}{2\pi ki}\int\limits_{\operatorname{Re} w=-k} (\cdot)dw. \tag{4.4}$$

Here we used the symbol of the residue v_q from (2.13), which does not depend on a vertex. The contour $\operatorname{Re} w = -k$ in (4.4) bypasses the pole of the function $\Gamma\left(\frac{w}{k}\right)$ at the point $\frac{w}{k} = -1$ along the right arc of radius $r = 1/\log N$ (note that the arbitrary parameter N is greater than e^2).

Thereafter we substitute the functional equation (2.24) into the kernel of the integral on the right-hand side of (4.4), taking into account the product

$$q^\omega \zeta_q(2\omega) R_{j,\infty}(\omega),$$

i.e., we divide the right-hand side of (2.24) by the product of Γ-factors

$$(2\pi^2)^{-\omega}\Gamma^2\left(\frac{\omega}{2}\right)\Gamma\left(\frac{\omega}{2} + i\varkappa_j\right)\Gamma\left(\frac{\omega}{2} - i\varkappa_j\right).$$

Moreover, the Rankin series $R_{j,\alpha}(1-\omega)$ will converge rapidly and absolutely, having order $\cosh \pi\varkappa_j$ with respect to each vertex α. This follows from the estimate $\operatorname{Re}(1-\omega) \ge k \ge 2$ that is valid on the contour $\operatorname{Re} w = -k$ for $\operatorname{Re} s \ge \frac{1}{2}$.

In order to simplify the estimation of the ratio of the Γ-factors in the kernel of (4.4), we assume that the parameter $\varkappa_j$ is contained in the interval

$$\frac{1}{2}X < \varkappa_j \le X, \tag{4.5}$$

where $X > 1$ is an arbitrary fixed parameter, which we bound from below by the condition

$$X \geq q^2 + |s|^2 \geq 2q|s|. \tag{4.6}$$

Then the integral on the right-hand side in (4.4) is of order

$$\ll \left(\frac{q|s|X}{\sqrt{N}}\right)^{2k} (q|s|X)^{1-2\sigma} \cosh \pi \varkappa_j \log N, \tag{4.7}$$

where $\sigma \geq \frac{1}{2}$ is taken from (4.2). Moreover, the constant in Vinoghradov's symbol $\ll$ depends only on k. In addition, condition (4.5) excludes the small eigenvalues $\lambda_j < 1/4$, therefore all the asymptotics and estimates are independnet of the Selberg conjecture.

The passage from the eigenvalues λ_j to the parameter $\varkappa_j = \sqrt{\lambda_j - 1/4}$ is associated with the passage from the Laplace equation $\Delta f_j(z) = \lambda_j f_j(z)$ to a modified Bessel equation:

$$-y^2 a_n''(y) + (2\pi n y)^2 a_n(y) = \lambda_j a_n(y), \quad n \neq 0,$$

$$-y^2 a_0''(y) = \lambda_j a_0(y), \quad n = 0,$$

for the Fourier coefficients of the function $f_j(z)$:

$$f_j(z) = a_0(y) + \sum_{n \neq 0} a_n(y) e^{2\pi i n x}.$$

The Bessel equations are classical equations in the theory of differential equations; therefore, we shall not dwell on the solution of them. We only note that the explicit form of the coefficients $a_n(y)$ is given in (0.7), (2.1), (2.1.^{a}), where the parameter $\varkappa_j$ arises. Since the majority of properties of the eigenfunctions $f_j(z)$ depends on these coefficients, it is more convenient to study their properties depending on the parameter $\varkappa_j$. This explains the passage from the eigenvalue λ_j to the parameter $\varkappa_j$ in the interval (4.5). For details, see [24, 25]. After this digression, we again return to estimate (4.7).

Since N is arbitrary, we define it by the relation

$$N = c(q|s|X)^2 \, X^{2\varepsilon}, \quad \text{where} \quad \varepsilon = \frac{1}{k}, \tag{4.8}$$

$c \geq 1$ is an arbitrary fixed constant. In this case, quantity (4.7) is of order

$$\ll \frac{\cosh \pi \varkappa_j}{X^2} \log X. \tag{4.9}$$

On the other hand, integral (4.1) has another representation. In order to obtain it, we write the product $\zeta_q(2\omega) R_{j,\infty}(\omega)$ in the region of absolute convergence $\operatorname{Re} \omega > 1$ in the form of a Dirichlet series:

$$\zeta_q(2\omega) R_{j,\infty}(\omega) = \sum_{m=1}^{\infty} a_j(m)/m^{\omega}, \tag{4.10}$$

where

$$a_j(m) = \sum_{\substack{m = nd^2, \\ (d,q)=1}} |\rho_{j,\infty}(n)|^2. \tag{4.11}$$

Substituting series (4.10) into the kernel of integral (4.1) and interchanging integration and summation (which is valid in the region $\mathrm{Re}\, w > 1$ and $\mathrm{Re}\, s \geq \frac{1}{2}$), we reduce the problem to the evaluation of the integral

$$J^{(1)}_{s,j}\left(\frac{m}{qN}\right) = \frac{1}{2\pi i} \int\limits_{\mathrm{Re}\, w>1} \left(\frac{m}{qN}\right)^{-w} \Gamma\left(\frac{w}{k}\right) \frac{1}{k}\left(w + s - \frac{1}{2}\right) dw. \qquad (4.12)$$

In the kernel of integral (4.12), we use the decomposition

$$\Gamma\left(\frac{w}{k}\right) \frac{1}{k}\left(w + s - \frac{1}{2}\right) = \Gamma\left(\frac{w}{k}\right) \frac{1}{k}\left(s - \frac{1}{2}\right) + \Gamma\left(\frac{w}{k}\right) \frac{w}{k},$$

and for the second summand we use the Gauss formula:

$$\Gamma\left(\frac{w}{k}\right) \frac{w}{k} = \Gamma\left(\frac{w}{k} + 1\right).$$

Thereafter, we use a known integral:

$$\frac{1}{2\pi i} \int\limits_{\mathrm{Re}\, s>0} x^{-s}\Gamma(s) ds = e^{-x}, \quad x > 0.$$

As a result, for integral (4.12) we obtain an explicit expression:

$$J^{(1)}_{s,j}\left(\frac{m}{qN}\right) = \left[\left(s - \frac{1}{2}\right) + k\left(\frac{m}{qN}\right)^{k}\right] \exp\left[-\left(\frac{m}{qN}\right)^{k}\right]. \qquad (4.13)$$

Thus, for integral (4.1) we obtain another representation:

$$J^{(1)}_{j}(N,s) = q^s \sum_{m=1}^{\infty} \frac{a_j(m)}{m^s}\, J^{(1)}_{s,j}\left(\frac{m}{qN}\right). \qquad (4.14)$$

Equating the right-hand sides in (4.3) and (4.14) and dividing the relation obtained by q^s, we arrive at the relation

$$v_q \cosh \pi\varkappa_j (qN)^{1-s} \frac{1}{2k} \Gamma\left(\frac{1-s}{k}\right) + \left(s - \frac{1}{2}\right) \zeta_q(2s) R_{j,\infty}(s)$$

$$= \sum_{m=1}^{\infty} \frac{a_j(m)}{m^s}\, J^{(1)}_{s,j}\left(\frac{m}{qN}\right) + O\left(\frac{\cosh \pi\varkappa_j}{q^\sigma X^2} \log X\right). \qquad (4.15)$$

Thus, we derived the first approximate equation for $R_{j,\infty}(s)$, in which there is only one approximating interval of length qN with a smooth cutting-off function. In fact, we can obtain two approximating intervals, each of length $\sqrt{qN}$, but in this case the second interval splits into $\tau(q)$ (the number of vertices) intervals with complicated Γ-coefficients before each of them. Therefore, we choose the simplest form at the cost of an increase in the length of one of them.

We also need the second approximate equation different from (4.15). In order to obtain it, we consider the second integral

$$J_j^{(2)}(N,s)=\frac{1}{2\pi ki}\int\limits_{\mathrm{Re}\,w>1} N^w\Gamma\left(\frac{w}{k}\right)q^{\omega}\zeta_q(2\omega)R_{j,\infty}(\omega)dw,\ \ \omega=w+s. \tag{4.16}$$

It is easy to see that the kernel of integral (4.16) differs from the kernel in (4.1) only by the absence of the suppressing factor $\left(\omega-\frac{1}{2}\right)$. Therefore, in the new equation the left-hand side obtained by the residues becomes more complicated and the right-hand side is simpler:

$$\begin{gathered}v_q\cosh\pi\varkappa_j(qN)^{1-s}\frac{1}{k}\Gamma\left(\frac{1-s}{k}\right)+\zeta_q(2s)R_{j,\infty}(s)\\+\frac{1}{2k}\Gamma\left(\frac{\frac{1}{2}-s}{k}\right)(qN)^{\frac{1}{2}-s}R_{j,\infty}\left(\frac{1}{2}\right)\frac{\varphi(q)}{q}=\\\sum_{m=1}^{\infty}\frac{a_j(m)}{m^s}\exp\left[-\left(\frac{m}{qN}\right)^k\right]+O\left(\frac{\cosh\pi\varkappa_j}{q^{\sigma}X^2}\log X\right).\end{gathered} \tag{4.17}$$

We use these two approximate equations in order to derive the Weyl asymptotics for the number of $\varkappa_j$ lying in interval (4.5).

5. Asymptotics due to Hermann Weyl

In order to obtain this asymptotics, it is necessary to use limit relation (4.17) as $s\to 1$. To this end, we use two Laurent expansions in the neighborhood $|s-1|\le\frac{1}{4}$:

$$\frac{1}{k}\Gamma\left(\frac{1-s}{k}\right)=-\frac{1}{s-1}-k^{-1}\gamma+O(|s-1|),$$

where γ is the Euler constant, and

$$\zeta_q(2s)R_{j,\infty}(s)=\frac{v_q\cosh\pi\varkappa_j}{s-1}+r_j+O(|s-1|).$$

Substituting these relations into (4.17), we reduce it to the form

$$\begin{gathered}\frac{1-(qN)^{-(s-1)}}{s-1}v_q\cosh\pi\varkappa_j=\Gamma\left(1-\frac{1}{2k}\right)\sqrt{\frac{q}{N}}R_{j,\infty}\left(\frac{1}{2}\right)\frac{\varphi(q)}{q}\\+c_j+S_{j,1}^{(2)}+O(|s-1|)+O\left(\frac{\cosh\pi\varkappa_j}{qX^2}\log X\right).\end{gathered} \tag{5.1}$$

In the first term of the right-hand side of (5), we use the Gauss formula

$$-\frac{1}{2k}\Gamma\left(-\frac{1}{2k}\right)=\Gamma\left(1-\frac{1}{2k}\right).$$

Moreover,

$$c_j=\gamma k^{-1}v_q\cosh\pi\varkappa_j-r_j,$$

$S^{(2)}_{j,1}(N)$ is the sum over m on the right-hand side in (4.17) at the point $s=1$.

Turning s to 1 in (5.1), we obtain the limit relation

$$v_q \cosh \pi\varkappa_j \log(qN) = c_j + \Gamma\left(1-\frac{1}{2k}\right)\sqrt{\frac{q}{N}}\, R_{j,\infty}\left(\frac{1}{2}\right)\frac{\varphi(q)}{q}$$
$$+S^{(2)}_{j,1}(N) + O\left(\frac{\cosh \pi\varkappa_j}{qX^2}\log X\right). \tag{5.2}$$

In (5.2), the quantity $R_{j,\infty}\left(\frac{1}{2}\right)$ is an obstacle for us. In order to get rid of it, we use approximate equation (4.15). Turning s to $\frac{1}{2}$ in it, we obtain the limit form

$$\frac{1}{2}\frac{\varphi(q)}{q}R_{j,\infty}\left(\frac{1}{2}\right) = -v_q\frac{1}{2k}\Gamma\left(\frac{1}{2k}\right)\sqrt{qN}\cosh\pi\varkappa_j + S^{(1)}_{j,\frac{1}{2}}(N) + O\left(\frac{\cosh\pi\varkappa_j}{\sqrt{q}X^2}\log X\right). \tag{5.3}$$

The symbol $S^{(1)}_{j,\frac{1}{2}}(N)$ on the right-hand side in (5.3) means the sum on the right-hand side in (4.15) for $s=\frac{1}{2}$. Substituting (5.3) into the right-hand side of (5.2) and dividing the two sides by $\cosh\pi\varkappa_j$, we reduce (5.2) to the form

$$v_q\log(qN) = \frac{S^{(2)}_{j,1}(N)}{\cosh\pi\varkappa_j} + 2\Gamma\left(1-\frac{1}{2k}\right)\sqrt{\frac{q}{N}}\frac{S^{(1)}_{j,\frac{1}{2}}(N)}{\cosh\pi\varkappa_j} + c'_j(k) + O\left(\frac{\log X}{\sqrt{q}X^2}\right), \tag{5.4}$$

where $c'_j(k)$ does not depend on N.

Now it should be remembered that there is an arbitrary constant $c\geq 1$ in the definition of the parameter N in (4.8). Putting $c=c_1=1$ and then $c=c_2=e$, we obtain two different relations (5.4), one of them has a parameter N_1 and the order has a parameter $N_2=eN_1$. Substituting one relation from the other term by term, we cancel the quantity $c'_j(k)$ in (5.4) that does not depend on N and obtain a new relation for the residue v_q (because $\log N_2 - \log N_1 = 1$):

$$v_q = [S^{(2)}_{j,1}(N_2) - S^{(2)}_{j,1}(N_1)](\cosh\pi\varkappa_j)^{-1} + \Gamma\left(1-\frac{1}{2k}\right)2\sqrt{q}$$
$$\times\left[\frac{1}{\sqrt{N_2}}S^{(1)}_{j,\frac{1}{2}}(N_2) - \frac{1}{\sqrt{N_1}}S^{(1)}_{j,\frac{1}{2}}(N_1)\right](\cosh\pi\varkappa_j)^{-1} + O\left(\frac{\log X}{\sqrt{q}X^2}\right). \tag{5.5}$$

Summing this relation over $\varkappa_j$ in interval (4.5), we obtain an averaged form of relation (5.5):

$$v_q W\left(X,\frac{1}{2}X\right) = [S^{(2)}_1(N_2) - S^{(2)}_1(N_1)]$$
$$+2\sqrt{q}\,\Gamma\left(1-\frac{1}{2k}\right)\left[\frac{1}{\sqrt{N_2}}S^{(1)}_{\frac{1}{2}}(N_2) - \frac{1}{\sqrt{N_1}}S^{(1)}_{\frac{1}{2}}(N_1)\right]$$
$$+O\left(W\left(X,\frac{1}{2}X\right)\frac{\log X}{\sqrt{q}\,X^2}\right), \tag{5.6}$$

where

$$W(X,Y) = \sum_{Y<\varkappa_j\leq X} 1$$

$$S_1^{(2)}(N) = \sum_{\frac{1}{2}X<\varkappa_j\le X} S_{j,1}^{(2)}(N)(\cosh\pi\varkappa_j)^{-1} = \sum_{m=1}^{\infty}\frac{S(m)}{m}\exp\left[-\left(\frac{m}{qN}\right)^k\right], \tag{5.7}$$

$$S(m) = \sum_{\frac{1}{2}X<\varkappa_j\le X} a_j(m)/\cosh\pi\varkappa_j. \tag{5.8}$$

The sum $S_{\frac{1}{2}}^{(1)}(N)$ is defined similarly. From definition (5.8), it follows that the two sums do not depend on small eigenvalues, therefore the estimates of them do not depend on the Selberg conjecture.

From the definition of the coefficient $a_j(m)$ in (4.11), we find an explicit form of the sum $S(m)$ from (5.8):

$$S(m) = \sum_{\substack{m=nd^2\\(d,q)=1}}\ \sum_{\frac{1}{2}X<\varkappa_j\le X} |\rho_{j,\infty}(n)|^2/\cosh\pi\varkappa_j. \tag{5.9}$$

Substituting (3.14) into (5.9) and then into (5.7), we obtain an asymptotics:

$$S_1^{(2)}(N) = \frac{1}{\pi^2}\left[X^2-\left(\frac{X}{2}\right)^2\right]\sum_{m=1}^{\infty}\frac{a(m)}{m}\exp\left[-\left(\frac{m}{qN}\right)^k\right] + O(\sqrt{N}\,\sigma_0(q)\log^6 N), \tag{5.10}$$

where

$$a(m) = \sum_{\substack{m=nd^2,\\(d,q)=1}} 1.$$

The asymptotic behavior of the sum over $m\ge 1$ on the right-hand side in (5.10) can easily be calculated by using the representation

$$\exp(-x) = \frac{1}{2\pi i}\int_{\mathrm{Re}\,w>0} x^{-w}\Gamma(w)dw, \quad x>0, \tag{5.11}$$

for $x=\left(\frac{m}{qN}\right)^k$. As a result, we obtain a relation:

$$\sum_{m=1}^{\infty}\frac{a(m)}{m}\exp\left[-\left(\frac{m}{qN}\right)^k\right] = \frac{1}{2\pi k i}\int_{\mathrm{Re}\,w>0}(qN)^w\Gamma\left(\frac{w}{k}\right)\zeta(1+w)\zeta_q(2(1+w))dw.$$

Moving the contour to the left, to the line $\mathrm{Re}\,w=-\frac{1}{2}$, we obtain an asymptotics of the sum under consideration:

$$\sum_{m=1}^{\infty}(\cdot) = \zeta_q(2)\log(qN) + c_q + O\left(\frac{\log N}{\sqrt{N}}\right).$$

Therefore, for the first difference on the right-hand side in (5.6) the following asymptotics is valid:

$$S_1^{(2)}(N_2) - S_1^{(2)}(N_1) = \frac{\zeta_q(2)}{\pi^2}\left[X^2-\left(\frac{X}{2}\right)^2\right] + O(\sigma_0(q)\sqrt{N_1}\,\log^6 N_1). \tag{5.12}$$

In order to find the asymptotic behavior of the sum $S^{(1)}_{\frac12}(N)$, we should use the explicit form (4.13) for the integral $J^{(1)}_{s,j}$ at the point $s=\frac12$. Further, applying the same procedure of calculation, we obtain the asymptotics

$$\frac{1}{\sqrt{N}}S^{(1)}_{\frac12}(N)=\frac{1}{k\pi^2}\left[X^2-\left(\frac{X}{2}\right)^2\right]\sqrt{q}\,\Gamma\left(1+\frac{1}{2k}\right)\zeta_q(2)+O\left(\frac{X^2}{\sqrt{N}}\log^2 N\right). \tag{5.13}$$

The right-hand side in (5.13) shows that the principal term of the asymptotics does not depend on N; therefore, the second difference of the right-hand side in (5.6) is estimated by the remainder in (5.13).

As a result, from (5.6) we obtain

$$v_q\,W\left(X,\frac12\,X\right)=\frac{\zeta_q(2)}{\pi^2}\left[X^2-\left(\frac{X}{2}\right)^2\right)+O(\sigma_0(q)\sqrt{N_1}\,\log^6 N_1). \tag{5.14}$$

Adding relations (5.14) for the parameters $X/2^k$, $k=0,1,2,\dots$, we obtain the asymptotics due to Hermann Weyl:

$$W(X)=\frac{\zeta_q(2)}{\pi^2 v_q}\,X^2+O(q^2X^{1+\varepsilon}), \tag{5.15}$$

where

$$W(X)=\sum_{0\le\varkappa_j\le X}1$$

$\varepsilon>0$ is arbitrarily small but fixed. The constant in the symbol O depends on ε. Asymptotics (5.15) is nontrivial for $X>q^2$. We also note that

$$\frac{\zeta_q(2)}{\pi^2 v_q}=\frac{1}{12}\,q\prod_{p|q}\left(1+\frac{1}{p}\right).$$

In subsequent applications we shall use only asymptotics (5.14), which does not depend on the Selberg conjecture, because only large eigenvalues occur in it, which follows from the condition $\varkappa_j>\frac12\,X$.

Formally, asymptotics (5.15) depends on the Selberg conjecture, because small eigenvalues should be added to it. But it makes sense only for $X>q$, because otherwise the remainder will be greater than the principal term of the asymptotics. Under this condition on the parameter X, the addition of small eigenvalues to the right-hand side of (5.15) does not influence the principal term since they are taken by the remainder, because the number of them $\ll\|F_q\|$. Having obtained (5.14), it is easy to estimate the average of $R_{j,\infty}\left(\frac12\right)$ over $\varkappa_j$. To this end, we should return again to relation (4.15) at the limit point $s=\frac12$ and calculate its average over $\varkappa_j$.

Using asymptotics (5.14), the principal term of the first summand of the left-hand side in (4.15) cancels with the principal term of the sum of the right-hand side in (4.15). As a result, we obtain an estimate:

$$\sum_{\frac12\,X<\varkappa_j\le X}R_{j,\infty}\left(\frac12\right)/\cosh\pi\varkappa_j\ll q^2X^{2+\varepsilon}, \tag{5.16}$$

where the constant in the symbol $\ll$ depends on ε. Estimate (5.16) does not depend on the Selberg conjecture, because summation on the left-hand side is taken over the interval $\left(\frac12\le\varkappa_j<X\right)$, i.e., only over large eigenvalues.

Now we have all necessary estimates and can prove the main theorem of the present paper.

6. The main theorem (Theorem 1)

We put $s = s_0 = \beta + i\gamma$ in relation (4.17), where s_0 is a zero of $L(s,\chi)$ with $\beta = \operatorname{Re} s_0 > \frac{1}{2}$. From Sec. 1 we know that s_0 is also a zero of the Rankin convolution $R_{j,\infty}(s)$ for any $j \geq 1$:

$$O = L(s_0,\chi) \Longrightarrow R_{j,\infty}(s_0) = 0. \tag{6.1}$$

Moreover, the condition $\operatorname{Re} s_0 = \beta > \frac{1}{2}$ ensures the estimate

$$\zeta_q(2s_0) \ll 1. \tag{6.2}$$

From (6.1) and (6.2), we have

$$\zeta_q(2s_0) R_{j,\infty}(s_0) = 0. \tag{6.3}$$

Substitute (6.3) into (4.17). As a result, we obtain

$$\begin{aligned} &v_q(qN)^{1-s_0}\frac{1}{k}\Gamma\left(\frac{1-s_0}{k}\right) + (qN)^{\frac{1}{2}-s_0} f_q(s_0)\frac{R_{j,\infty}\left(\frac{1}{2}\right)}{\cosh \pi\varkappa_j} \\ &= \frac{1}{\cosh \pi\varkappa_j}\sum_{m=1}^{\infty}\frac{a_j(m)}{m^{s_0}}\exp\left[-\left(\frac{m}{qN}\right)^k\right] + O\left(\frac{\log X}{\sqrt{q}\,X^2}\right), \end{aligned} \tag{6.4}$$

where

$$f_q(s_0) = \frac{1}{2k}\Gamma\left(\frac{\frac{1}{2}-s_0}{k}\right)\frac{\varphi(q)}{q}.$$

Then we average relation (6.4) over all $\varkappa_j$ from the interval $\frac{1}{2}X < \varkappa_j \leq X$, and in the left-hand side we use the Weyl asymptotics (5.14) for the first term and estimate (5.16) for the second term. In the right-hand side we apply asymptotics (3.14). Then we divide the two sides of the relation obtained by $\frac{1}{\pi^2}\left[X^2 - \left(\frac{X}{2}\right)^2\right]$ and reduce the averaged relation (6.4) to the form

$$\zeta_q(2)(qN)^{1-s_0}\frac{1}{k}\Gamma\left(\frac{1-s_0}{k}\right) + O((qN)^{\frac{1}{2}-\beta}q^2N^{\varepsilon}) = \sum_{m=1}^{\infty}\frac{a(m)}{m^{s_0}}\exp\left[-\left(\frac{m}{qN}\right)^k\right]. \tag{6.5}$$

Using again integral operator (5.11), we reduce the sum of the right-hand side in (6.5) to the integral form

$$\sum_{m=1}^{\infty}(\cdot) = \frac{1}{2\pi ik}\int_{\operatorname{Re} w>\frac{1}{2}}(qN)^w\Gamma\left(\frac{w}{k}\right)\zeta(\omega_0)\zeta_q(2\omega_0)dw, \quad \omega_0 = w + s_0. \tag{6.6}$$

Moving the contour of integral (6.6) to the left, to the line $\operatorname{Re} w = \frac{1}{2} - \beta$ with $\beta = \operatorname{Re} s_0$, we met the pole of $\zeta(w+s_0)$ at the point $w = 1 - s_0$ and the pole of $\Gamma\left(\frac{w}{k}\right)$ at the point $w = 0$. By the Cauchy theorem, we conclude that integral (6.6) is equal to the sum

$$\zeta_q(2)(qN)^{1-s_0}\frac{1}{k}\Gamma\left(\frac{1-s_0}{k}\right) + \zeta_q(2s_0)\zeta(s_0) + \frac{1}{2\pi ki}\int_{\operatorname{Re} w=\frac{1}{2}-\beta}(\cdot)dw. \tag{6.7}$$

In integral (6.7) we draw a contour bypassing the point $w + s_0 = \frac{1}{2}$ along the right arc of radius $r = 1/\log(qN)$. This integral is of order

$$(qN)^{\frac{1}{2}-\beta}\log(qN|s_0|),$$

i.e., it is not greater than the order of the remainder on the left-hand side in (6.5). Substituting (6.7) into the right-hand side of (6.5) and canceling similar terms with Γ-function, we obtain the estimate

$$\zeta_q(2s_0)\zeta(s_0) \ll (qN)^{\frac{1}{2}-\beta}q^2N^{\varepsilon}. \tag{6.8}$$

For $\beta > \frac{1}{2}$, the following estimate holds:

$$\zeta_q^{-1}(2s_0) \ll_{s_0} 1;$$

therefore, (6.8) implies an estimate for the pure zeta function:

$$\zeta(s_0) \ll_{s_0} (qN)^{\frac{1}{2}-\beta}q^2N^{\varepsilon}. \tag{6.9}$$

Now we should remember that the parameter N is defined by relation (4.8), where X is arbitrary; therefore, we can choose X so large that we have $q^2 < N^{\varepsilon}$. Consequently, estimate (6.9) can be rewritten in the form

$$\zeta(s_0) \ll_{s_0} (qN)^{\frac{1}{2}-\beta}N^{2\varepsilon}. \tag{6.10}$$

Assume that the real part of the zero s_0 satisfies the condition

$$\beta \geq \frac{1}{2} + 3\varepsilon. \tag{6.11}$$

Substituting this estimate into (6.10), we obtain

$$\zeta(s_0) \ll (qN)^{-\varepsilon}, \tag{6.12}$$

where the constant in Vinogradov's symbol $\ll$ depends on ε and s_0.

As $X \to \infty$ and, thus, $N \to \infty$, (6.12) implies the relation

$$\zeta(s_0) = 0. \tag{6.13}$$

But $s_0 = \beta + i\gamma$ is a fixed zero of $L(s,\chi)$ with condition (6.11), where $\varepsilon > 0$ is arbitrarily small. Therefore, (6.11) and (6.13) yield the main theorem.

Theorem 1. *If a zero of $L(s,\chi)$ from the critical strip does not lie on the line $\operatorname{Re} s_0 = \frac{1}{2}$, then it is also a zero of the Riemann zeta function.*

The above results imply all the consequences cited in the introduction. At the end of this section, we have used Weyl's asymptotics (5.14) and Kuznetsov's asymptotics (3.14), which do not depend on the Selberg conjecture; therefore, the main theorem (Theorem 1) also does not depend on it.

7. On some assertions from the introduction

Relation (0.4) in the introduction is a serious problem only for the general Laplace–Beltrami operator. In the case of the classical Laplace operator Δ, which we need, no difficulties arise.

In order to convince ourselves in this, it is sufficient to verify the validity of the commutation relation in the space under consideration:

$$T_{-1} \otimes \Delta = \Delta \otimes T_{-1}. \tag{7.1}$$

To this end, we must use the factorization of the Laplace operator Δ into two linear complex conjugate multipliers:

$$\Delta = -L \cdot \bar{L}, \quad L = y \cdot \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \tag{7.2}$$

From relation (7.2) it follows that any basis $\{u_j(z)\}_{j=1}^{\infty}$ of the space under consideration, generated by all linear independent eigenfunctions of the discrete spectrum of the Laplace operator Δ, contains the complex conjugate function $\overline{u(z)}$ together with each function $u(z)$.

In addition, the self-adjointness of the operator Δ and its negative normalization (7.2) imply the reality and positiveness of all eigenvalues of Δ. Therefore, eigenfunctions $u(z)$ and $\overline{u(z)}$ belong to one and the same eigenvalue $\lambda > 0$:

$$\Delta \cdot u = \lambda \cdot u, \quad \Delta \cdot \bar{u} = \lambda \cdot \bar{u}. \tag{7.3}$$

Hence we have the standard reduction of an arbitrary u-basis to the canonical basis $\{v_j(z)\}_{j=1}^{\infty}$ relative to the reflection operator T_{-1}:

$$(T_{-1} v_j)(z) = v_j(-\bar{z}) = \xi_j \cdot v_j(z), \tag{7.4}$$

where $\xi_j = \pm 1$ depending on the eveness or oddness of the function $v_j(z)$.

To this end, for each pair $u(z)$ and $\overline{u(z)}$ of functions of the initial u-basis it is necessary to construct two new functions in a linear may: 1) the real function

$$v_1(z) = \frac{1}{2}[u(z) + \overline{u(z)}] \tag{7.5}$$

and 2) the pure imaginary function

$$v_{-1}(z) = \frac{1}{2}[u(z) - \overline{u(z)}]. \tag{7.6}$$

This passage from a u-basis to the v-basis is a precise analog of obtaining the real even cosines

$$\{\cos 2\pi n x\}_{n=0}^{\infty}$$

and the pure imaginary odd sines

$$\{i \sin 2\pi n x\}_{n=1}^{\infty}$$

from the classical basis of Fourier series

$$\{e^{2\pi i n x}\}_{n \in \mathbb{Z}}$$

with the help of the Euler relation

$$e^{2\pi inx} = \cos 2\pi nx + i \sin 2\pi nx.$$

This analogy was mentioned in the introduction for the main basis $\{f_j(z)\}$, for which it remains valid.

Therefore, the spaces of the classical Laplace operator Δ are intermediate between the classical Fourier series and the general Hilbert spaces for the Laplace–Beltrami operator.

From relations (7.5) and (7.6) it follows that the functions $v_{\pm 1}(z)$ are also eigenfunctions of the Laplace operator Δ with the same eigenvalue λ:

$$\Delta v_j = \lambda \cdot v_j, \quad j = \pm 1. \tag{7.7}$$

Hence it is clear that the invariant relation for u-functions of the initial basis

$$u_j(\sigma z) = \chi(\sigma) \cdot u_j(z), \quad \sigma \in \Gamma_0(q),$$

remains valid for v-functions if the multiplier χ is pure real as is the eigenvalue λ:

$$v_j(\sigma z) = \chi(\sigma) \cdot v_j(z), \quad \sigma \in \Gamma_0(q).$$

In the case under consideration, $\chi = \chi_q$ is a quadratic character of modulus q, satisfying this property of being real. Therefore, the following Fourier expansion holds for v-functions at any vertex α of the fundamental domain F_q of the group $\Gamma_0(q)$, typical of the eigenfunctions:

$$v_j(\sigma_\alpha z) = \sum_{n \neq 0} \psi_{j,\alpha}(n) \cdot \sqrt{y} \cdot K_{i\varkappa_j}(2\pi |n| y) e^{2\pi inx}, \tag{7.8}$$

where σ_α is a normed matrix taking ∞ to α: $\sigma_\alpha(\infty) = \alpha$.

Putting $j = 1$ in relation (7.8) and using the reality of v_1, we obtain a relation for the Fourier coefficients of the right-hand side in (7.8):

$$\psi_{1,\alpha}(-n) = \overline{\psi_{1,\alpha}(n)}. \tag{7.9}$$

Relations (7.8) and (7.9) imply the eveness of the function $v_1(z)$:

$$(T_{-1} \cdot v_1)(z) = v_1(-\bar{z}) = v_1(z). \tag{7.10}$$

If $j = -1$, then the pure imaginarity of v_{-1} and expansion (7.8) yield the following relation for the Fourier coefficients of v_{-1}:

$$\psi_{-1,\alpha}(-n) = -\overline{\psi_{-1,\alpha}(n)}. \tag{7.11}$$

Relations (7.11) and (7.8) imply the oddness of the function v_{-1}:

$$(T_{-1} \cdot v_{-1})(z) = v_{-1}(-\bar{z}) = -v_{-1}(z). \tag{7.12}$$

Combining relations (7.9) and (7.11), we obtain the following system of relations for the Fourier coefficients of the functions $v_j(z)$:

$$\psi_{j,\alpha}(-n) = \xi_j \cdot \overline{\psi_{j,\alpha}(n)}, \tag{7.13}$$

where $j = \pm 1$, $\xi_j = \pm 1$.

In the case of degeneracy, where the eigenvalue λ is of multiplicity one and, thus, the function $u(z)$ is either real or pure imaginary, the reduction of it to v-functions is trivial. We put $u(z) = v_1(z)$ in the first case and $u(z) = v_{-1}(z)$ in the second one.

The zeroth Fourier coefficient is an analog of this case in the classical Fourier series.

Relations (7.10) and (7.12), combined together, show that canonical relation (7.4) does hold for the v-basis. Therefore, it is also valid for the whole space.

The commutation relations for the Hecke operators

$$T_{-1} \otimes T_n = T_n \otimes T_{-1}, \tag{7.14}$$

$$T_n \otimes \Delta = \Delta \otimes T_n, \tag{7.15}$$

$$T_n \otimes T_m = T_m \otimes T_n \tag{7.15a}$$

are well known for all $n \geq 1$ and $m \geq 1$. They can easily be verified for the v-basis, using definition (0.3) for the Hecke operator.

Consequently, all of the commutation relations hold for the set of operators $\{\Delta \cup T_{-1} \cup \{T_n\}_{n=1}^{\infty}\}$ of the space under consideration. Therefore, using the main theorem on commuting operators in a Hilbert space, we prove the existence of a general canonical basis $\{f_j(z)\}_{j=1}^{\infty}$ consisting of eigenfunctions of all operators of the set under consideration:

$$\Delta \cdot f_j(z) = \lambda_j \cdot f_j(z), \tag{7.16}$$

$$T_{-1} f_j(z) = \xi_j \cdot f_j(z), \tag{7.17}$$

$$T_n \cdot f_j(z) = t_j(n) \cdot f_j(z), \quad n \in \mathbb{N}. \tag{7.18}$$

Relation (7.17) is the required relation (0.4). Together with relations (7.18), we obtain an extended ring of Hecke operators, which was postulated in the introduction.

Moreover, the even functions $f_j(z)$ of the basis under consideration, for which $\xi_j = +1$ in relation (7.17), are, as before, real and the odd functions are pure imaginary, as was proved for even and odd v-functions in the initial v-basis. Although this fact cannot be obtained for an f-basis with the help of the theorem on commuting operators, it can easily be proved with the help of the same analogy with Fourier series that was used for a v-basis. To this end, it is sufficient to substitute the f-functions of the new basis $\{f_j(z)\}$ for the u-functions of the initial basis $\{u_j(z)\}$. In this case, relations (7.5) and (7.6) for a fixed function $f_j(z)$ take the form

$$v_{\pm 1}(z) = \frac{1}{2}[f_j(z) \pm \overline{f_j(z)}], \tag{7.19}$$

and the Fourier coefficients of expansion (7.8) for any vertex α are determined by the relation

$$\psi_{\pm 1, \alpha}(n) = \frac{1}{2}[\rho_{j,\alpha}(n) \pm \overline{\rho_{j,\alpha}(-n)}]. \tag{7.20}$$

Of course, the left-hand sides of relations (7.19) and (7.20) depend on the index j of the function $f_j(z)$, but we omit this index both on the right and on the left for simplicity. Since we study these relations for a fixed number j, we hope that this will not cause confusion. Relation (7.20) follows from relation (7.19) if we rewrite (7.19) in the form

$$v_{\pm 1}(\sigma_\alpha z) = \frac{1}{2}[f_j(\sigma_\alpha z) \pm \overline{f_j(\sigma_\alpha z)}]$$

and substitute expansion (7.8) into the left-hand side of (7.19) and expansion (2.1) from Sec. 2 into the right-hand side.

In addition to (7.20), we derive another representation for the coefficients $\psi_{\nu,\alpha}(n)$, $\nu = \pm 1$, using the orthogonality of Fourier series (7.8) on the unit interval $\left(-\frac{1}{2} \leq x \leq \frac{1}{2}\right)$:

$$\psi_{\nu,\alpha}(n) \cdot \sqrt{y} \cdot K_{i\varkappa_j}(2\pi|n|y) = \int_{-\frac{1}{2}}^{\frac{1}{2}} v_\nu(\sigma_\alpha z) \cdot e^{-2\pi i n x}\, dx. \tag{7.21}$$

This relation is of special interest in the case of the vertex $\alpha = \infty$. In this case, $\sigma_\alpha = 1$ (the identity matrix), therefore for $\alpha = \infty$ relation (7.21) takes the form

$$\psi_{\nu,\infty}(n) \cdot \sqrt{y} \cdot K_{i\varkappa_j}(2\pi|n|y) = \int_{-\frac{1}{2}}^{\frac{1}{2}} v_\nu(z) \cdot e^{-2\pi i n x}\, dx. \tag{7.22}$$

We represent the integral in (7.22) as the sum of two integrals

$$\int_0^{\frac{1}{2}} [v_\nu(z)e^{-2\pi i n x} + v_\nu(-\bar{z}) \cdot e^{2\pi i n x}]dx. \tag{7.23}$$

Substituting into (7.23) the second part of relation (7.4), we express this integral as

$$\int_0^{\frac{1}{2}} v_\nu(z) \cdot [e^{2\pi i n x} + \xi_\nu \cdot e^{-2\pi i n x}]dx. \tag{7.24}$$

For $\nu = +1$ the function $v_{+1}(z)$ is even, i.e., $\xi_1 = +1$, and integral (7.24) is equal to

$$2 \cdot \int_0^{\frac{1}{2}} v_1(z) \cdot \cos 2\pi n x\, dx. \tag{7.25}$$

From (7.19) it follows that the function $v_{+1}(z)$ is real. Therefore, integral (7.25) is also real.

For $\nu = -1$, the function $v_{-1}(z)$ is odd, i.e., $\xi_{-1} = -1$, and integral (7.24) is equal to

$$-2 \cdot i \cdot \int_0^{\frac{1}{2}} v_{-1}(z) \cdot \sin 2\pi n x\, dx. \tag{7.26}$$

From (7.19) it follows that the function $v_{-1}(z)$ is pure imaginary. Therefore, integral (7.26) is real.

Consequently, the right-hand side of (7.26) is real both for $\nu = +1$ and $\nu = -1$, for any $n \neq 0$.

Moreover, on the left-hand side in (7.22), $\sqrt{y}$ is also real (and positive), because, by condition, $y > 0$, and we consider the principal branch of the square root.

MacDonald's K-function on the left-hand side in (7.22):

$$K_{i\varkappa_j}(2a) = \frac{1}{2}\int\limits_0^\infty u^{i\varkappa_j} \cdot e^{-a\left(u+\frac{1}{u}\right)} \frac{du}{u} \tag{7.27}$$

is also real for any $a > 0$. In the case under consideration, $a = \pi|n| \cdot y > 0$, because, by condition, $n \neq 0$ and $y > 0$.

The reality of the K-function follows from the partition of the interval of integration in (7.27) into two parts:

$$(0, \infty) = (0, 1) \cup (1, \infty)$$

and from the change of the variable of integration $u \to \frac{1}{u}$ in the first interval (0.1). As a result, the right-hand side in (7.27) transforms to

$$\frac{1}{2}\int\limits_1^\infty [u^{i\varkappa_j} + u^{-i\varkappa_j}] \cdot e^{-a\left(u+\frac{1}{u}\right)} \frac{du}{u}. \tag{7.28}$$

By condition, $\varkappa_j = \sqrt{\lambda_j - 1/4}$, where $\lambda_j > 0$ is an eigenvalue to which the fixed function $f_j(z)$ belongs. Therefore, $i\varkappa_j$ is pure imaginary for large eigenvalues $\lambda_j > 1/4$ and is real for small $\lambda_j \leq 1/4$. In either case, the square bracket in the kernel of integral (7.28) is real.

Therefore, the coefficient $\psi_{\nu,\infty}(n)$ on the left-hand side in (7.22) is real both for $\nu = +1$ and $\nu = -1$. Consequently, the two relations in (7.20) imply the reality of $\rho_{j,\infty}(n)$ and $\overline{\rho_{j,\infty}(-n)}$.

Thus, we have established that Fourier expansion (2.1) from Sec. 2 for the vertex ∞ enjoys a special property: all the Fourier coefficients $\rho_{j,\infty}(n)$ are real for any $n \neq 0$.

Now we use relation (7.17), which we write in the form

$$f_j(-\bar{z}) = \xi_j \cdot f_j(z). \tag{7.29}$$

Substituting Fourier expansions (2.1) from Sec. 2 for $\alpha = \infty$ into the left- and right-hand sides of (7.29), we obtain the following relation between the Fourier coefficients of this vertex:

$$\rho_{j,\infty}(-n) = \xi_j \cdot \rho_{j,\infty}(n). \tag{7.30}$$

But the reality of the coefficients $\rho_{j,\infty}(n)$ implies that (7.30) is equivalent to the relation

$$\rho_{j,\infty}(-n) = \xi_j \cdot \overline{\rho_{j,\infty}(n)},$$

which implies, in turn, that all even functions of the f-basis are real and all odd functions of this basis are pure imaginary if we make use of the Fourier expansion at the vertex $\alpha = \infty$ for each of them. But these properties of the basis are common for the whole fundamental domain F_q and do not depend on a vertex. Therefore, we may return to the general Fourier expansion (2.1) from Sec. 2 at any vertex α, and, using the fact that the even functions $f_j(\sigma_\alpha z)$ are real, we can obtain the following relation for their Fourier coefficients:

$$\rho_{j,\alpha}(-n) = \overline{\rho_{j,\alpha}(n)}.$$

Since the odd functions $f_j(\sigma_\alpha z)$ are pure imaginary, we have

$$\rho_{j,\alpha}(-n) = -\overline{\rho_{j,\alpha}(n)}.$$

Combining these two relations, we derive the second relation

$$\rho_{j,\alpha}(-n) = \xi_j \cdot \overline{\rho_{j,\alpha}(n)} \tag{7.31}$$

that we have postulated in the introduction.

In the explicit form, we have used this relation in the introduction for $\alpha = \infty$ in order to show that all even functions of the f-basis are real and all odd functions of this basis are pure imaginary. But in the implicit form, we used this relation very often in the general Rankin convolutions $R_{j,\alpha}(s)$, beginning with Sec. 2.

Now we know that in (7.31) for $\alpha = \infty$ we may omit the bar of complex conjugation, because all Fourier coefficients $\rho_{j,\infty}(n)$ are real. But for the remaining vertices, we cannot omit the bar, because the proof of the reality of $\rho_{j,\infty}(n)$ based on integral representation (7.21) does not work at the other vertices.

At first glance, it seems that the reality of $\rho_{j,\infty}(n)$ contradicts the fact that the Hecke operators T_n are anti-Hermitian for nonresidues $n : \chi(n) = -1$, because in this case the eigenvalues $t_j(n)$ are pure imaginary.

But this contradiction can easily be removed by the condition $t_j(n) = 0$ for $\chi(n) = -1$. This fact is rather natural if we recall that the Rankin convolution $R_{j,\infty}(s)$ is divisible by the zeta function of the quadratic field k:

$$\zeta_k(s) = \sum_{\mathfrak{a} \in \mathcal{O}_k} \frac{1}{(N\mathfrak{a})^s}, \tag{7.32}$$

where $\mathcal{O}_k$ is the ring of integral ideals $\mathfrak{a}$ of the field k. Therefore, in the Rankin convolution

$$R_{j,\infty}(s) = \sum_{n=1}^{\infty} |\rho_{j,\infty}(n)|^2 \cdot n^{-s}$$

only the coefficients $\rho_{j,\infty}(n)$ with $n = N(\mathfrak{a})$ for some $\mathfrak{a} \in \mathcal{O}_k$ are different from zero. But all such numbers are quadratic residues: $\chi(N\mathfrak{a}) = +1$. Therefore, the Hecke operators T_n for $n = N(\mathfrak{a})$ are Hermitian, and, thus, $t_j(N\mathfrak{a})$ are real.

We note that the eigenvalues of the Hecke operators T_n of Eisenstein series whose explicit form is equal to $\chi(n) \cdot \tau_s(n, \chi)$ confirm (for $s = \frac{1}{2}$) all that we have said above.

This is an algebraic-arithmetical explanation of the phenomenon of the reality of all the coefficients $\rho_{j,\infty}(n)$.

But for the other vertices this argument does not work, at these vertices we does not have Hecke operators and the multiplicative structure of Fourier coefficients. (Recall that we have the theory of Hecke operators for the vertex ∞). Most likely, at the other vertices we cannot omit the bar of complex conjugation in relation (7.31) because of the fact that the Hecke operators are anti-Hermitian for nonresidues, and this property manifests itself in the Fourier coefficients of the other vertices.

At the end of this section we return once more to Sec. 3, where we used the Kuznetsov summation formulas from papers [8, 12]. Formally, in those papers an idea of a real basis was used, whereas the basis $\{f_j(z)\}$ is real only for even functions and is imaginary for odd functions. But this (apparent) contradiction can easily be removed, because of the fact that a basis is real is the question of its normalization.

Indeed, if we define a new basis $\{f_j(z)\}$ by using the basis $\{u_j(z)\}$ with the help of the relations

$$u_j(z) = \begin{cases} f_j(z), & \xi_j = +1, \\ -i \cdot f_j(z), & \xi_j = -1, \end{cases} \tag{7.33}$$

then we conclude that the basis $\{u_j(z)\}$ is real in the sense of papers [8, 12].

Now if we write the Fourier series for the function $u_j(z)$ from (7.33) with respect to any vertex α:

$$u(\sigma_\alpha z) = \sum_{n\neq 0} \varphi_{j,\alpha}(n) \cdot \sqrt{y} \cdot K_{i\varkappa_j}(2\pi|n|y)e^{2\pi i n x},$$

then the Fourier coefficients $\varphi_{j,\alpha}(n)$ are linearly related with the coefficients $\rho_{j,\alpha}(n)$ of the functions $f_j(\sigma_\alpha z)$:

$$\varphi_{j,\alpha}(n) = \frac{1}{2}[(1+\xi_j) - i(1-\xi_j)]\rho_{j,\alpha}(n). \tag{7.34}$$

This relation follows from relations (7.33).

Relation (7.34) implies the following equality of Hermitian products of these coefficients:

$$\varphi_{j,\alpha}(n) \cdot \overline{\varphi_{j,\beta}(m)} = \rho_{j,\alpha}(n) \cdot \overline{\rho_{j,\beta}(m)}. \tag{7.35}$$

Therefore, the Kuznetsov summation formulas from [8, 12] for the Hermitian products are the same both for $\{u_j(z)\}$ and $\{f_j(z)\}$.

All that we have said in this section concerning the basis $\{f_j(z)\}$ and the operator T_{-1} is valid if the Selberg conjecture from [22] is true. If we do not use this conjecture, then a difficulty arises, which may be revealed as follows. First we subdivide the basis of the space under consideration into three subsets:

$$\{E_\alpha\}_\alpha \otimes \{f_j\}_{\lambda_j<1/4} \otimes \{f_j\}_{\lambda_j\geq 1/4}. \tag{7.36}$$

Under such a partition, the finite set of functions for small eigenvalues $\lambda_j < 1/4$ behaves with respect to the reflection operator T_{-1} in the same way as the subset of Eisenstein series $\{E_\alpha\}_\alpha$, i.e., the operator T_{-1} is trivial on the zeroth Fourier coefficients and is nontrivial on the additively normalized functions of this subset:

$$f_j^*(z) = f_j(z) - \rho_{j,\infty}(0)y^{\frac{1}{2}+i\varkappa_j}.$$

Under such a normalization, the functions $f_j^*(z)$ also are eigenfunctions of the reflection operator:

$$T_{-1}f_j^*(z) = f_j^*(-\bar{z}) = \xi_j f_j^*(z). \tag{7.37}$$

Relation (7.37) is valid if for all vertices α the zeroth coefficients $\rho_{j,\alpha}(0)$ are real for real functions $f_j(z)$ and are imaginary for pure imaginary $f_j(z)$. But this property follows immediately from the relation

$$\rho_{j,\alpha}(0)y^{\frac{1}{2}+i\varkappa_j} = \int_{-\frac{1}{2}}^{\frac{1}{2}} f_j(\sigma_\alpha z)dx. \tag{7.38}$$

The right-hand side in (7.38) is real in the first case and pure imaginary in the second one. The factor $y^{\frac{1}{2}+i\varkappa_j}$ on the left in (7.38) is real for small $\lambda_j < 1/4$, because in this case $i\varkappa_j = -\sqrt{\frac{1}{4}-\lambda_j} \in \mathbb{R}$. Therefore, together with relations (7.37), the following main relation is valid for nonzeroth Fourier coefficients of the function $f_j(\sigma_\alpha z)$ at all vertices α:

$$\rho_{j,\alpha}(-n) = \xi_j\overline{\rho_{j,\alpha}(n)}, \quad n \neq 0.$$

When partitioning the basis into three subsets (7.36), the second set has some properties of the first one (the f_j tend to ∞ at all vertices of the fundamental domain F_q) and of the third one (the f_j are square integrable on F_q). It is more convenient to work with each subset individually.

It is worthwhile noting that if in the zeroth summand of (7.38), the condition $\rho_{j,\infty}(0) \neq 0$ is valid for the vertex $\alpha = \infty$, then the eigenvalues $t_j(m)$ of the Hecke operators T_m in the relation

$$T_m f_j(z) = t_j(m) f_j(z) \tag{7.39}$$

has a simple form. In order to obtain this form, it is necessary to apply the explicit form of the Hecke operator to the function $f_j(z)$:

$$T_m f_j(z) = \frac{1}{\sqrt{m}} \sum_{m=ad} \sum_{b=1}^{d} \chi(a) f_j\left(\frac{az+b}{d}\right) \tag{7.40}$$

and to compare the zeroth Fourier coefficients in (7.39) and (7.40). As a result, we obtain a relation:

$$t_j(m) = m^{-i\varkappa_j} \sum_{a/m} \chi(a) a^{2i\varkappa_j}.$$

From this relation it is seen that for small eigenvalues $\lambda_j < 1/4$, the eigenvalue of the Hecke operator is the Fourier coefficient of the Eisenstein series moved to the line $\operatorname{Re} s = \frac{1}{2} + \delta_j$:

$$t_j(m) = m^{\delta_j} \sum_{a/m} \chi(a) a^{-2\delta_j}, \tag{7.41}$$

where

$$\delta_j = \sqrt{\frac{1}{4} - \lambda_j} \in \left(0, \frac{1}{4}\right].$$

Relations (7.41) imply the following estimate from below for the eigenvalue $\lambda_j = 3/16$

$$t_j(m) \gg \sqrt[4]{m}.$$

Therefore, without the Selberg conjecture from [22], the upper bound in the general estimate

$$\rho_j(m) \ll m^{\frac{1}{4}+\varepsilon}$$

is achieved. Here we see an interesting relationship between the Petersson conjecture for the weight 0 and the Selberg conjecture: one of them implies the other.

If we assume that the Selberg conjecture [22] holds, then the second subset in (7.36) is empty, and we obtain the classical partition of the basis into the continuous and discrete spectrum.

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