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ON RECONSTRUCTION OF GRAVITY FIELD VIA EXTERNAL ELECTROMAGNETIC MEASUREMENTS

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0. On the paper. An approach to inverse problems (IP's) using their relationship to boundary control theory (the BC-method [1]) is developed. A dynamical IP for the Maxwell system in a bounded domain of a 3-D curved space is considered. The domain is prospected by boundary electromagnetic sources; the waves scattered by inhomogeneities of metric (gravity field) are detected at the same boundary. The problem is to recover the metric from these measurements. The proposed procedure solving the IP is time-optimal, i.e., the metric in the near-boundary subdomain of the optical thickness T is recovered by measurements on times $0 < t < 2T$. We restrict ourselves to considering a subdomain covered by the regular system of semigeodesical coordinates with the base at the boundary. Among known results, let us mention uniqueness theorems for the Maxwell system in the frequency domain [6, 9].

1. Geometry. Let (Ω, g) be a smooth (of class C^∞) oriented compact 3-dimensional Riemannian manifold with the boundary Γ. Define *the eikonal* $\tau : \Omega \to \mathbf{R}$, $\tau(x) = \operatorname{dist}(x, \Gamma)$; denote

$$\Omega^\xi = \{x \in \Omega \mid \tau(x) < \xi\} \bigcup \Gamma, \quad \Gamma^\xi = \{x \in \Omega \mid \tau(x) = \xi\} \text{ for } \xi \geq 0,$$

$(\Gamma^0 = \Gamma)$, and $\Theta^T = \Gamma \times [0, T)$. Let T_ω be the supremum of such $T > 0$ that the map $\exp_\Gamma : \Theta^T \to \Omega^T$ is a diffeomorphism.

We always assume that $T < T_\omega$.

The vector field of normals to surfaces Γ^ξ, $\nu = \nabla\tau$ is well defined in Ω^T. Denoting $i = \exp_\Gamma^{-1}$, the paire $(\gamma, \tau) = i(x)$ is called *the semigeodesical coordinates* (s.g.c.) of a point $x \in \Omega^T$. The length element in a s.g.c. has the known form: $ds^2 = g_{\alpha\beta} d\gamma^\alpha d\gamma^\beta + d\tau^2$, where γ^1, γ^2 are local coordinates on Γ.

2. The Maxwell system. We consider the system

$$e_t = \operatorname{rot} h, \quad h_t = -\operatorname{rot} e \text{ in } \Omega \times (0, T); \tag{1}$$

$$e|_{t=0} = 0, \quad h|_{t=0} = 0; \tag{2}$$

$$e_\theta|_{\Gamma \times [0,T]} = f \tag{3}$$

(see the definitions of rot and other operations of vector analysis in [7]), where $e_\theta = e - \langle e, \nu\rangle\nu$ is the tangent component of e, and f is *a boundary control.* Let $e = e^f(x, t)$, $\quad h = h^f(x, t)$ be the solution (*wave*). The system (1)–(3) describes propagation of electromagnetic waves initiated by boundary sources (controls) in a weak stationary static gravity field. The "input-output" correspondence in the system is realized by *the response operator* $R^T : f \mapsto -\nu \times h^f|_{\Gamma \times [0,T]}$. The relation

$$h^f = e^{f_\star} \tag{4}$$

holds with $f_\star = \nu \times R^T f$.

3. The main result. By virtue of finiteness of the wave velocity, the operator R^{2T} corresponding to system (1)–(3) with the double final moment is determined by the submanifold (Ω^T, g). *The inverse problem* is to recover (Ω^T, g) from R^{2T}.

Theorem 1. *The operator R^{2T} determines the submanifold (Ω^T, g) uniquely up to isometry.*

Given R^{2T}, we propose a procedure which permits us to equip the cylinder Θ^T with the metric i^*g turning it into an isometrical copy of (Ω^T, g).

4. Controllability. The space of controls

$$\mathcal{F}^T=\{f\in\mathbf{L}_2(\Gamma\times[0,T])\mid\langle f(\gamma,t),\nu(\gamma)\rangle=0,\ \text{for all }(\gamma,t)\}$$

is *the outer space* of system (1)–(3). The operator $R^T:\mathcal{F}^T\to\mathcal{F}^T$ is densely defined and closable. In $\mathcal{F}^T$, we select the family of subspaces $\mathcal{F}^{T,\xi}=\{f\in\mathcal{F}^T\mid \operatorname{supp}f\subset\Gamma\times[T-\xi,T]\}$, $0\le\xi\le T$ formed by delayed controls.

The role of *an inner space* of system (1)–(3) is played by the space of solenoidal vector fields

$$\mathcal{J}^T=\{y\in\mathbf{L}_2(\Omega)\mid \operatorname{div}y=0\ \text{ in }\ \Omega;\quad \operatorname{supp}y\subset\bar\Omega^T\},$$

whose elements are waves $e^f(\cdot,t),h^f(\cdot,t)$. In the latter space, we select the family of subspaces $\mathcal{J}^\xi=\{y\in\mathcal{J}^T\mid \operatorname{supp}y\subset\bar\Omega^\xi\}$, $\quad 0\le\xi\le T$.

The operator $W^T:\mathcal{F}^T\to\mathcal{J}^T$, $W^Tf=e^f(\cdot,T)$ is densely defined, closable, and injective [2, 4]. The subspace $\mathcal{E}^\xi=\operatorname{clos}W^T\big(\mathcal{F}^{T,\xi}\bigcap\operatorname{Dom}W^T\big)\subset\mathcal{J}^\xi$ is called reachable at the moment $t=\xi$.

Theorem 2. *The equality*

$$\mathcal{E}^\xi=\mathcal{J}^\xi \tag{5}$$

holds for every $0\le\xi\le T\quad(T<T_\omega)$.

Property (5) is interpreted as *an approximate controllability* of system (1)–(3); its justification (see [2, 4]) is based upon the vectorial version [5] of the Holmgren–John–Tataru's uniqueness theorem [8]. Controllability is an analog of the completeness property of the wave operators in scattering theory.

By (4), we have $h^f=W^T\big(\nu\times R^Tf\big)$; since operators in (1) are independent of time, one has:

$$\operatorname{rot}e^f(\cdot,T)=-h^f_t(\cdot,T)=-h^{f_t}(\cdot,T)=-W^T\big(\nu\times R^Tf_t\big). \tag{6}$$

5. Images. Let P^ξ be the projector in $\mathcal{J}^T$ onto $\mathcal{J}^\xi$. If $y\in\mathcal{J}^T$ is a smooth field, then the projection $P^\xi y$ is discontinuous on Γ^ξ, the discontinuity $P^\xi y\,|_{\Gamma^\xi-0}$ turning out to be tangent to Γ^ξ [2–4]. Consider the space of *transversal fields*

$$\mathcal{L}^T_\theta=\{y\in\mathbf{L}_2(\Omega^T)\mid\langle y,\nu\rangle=0\text{ in }\Omega^T\}$$

and introduce the map $M^T:\mathcal{J}^T\to\mathcal{L}^T_\theta$ by the rule $(M^Ty)|_{\Gamma^\xi}=(P^\xi y)|_{\Gamma^\xi-0}$ for $0\le\xi\le T$.

Theorem 3. M^T *is a unitary operator.*

This fact is proved with the help of the Riccati equation for the Neumann–to–Dirichlet map related to an elliptic boundary value problem [2–4].

Introduce the space

$$\mathcal{T}(\Gamma)=\{a\in\mathbf{L}_2(\Gamma)\mid\langle a(\gamma),\nu(\gamma)\rangle=0\text{ for all }\gamma\},$$

and define *the image space*

$$\widetilde{\mathcal{F}}^T=\oplus\int_0^T\mathcal{T}(\Gamma)\mathrm{d}\tau=\mathbf{L}_2((0,T);\mathcal{T}(\Gamma)).$$

Note that $\mathcal{F}^T$ and $\widetilde{\mathcal{F}}^T$ are two copies of the same space. Taking $v\in\mathcal{L}^T_\theta$, we define $\pi[v\,|_{\Gamma^\tau}]$ as the result of parallel translating the tangent field $v|_{\Gamma^\tau}$ from Γ^τ onto Γ along the shortest geodesics connecting these surfaces. Define the map $\Pi^T:\mathcal{L}^T_\theta\to\widetilde{\mathcal{F}}^T$ by the equality $(\Pi^Tv)(\tau)=\pi[(J^{1/2}v)|_{\Gamma^\tau}]$, $0\le\tau\le T$, where $J:J|_{\Gamma^\tau}=\frac{d\Gamma^\tau}{d\Gamma}$ is *the geometrical spreading* of the field ν. The operator Π^T is unitary.

The transform $I^T=\Pi^TM^T:\mathcal{J}^T\to\widetilde{\mathcal{F}}^T$ is unitary; an element $\tilde y=I^Ty$ is said to be *image* of a field y. For a smooth y, the equality $\tilde y\,|_{\tau=0}=y_\theta|_\Gamma\in\mathcal{T}(\Gamma)$ holds [4].

6. The operator $\widetilde{\mathrm{rot}}$. We denote by φ the operator in the space $\mathcal{T}(\Gamma)$ which rotates tangent fields on the angle $-\frac{\pi}{2}$ with respect to the induced orientation of the boundary. Denote $\Phi : \widetilde{\mathcal{F}}^T \to \widetilde{\mathcal{F}}^T$ by the equality $\Phi = \oplus \int_0^T \varphi d\tau$. On smooth elements of $\widetilde{\mathcal{F}}^T$, the operator $\widetilde{\mathrm{rot}} = I^T \mathrm{rot}(I^T)^* : \widetilde{\mathcal{F}}^T \to \widetilde{\mathcal{F}}^T$ is well defined.

Theorem 4. *The representation*

$$\widetilde{\mathrm{rot}} = \Phi \frac{d}{d\tau} + A \tag{7}$$

is valid, where $A = \oplus \int_0^T A(\tau) d\tau$, and $A(\tau)$ is a 1-st order PDO in $\mathcal{T}(\Gamma)$ with the principal symbol $\sigma_{A(\tau)}$:

$$\det \sigma_{A(\tau)}(\gamma, k) = [g^{\alpha\beta}(\gamma,\tau) k_\alpha k_\beta]^{\frac{3}{2}}, \quad (\gamma, k) \in \Gamma \times \mathbf{R}^2.$$

Writing (1)–(3) in terms of images, we get the system:

$$\tilde{e}_t = \widetilde{\mathrm{rot}} \tilde{h}, \quad \tilde{h}_t = -\widetilde{\mathrm{rot}} \tilde{e}, \quad (\tau, t) \in (0,T) \times (0,T); \tag{8}$$

$$\tilde{e}\,|_{t=0} = 0, \quad \tilde{h}|_{t=0} = 0, \quad 0 \le \tau \le T; \tag{9}$$

$$\tilde{e}\,|_{\tau=0} = f; \tag{10}$$

in which $\tilde{e} = \tilde{e}^f(\tau,t) = (I^T e^f(\cdot,t))(\tau)$ and $\tilde{h} = \tilde{h}^f(\tau,t) = (I^T h^f(\cdot,t))(\tau)$ are the wave images.

7. The amplitude formula. In the case of the wave equation, the main tool of solving the IP is some relation of geometrical optics which describes a propagation of wave discontinuities; this is *the amplitude formula* (see [1]; Sec. 5.5). In the case of the Maxwell system, a direct analog of the amplitude formula is derived along the same line, and takes the same form:

$$\left((W^T)^* P_\perp^\xi y\right)\Big|_{t=T-\xi-0} = \tilde{y}(\xi), \quad 0 \le \xi \le T$$

(the equality holds in $\mathcal{T}(\Gamma)$) where $y \in \mathcal{J}^T$, $P_\perp^\xi = \mathbf{1} - P^\xi$, and the limit is understood in an appropriate weak sense [4]. Putting $y = e^f(\cdot,T) = W^T f$, we obtain:

$$\left((W^T)^* P_\perp^\xi W^T f\right)\Big|_{t=T-\xi-0} = \tilde{e}^f(\xi, T). \tag{11}$$

8. Visualization of waves. For the IP, the crucial fact is that the operator block on (11) is determined by the response operator.

The bilinear form $c^T[f,g] = (e^f(\cdot,T), e^g(\cdot,T))_{\mathcal{J}^T} = (W^T f, W^T g)_{\mathcal{J}^T}$ is called *the connecting form*; it is densely defined in $\mathcal{F}^T \times \mathcal{F}^T$, positive, and closable.

Lemma 1. *The representation*

$$c^T[f,g] = 2^{-1} \left((S^T)^* R^{2T} S^T f, g\right)_{\mathcal{F}^T} \tag{12}$$

holds, where S^T is the operator extending controls in time from $[0,T]$ onto $[0,2T]$ as odd functions with respect to $t = T$.

The proof is carried on the same scheme as [1] (see [4]).

Let $\mathcal{E}_\flat^T$ be the completion of $\mathrm{Dom}\, W^T$ in the metric $c^T[f,g]$, and $W_\flat^T : \mathcal{F}^T \to \mathcal{E}_\flat^T$ be the embedding of $\mathrm{Dom}\, W^T$ into $\mathcal{E}_\flat^T$. Denoting $\mathcal{E}_\flat^\xi = \mathrm{clos}\, W_\flat^T\left(\mathcal{F}^{T,\xi} \bigcap \mathrm{Dom}\, W^T\right)$, let $E_\flat^\xi$ be the projector in $\mathcal{E}_\flat^T$ onto $\mathcal{E}_\flat^\xi$ and $E_{\flat\perp}^\xi = \mathbf{1} - E_\flat^\xi$.

By definition of the metric in $\mathcal{E}_\flat^T$, the operator $U^T = W^T(W_\flat^T)^{-1}$ is isometric and it may be extended to an isometry of $\mathcal{E}_\flat^T$ onto $\mathcal{E}^T$. The relations $U^T\mathcal{E}_\flat^\xi = \mathcal{E}^\xi$, $U^T E_\flat^\xi = E^\xi U^T = \langle \textit{see } (5)\rangle = P^\xi U^T$ hold and imply that

$$U^T E_{\flat\perp}^\xi (U^T)^* = P_\perp^\xi. \tag{13}$$

Substituting $W^T = U^T W_\flat^T$ and (13) into (11), we obtain

$$\left((W_\flat^T)^* E_{\flat\perp}^\xi W_\flat^T f\right)\Big|_{t=T-\xi-0} = \tilde{e}^f(\xi, T), \quad 0 \le \xi \le T. \tag{14}$$

Since the form c^T and, after it, the operators $W_\flat^T$, $E_{\flat\perp}^\xi$ are determined by the operator R^{2T}, the representation (13) enables an external observer to recover ("to see") the wave image $\tilde{e}^f(\cdot, T)$ for every f. Moreover, on using (6) the observer can visualize simultaneously the image $\widetilde{\operatorname{rot}}\,\tilde{e}^f(\cdot, T)$ as follows:

$$\left[(W^T)^* E_{\flat\perp}^\xi W_\flat^T(\nu \times R^T f_t)\right]\Big|_{t=T-\xi-0} = \left(\widetilde{\operatorname{rot}}\,\tilde{e}^f(\cdot, T)\right)(\xi), \quad 0 \le \xi \le T. \tag{15}$$

9. Reconstruction of the metric. The considerations are summarized by the following procedure solving the IP:

(i) on using (12), find c^T through R^{2T}; then determine the operators $W_\flat^T$, $(W_\flat^T)^*$, and $E_{\flat\perp}^\xi$ for all $0 \le \xi \le T$;

(ii) with the help of representations (14) and (15), recover the operator $\widetilde{\operatorname{rot}}$ by varying f and finding $\tilde{e}^f(\cdot, T)$ and $\widetilde{\operatorname{rot}}\,\tilde{e}^f(\cdot, T)$; then find the operator A by (7);

(iii) from the symbols $\sigma_{A(\tau)}$, recover the components of the tensor i^*g on the cylinder $\Gamma \times [0, T] = \Theta^T$.

The manifold (Θ^T, i^*g) is isometric to (Ω^T, g). The inverse problem is thus solved.

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