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Measurability of self-similar spectral geometries

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Abstract

The notion of a spectral geometry on a compact metric space X is motivated, on the one hand, by tasks of calculus on general metric spaces and, on the other hand, by the notion of a spectral triple playing the role of a "Riemannian manifold" in noncommutative geometry. A spectral geometry M is given by a symmetric subset $B \subset X^2 \setminus \Delta$ which is finite outside of each neighborhood of the diagonal Δ, and defines, via the Dixmier trace Tr_ω, a Radon measure $d_\omega M$ on X converting X into a metric measure space. The geometry M is called ω-measurable, if the measure $d_\omega M$ is finite and independent of the choice of the limiting procedure ω. We prove the ω-measurability of a broad class of self-similar geometries including geometries on each self-similar compact subset in R^n satisfying the standard OSC (Open Set Condition).

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§ 0. Introduction

A spectral geometry on a compact metric space X is given by a symmetric subset $B \subset X^2 \backslash \Delta$ which is finite outside of each neighborhood of the diagonal Δ. This notion is motivated, on the one hand, by tasks of calculus on general metric spaces (see, for example, [BP], [Che], [Ha], [Se]), and, on the other hand, by the notion of a spectral triple playing the role of a "Riemannian manifold" in noncommutative geometry (see [Co], [La2]). Using B, we define a spectral triple (geometry) $M = \{\mathcal{A}, \mathcal{H}, ds\}$ on X (see 1.1), where the algebra $\mathcal{A} = \mathcal{C}(X)$ of the continuous functions $X \to \mathbb{C}$ is represented by multiplication operators $\pi(a)$, $a \in \mathcal{A}$ in a Hilbert space $\mathcal{H}$, the set B is identified with an orthonormal basis of $\mathcal{H}$, and the spectrum of the compact self-adjoint operator $ds : \mathcal{H} \to \mathcal{H}$ (the unit length operator) coincides with $\{\pm|b| \mid b \in B\}$ (here $|b| = d(y, y')$ is the distance between y, y' for $b = (y, y'))$.

In terms of the geometry M one defines: the gradient of a Lipschitz funciton $a \in \mathcal{A}$ as a (bounded) operator $\nabla a = i[\mathcal{D}, \pi(a)]$ in $\mathcal{H}$, where $\mathcal{D} = ds^{-1}$; the Connes metric

$$d_M(x, x') = \sup\{|a(x) - a(x')| \mid a \in \mathcal{A}, \|\nabla a\| \leq 1\}$$

on X; the dimension of M,

$$\dim M = \sup\{p \geq 0 \mid \mathrm{Tr}_\omega(|ds|^p) = \infty\} = \inf\{p \geq 0 \mid \mathrm{Tr}_\omega(|ds|^p) = 0\},$$

where Tr_ω is the Dixmier trace (see § 1.2); the (Radon) measure $d_\omega M$ on X,

$$d_\omega M(a) = \mathrm{Tr}_\omega(\pi(a)|ds|^p)$$

for $a \in \mathcal{A}$, where $p = \dim M$.

The Connes metric d_M dominates the initial metric d on X, $d_M \geq d$, and coincides with d on B, $d_M(y, y') = d(y, y')$ for every $(y, y') \in B$. The natural problem to find B such that $d_M = d$ has an easy solution if one neglects the condition for B to be finite outside of the neighborhoods of the diagonal (see [Pa]), i.e., the compactness of the operator ds. The compactness of ds which plays a fundamental role in this approach might be an obstacle for the coincidence $d_M = d$ of the metrics, and in general case we can only prove the existence of spectral geometries M on X such that the identity map $(X, d) \to (X, d_M)$ is a bi-Lipschitz homeomorphism with the Lipschitz constant arbitrarily close to 1. In the case of an intrinsic metric d the equality $d_M = d$ can always be achieved. This means, in particular, that the spectral geometry M in that case contains the whole information on the geometry of the space X. A spectral geometry contains also an additional information, namely, the dimensions of geometries M, M' and the measures $d_\omega M$, $d_\omega M'$ may be essentially distinct even if their Connes metrics coincide, $d_M = d_{M'}$. In other words, a spectral geometry on X converts X into a metric measure space.

The problem of approximation of d by the Connes metrics d_M of spectral geometries M on X as well as the relations between $\dim M$ and more traditional dimensions of a metric space (the Hausdorff dimension, the Bouligand-Minkowski dimension, the upper and lower box-dimensions) are considered in [Bu]. In this paper, we study the problem of measurability of spectral geometries which can be described as follows. The Dixmier trace Tr_ω is defined with help of a limiting procedure ω which might be, for instance, a nonprincipal ultrafilter on $\mathbb{N}$. Thus the measure $d_\omega M$ might depend on the choice of ω and hence be not computable.

0.1. Definition. A spectral geometry M of a finite dimension p is called *ω-measurable*, if $d_\omega M(a) = \mathrm{Tr}_\omega(\pi(a)|ds|^p)$ is finite and independent of the choice of the limiting procedure ω for each function $a \in \mathcal{C}(X)$.

The problem of ω-measurability of spectral triples is not new, see [Co, Ch.IV], [La2, Remark 4.48(b)]. Here we establish ω-measurability of a broad class of self-similar spectral geometries. In this context, the following notion of measurability of a spectral geometry turns out to be useful and possesses an independent interest.

0.2. Definition. A spectral geometry M of a finite dimension p is called *measurable* if there exists a finite limit

$$dM(a) = \lim_{t\to 0+} t \cdot \mathrm{Trace}\,(\pi(a)|ds|^{p+t})$$

for each $a \in \mathcal{C}(X)$ (here Trace is the usual trace of operators in a Hilbert space).

In this case, dM is, obviously, a positive Radon measure on X. The main results of the paper are the following.

0.3. Theorem. *Every ample self-similar almost free spectral geometry is measurable.*

0.4. Theorem. *Every ample self-similar almost free spectral geometry is ω-measurable.*

A spectral geometry M is *ample* if the diameter of X with respect to the Connes metric d_M is finite. For the definitions of a self-similar and almost free self-similar (a.f.s.s. for brevity) spectral geometry see § 2.1. We note here that a.f.s.s. geometries always exist on each self-similar compact space X satisfying OSC (Open Set Condition, which is a standard separation condition in theory of self-similar sets, see [Hu], [Fa2]), in particular, they exist on the standard 1/3-Cantor set, the von Koch curve, the Sierpinski gasket and carpet. At the same time, the a.f.s.s.-condition being analogous to OSC is considerably less restrictive, and a.f.s.s. geometries may exist on self-similar sets with substantial overlaps, i.e., which are far from satisfying OSC. The corresponding examples are given in § 4.

Furthermore, one can weaken the a.f.s.s.-condition in theorems 0.3, 0.4. However, this condition is essential for the proof of the following theorem. Namely, we obtain an analog of the Weyl's asymptotic formula for the eigenvalue counting function of the operator $|\mathcal{D}| = |ds|^{-1}$ of an a.f.s.s. spectral geometry. Let $N(t)$ be the number of eigenvalues (counted according to multiplicity) of $|\mathcal{D}|$ not exceeding t. In other words, $N(t)$ is the number of elements of $B_t = \{b \in B \,|\, |b| \geq 1/t\}$. Here, for simplicity, we formulate the result for the non-arithmetic case (see § 7), i.e., for the generic case. In general case see theorem 7.5.

0.5. Theorem. *Let $M = \{\mathcal{A}, \mathcal{H}, ds\}$ be an a.f.s.s. spectral geometry on a compact metric space X, $p = \dim M$. Then for the eigenvalue counting function $N(t)$ of $|\mathcal{D}|$ in the non-arithmetic case we have*

$$N(t) = d_\omega M(X) \cdot t^p + o(t^p)$$

as $t \to \infty$, where $d_\omega M(X) = d_\omega M(1)$ is the p-dimensional volume of X with respect to the geometry M.

The preceding results are more related to the properties of a spectral geometry than to the space X itself. The following result establishes a relation between the measure $d_\omega M$ and the Hausdorff measure H^p (of dimension p) on X.

0.6. Theorem. *Let M be an ample a.f.s.s. spectral geometry of dimension p on a compact metric space X such that the support of $d_\omega M$ coincides with X. Then $H^p \leq c \cdot d_\omega M$ for some constant $c \geq 0$, in particular, the Hausdorff dimension $\dim_{\mathrm{H}} X \leq p$. In that case if $d_M = d$ and $H^p(X) > 0$, then $H^p = c \cdot d_\omega M$, where the constant $c > 0$ is independent of the choice of the limiting procedure ω.*

The proofs of these theorems use the tools traditional for the geometry of self-similar sets: the contraction method [Hu] adapted to spectral geometries, the renewal theorem (see, for instance, [Fa2]), and also a Tauberian type theorem.

The paper is organized as follows. In §1, we give definitions of a spectral geometry on a compact metric space, its dimension and discuss their properties necessary for the work. The definition and examples of self-similar geometries are considered in §2. In §3, we compile technical lemmas needed for the proof of the main results. §4 is devoted to computation of the dimension of a self-similar geometry. In §5, using the contraction method, we prove theorem 0.3. Theorem 0.4 is proved in §6. The key Tauberian condition needed for its proof (proposition 6.2.1) is established in the next §7 with help of the renewal theorem. Here we also prove theorem 0.5. Finally theorem 0.6 is proved in §8.

§ 1. Spectral geometries on a compact metric space

A spectral geometry on a compact metric space X can loosely be described as a discrete approximation of X, for which geometric properties of X are studied using the asymptotic behavior of various characteristics defined via finite subsets. The main element of a spectral geometry is its *support,* that is a subset $B \subset X^2 \setminus \Delta$ satisfying the following conditions

(i) B is symmetric, i.e., $b \in B$ if and only if $\gamma(b) \in B$, where $\gamma : X^2 \to X^2$ is the canonical involution, $\gamma(x, x') = (x', x)$;

(ii) the set $B_t = \{b \in B \,|\, |b| \geq t\}$ is finite for every $t > 0$, where $|b| = d(y, y')$ is the distance between y, $y' \in X$ for $b = (y, y')$.

Sets with these properties naturally arise in various situations. Let us give two examples related to the limit sets of discrete groups.

Let $F = H^2/G$ be a closed surface of constant curvature -1. Every nontrivial element $g \in G$ of its fundamental group considered as an isometry $g : H^2 \to H^2$, possesses two fixed points at the boundary at infinity $X = \partial_\infty H^2$, which are the ends of the axis of g. Then the set $B = B(g)$ consisting of pairs of the axes ends of the isometries $\sigma g \sigma^{-1}$, $\sigma \in G$ satisfies the conditions (i), (ii) (we assume that a metric, for example, the standard metric, on $S^1 = X$ is fixed). Here, instead of one element g, one can take any finite collection of elements.

Now let F be a compact 3-dimensional hyperbolic manifold with nonempty geodesic boundary. Its universal cover $\widetilde{F} \subset H^3$ can be obtained by removing from H^3 of a countable disjoint collection of open half-spaces (whose boundaries cover the boundary ∂F). Thus the limit set $X \subset \partial_\infty H^3$ of the group $G = \pi_1(F)$ is homeomorphic to the Sierpinski carpet. Choosing as in the former example a finite collection of closed geodesics on each boundary component of F, we arrive at the set $B \subset X^2 \setminus \Delta$ with properties (i), (ii).

1. Definition of a spectral geometry. Let X be a compact metric space. Assume that a set $B \subset X^2 \setminus \Delta$ satisfies the conditions (i), (ii). We identify B with an orthonormal basis of a Hilbert space $\mathcal{H}$. The spectral geometry M with the support B is given by the representation π in $\mathcal{H}$ of the algebra $\widetilde{\mathcal{A}} = \mathcal{C}(X^2)$ of continuous functions on X^2, $\pi(a)(b) = a(b)b$ for $a \in \widetilde{\mathcal{A}}$, $b \in B$ and by the (*unit length*) operator $ds : \mathcal{H} \to \mathcal{H}$, $ds(b) = |b| \cdot \gamma(b)$.

It follows from the properties of B that ds is an injective, compact, self-adjoint operator whose spectrum coincides with $\{\pm|b| \,|\, b \in B\}$. Though the self-adjoint operator $\mathcal{D} = ds^{-1}$ is unbounded if B is infinite, the operator $\nabla a = i[\mathcal{D}, \pi(a)]$, $\nabla a(b) = i|b|^{-1}\left(a(b) - a \circ \gamma(b)\right) \cdot \gamma(b)$ for a Lipschitz function $a \in \widetilde{\mathcal{A}}$ can be extended to a bounded operator in $\mathcal{H}$ since

$$\|\nabla a\| = \sup_{b \in B} |a(b) - a \circ \gamma(b)| \,/ |b| \leq \mathrm{Lip}\ (a)$$

(here we consider the metric $d((x,x'),(y,y')) = \max\{d(x,y), d(x',y')\}$ on X^2).

Identifying $\mathcal{C}(X)$ with the subalgebra $\mathcal{A}_1 \subset \widetilde{\mathcal{A}}$ of functions $a(x,x') = a(x)$, we obtain the Connes metric on X (see [Co]),

$$d_M(x,x') = \sup\{|a(x) - a(x')| \mid a \in \mathcal{C}(X), \|\nabla a\| \leq 1\},$$

which, obviously, dominates over the initial metric d, $d_M \geq d$. It is shown in [Bu] that for each $\epsilon > 0$ there exists a spectral geometry M on X with $d_M \leq (1+\epsilon)d$ (see remark 1.3), and if d is intrinsic, then also with $d_M = d$.

1.1. Example. For $X = [0,1]$, let us consider similitudes j_α, $j_\beta : X \to X$, $j_\alpha = hx$, $j_\beta = hx + 1 - h$, where $h \in (0,1)$. We define the spectral geometry M on X with the support

$$B = \cup_{w \in W} \mathbf{j}_w(\Delta B),$$

where ΔB consists of two pairs $(0,1)$ and $(1,0)$, W is the set of all (finite) words in the alphabet $\{\alpha, \beta\}$, $\mathbf{j}_w = (j_w, j_w) : X^2 \to X^2$ and j_w is the composition of the maps j_α, j_β corresponding to the word w. Then for $h \geq 1/2$, as easy to see, we have $d_M = d$.

2. The Dixmier trace. We define the dimension of a spectral geometry M and the associated with M Radon measure $d_\omega M$ on X via the Dixmier trace which serves as an integral in noncommutative geometry. Here we collect some of the basic properties of the Dixmier trace needed for the paper. For more details see [Di] and [Co]. Roughly speaking, the Dixmier trace of a positive compact operator T in $\mathcal{H}$ is the coefficient of logarithmic divergency of partial sums $\sigma_N(T) = \sum_{n=1}^N \mu_n$ for the trace of T, where eigenvalues μ_n of T arranged in decreasing order and repeated as many times as their multiplicity,

$$\mathrm{Tr}_\omega(T) = \lim_{N\to\infty} \sigma_N(T)/\log N,$$

if the limit on the right hand side exists. In general case, the limit may not exist, and one applies a limiting procedure ω to the sequence $s_N(T) = \sigma_N(T)/\log N$, which is an invariant mean ω on $\mathbb{R}_+^*$, the multiplicative group of positive real numbers, applied to the function $f_s(t) = s_N(T)$, $t \in (N-1, N]$. If $\sigma_N(T) = O(\log N)$, then it gives a finite number

$$\mathrm{Tr}_\omega(T) = \lim_\omega s_N(T) = \omega(f_s).$$

Application of the mean ω provides additivity $\mathrm{Tr}_\omega(T+T') = \mathrm{Tr}_\omega(T) + \mathrm{Tr}_\omega(T')$ for positive T, $T' \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$. Here $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ is the ideal of compact operators T in $\mathcal{H}$, whose charateristic numbers $\mu_n(T)$ (that is the eigenvalues of $|T| = \sqrt{(T^*T)}$) satisfy the condition $\sigma_N(T) = O(\log N)$. Then Tr_ω extends by linearity to all of $\mathcal{L}^{(1,\infty)}(\mathcal{H})$, and the following properties hold true

(i) positivity: if $T \geq 0$, then $\mathrm{Tr}_\omega(T) \geq 0$;
(ii) linearity: $\mathrm{Tr}_\omega(\lambda_1 T_1 + \lambda_2 T_2) = \lambda_1 \mathrm{Tr}_\omega(T_1) + \lambda_2 \mathrm{Tr}_\omega(T_2)$ for each numbers λ_1, $\lambda_2 \in \mathbb{C}$ and operators T_1, $T_2 \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$;
(iii) covariance: if S is any bounded operator and $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, then $\mathrm{Tr}_\omega(ST) = \mathrm{Tr}_\omega(TS)$;
(iv) locality: $\mathrm{Tr}_\omega(T) = 0$, if $\sigma_N(T) = o(\log N)$.

The Dixmier trace playing due (iv) the role of an integral may depend on the choice of the limiting procedure ω, and the aim of the paper is to prove that a number of characteristics of spectral geometries defined via Tr_ω are independend of ω. This is easy to show for the dimension.

3. Dimension of a spectral geometry. We present three definition of the dimension of a spectral geometry M, which all give one and the same number. The first definition is similar to the one of the Hausdorff dimension as the rarefaction index of a family of measures

$$\dim_1 M = \sup\{p \geq 0 \mid \mathrm{Tr}_\omega(|ds|^p) = \infty\} = \inf\{p \geq 0 \mid \mathrm{Tr}_\omega(|ds|^p) = 0\}.$$

The second definition is analogous to the one of the upper box-dimension (see [KT], [Fa2], [Tr]), and related to the asymptotic behavior of the cardinality of the set B_t, namely

$$\dim_2 M = \limsup_{t\to 0} \frac{\log|B_t|}{\log 1/t}.$$

Finally we put (cp. [La1, Remark 3.2(b)])

$$\dim_3 M = \inf\{p \geq 0 \mid \mathrm{Trace}\,(|ds|^p) < \infty\}.$$

1.2. Proposition. *All of the numbers* $\dim_1 M$, $\dim_2 M$, $\dim_3 M$ *coincide. Their common value* $\dim M$ *is called the dimension of the geometry* M.

Proof. Put $p_i = \dim_i M$, $i = 1, 2, 3$. Since $\mathrm{Trace}\,(|ds|^{p_3+\epsilon}) < \infty$ for each $\epsilon > 0$, we have due to the property (iv) that $\mathrm{Tr}_\omega(|ds|^{p_3+\epsilon}) = 0$. Hence, $p_1 \leq p_3$.

The eigenvalue set of $|ds|$ coincides with $\{|b| \mid b \in B\}$. We arrange it in decreasing order according to multiplicity $\mu_1 \geq \mu_2 \geq \ldots$. Fix $\epsilon > 0$ and put $t = \mu_n$. Then $n \leq |B_t| \leq 1/t^{p_2(1+\epsilon)}$, therefore $\mu_n^{p_2(1+\epsilon)} \leq 1/n$ for all sufficiently large n. Thus $\mathrm{Trace}\,(|ds|^{p_2(1+\epsilon)^2}) < \infty$ and hence $p_3 \leq p_2$.

Furthermore, $\sigma_N(|ds|^{p_1+\epsilon}) = \sum_{n=1}^N \mu_n^{p_1+\epsilon} \leq C \cdot \log N$. Using monotonicity of the sequence μ_n, we obtain $\sigma_N(|ds|^{p_1+\epsilon}) \geq N \cdot \mu_N^{p_1+\epsilon}$, hence $\mu_n^{p_1+\epsilon} = O(\log n/n)$ for each $\epsilon > 0$. It immediately follows that $\mu_n^{p_1+\epsilon} = o(1/n)$, and therefore $p_2 \leq p_1$. $\square$

1.3. Remark. We define the density of a spectral geometry M putting

$$\mathrm{dens} M = \inf_{z\in X^2\setminus\Delta} \sup_{b\in B} |z|/d(z,b),$$

where B is the support of M, and the local density $\mathrm{dens}_{\mathrm{loc}} M = \lim_{t\to 0} \mathrm{dens}(M, U_t)$, where $\mathrm{dens}(M, U_t)$ is the density of M in the neighborhood $U_t = \{z \in X^2 \mid |z| \leq t\}$ of the diagonal Δ. For infinite X the density of M is finite. It is shown in [Bu] that there exist spectral geometries on X with arbitrarily large density as well as with infinite local density. Furthermore, if $D = \mathrm{dens} M > 2$, then for the initial metric d of X and its Connes metric d_M one has

$$d \leq d_M \leq \frac{D+2}{D-2} d.$$

Putting $\dim_{\mathrm{S}} X = \liminf\{\dim M \mid \mathrm{dens}_{\mathrm{loc}}(M) \to \infty\}$, we obtain a bi-Lipschitz invariant of X, which for a manifold coincides with its dimension, and in general case dominates well known dimensions: the Hausdorff dimension, the Bouligand-Minkowski dimension, the packing, upper and lower box-dimensions.

The notion of self-similarity plays an important role in various fields, and there is a number of different definitions of that notion, see, for instance, [Mo], [Hu], [Fa1,2]. Here we introduce the notion of self-similar spectral geometry and discuss a number of examples.

1. Definition of a self-similar geometry. We start with the notion of a similar contraction of a spectral geometry. A linear bounded operator $J : \mathcal{H} \to \mathcal{H}$ in a Hilbert space $\mathcal{H}$ is said to be *isometric* if $J^*J = 1$. This is equivalent to $\|J\xi\| = \|\xi\|$ for each $\xi \in \mathcal{H}$. (The distinction from a unitary operator is that for an isometric operator one does not require $JJ^* = 1$, i.e., surjectivity of J).

Let $M = \{\mathcal{A}, \mathcal{H}, ds\}$ be a spectral geometry on a compact metric space X. There are two ways to identify the algebra $\mathcal{A} = \mathcal{C}(X)$ of the continuous functions on X with a subalgebra in $\widetilde{\mathcal{A}} = \mathcal{C}(X^2)$. Namely, they are associated with the projections $X \times X \to X$ on the first and the second factor accordingly. We denote these subalgebras as $\mathcal{A}_1$, $\mathcal{A}_2$. In other words, a function $a \in \widetilde{\mathcal{A}}$ lies in $\mathcal{A}_1$ if and only if $a(x, x') = a(x)$ is independent of the second argument. A similar condition is fulfilled for $\mathcal{A}_2$. The canonical involution γ interchanges these subalgebras, $a \mapsto a \circ \gamma$.

An isometric operator $J : \mathcal{H} \to \mathcal{H}$ is called *a similar contraction* of a spectral geometry M, if

(1) $ds \circ J = h \cdot J \circ ds$ for some *contraction coefficient* $h \in (0, 1)$;
(2) the map $\pi(a) \mapsto J^* \circ \pi(a) \circ J$ is a homomorphism of the algebra $\pi(\widetilde{\mathcal{A}})$ leaving invariant the subalgebra $\pi(\mathcal{A}_1)$.

A subspace $L \subset \mathcal{H}$ is said to be *p-negligible*, $p \geq 0$, if the operator $|ds|^p \circ P_L : \mathcal{H} \to \mathcal{H}$ belongs to the ideal $\mathcal{L}^1(\mathcal{H})$ of compact operators T in $\mathcal{H}$ with $\mathrm{Trace}\,(T) < \infty$. Here $P_L : \mathcal{H} \to \mathcal{H}$ is the orthogonal projection on L. In that case we have $\mathrm{Tr}_\omega(|ds|^p \circ P_L) = 0$, and this property is independend of the choice of the limiting procedure ω.

For a finite nonempty collection $\mathcal{S} = \{J_\alpha \,|\, \alpha \in A\}$ of similar contractions of a geometry M we define $p = p(\mathcal{S})$ as a unique nonnegative number with

$$\sum_{\alpha \in A} h_\alpha^p = 1, \tag{\#}$$

where h_α is the contraction coefficient for J_α, $0 < h_\alpha < 1$.

2.1. Definition. A spectral geometry $M = \{\mathcal{A}, \mathcal{H}, ds\}$ on a compact metric space X is said to be *self-similar* if there is a collection of its similar contractions $\mathcal{S} = \{J_\alpha \,|\, \alpha \in A\}$ such that

(2.1.1) the subspace $\cap_{\alpha \in A} \ker J_\alpha^*$ is p-negligible for $p = p(\mathcal{S})$.

The collection $\mathcal{S}$ is called *the self-similarity structure* of M, the number $\dim_{\mathcal{S}} M = p(\mathcal{S})$ is called *the similarity dimension* of M, and the set A is the alphabet of the self-similarity structure. In the case when all contraction coefficients of $\mathcal{S}$ coincide, $h_\alpha = h$ for all $\alpha \in A$, for the similarity dimension we, obviously, have

$$\dim_{\mathcal{S}} M = \frac{\log |A|}{\log(1/h)}.$$

As we shell see, the subspace $\cap_{\alpha \in A} \ker J_\alpha^*$ in a sense governs the self-similarity structure, and the condition for it to be p-negligible bounds the size of M from above. The size of M is bounded from below if, for example, M possesses a free subspace, which is defined as follows.

Any spectral geometry M defines *a spectral* measure P on X^2 (see, for example, [N, Ch. IV]), which assigns to each Borel subset $U \subset X^2$ the orthogonal projection $P(U)$ on the subspace in $\mathcal{H}$

spanned by $B \cap U$. The subspaces in $\mathcal{H}$ of type $P(U)(\mathcal{H})$ are called *eigenspaces* for M. Eigenspace $L \subset \mathcal{H}$ for M is said to be *free* if all subspaces $J_\alpha(L)$ lie in L and are mutually orthogonal for different $\alpha \in A$.

2.2. Definition. A self-similar geometry M is said to be *free* if $\mathcal{H}$ is free, and *almost free* (a.f.s.s. for brevity) if there exists a nontrivial free subspace $L \subset \mathcal{H}$ such that its orthogonal complement is p-negligible, where $p = \dim_{\mathcal{S}} M$.

These freedom conditions are variants of the standard OSC in theory of self-similar sets (see [Hu]) and they bound the size of M from below.

As a motivation of these definitions let us consider the following example. Assume that there is a finite collection $\mathcal{S} = \{j_\alpha \mid \alpha \in A\}$ of similar contractions $j_\alpha : X \to X$ of X with coefficients $h_\alpha \in (0,1)$. These contractions induce maps $\mathbf{j}_\alpha : X^2 \to X^2$, $\mathbf{j}_\alpha(x,x') = (j_\alpha(x), j_\alpha(x'))$. Let p be the similarity dimension, i.e., the solution to (#). Choosing a p-negligible subset $\Delta B \subset X^2$, we obtain a geometry M with the support $B = \cup_{w \in W} \mathbf{j}_w(\Delta B)$ (we suppose that the set W of words in the alphabet A contains the empty word). Isometric operators $J_\alpha : \mathcal{H} \to \mathcal{H}$, $J_\alpha(b) = \mathbf{j}_\alpha(b)$, $b \in B$, satisfy the conditions $ds \circ J_\alpha = h_\alpha \cdot J_\alpha \circ ds$ and $J_\alpha^* \circ \pi(a) \circ J_\alpha = \pi(a \circ \mathbf{j}_\alpha) \in \pi(\mathcal{A}_1)$ for each function $a \in \mathcal{A}_1$, and they are, therefore, similar contractions (with the coefficients h_α) of M. The subspace $\cap_{\alpha \in A} \ker J_\alpha^* = P(\Delta B)(\mathcal{H})$ is p-negligible according to the choice of ΔB.

OSC for $\mathcal{S}$ means the existence of a nonempty open subset $U \subset X$ with disjoint $j_\alpha(U) \subset U$ for different $\alpha \in A$. If this condition is fulfilled, then choosing $\Delta B \subset U^2$ we obtain a free self-similar geometry M.

2. Examples of self-similar geometries. We give here a couple of examples of self-similar geometries. Their construction follows the scheme above.

2.3. Example. We fix $p, q \in \mathbb{N}$, $q \geq 2$ and for $n = 0, 1, \ldots$ consider the standard cell subdivision of the cube $X = I^p$ into q^{np} congruent p-dimensional subcubes with the edge lengths $1/q^n$. Let $B(n) \subset X^2 \setminus \Delta$ be the set of the pairs representing the oriented edges of that subcubes. We put $B = \cup_{n \geq 0} B(n)$ and consider the spectral geometry $M = M(p,q)$ on X with the support B. It is self-similar: as an alphabet A we take the set $\{1, 2, \ldots, q^p\}$ and assign to each $\alpha \in A$ a similitude $j_\alpha : X \to X$ mapping X into one of q^p subcubes of its first subdivision. We assume that each similitude j_α is composed from a homothety and a parallel translation. The contraction coefficient h_α for each similitude is equal to $1/q$. Thus $\dim_{\mathcal{S}} M = p$.

For $n \geq 1$ the set $B(n)$ can be described recurrently as

$$(\star) \qquad B(n) = \bigcup_{\alpha \in A} \mathbf{j}_\alpha(B(n-1)).$$

It follows from $(\star)$ that the generating set $\Delta B = B(0)$ is finite and thus p-negligible.

The geometry M is almost free: one can take as a free set U the set of all elements $b = (y, y') \in B$ with at least one of the points y, y' lying inside of I^p. Its complement $V = B \setminus U$ consists of the pairs of points lying on the boundary ∂I^p and thus it is p-negligible.

A disadvantage of $M(p,q)$ for $p \geq 2$ is that its Connes metric is different from the initial metric of X since this geometry has a weak control over the functions whose gradients directed, for instance, along diagonals of the cube. At the same time, there is a wide range of possibilities to construct a self-similar geometry on $X = I^p$ whose Connes metric coincides with the initial one. To this end, it suffices to choose a generating subset $\Delta B \subset B$ such that it would be p-negligible and the directions of its elements would be everywhere dense. For simplicity, we consider an appropriate modification of $M(p,q)$.

We define a set $E = \cup_{n\geq 0} E(n)$, which will control all directions in $X = I^p$, recurrently by

$$E(n) = E'(n) \cup \cup_{\alpha\in A} \mathbf{j}_\alpha \left(E(n-1)\right), \quad n \geq 0,$$

where $E(-1) = E'(0) = \emptyset$, and for $n \geq 1$ the set $E'(n) \subset X^2 \setminus \Delta$ consists of pairs (y_0, y), (y, y_0) with $y \in Y_n$, $|y_0 y|_\infty \leq n/2q^n$, $y_0 = (1/2, \dots, 1/2)$ is the center of X. Here Y_n is the vertex set of nth subdivision, $|\ |_\infty$ the L^∞-metric on X.

Now consider the modification $M'(p,q)$ of the spectral geometry $M(p,q)$, whose support is $B' = B \cup E$. The self-similarity structure for $M'(p,q)$ is an obvious modification of the corresponding structure for $M(p,q)$, in particular, $\dim_{\mathcal{S}} M = p$. The condition to be almost free is, obviously, kept unchanged. Let us check that its generating set $\Delta B = B(0) \cup \cup_{n\geq 0} E'(n)$ is p-negligible. The set $E'(n)$ contains at most $2(n+1)^p$ elements, and their lengths are at most $p^{1/2} n/2q^n$. Thus

$$\text{Trace}\left(|ds|^p \circ P(\Delta B)\right) \leq C(p) \sum_{n\geq 0} n^{2p}/q^{np} < \infty,$$

and the set ΔB is p-negligible.

Clearly, the directions of elements of B' are everywhere dense in the direction set of X. This easy implies (cp. example 1.1) that the Connes metric of $M'(p,q)$ coincides with the initial metric on X.

2.4. Example: the Sierpinski carpet. Let $X_0 \subset \mathbb{R}^2$ be the unit square. Consider its subdivision into 9 congruent subsquares with the side length 1/3. Removing the open middle square Q_0, we obtain the compact set $X_1 \subset X_0$ consisting of 8 copies of $3^{-1} X_0$. Applying to each of them this procedure and iterating it, we obtain the Sierpinski carpet $X = \cap_{n\geq 0} X_n$. We consider on X the intrinsic metric induced from $\mathbb{R}^2$. For each element α from the alphabet $A = \{1, 2, \dots, 8\}$ we assign a similitude $j_\alpha : X_0 \to X_0$ mapping X_0 into one of 8 squares, which form X_1. Let $\mathbf{j}_\alpha : X_0^2 \to X_0^2$ be the induced map. We take as $B(0)$ the set of the ends of the oriented sides of the boundary of X_0, and for $n \geq 1$ put $B(n) = \cup_{\alpha\in A} \mathbf{j}_\alpha \left(B(n-1)\right)$. Now we define M as the spectral geometry on X with the support $B = \cup_{n\geq 0} B(n)$. Its generating set $\Delta B = B(0)$ is finite and thus negligible. As in example 2.3, one can take as a free set U the set of all elements $b = (y, y') \in B$ with at least one of the points y, y' lying inside of X_0. Thus M is self-similar and almost free. The contraction coefficient is $h_\alpha = 1/3$ for all $\alpha \in A$, hence $\dim_{\mathcal{S}} M = \log_3 8$. To compute the dimension p of M, we use the recurrent formula

$$|B(n)| = 8\left(|B(n-1)| - 2 \cdot 3^{n-1}\right).$$

With its help we obtain $|B(n)| = 8^{n+1} - 2\sum_{k=1}^n 3^{n-k} \cdot 8^k$, therefore $|B(n)| \sim 4.8 \cdot 8^n$. Since the length of each element $b \in B(n)$ equals 3^{-n}, we find using the definition of $\dim_2 M$ that $p = \log_3 8 = \dim_{\mathcal{S}} M$ coincides with the Hausdorff dimension of X. Applying the asymptotic formula for $|B(n)|$ we compute the p-dimensional volume of X with respect to the measure dM,

$$\begin{aligned} dM(1) &= \lim_{t\to 0+} t \cdot \text{Trace}\left(|ds|^{p+t}\right) \\ &= \lim_{t\to 0+} t \cdot \sum_{n\geq 0} (3^{-n})^{p+t} \cdot 4.8 \cdot 8^n = 4.8/\log 3. \end{aligned}$$

The Sierpinski carpet contains segments, which are parallel to no side of the initial square. For example, any segment connecting a vertex of the square with the middle point of a not adjacent side lies in X. It follows easily that the Connes metric of M is different from the intrinsic metric on X. The reason is that the Sierpinski carpet is not a “finitely ramified” fractal (see [Ki1,2], [La2]). Nevertheless, it is not difficult to find a self-similar geometry on X whose Connes metric coincides with the one of X using a construction analogous to example 2.3. We leave details to the reader.

§ 3. Properties of similar contractions and self-similar geometries

Here we discuss various properties of similar contractions and self-similar geometries which we need to compute the dimension and prove measurability of a.f.s.s. geometries.

3.1. Lemma. *Let $J : \mathcal{H} \to \mathcal{H}$ be a similar contraction of a spectral geometry $M = \{\mathcal{A}, \mathcal{H}, ds\}$. Then $J(\mathcal{H})$ is an eigenspace for M.*

Proof. Recall that $\mathcal{H} = J(\mathcal{H}) \oplus \ker J^*$. If the assertion is false, then there is $b \in B$ such that the subspace $L_b = P(b)(\mathcal{H})$ lies neither in $J(\mathcal{H})$ not in its orthogonal complement $\ker J^*$. Choose $\xi \in L_b$ with nonzero orthogonal projections ξ_J on $J(\mathcal{H})$ and ξ_{ker} on $\ker J^*$, $\xi = \xi_J + \xi_{\text{ker}}$. Let L be the *real* linear hull of ξ_J and ξ_{ker}. Then, obviously, we have $\xi_J = \lambda\xi + \eta$, $\xi_{\text{ker}} = (1-\lambda)\xi - \eta$, where $0 < \lambda < 1$ and $\eta \in L$ is orthogonal to ξ.

The points of B are isolated in X^2, thus there are functions a, $a' \in \widetilde{\mathcal{A}}$ with $a(b) = (\lambda-1)/\lambda$, $a'(b) = \lambda/(\lambda-1)$ and $a(b') = a'(b') = 1$ for the remaining $b' \in B$. Then $\pi(a)\xi = \frac{\lambda-1}{\lambda}\xi$, $\pi(a')\xi = \frac{\lambda}{\lambda-1}\xi$. Since η is orthogonal to ξ, we have $\eta \in L_{B\setminus b}$ and, consequently, $\pi(a)\eta = \pi(a')\eta = \eta$. Whence $\pi(a)\xi_J = (\lambda-1)\xi + \eta = -\xi_{\text{ker}}$ and $\pi(a')\pi(a)\xi_J = \xi_J$.

Since $\xi_J \in J(\mathcal{H})$ is nonzero, there is a nonzero $\zeta \in \mathcal{H}$ with $J(\zeta) = \xi_J$. Therefore, $J^*\pi(a)J(\zeta) = -J^*(\xi_{\text{ker}}) = 0$ and $J^*\pi(a')\pi(a)J(\zeta) = J^*(\xi_J) = \zeta \neq 0$. This contradicts the fact that the map $\pi(a) \mapsto J^* \circ \pi(a) \circ J$ is a homomorphism of $\widetilde{\mathcal{A}}$. $\square$

The canonical involution $\gamma : X^2 \to X^2$ induces a self-adjoint involution $\mathcal{H} \to \mathcal{H}$, for which we use the same notation γ. It easily follows from definitions that $\gamma = |ds|^{-1} \circ ds$.

3.2. Lemma. *Every similar contraction $J : \mathcal{H} \to \mathcal{H}$ commutes with the involution γ.*

Proof. We first note that $ds^{-1} \circ J = h^{-1} \cdot J \circ ds^{-1}$. Indeed,

$$ds \circ ds^{-1} \circ J = J = J \circ ds \circ ds^{-1} = h^{-1} \cdot ds \circ J \circ ds^{-1}.$$

Multiplying by ds^{-1} from the left, we obtain what is required. Therefore, $|ds|^{-1} \circ J = h^{-1} \cdot J \circ |ds|^{-1}$. Thus

$$\gamma J = |ds|^{-1} ds J = |ds|^{-1} h J ds = J |ds|^{-1} ds = J\gamma. \quad \square$$

3.3. Corollary. *The map $\pi(a) \mapsto J^* \circ \pi(a) \circ J$ leaves invariant the subalgebra $\pi(\mathcal{A}_2) \subset \pi(\widetilde{\mathcal{A}})$.*

Proof. Indeed, $\pi(\mathcal{A}_2) = \gamma^*\pi(\mathcal{A}_1)\gamma$, and the assertion follows from property (2) of similar contractions and lemma 3.2. $\square$

3.4. Lemma. *The subspace $J(L)$ is an eigenspace for every eigenspace $L \subset \mathcal{H}$ for M.*

Proof. A least eigenspace $\mathrm{E}(L')$ containing L' is called *the eigenhull* of a subspace $L' \subset \mathcal{H}$. A subspace L' is an eigenspace for M if and only if $\pi(a)(L') \subset L'$ for every function $a \in \widetilde{\mathcal{A}}$. Thus $\mathrm{E}(L')$ coincides with the linear hull of the subspaces $\pi(a)(L')$, $a \in \widetilde{\mathcal{A}}$.

It follows from lemma 3.1 that $\mathrm{E}(J(L)) \subset J(\mathcal{H})$, in particular, the operator J^* is injective on $\mathrm{E}(J(L))$. On the other hand, $J^*\pi(a)J(L) = \pi(a')(L) \subset L$. Thus

$$L \supset J^*(\mathrm{E}(J(L))) \supset J^*J(L) = L$$

and, hence, $J^* : \mathrm{E}(J(L)) \to L$ is an isomorphism. Therefore $\mathrm{E}(J(L)) = J(L)$. $\square$

3.5. Lemma. *There exists an injective map* $\mathbf{j} : B \to B$ *commuting with the canonical involution* γ *on* B *such that* $J \circ P(U) = P(\mathbf{j}(U)) \circ J$ *for each* $U \subset B$.

Proof. It follows from the properties of the support B of the spectral measure P that for each U, $U' \subset B$ the projections $P(U)$, $P(U')$ coincide if and only if $U = U'$. Thus by lemma 3.4, for every $b \in B$ there is a nonempty $\mathbf{j}(b) \subset B$ with $J \circ P(b) = P(\mathbf{j}(b)) \circ J$. We show that $\mathbf{j}(b)$ is an one-point subset. For brevity, we use the notation $L_U = P(U)(\mathcal{H})$ for $U \subset B$. Given $a \in \widetilde{\mathcal{A}}$, the operator $\pi(a') = J^* \circ \pi(a) \circ J$ belongs to $\pi(\widetilde{\mathcal{A}})$ and thus acts on L_b as a homothety with the coefficient $a'(b)$, $\pi(a')\xi = a'(b)\xi$ for each $\xi \in L_b$. Assume that $\mathbf{j}(b)$ contains *different* points b', b''. There is a function $a \in \widetilde{\mathcal{A}}$ with $a(b') \neq a(b'')$. Then $\pi(a)$ is not a homothety on $L_{b' \cup b''} = L_{b'} \oplus L_{b''} \subset J(L_b)$ being the homothety with the coefficient $a(b')$ on $L_{b'}$ and the homothety with the coefficient $a(b'')$ on $L_{b''}$. The operator J^* isometrically maps back $L_{b' \cup b''}$ into L_b, $J^*(L_{b' \cup b''}) = L' \oplus L'' \subset L_b$. Thus $\pi(a')$ cannot be a homothety on L_b, a contradiction. Therefore, $\mathbf{j}(b)$ is an one-point set, and the map $\mathbf{j} : B \to B$ satisfies the condition $J \circ P(U) = P(\mathbf{j}(U)) \circ J$ for each $U \subset B$.

The map $\mathbf{j}$ is injective since if $\mathbf{j}(b) = \mathbf{j}(b')$, then

$$J \circ P(b) = P(\mathbf{j}(b)) \circ J = P(\mathbf{j}(b')) \circ J = J \circ P(b'),$$

whence $P(b) = P(b')$ by isometricity of J and, consequently, $b = b'$. The fact that $\mathbf{j}$ commutes with the canonical involution γ follows from lemma 3.2. □

The map $\mathbf{j}$ is said to be *adjoint* to J.

3.6. Lemma. *Assume that a geometry* M *is ample, i.e.,* $\operatorname{diam}_M(X) < \infty$. *Let* J *be a similar contraction of* M, $\mathbf{j} : B \to B$ *the adjoint to it map. Then there exists a continuous* $j : X \to X$, *which is a homothety with respect to the Connes metric* d_M, *such that* $\mathbf{j}(b) = (j(y), j(y'))$ *for each* $b = (y, y') \in B$.

Proof. The projection $Y \subset X$ of B on a factor X is independent of the choice of a factor in $X \times X$. Since M is ample, it easily follows that Y is dense in X.

The operator $\pi(a)$ depends only on the restriction $a|B$ for each $a \in \widetilde{\mathcal{A}}$. Thus for $a' \in \widetilde{\mathcal{A}}$ with $J^* \circ \pi(a) \circ J = \pi(a')$ we have $a'|B = a \circ \mathbf{j}$. If $a \in \mathcal{A}_i$, then $a' \in \mathcal{A}_i$, $i = 1, 2$, hence the first (second) coordinate of $\mathbf{j}(b)$ depends only on the first (second) coordinate of b. Since $\mathbf{j}$ commutes with γ, there is a well defined map $j : Y \to Y$ with $\mathbf{j}(b) = (j(y), j(y'))$ for each $b = (y, y') \in B$. Therefore, for every function $a \in \mathcal{C}(X)$ there is a function $a' \in \mathcal{C}(X)$ such that $a'(y) = a \circ j(y)$ for each $y \in Y$. Since Y is dense in X, it follows that j can be extended to a continuous map $X \to X$, for which we use the same notation j. Furthermore, $d(j(y), j(y')) = h \cdot d(y, y')$ for every $b = (y, y') \in B$, since $|\mathbf{j}(b)| = h|b|$, where h is the contraction coefficient of J. This gives

$$\|\nabla a'\| = \sup_{(y,y') \in B} \frac{|a'(y) - a'(y')|}{d(y, y')} = h \cdot \|\nabla a\|$$

and whence $d_M(j(x), j(x')) = h \cdot d_M(x, x')$ for each x, $x' \in X$. □

Any self-similarity structure $\mathcal{S} = \{J_\alpha \,|\, \alpha \in A\}$ defines the collection of the adjoint maps $\{\mathbf{j}_\alpha \,|\, \alpha \in A\}$. It is convenient to use the following notations. Given $n \geq 0$ let $W(n)$ be the set of the words of length n in the alphabet A (we assume that $W(0)$ consists of the empty word). For a finite alphabet A we have $|W(n)| = |A|^n$. We put $W = \cup_{n \geq 0} W(n)$. Given $w = \alpha_1 \dots \alpha_n \in W(n)$, the composition $J_w = J_{\alpha_1} \circ \cdots \circ J_{\alpha_n}$ is, obviously, a similar contraction with the coefficient $h_w = h_{\alpha_1} \cdot \ldots \cdot h_{\alpha_n}$ and the map $\mathbf{j}_w = \mathbf{j}_{\alpha_1} \circ \cdots \circ \mathbf{j}_{\alpha_n}$ is adjoint to J_w (we assume that J_w is the identity map for $w \in W(0)$). *The orbit* of $U \subset B$ is the set $O(U) = \cup_{w\in W}\mathbf{j}_w(U)$. Each orbit consists, obviously, of *blocks* $O_n(U) = \cup_{w\in W(n)}\mathbf{j}_w(U)$, $n \geq 0$. The set $O_n(U)$ is called the block of the level n of $O(U)$.

By lemma 3.1, $J_\alpha(\mathcal{H})$ is an eigenspace for each $\alpha \in A$, $J_\alpha(\mathcal{H}) = P\left(\mathbf{j}_\alpha(B)\right)(\mathcal{H})$. Thus $\ker J_\alpha^*$ is also an eigenspace, $\ker J_\alpha^* = P\left(B \setminus \mathbf{j}_\alpha(B)\right)(\mathcal{H})$. Hence, $L = \cap_{\alpha\in A}\ker J_\alpha^*$ is an eigenspace, $L = P(\Delta B)(\mathcal{H})$, where $\Delta B = B \setminus \cup_{\alpha\in A}j_\alpha(B)$. The set ΔB is said to be *generating*. This terminology is justified by the following.

3.7. Lemma. *The generating set is nonempty and its orbit coincides with the support B.*

Proof. It follows from the definition of a similar contraction (condition (2)) that the adjoint maps are strictly monotone, $|b'| = h_\alpha|b| < |b|$ for each $b \in B$, $b' \in \mathbf{j}_\alpha(b)$ and $\alpha \in A$. This and the properties of B implies the claim. $\square$

§ 4. Dimension of self-similar geometries

Here we compute the dimension of a self-similar geometry.

4.1. Theorem. *Let M be a self-similar spectral geometry on a compact metric space X. If M possesses a nontrivial free subspace, then its dimension coincides with the similarity dimension, $\dim M = \dim_{\mathcal{S}} M$, the number defined by the condition (#).*

4.1.1. Remark. The formula $\dim_{\mathrm{H}} X = p$, where $\sum_{\alpha\in A} h_\alpha^p = 1$, apparently first had appeared in [Mo] for computing the Hausdorff dimension of self-similar subsets of an interval satisfying OSC (see also [Hu], [Fa1,2], [Ki2]). The condition of the theorem is a weak analog of OSC with an *arbitrary* nonempty free subset $U \subset B$.

Proof of theorem 4.1. Put $p = \dim_{\mathcal{S}} M$. If $|A| = 1$, then $p = 0$, the set ΔB is finite and generates B by iterating one and the same $\mathbf{j}_\alpha$. It gives immediately that $\dim M = 0$. Thus we assume that $|A| \geq 2$.

a. The estimate from above. Put $\sigma(q) = \sum_{a\in A} h_a^q$. Then $\sigma(p) = 1$ and $0 < \sigma(q) < 1$ for $q > p$. Furthermore, for each $n \geq 1$ we have

$$\sum_{w\in W(n)} h_w^q = \sum_{\alpha\in A} h_\alpha^q \sum_{w\in W(n-1)} h_w^q,$$

thus $\sum_{w\in W(n)} h_w^q = (\sigma(q))^n$ for each $n \geq 0$. Hence, for $q > p$ the series $\sum_{w\in W} h_w^q$ converges and its sum equals $\sum_{n\geq 0} (\sigma(q))^n = 1/\left(1 - \sigma(q)\right)$. Using lemma 3.7, we obtain

$$\mathrm{Trace}\left(|ds|^q\right) \leq \sum_{b\in\Delta B} |b|^q \sum_{w\in W} h_w^q = \mathrm{Trace}\left(|ds|^q \circ P(\Delta B)\right)/\left(1 - \sigma(q)\right) < \infty,$$

since the series $\mathrm{Trace}\left(|ds|^p \circ P(\Delta B)\right)$ and, moreover, the series $\mathrm{Trace}\left(|ds|^q \circ P(\Delta B)\right)$ converges because the subspace $\cap_{\alpha\in A}\ker J_\alpha^* = P(\Delta B)(\mathcal{H})$ is p-negligible. Thus $\dim M \leq p$ by proposition 1.2.

b. The estimate from below. We first note that any permutation $f : \mathbb{N} \to \mathbb{N}$ does not increase the partial sums $\sigma_N(\mu) = \sum_{n=0}^{N-1} \mu(n)$, $\sigma_N(\mu \circ f) \leq \sigma_N(\mu)$ for each N, where $\mu : \mathbb{N} \to \mathbb{R}$ is a positive sequence, which tends monotonically to zero. This is obvious for the transposition of neighboring elements $f(n) = n + 1$, $f(n + 1) = n$, for which $\mu(n) \geq \mu(n + 1)$, $f(k) = k$ for remaining k. In general case, we convert the permuted sequence into the initial one bringing the biggest element on the first place by transpositions as above, then the biggest remaining element on the second place etc.

By the condition of the theorem, there is a nonempty free subset $U \subset B$. It follows from its properties and injectivity of adjoint maps (lemma 3.5) that $\mathbf{j}_w(b) \neq \mathbf{j}_{w'}(b)$ for each $b \in U$ and every distinct words w, $w' \in W$. Thus to estimate the dimension $\dim M$ from below, it suffices to consider the orbit $O(b) = \cup_{w \in W} \mathbf{j}_w(b)$ of an element $b \in U$ and to assume that $B = O(b)$. Let $\mu : \mathbb{N} \to \mathbb{R}$ be the sequence of eigenvalues of $|ds|^p$ written in decreasing order according to multiplicity. We permute this sequence according to the decomposition into the blocks $B = \cup_n O_n(b)$, where $O_n(b) = \cup_{w \in W(n)} \mathbf{j}_w(b)$, assuming that inside of each block the permuted sequence $\mu' = \mu \circ f$ does not increase. Then by the former remark, the partial sums do not increase, $\sigma_N(\mu) \geq \sigma_N(\mu')$ for all N.

The block $O_n(b)$ consists of $|O_n(b)| = |A|^n$ elements. Thus if $N+1$ is the index of the first element of the block, then $N = \sum_{k=0}^{n-1} |A|^n = (|A|^n - 1)/(|A| - 1)$. Hence, width of the block is of order N, $|A|^n = (|A| - 1)N + 1$.

Now we use the key relation

$$\sum_{w \in W(n)} |\mathbf{j}_w(b)|^p = \sum_{\alpha \in A} |\mathbf{j}_\alpha(b)|^p = |b|^p \sum_{\alpha \in A} h_\alpha^p = |b|^p$$

for the sum of eigenvalues of $|ds|^p$ corresponding to the block $O_n(b)$. This relation easily follows from the definition (#) of p. This immediately implies that

$$\sigma_N(\mu') = |b|^p \cdot n \sim \frac{|b|^p}{\log |A|} \log N$$

for the index N of the first element of nth block. For the remaining elements of that block we have

$$\frac{\sigma_{N+k}(\mu')}{\log(N+k)} - \frac{\sigma_N(\mu')}{\log N} \geq \sigma_N(\mu') \left(\frac{1}{\log(N+k)} - \frac{1}{\log N} \right) = O\left(\frac{1}{\log N} \right),$$

where $1 \leq k \leq |A|^n$. It follows from what is said above that $\mathrm{Tr}_\omega(|ds|^p) \geq |b|^p / \log |A| > 0$. Hence, $\dim M \geq p$. $\square$

4.2. Example. Let us come back to the geometry $M = M(h)$ described in example 1.1. It is, obviously, self-similar for each $h \in (0,1)$. For $h \leq 1/2$, the collection of similitudes $\{j_\alpha, j_\beta\}$ of $X = [0,1]$ satisfies OSC, since one can take the interval $(0,1)$ as a free open subset $U \subset X$. In particular, $M(h)$ is free for $h \in (0, 1/2]$. For example, for $h = 1/3$ we obtain a geometry of the standard 1/3-Cantor set. If $h > 1/2$, then the images of the maps j_α, j_β have an essential overlap, and OSC is not fulfilled in that case.

Nevertheless, the geometry $M(h)$ is not free only for a countable set $H \subset (1/2, 1)$ of values of the parameter h. Indeed, the geometry $M(h)$ is not free if and only if $j_w = j_{w'}$ for some distinct words w, w' in the alphabet $\{\alpha, \beta\}$. The map $j_w : X \to X$ has a form $j_w(x) = h^{|w|}x + P_w(h)$, where the polynomial P_w is defined recurrently as follows $P_{\alpha w}(h) = hP_w(h)$, $P_{\beta w}(h) = hP_w(h) + 1 - h$ and $P_\alpha(h) = 0$, $P_\beta(h) = 1 - h$. From this, we easily obtain that $P_w(h) = (1-h)(a_1 h^{n-1} + \cdots + a_n)$ for $|w| = n$, where $a_i = a_i(w) = 0$ or 1 and any collection of 0's and 1's can be realized. Therefore, the equality $j_w = j_{w'}$ is equivalent to that $|w| = |w'| = n$ and $P_w(h) = P_{w'}(h)$, i.e., $h > 1/2$ is a root of an equation

$$(a_1 - a_1')h^{n-1} + \cdots + a_n - a_n' = 0.$$

Namely these conditions define the set H. For $h \notin H$ the geometry $M(h)$ is free and by theorem 4.1 its dimension is $\dim M(h) = -\log 2/\log h$. In particular, this gives examples of free self-similar geometries on $[0,1]$ of arbitrarily large dimension, for which the Connes metrics coincide with the initial one.

The geometry $M(1/\tau)$, where $\tau = (\sqrt{5}+1)/2$ is the "golden number", is not free, since $j_w = j_{w'}$ for $w = \beta\alpha^2$, $w' = \alpha\beta^2$ (these are the shortest distinct words, for which the coincidence $j_w = j_{w'}$ is possible). Moreover, $M(1/\tau)$ possesses no free subspace, and theorem 4.1 cannot be applied to it. One can show that for $t = (1/\tau)^n$ one has $|B_t| \asymp \tau^n$ as $n \to \infty$, and thus $\dim M(1/\tau) = 1 < \dim_{\mathcal{S}} M(1/\tau)$.

§ 5. Measurability of a self-similar geometry

Here we prove theorem 0.1. A self-similarity structure induces a map $\mathcal{S}$ of the space $\mathcal{M}$ of the Radon measures on X (with one and the same mass equals, for example, 1) into itself, which is a contraction with respect to the Kantorovich metric and whence possesses a unique fixed point μ_0. (This *contraction* method is standard in theory of fractal sets and was first applied in [Hu]). The property to be ample for M implies that the space $\mathcal{M}$ is bounded with respect to the Kantorovich metric. On the other hand, we prove for positive Radon measures $d_t M(a) = t \cdot \operatorname{Trace}\,(\pi(a)|ds|^{p+t})$ the existence of a finite positive limit $dM(1) = \lim_{t\to 0+} d_t M(1)$ (theorem 5.2) and the weak convergence of measures $d_t M \to dM(1)\cdot\mu_0$ as $t \to 0+$, which implies the measurability of M. The key point of the proofs is the existence of a free subset $U \subset B$ with the negligible complement $B \setminus U$, and also that X is bounded with respect to the Connes metric d_M.

The contraction method. We denote by $\mathcal{M}$ the space of Radon measures on X with the mass $\mu(X) = 1$ for all $\mu \in \mathcal{M}$. Consider on $\mathcal{M}$ the metric

$$d_M(\mu, \mu') = \sup\{|\mu(a) - \mu'(a)| \mid a \in \mathcal{A}_1, \|\nabla a\| \le 1\},$$

where the algebra $\mathcal{C}(X)$ of the continuous functions on X is identified with the subalgebra $\mathcal{A}_1 \subset \widetilde{\mathcal{A}}$. One calls d_M *the Kantorovich metric.* On the subset of delta-measures, which can be identified with X, d_M, obviously, coincides with the Connes metric. The condition $\mu(X) = 1$ (instead of 1 one can take any positive number) for all $\mu \in \mathcal{M}$ implies that the distances with respect to d_M are finite.

5.1.1. Lemma. *If a geometry M is ample and $a \in \mathcal{A}_1$ satisfies the condition $\|\nabla a\| \le 1$, then for each measures μ, $\mu' \in \mathcal{M}$ and any point $x_0 \in X$ we have*

$$|\mu(a) - \mu'(a)| \le 2 \max |a(x) - a(x_0)| \cdot \mu(1) \le 2 \cdot \operatorname{diam}_M X,$$

in particular, $\operatorname{diam}_M \mathcal{M} \le 2 \cdot \operatorname{diam}_M X$.

Proof. We have $\mu(a) - \mu'(a) = \mu(a - a_0) - \mu'(a - a_0)$ for each constant function a_0, since the masses of μ, μ' are equal. Adding a constant does not change the operator ∇a. Furthermore, $|a(x) - a(x_0)| \le d_M(x, x_0) \le \operatorname{diam}_M X$. Finally

$$|\mu(a) - \mu'(a)| = |\mu(a - a(x_0)) - \mu'(a - a(x_0))| \le 2 \max |a(x) - a(x_0)| \cdot \mu(1). \quad \square$$

The space $\mathcal{M}$ is complete with respect to the Kantorvich metric: clearly that the convergence w.r.t. d_M implies the point-wise convergence on the Lipschitz functions; the continuos functions can be appoximated in the max-norm by Lipschitz ones, for instance, using a Lipschitz unity decomposition. Thus any (strictly) contracting map $\mathcal{M} \to \mathcal{M}$ possesses a unique fixed point.

If the geometry is ample, then the self-similarity structure $\mathcal{S} = \{J_\alpha \,|\, \alpha \in A\}$ induces the map $\mathcal{S} : \mathcal{M} \to \mathcal{M}$ defined as follows. For $\mu \in \mathcal{M}$, we put

$$\mathcal{S}(\mu) = \sum_{\alpha \in A} h_\alpha^p J_\alpha^\#(\mu),$$

where $p = p(\mathcal{S})$ is the similarity dimension, $J_\alpha^\#(\mu)(a) = \mu(a_\alpha)$, and the function $a_\alpha \in \widetilde{\mathcal{A}}_1$ is defined by $J_\alpha^* \circ \pi(a) \circ J_\alpha = \pi(a_\alpha)$. This relation defines a_α up to adding a function $a' \in \mathcal{A}_1$ with $\pi(a') = 0$. Since M is ample, the projection of its support on X is everywhere dense. Thus $a' \equiv 0$. If $a \equiv 1$, then $a_\alpha \equiv 1$, and thus $\mathcal{S}(\mu)(X) = 1$, i.e., $\mathcal{S}(\mu) \in \mathcal{M}$.

The following assertion is well known in different forms (see [Hu], [Fa1,2], [HR], [La2]).

5.1.2. Lemma. *The map $\mathcal{S}$ is strictly contracting,*

$$d_M\left(\mathcal{S}(\mu), \mathcal{S}(\mu')\right) \le h \cdot d_M(\mu, \mu')$$

for each μ, $\mu' \in \mathcal{M}$, where $h = \sum_{\alpha \in A} h_\alpha^{p+1} \in (0,1)$. In particular, there exists a unique invariant for $\mathcal{S}$ measure $\mu_0 \in \mathcal{M}$.

Proof. It follows from lemma 3.5 that $a_\alpha(b) = a \circ \mathbf{j}_\alpha(b)$ for each $b \in B$, where $\mathbf{j}_\alpha$ is the adjoint map to the similar contraction J_α. In that case one has $|\mathbf{j}_\alpha(b)| = h_\alpha |b|$. Thus $\|\nabla a_\alpha\| \le h_\alpha \|\nabla a\|$. Hence, if $\|\nabla a\| \le 1$, then

$$\left|J_\alpha^\#(\mu)(a) - J_\alpha^\#(\mu')(a)\right| = h_\alpha \left|\mu(a_\alpha/h_\alpha) - \mu'(a_\alpha/h_\alpha)\right| \le h_\alpha d_M(\mu, \mu')$$

for each $\alpha \in A$. This immediately implies the required estimate. $\square$

In the following theorem, which is a part of the proof of theorem 0.1, we do not use the condition that M is ample.

5.2. Theorem. *Let $M = \{\mathcal{A}, \mathcal{H}, ds\}$ be an a.f.s.s. spectral geometry on a compact metric space X, $\mathcal{S} = \{J_\alpha \,|\, \alpha \in A\}$ its self-similarity structure, $p = \dim_{\mathcal{S}} M$ the similarity dimension. Then there exists a finite positive limit $dM(1) = \lim_{t \to 0+} t \cdot \mathrm{Trace}\left(|ds|^{p+t}\right)$.*

Proof. We put

$$H = -\sum_{\alpha \in A} h_\alpha^p \log h_\alpha = \lim_{t \to 0+} \left(1 - \sigma(p+t)\right)/t.$$

Since M is almost free, there is a nonempty free subset $U \subset B$, whose complement $V = B \setminus U$ is p-negligible. We prove that

$$dM(1) = H^{-1} \mathrm{Trace}\left(|ds|^p \circ P(\Delta U)\right),$$

where the set $\Delta U = U \setminus \cup_{\alpha \in A} \mathbf{j}_\alpha(U)$ is nonempty and p-negligible.

The set ΔU is nonempty, because U is nonempty. Since U is free, it follows that if $\mathbf{j}_w(b) = \mathbf{j}_{w'}(b')$ for b, $b' \in U$, then one of the words w, $w' \in W$ is an initial segment of the other, to be definite we assume that $w' = wv$, and $b = \mathbf{j}_v(b')$. It follows that

$$d_t M(U) = t \cdot \sum_{b \in \Delta U} |b|^{p+t} \sum_{w \in W} h_w^{p+t} = t \cdot \mathrm{Trace}\left(|ds|^{p+t} \circ P(\Delta U)\right)/\left(1 - \sigma(p+t)\right)$$

for $t > 0$.

Furthermore, $(1-\sigma(p+t))\, t^{-1} d_t M(B) \le \text{Trace}\,(|ds|^{p+t} \circ P(\Delta B))$ by sect.**a** of the proof of theorem 4.1. Since V is p-negligible, we have $d_t M(B \setminus U) = o(1)$. Using $d_t M(B) = d_t M(U) + d_t M(B \setminus U)$, we obtain that ΔU is p-negligible

$$\text{Trace}\,(|ds|^p \circ P(\Delta U)) \le \text{Trace}\,(|ds|^p \circ P(\Delta B)) < \infty,$$

and there exists the finite positive limit

$$dM(1) = \lim_{t\to 0+} d_t M(B) = H^{-1}\text{Trace}\,(|ds|^p \circ P(\Delta U))\,. \quad \square$$

We call the number $\mathbf{h} = p \cdot H = -\sum_{\alpha\in A} h_\alpha^p \log h_\alpha^p$ *the entropy* of the self-similar geometry M. If all of the contraction coefficients h_α coincide, then $\mathbf{h} = \log|A|$. In general case, $\mathbf{h} \le \log|A|$. Thus the entropy vanishes if $p = \dim_{\mathcal{S}} M = 0$.

Therefore, we have also computed the mass

$$(*) \qquad dM(1) = p \cdot \mathbf{h}^{-1} \cdot \text{Trace}\,(|ds|^p \circ P(\Delta U))\,,$$

where p is the dimension of the geometry, $\mathbf{h}$ its entropy, the set ΔU generates the free subset U. This formula makes also sense for $p = 0$, since $\mathbf{h}/p = H$. Turning back to example 2.4 (the Sierpinski carpet), we see that the maximal generating free set ΔU is infinite and $\text{Trace}\,(|ds|^p \circ P(\Delta U)) = 4.8$.

Proof of theorem 0.1. For $\mathcal{S}$-invariant measure $\mu_0 \in \mathcal{M}$ (see lemma 5.1.1) we put $\mu_1 = dM(1) \cdot \mu_0$, $\mu_t = d_t M(1) \cdot \mu_0$. By theorem 5.2, measures μ_t weakly converge to μ_1 as $t \to 0+$. It will be shown that the distance with respect to the Kantorovich metric between the measures $d_t M$ and μ_t tends to zero as $t \to 0+$ (for distinct t these measures, in general, are at infinite distance from each other). This implies that $d_t M \to \mu_1$ weakly as $t \to 0+$, i.e., measurability of M.

5.3.1. Lemma. *For each function $a \in \mathcal{A}_1$ we have*

$$\mathcal{S}(d_t M)(a) - d_t M(a) = o(1) \cdot \max|a|,$$

where $o(1)$ as $t \to 0+$ is independent of a.

Proof. Recall that

$$d_t M(a) = t \cdot \text{Trace}\,\big(\pi(a)|ds|^{p+t}\big) = t \cdot \sum_{b\in B} a(b)|b|^{p+t}.$$

Using that, according lemma 3.5, $|\mathbf{j}_\alpha(b)| = h_\alpha |b|$ for each $\alpha \in A$, $b \in B$, we obtain

$$\begin{aligned}
\mathcal{S}(d_t M)(a) &= \sum_{\alpha\in A} h_\alpha^p \cdot t \cdot \sum_{b\in B} a \circ \mathbf{j}_\alpha(b)|b|^{p+t} \\
&= \sum_{\alpha\in A} h_\alpha^{-t} \cdot t \cdot \sum_{b\in \mathbf{j}_\alpha(B)} a(b)|b|^{p+t}.
\end{aligned}$$

Using the existence of the free set $U \subset B$, we represent the difference $\mathcal{S}(d_tM)(a) - d_tM(a)$ as the sum of four summands $S_1 + S_2 + S_3 + S_4$, where

$$S_1 = \sum_{\alpha \in A} (h_\alpha^{-t} - 1) \cdot t \sum_{b \in \mathbf{j}_\alpha(B)} a(b)|b|^{p+t},$$

$$S_2 = \sum_{\alpha \in A} t \sum_{b \in \mathbf{j}_\alpha(B \setminus U)} a(b)|b|^{p+t},$$

$$S_3 = \sum_{\alpha \in A} t \sum_{b \in \mathbf{j}_\alpha(U)} a(b)|b|^{p+t} - t \sum_{b \in U} a(b)|b|^{p+t},$$

$$S_4 = -t \sum_{b \in B \setminus U} a(b)|b|^{p+t}.$$

The first summand is, obviously, estimated as

$$|S_1| \leq \sum_{\alpha \in A} (h_\alpha^{-t} - 1) \cdot \max|a| \cdot d_tM(1) = o(1) \max|a|.$$

For the estimate of the second and fourth summands

$$|S_2|, |S_4| \leq t \sum_{b \in B \setminus U} |b|^{p+t} \cdot \max|a| = o(1) \max|a|,$$

we use the fact that $B \setminus U$ is p-negligible.

Finally S_3 can be represented as $S_3 = -t \cdot \sum_{b \in \Delta U} a(b)|b|^{p+t}$, where $\Delta U = U \setminus \cup_{\alpha \in A} \mathbf{j}_\alpha(U)$ is the generating U set. It was shown in the proof of theorem 5.2 that ΔU is p-negligible, in particular, $d_tM(\Delta U) = o(1)$. It is clear that

$$|S_3| \leq \max|a| \cdot t \sum_{b \in \Delta U} |b|^{p+t} = \max|a| \cdot d_tM(\Delta U) = o(1) \cdot \max|a|.$$

This completes the proof. $\square$

5.3.2. Lemma. *For the measures d_tM, μ_t we have*

$$d_M(d_tM, \mu_t) = o(1)$$

as $t \to 0+$.

Proof. The measure $\mathcal{S}(d_tM)$ has the same mass as d_tM, thus for each function $a \in \mathcal{A}_1$ with $\|\nabla a\| \leq 1$ we have

$$\begin{aligned} d_tM(a) - \mathcal{S}(d_tM)(a) &= d_tM(a - a(x_0)) - \mathcal{S}(d_tM)(a - a(x_0)) \\ &= o(1) \cdot \max|a(x) - a(x_0)| \leq o(1) \cdot \operatorname{diam}_M X = o(1) \end{aligned}$$

by lemma 5.3.1. In other words, $d_M(d_tM, \mathcal{S}(d_tM)) = o(1)$ as $t \to 0+$. The measure μ_t is $\mathcal{S}$-invariant by the construction, $\mathcal{S}(\mu_t) = \mu_t$. Thus

$$\begin{aligned} d_M(d_tM, \mu_t) &\leq d_M(d_tM, \mathcal{S}(d_tM)) + d_M(\mathcal{S}(d_tM), \mathcal{S}(\mu_t)) \\ &\leq d_M(d_tM, \mathcal{S}(d_tM)) + h \cdot d_M(d_tM, \mu_t) \end{aligned}$$

by lemma 5.1.2, where $h \in (0, 1)$. This immediately implies the claim. $\square$

The weak convergence of the measures $d_tM \to \mu_1$ and, hence, the measurability of M now follows from theorem 5.2 and lemma 5.3.2. This completes the proof of theorem 0.1. $\square$

We obtain the ω-measurability of self-similar geometries from their measurability using theorem 6.1 below. A close to that theorem assertion is formulated in [Co, Ch. IV, Prop. 4] without proof referring to the book [Har]. I was not able to find the corresponding result in [Har]. However, theorem 6.1 is an easy corollary of theorem 98 from [Har].

Let $\mu_1 \geq \mu_2 \geq \dots$ be the sequence of the eigenvalues of a compact positive operator T in $\mathcal{H}$ written in decreasing order according to multiplicity.

6.1. Theorem. *Assume that for a compact positive operator T in $\mathcal{H}$ and $p > 0$ the following conditions are fulfilled*

(1) $-p \log \mu_n \sim \log n$;
(2) *there exists a finite limit* $\lim_{t \to 0+} t \cdot \mathrm{Trace}\,(T^{p+t}) = L$.

Then $T^p \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ and moreover $\sigma_N(T^p) = \sum_{n=1}^N \mu_n^p \sim (L/p) \log N$. In particular, the Dixmier trace $\mathrm{Tr}_\omega(T^p) = L/p$ is independent of the choice of the limiting procedure ω.

Theorem 6.1 is a Tauberian type theorem. Here the Tauberian condition (1) is weaker than $\mu_n^p = O(1/n)$ since it does not exclude that $\liminf_n (n\mu_n^p) = 0$ or $\limsup_n (n\mu_n^p) = \infty$. Theorem 6.1 is an easy corollary of the following theorem about Stieltjes' integrals. For its proof and authorship see [Har].

6.1.1. Theorem ([Har, theorem 98]). *If $\alpha(t)$ is an increasing function on $[0, \infty)$, the intergral*

$$I(y) = \int_0^\infty e^{-yt} d\alpha(t)$$

converges for $y > 0$ and $I(y) \sim L/y$ as $y \to 0$, then $\alpha(t) \sim L \cdot t$ as $t \to \infty$. $\square$

Proof of theorem 6.1. The spectrum $\mu_0 \geq \mu_1 \geq \dots$ of T can be represented as the union of blocks L_n, $n \geq 0$, corresponding to its eigenspaces. In other words, L_n consists of k_n copies of an eigenvalue μ, where k_n is the multiplicity of μ,

$$L_n = \{\mu_{k_0+\dots+k_{n-1}}, \mu_{k_0+\dots+k_{n-1}+1}, \dots, \mu_{k_0+\dots+k_n-1}\}.$$

Let $N(n)$ be the index of the last eigenvalue from L_n, i.e., $N(n) = k_0 + \dots + k_n - 1$. We define $\alpha : [0, \infty) \to \mathbb{R}$ as a piecewise constant continuous from the right function with positive skips $k_n \cdot \mu_{N(n)}^p$ at $t_n = \log(1/\mu_{N(n)})$ (one can assume that $\mu_0 \leq 1$). Then

$$I(y) = \sum_{n \geq 0} k_n \cdot \mu_{N(n)}^p e^{-yt_n} = \sum_{n \geq 0} \mu_n^{p+y} = \mathrm{Trace}\,(T^{p+y}).$$

According to (2), the integral $I(y)$ converges for $y > 0$ and $I(y) \sim L/y$ as $y \to 0$. It follows from theorem 6.1.1 that $\alpha(t) \sim L \cdot t$ as $t \to \infty$, in particular, $\alpha(t_n) \sim L \cdot t_n$. On the other hand, $\alpha(t_n) = \sum_{n=1}^{N(n)} \mu_n^p = \sigma_{N(n)}(T^p)$, and it follows from (1) that $t_n \sim p^{-1} \log N(n)$. Therefore,

$$\lim_{n \to \infty} \frac{\sigma_{N(n)}(T^p)}{\log N(n)} = L/p. \tag{+}$$

We show that this already implies the assertion, i.e., that there exists the limit

$$\lim_{N \to \infty} \sigma_N(T^p)/\log N = L/p.$$

To this end we use the fact that $\mu_{N(n)+k} = \mu_{N(n+1)}$ for all $1 \le k \le k_{n+1}$ and condition (1). We have

$$\log N(n+1) \sim \log \mu_{N(n+1)}^{-p} = \log \mu_{N(n)+1}^{-p} \sim \log(N(n)+1).$$

Thus

$$1 \le \log N(n+1)/\log(N(n)+k) \le \log N(n+1)/\log N(n) \to 1$$

as $n \to \infty$ for all $1 \le k \le k_{n+1}$. It remains to note that

$$\sigma_{N(n)}(T^p) \le \sigma_{N(n)+k}(T^p) \le \sigma_{N(n+1)}(T^p),$$

and to use the existence of limit (+). $\square$

The proof of theorem 0.2 is based on the existence of the limit in the definition of $\dim_2 M$.

6.2.1. Proposition. *Let M be an a.f.s.s. geometry, $p = \dim M = \dim_{\mathcal{S}} M$ its dimension. Then there exists the limit*

$$\lim_{t\to 0+} \frac{\log |B_t|}{\log(1/t)} = p.$$

This proposition is an easy consequence of the renewal theorem (see § 7.7), and we postpone its proof to the next chapter. Theorem 0.2 now can be proved as follows. Using proposition 6.2.1, we obtain in lemma 6.2.2 that for $|ds|$ the Tauberian condition (1) from theorem 6.1 is fulfilled. Then, according that theorem and theorem 5.2, we have

$$d_\omega M(1) = \mathrm{Tr}_\omega(|ds|^p) = \mathbf{h}^{-1} \cdot \mathrm{Trace}\,(|ds|^p \circ P(\Delta U)),$$

in particular, $d_\omega M$ is a positive Radon measure on X, whose mass is independent of the choice of ω. To prove the ω-measurability of M, it remains only to show that the measure $d_\omega M$ is $\mathcal{S}$-invariant (lemma 6.2.4).

6.2.2. Lemma. *Let $\mu_0 \ge \mu_1 \ge \dots$ be the sequence of the eigenvalues of the operator $|ds|$ of M written in decreasing order according to multiplicity. Then $\log \mu_n^{-p} \sim \log n$ as $n \to \infty$, where $p = \dim M$.*

Proof. The points of the spectrum of $|ds|$ are isolated. Thus

$$|B_{\mu_n - \epsilon}| \le n \le |B_{\mu_n}|$$

for a fixed n for all sufficiently small $\epsilon > 0$. Since $\log |B_t| \sim \log t^{-p}$ by proposition 6.2.1, we obtain $\log n \sim \log \mu_n^{-p}$ as $n \to \infty$. $\square$

Now it immediately follows from theorems 6.1 and 5.2

6.2.3. Corollary. *Given an a.f.s.s. spectral geometry M on X, there exists the finite positive Radon measure $d_\omega M$ whose mass*

$$d_\omega M(1) = \mathrm{Tr}_\omega(|ds|^p) = \mathbf{h}^{-1} \cdot \mathrm{Trace}\,(|ds|^p \circ P(\Delta U))$$

is independent of the choice of the limiting procedure ω. $\square$

6.2.4. Lemma. *The measure $d_\omega M$ is $\mathcal{S}$-invariant.*

Proof. Let $U \subset B$ be a free subset with p-negligible complement. It follows from this and the proof of theorem 5.2 that the sets $B \setminus U$ and ΔU in the disjoint decompositions $B = U \cup (B \setminus U)$ and $U = \Delta U \cup \cup_{\alpha \in A} \mathbf{j}_\alpha(U)$ are p-negligible. Thus for each operator $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ we have

$$\mathrm{Tr}_\omega(T) = \sum_{\alpha \in A} \mathrm{Tr}_\omega(T \circ P(\mathbf{j}_\alpha(U))).$$

Using this and also the covariance of the Dixmier trace and properties of similar contraction J_α, we obtain for $a \in \mathcal{C}(X)$

$$\begin{aligned}
\mathcal{S}(d_\omega M)(a) &= \sum_{\alpha \in A} h_\alpha^p J_\alpha^{\#}(d_\omega M)(a) \\
&= \sum_{\alpha \in A} h_\alpha^p \mathrm{Tr}_\omega(J_\alpha^* \circ \pi(a) \circ J_\alpha |ds|^p) \\
&= \sum_{\alpha \in A} \mathrm{Tr}_\omega(J_\alpha^* \circ \pi(a)|ds|^p \circ J_\alpha) \\
&= \sum_{\alpha \in A} \mathrm{Tr}_\omega(J_\alpha^* \circ \pi(a)|ds|^p \circ J_\alpha \circ P(U)) \\
&= \sum_{\alpha \in A} \mathrm{Tr}_\omega(J_\alpha^* \circ \pi(a)|ds|^p P(\mathbf{j}_\alpha(U)) \circ J_\alpha) \\
&= \sum_{\alpha \in A} \mathrm{Tr}_\omega(\pi(a)|ds|^p \circ P(\mathbf{j}_\alpha(U))) \\
&= \mathrm{Tr}_\omega(\pi(a)|ds|^p) = d_\omega M(a).
\end{aligned}$$

This proves the $\mathcal{S}$-invariance of $d_\omega M$. $\square$

So far we did not use in the proof the condition that M is ample. The map $\mathcal{S}$ is well defined on $d_\omega M$ without that condition since the value $d_\omega M(a)$ depends only on the operator $\pi(a)$. However, that condition we need to apply the contraction method and lemma 5.1.2 because it is necessary to define the map $\mathcal{S}$ on all of the space $\mathcal{M}$.

Proof of theorem 0.2. It follows from lemmas 5.1.2 and 6.2.4 that $d_\omega M = d_\omega M(1) \cdot \mu_0$, where $\mu_0 \in \mathcal{M}$ is the unique invariant for $\mathcal{S}$ measure of mass 1. Corollary 6.2.3 now allows to claim that the measure $d_\omega M$ is finite and independent of the choice of ω, i.e., the spectral geometry M is ω-measurable. $\square$

§ 7. The renewal theorem

Different variants of the renewal theorem have a wide range of applications for study of self-similar sets, see, for instance, [Fa2], [Lall], [LV]. Here we use a simplest version which gives nonetheless much more information about the asymptotic behavior of the function $|B_t|$ as $t \to 0+$ than one needs for the proof of proposition 6.2.1.

Given a function $g : \mathbb{R} \to \mathbb{R}$, finite collections of times $T = \{s_\alpha > 0 \,|\, \alpha \in A\}$ and probabilities $Q = \{q_\alpha \in (0,1) \,|\, \alpha \in A\}$, $\sum_{\alpha \in A} q_\alpha = 1$, $|A| \geq 2$, the renewal equation for $f : \mathbb{R} \to \mathbb{R}$ has the form

$$f(s) = \sum_{\alpha \in A} q_\alpha f(s - s_\alpha) + g(s). \tag{7.1}$$

The renewal theorem describes the limiting behavior of solutions of this equation as $s \to \infty$. One seeks the solutions in the space $\mathcal{F}$ of Borel measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(s) \to 0$ as $s \to -\infty$ and such that f is bounded on the half-line $(-\infty, a]$ for every $a \in \mathbb{R}$. One also assumes that the set of discontinuities of g is discrete, and for some c, $\beta > 0$ the condition

$$|g(s)| \leq c \cdot e^{-\beta|s|} \tag{7.1.1}$$

is fulfilled for all $s \in \mathbb{R}$.

The set of times T is called τ-*arithmetic* if $\tau > 0$ is the greatest positive number such that each $s_\alpha \in T$ is an integer multiple of τ, and *non-arithmetic* if no such $\tau > 0$ exists.

7.2. Renewal theorem. *Let $f \in \mathcal{F}$ satisfy (7.1) for g satisfying (7.1.1). If the set T is non-arithmetic then*

$$\lim_{s\to\infty} f(s) = H^{-1} \int_{-\infty}^{\infty} g(x)dx,$$

and if T is τ-arithmetic then

$$\lim_{k\to\infty} f(k\tau + s) = H^{-1}\tau \sum_{j\in\mathbb{Z}} g(j\tau + s)$$

for all $s \in [0, \tau)$, where $H = \sum_{\alpha\in A} q_\alpha s_\alpha$.

The proof of this and more general theorems can be found in [Fa2]; see also [Ru, Chapter 9], [Lall], [LV] and references therein.

Reduction to the renewal theorem. We first show how to obtain proposition 6.2.1 from the renewal theorem. Assume that a finite alphabet A, $|A| \geq 2$ and the collection of contraction coefficients $\{h_\alpha \in (0,1) \,|\, \alpha \in A\}$ are given. Let $p > 0$ be the unique solution to the equation $\sum_{\alpha\in A} h_\alpha^p = 1$. Given a word $w = \alpha_1 \dots \alpha_k \in W$, we put as usual $h_w = h_{\alpha_1} \dots h_{\alpha_k}$, $k \geq 1$. Let $W^t = \{w \in W \,|\, h_w^{-1} \leq t\}$. We are interested in the asymptotic behavior of $G(t) = |W^t|$ as $t \to \infty$. For each $t \geq 0$, we have the disjoint decomposition

$$W^t = \cup_{\alpha\in A} \alpha W^{th_\alpha} \cup \{\alpha \in A \,|\, h_\alpha^{-1} \leq t\},$$

where one supposes that if W^t is empty then αW^t is empty for all $\alpha \in A$. This decomposition is a key relation for application of the renewal theorem. It gives

$$G(t) = \sum_{\alpha\in A} G(th_\alpha) + \#\{\alpha \in A \,|\, h_\alpha^{-1} \leq t\} \tag{7.3}$$

for each $t \geq 0$. Put $t = e^s$, $f(s) = e^{-ps}G(e^s)$, $g(s) = e^{-ps}\#\{\alpha \in A \,|\, h_\alpha^{-1} \leq e^s\}$. Then equation (7.3) becomes renewal equation (7.1) with $s_\alpha = \log h_\alpha^{-1}$, $q_\alpha = h_\alpha^p$ for $\alpha \in A$.

7.4. Corollary. *If $T = \{\log h_\alpha^{-1} \,|\, \alpha \in A\}$ is non-arithmetic then $G(t) \sim \mathbf{h}^{-1} \cdot t^p$ as $t \to \infty$, where $\mathbf{h} = -\sum_{\alpha\in A} h_\alpha^p \log h_\alpha^p$. If T is τ-arithmetic, then $G(t) \sim \mathbf{h}^{-1} \cdot t^p \cdot G_0(t)$, where G_0 is a positive bounded function τ-periodic on the logarithmic scale, $G_0(te^\tau) = G_0(t)$ for each $t > 0$.*

Proof. In the non-arithmetic case we have

$$\int_{-\infty}^{\infty} g(s)ds = \sum_{\alpha\in A} \int_{\log h_\alpha^{-1}}^{\infty} e^{-ps}ds = p^{-1} \sum_{\alpha\in A} h_\alpha^p = p^{-1}.$$

Now the required asymptotic expansion follows from the renewal theorem. In the arithmetic case, the right hand side of the corresponding equality in theorem 7.2 is a τ-periodic function of s. It follows from the expression $g(s) = e^{-ps}\#\{\alpha \in A \,|\, h_\alpha^{-1} \le e^s\}$ that this function is positive and bounded. This gives the required asymptotic behavior in the arithmetic case. $\square$

Proof of proposition 6.2.1. For simplicity, we restrict only to the non-arithmetic case. Assuming that $t \to \infty$, we put $N(t) = |B_{1/t}|$. It will be shown that

$$\limsup t^{-p} N(t) \le \mathbf{h}^{-1} \mathrm{Trace}\,(|ds|^p \circ P(\Delta B))$$

and

$$\liminf t^{-p} N(t) \ge \mathbf{h}^{-1} \mathrm{Trace}\,(|ds|^p \circ P(\Delta U))$$

(we use the notations of § 6). This is sufficient for the proof of proposition 6.2.1, since the right hand sides of these inequalities are positive and finite.

One can represent each $b \in B$ as $b = \mathbf{j}_w(b')$, where $b' \in \Delta B$, $w \in W$. Such a representation is in general not unique. Thus

$$N(t) \le \#\,\{\mathbf{j}_w(b) \,|\, b \in \Delta B, w \in W, |\mathbf{j}_w(b)| \ge 1/t\} = \sum_{b \in \Delta B} G(t|b|).$$

The right hand side $\widetilde{G}(t)$ of this inequality satisfies the equation 7.3 with the additional function $\sum_{b\in\Delta B} \#\{\alpha \in A \,|\, h_\alpha^{-1} \le t|b|\}$. As in corollary 7.4, we obtain $t^{-p}\widetilde{G}(t) \sim \mathbf{h}^{-1}\sum_{b\in\Delta B}|b|^p$. This gives the first of the required inequalities. A similar argument together with the fact that U is free gives the second one. $\square$

The asymptotic behavior of the eigenvalue counting function for $|\mathcal{D}|$. The function $N(t)$, which has appeared above, is the eigenvalue counting function of $|\mathcal{D}|$,

$$N(t) = \#\{\lambda \,|\, \lambda \text{ is an eigenvalue of } |\mathcal{D}|, \lambda \le t\}.$$

Here we give a more precise information about its asymptotic behavior than that was used in the proof of proposition 6.2.1.

7.5. Theorem. *Let $M = \{\mathcal{A}, \mathcal{H}, ds\}$ be an a.f.s.s. spectral geometry on a compact metric space X, $p = \dim M$. Then for the eigenvalue counting function $N(t)$ of $|\mathcal{D}| = |ds|^{-1}$ there is the asymptotic expansion as $t \to \infty$*

$$N(t) = d_\omega M(X) \cdot t^p + o(t^p)$$

in the non-arithmetic case and

$$N(t) = q(\log t) \cdot t^p + o(t^p)$$

in the τ-arithmetic case, where q is a bounded positive τ-periodic function. $\square$

7.5.1. Remarks. (i) A particular case of this theorem for a spectral geometry of a special type is considered in [Fa2, Prop. 7.5].

(ii) The function q for the τ-arithmetic case can be written as

$$q(s) = \mathbf{h}^{-1} p\tau (1 - e^{-p\tau})^{-1} \sum_{b\in\Delta U} |b|^p \sum_{\alpha \in A} h_\alpha^p \cdot q_{\alpha,b}(s),$$

where the τ-periodic function $q_{\alpha,b}$ satisfies the condition $e^{-p\tau} < q_{\alpha,b}(s) \le 1$ for each $\alpha \in A$, $b \in \Delta U$, $s \in \mathbb{R}$.

Theorem 7.5 in the case $B = U$ (we use the notations of § 6) easily follows from the renewal theorem (cp. the proof of proposition 6.2.1 and [Fa2, Prop. 7.5]). In general case we need the following fact.

7.5.2. Lemma. *Assume that a monotone function $f : [0,\infty) \to \mathbb{R}_+$ satisfies the condition $\int_0^\infty f(t)dt < \infty$. Then $t \cdot f(t) = o(1)$ as $t \to \infty$.*

Proof. If this is not the case, then there exists a sequence $t_k \to \infty$ such that $t_k f(t_k) \geq c > 0$. We define a piecewise constant function $g : [0,\infty) \to \mathbb{R}_+$, which is constant on each interval $(t_{k-1}, t_k]$, putting $g(t_k) = c/t_k$. Then g is not increasing and $g(t) \leq f(t)$ for all $t \geq 0$. In particular, $\int_0^\infty g(t)dt < \infty$. On the other hand, $\int_0^{t_k} g(t)dt = \sum_1^k a_i$, where $a_i = 1 - t_{i-1}/t_i$. Hence $a_i \to 0$ as $i \to \infty$, $t_0/t_k = \prod_1^k (1 - a_i)$ and

$$\log t_k = \log t_0 - \sum_1^k \log(1 - a_i) \leq c'(1 + \sum_i a_i) < \infty.$$

This contradicts the fact that $t_k \to \infty$ as $k \to \infty$. $\square$

Proof of theorem 7.5. Using the notations of § 6, we represent the support B of M as the disjoint union $B = U \cup V$, where U is free and V p-negligible. We show that the contribution of V in the asymptotic expansion of $N(t)$ is of order $o(t^p)$, and find the required asymptotic behavior using the properties of U and the renewal theorem.

Given $B' \subset B$ and $t > 0$, we put $B'_{1/t} = \{b \in B' \,|\, |b| \geq 1/t\}$, $N(t, B') = |B'_{1/t}|$. In particular, $N(t) = N(t, B)$ and $N(t) = N(t, U) + N(t, V)$. First we show that $N(t, V) = o(t^p)$. The set $V_{1/t}$ consists of $N(t, V)$ elements $b_1, \dots, b_{N(t,V)}$, written in decreasing order (according to multiplicity) such that $|b_1| \geq \dots \geq |b_{N(t,V)}| \geq 1/t$. This defines a piecewise constant function f, which is constant on the intervals $(k-1, k]$, and $f(k) = |b_k|^p$. Then $\int_0^\infty f(s)ds = \sum_{b \in V} |b|^p < \infty$. By lemma 7.5.2 we have

$$t^{-p} N(t, V) \leq N(t, V) f(N(t, V)) = o(1).$$

Thus $N(t, V) = o(t^p)$ as $t \to \infty$.

Since U is free, each $b \in U$ can *uniquelly* be represented as $b = \mathbf{j}_w(b')$, where $b' \in \Delta U$, $w \in W$. Thus as in the proof of proposition 6.2.1 we obtain that

$$N(t, U) = \sum_{b \in \Delta U} G(t|b|)$$

satisfies (7.3) with the additional function $\sum_{b \in \Delta U} \#\{\alpha \in A \,|\, h_\alpha^{-1} \leq t|b|\}$. Now taking into account that $\mathbf{h}^{-1} \sum_{b \in \Delta U} |b|^p = \mathbf{h}^{-1} \cdot \mathrm{Trace}\,(|ds|^p \circ P(\Delta U)) = d_\omega M(X)$, the theorem follows from the renewal theorem. We leave to the reader details how to compute q in the τ-arithmetic case. $\square$

§ 8. The measure $d_\omega M$ and the Hausdorff measure H^p

Here we prove theorem 0.6. Before to start the proof, we make some remarks on the conditions of that theorem.

8.1. Remarks. (1) The condition $X = \mathrm{supp}\, d_\omega M$ is not an essential restriction. If one omits it, then the conclusions of the theorem remain valid for the restrictions of the measures on $X' = \mathrm{supp}\, d_\omega M$. One can easily construct examples of ample a.f.s.s. geometries with $X' \neq X$ using similitudes of an Euclidean space.

(2) The condition $H^p(X) > 0$ is essential for the conclusion that $c > 0$ in the equality $H^p = c \cdot d_\omega M$. For example, the condition $\dim_{\mathrm{H}} X = p$ is not sufficient for that, since there

are known examples of self-similar compact spaces X with $\dim_{\mathrm{H}} X = 1$ and $H^1(X) = 0$ (see [Sch]). For self-similar $X \subset \mathbb{R}^n$ the condition $H^p(X) > 0$, where p is the similarity dimension, is equivalent to OSC as it is proved in [Sch].

Proof of theorem 0.6. We prove following to [Hu, Theorem 5.3(1)], that the lower density of the measure $d_\omega M$ is separated from zero uniformly for each $x \in X$. This gives the first conclusion of the theorem and, in particular, the finiteness of H^p. The second conclusion follows from $\mathcal{S}$-invariance of H^p, the proof of which is based on the finitenes of H^p and the equality $H^p \circ j = h^p H^p$ for a similar contraction $j : X \to X$ with the coefficient h.

According to lemma 3.6, the self-similarity structure $\mathcal{S} = \{J_\alpha \,|\, \alpha \in A\}$ of M induces the continuous maps $j_\alpha : X \to X$, which are homotheties with the coefficients h_α, $\alpha \in A$ with respect to the Connes metric d_M. Moreover, $d\,(j_\alpha(y), j_\alpha(y')) = h_\alpha d(y, y')$ for each pair $b = (y, y')$ lying in the support B of M.

It follows from the condition $X = \operatorname{supp} d_\omega M$ that $X = \cup_{\alpha\in A} j_\alpha(X)$. Indeed, the set $\Delta X = X \setminus \cup_{\alpha\in A} j_\alpha(X)$ is open. Suppose that the support of a function $a \in \mathcal{C}(X)$ lies in ΔX. Viewing a as a function on X^2, $a(x, x') = a(x)$, we obtain that $a(b) = 0$ for all $b \notin \Delta B = B \setminus \cup_{\alpha\in A} \mathbf{j}_\alpha(B)$. Since ΔB is p-negligible, we have $d_\omega M(a) = 0$ and, consequently, ΔX contains no point of the support of the measure $d_\omega M$. Thus $\Delta X = \emptyset$.

Since $\operatorname{diam}_M(X) < \infty$, $d_M \geq d$ and j_α is a homothety of d_M with the coefficient $h_\alpha \in (0, 1)$, we have $\operatorname{diam}(j_w(X)) \to 0$ as $|w| \to \infty$. Thus the equality $X = \cup_{\alpha\in A} j_\alpha(X)$ implies that for each $x \in X$ and $\rho > 0$ there is a word $w \in W$ with $x \in j_w(X) \subset D_\rho(x)$ and $j_w(X) \subset j_{w'}(X) \not\subset D_\rho(x)$, where $D_\rho(x) = \{x' \in X \,|\, d(x, x') \leq \rho\}$, $w = w'\alpha$ for some $\alpha \in A$. In particular, $d_\omega M(D_\rho(x)) \geq d_\omega M(j_w(X))$ and $\operatorname{diam}(j_{w'}(X)) \geq \rho$. By properties of the maps j_α we have $\max_{b\in B} |b|/\operatorname{diam}_M(X) = \max_{b\in B} |\mathbf{j}_w(b)|/\operatorname{diam}_M(j_w(X))$ for each $w \in W$. Thus

$$\begin{aligned}
h_w \operatorname{diam}_M(X) &= \operatorname{diam}_M(j_w(X)) \geq \operatorname{diam}(j_w(X)) \\
&\geq \max_{b\in B} |\mathbf{j}_w(b)| \geq \min_{\alpha\in A} h_\alpha \max_{b\in B} |\mathbf{j}_{w'}(b)| \\
&\geq \min_{\alpha\in A} h_\alpha (\max_{b\in B} |b|/\operatorname{diam}_M(X)) \operatorname{diam}_M(j_{w'}(X)) \\
&\geq c' \operatorname{diam}(j_{w'}(X)) \geq c'\rho,
\end{aligned}$$

where $c' = \min_{\alpha\in A} h_\alpha(\max_{b\in B} |b|/\operatorname{diam}_M(X))$. Besides, $d_\omega M(j_w(X)) \geq h_w^p d_\omega M(X)$ and hence

$$d_\omega M(D_\rho(x)) \geq h_w^p d_\omega M(X) \geq c''\rho^p,$$

where the constant $c'' = d_\omega M(X)(c'/\operatorname{diam}_M(X))^p > 0$ depends neither on ρ, nor on x. Therefore, the lower density of $d_\omega M$ is separated from zero uniformly on X. Now it follows from [Fe, 2.10.19(3)] that $H^p \leq c \cdot d_\omega M$ for some $c \geq 0$. This proves the first assertion of the theorem.

Now we prove the second one. The equality $d_M = d$ implies that the maps j_α are homotheties and, hence, $H^p(j_\alpha(V)) = h_\alpha^p H^p(V)$ for each $V \subset X$. Thus

$$H^p(X) \leq \sum_{\alpha\in A} H^p(j_\alpha(X)) = \left(\sum_{\alpha\in A} h_\alpha^p\right) H^p(X) = H^p(X).$$

Since H^p is finite, this impies that $H^p(j_\alpha(X) \cap j_{\alpha'}(X)) = 0$ for each $\alpha \neq \alpha'$. Furthermore, it follows from $H^p \circ j_\alpha = h_\alpha^p H^p$ that $h_\alpha^p \int_X a \circ j_\alpha(X) dH^p = \int_{j_\alpha(X)} a \cdot dH^p$ for each $a \in \mathcal{C}(X)$. Thus

$$\begin{aligned}
\mathcal{S}(H^p)(a) &= \sum_{\alpha\in A} h_\alpha^p H^p(a \circ j_\alpha) = \sum_{\alpha\in A} h_\alpha^p \int_X a \circ j_\alpha dH^p \\
&= \sum_{\alpha\in A} \int_{j_\alpha(X)} a \cdot dH^p = \int_X a \cdot dH^p = H^p(a).
\end{aligned}$$

Therefore, the measure H^p is $\mathcal{S}$-invariant. It follows from the condition $H^p(X) > 0$ and lemmas 5.1.2, 6.2.4 that $H^p = c \cdot d_\omega M$, where $c > 0$ is independent of the choice of the limiting procedure ω by the ω-measurability of M. $\square$

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