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ON CUBIC METAPLECTIC EISENSTEIN SERIES ATTACHED TO THE JACOBI SUBGROUP OF $\mathbf{Sp_4(C)}$

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ABSTRACT

This paper is devoted to the study of cubic metaplectic Eisenstein series attached to the Jacobi maximal parabolic subgroup of the symplectic group $\mathrm{Sp}_4(\mathbb{C})$. One has found formulae of Eisenstein series Fourier coefficients. These formulae contains the summation over some double cosets, which representative system is described explicitly.

0. Introduction.

The aim of the present paper is the study of cubic metaplectic Eisenstein series on the symplectic group $\mathrm{Sp}_4(\mathbb{C})$ of the rank 2. The Eisenstein series we mean are cubic metaplectic forms under the congruence subgroup $\Gamma(\mathfrak{q})$ of $\mathrm{Sp}_4(\mathfrak{o})$ where $\mathfrak{o}$ is the integer ring of the field $\mathbb{Q}(\sqrt{-3})$. Cubic metaplectic Eisenstein series were studied by Proskurin (see [4], [5]), but Proskurin dealt with Eisenstein series attached to the Siegel maximal parabolic subgroup of the symplectic group $\mathrm{Sp}_4(\mathbb{C})$. We deal with Eisenstein series attached to the Jacobi subgroup of the symplectic group and we use the so-called *"sl(2)-triples"* technique (see [4], [5]) in order to evaluate their Fourier coefficients.

The paper is organized into four sections. The first section contains notations, definitions and an auxiliary proposition. In the section 2 we study in details the group $\Gamma(\mathfrak{q})$ and its subgroups and cosets. The most essential statements of this section are collected in Proposition 2.4 and Proposition 2.5. In the beginning of section 3 we give the necessary definitions concerning the metaplectic forms on $\mathrm{Sp}_4(\mathbb{C})/\mathrm{Sp}_4$, the Eisenstein series and their Fourier coefficients. Then our main result is stated in the Theorem. The rest of this section is devoted to the proof of this Theorem. In the section 4 we restrict our attention to the case of Bass–Milnor–Serre multiplier system and we use results of the section 2 for the study of series $\tilde{\mathcal{D}}_{\mu,\nu}$ that was obtained in the Theorem.

1. Preliminaries.

Let $\mathrm{Sp}_4(\mathbb{C}) = \{\gamma \in \mathrm{SL}_4(\mathbb{C}) | \gamma\sigma{}^t\gamma = \sigma\}$ be the symplectic group and

$$\sigma = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{1.1}$$

Given 4×4 matrix $\gamma \in \mathrm{SL}_4(\mathbb{C})$ let us write $\gamma = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$, where

$$\mathcal{A} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}, \quad \mathcal{D} = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} \tag{1.2}$$

are 2×2 matrices. Also we take e_n to be a $n \times n$ identity matrix. For $\gamma \in \mathrm{Sp}_4(\mathbb{C})$ we have $\gamma^{-1} = \begin{pmatrix} {}^t\mathcal{D} & -{}^t\mathcal{B} \\ -{}^t\mathcal{C} & {}^t\mathcal{A} \end{pmatrix}$. Hence we get that $\gamma^{-1} \in \mathrm{Sp}_4(\mathbb{C})$ if and only if

$${}^t\mathcal{B}\mathcal{D} = {}^t\mathcal{D}\mathcal{B}, \quad {}^t\mathcal{C}\mathcal{A} = {}^t\mathcal{A}\mathcal{C}, \quad {}^t\mathcal{A}\mathcal{D} - {}^t\mathcal{C}\mathcal{B} = {}^t\mathcal{D}\mathcal{A} - {}^t\mathcal{B}\mathcal{C} = e_2. \tag{1.3}$$

One has so-called Iwasawa decomposition $\mathrm{Sp}_4(\mathbb{C}) = NAK$, where

$$N = \{n(l) | l = (l_1, l_2, l_3, l_4) \in \mathbb{C}^4\}, \quad n(l) = \begin{pmatrix} 1 & l_1 & l_2 - l_1 l_4 & l_4 \\ 0 & 1 & l_4 - l_1 l_3 & l_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -l_1 & 1 \end{pmatrix}, \tag{1.4}$$

A is the group of $\mathrm{diag}(t_1, t_2, t_1^{-1}, t_2^{-1})$, $t_1, t_2 \in \mathbb{R}_+^*$, (1.5)

$K = \mathrm{Sp}_4 = \mathrm{Sp}_4(\mathbb{C}) \cap \mathrm{SU}_4$ is a maximal compact subgroup of $\mathrm{Sp}_4(\mathbb{C})$.

and the product mapping $N \times A \times K \to NAK$ is an analytic diffeomorphism.

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ we set

$$\begin{aligned} h_1(\gamma) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}, \quad h_2(\gamma) = \begin{pmatrix} a & 0 & 0 & b \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ c & 0 & 0 & d \end{pmatrix}, \\ h_3(\gamma) &= \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad h_4(\gamma) = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & d & -c \\ 0 & 0 & -b & a \end{pmatrix}. \end{aligned} \tag{1.6}$$

It is obvious that h_i are group monomorphisms $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_4(\mathbb{C})$.

Let T be the subgroup of all diagonal matrices in $\mathrm{Sp}_4(\mathbb{C})$ and $B = NT = TN$. Then B is a minimal parabolic subgroup of $\mathrm{Sp}_4(\mathbb{C})$ and we have Bruhat decomposition

$$\mathrm{Sp}_4(\mathbb{C}) = \bigcup_{r \in S} BrB, \tag{1.7}$$

where one can take $S = \{e_4,\ p,\ \tilde{p},\ p\tilde{p},\ \tilde{p}p,\ p\tilde{p}p,\ \tilde{p}p\tilde{p},\ \sigma\}$. Here

$$p = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \tilde{p} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{1.8}$$

and σ is the matrix in (1.1), which we can express also as $\sigma = (p\tilde{p})^2$.

Let $\tilde{P}$ be the group of all matrices in $\mathrm{Sp}_4(\mathbb{C})$ of the form

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & * & * & * \end{pmatrix}. \tag{1.9}$$

The so-defined group $\tilde{P}$ is a maximal parabolic subgroup of $\mathrm{Sp}_4(\mathbb{C})$ and it is known also as Jacobi subgroup.

The following proposition is auxiliary and it uses notations in (1.2), (1.4).

Proposition 1.1. *Let* $\gamma = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{C})$ *and* $d_1 \neq 0,\ d_1 d_4 - d_2 d_3 \neq 0$. *If* σ *is as in* (1.1) *then one has the decomposition*

$$\gamma\sigma = n(l)\mathrm{diag}(t_1, t_2, t_1^{-1}, t_2^{-1})\sigma n(l'), \tag{1.10}$$

where $l = (l_1, l_2, l_3, l_4) \in \mathbb{C}^4,\ l' = (l_1', l_2', l_3', l_4') \in \mathbb{C}^4$ *and*

$$\begin{aligned} t_1 &= \frac{1}{d_1}, \quad t_2 = \frac{d_1}{d_1 d_4 - d_2 d_3}, \quad l_1' = \frac{d_2}{d_1}, \\ l_3' &= \frac{c_2 d_3 - c_4 d_1}{d_1 d_4 - d_2 d_3}, \quad l_3 = \frac{b_4 d_1 - b_3 d_2}{d_1 d_4 - d_2 d_3}. \end{aligned} \tag{1.11}$$

Proof. It is not too difficult but very tedious to prove that the matrix $\gamma\sigma$ lies in the double coset $B\sigma B$. Hence we get the decomposition in (1.10). Then formulae in (1.11) can be obtained by direct multiplying of matrices. $\square$

2. Discrete subgroups.

Let $\mathfrak{q} \subset (3)$ be an ideal of integer ring $\mathfrak{o}$ of the field $\mathbb{Q}(\sqrt{-3})$ and let $\Gamma_n(\mathfrak{q})$ be the congruence subgroup mod $\mathfrak{q}$ of the group $\mathrm{SL}_n(\mathfrak{o})$, i.e. $\Gamma_n(\mathfrak{q})$ is the group of all matrices $(a_{ij}) \in \mathrm{SL}_n(\mathfrak{o})$ which satisfy the conditions

$$\begin{aligned} &a_{ii} \equiv 1 \pmod 3, \quad i = 1, \dots, n; \\ &a_{ij} \in \mathfrak{q}, \quad i, j = 1, \dots, n, \quad i \neq j. \end{aligned} \tag{2.1}$$

We define group $\Gamma(\mathfrak{q})$ as $\Gamma(\mathfrak{q}) = \Gamma_4(\mathfrak{q}) \cap \mathrm{Sp}_4(\mathbb{C})$ and we put $\Delta = \{n(l) | l \in \mathfrak{q}^4\}$, $\tilde{\Delta}_\times = \Gamma(\mathfrak{q}) \cap \tilde{P}$.

Proposition 2.1. *Each $\tilde{\gamma} \in \tilde{\Delta}_\times$ can be factored uniquely as $\tilde{\gamma} = \gamma'\gamma''$ with $\gamma' = n(*, *, 0, *) \in \Delta$, $\gamma'' = h_1(\lambda)$, $\lambda \in \Gamma_2(\mathfrak{q})$ and $\tilde{\Delta}_\times = \Delta h_1(\Gamma_2(\mathfrak{q}))$.*

Proof. Obviously the upper left entry of the matrix $\tilde{\gamma}$ is equal to 1. Conditions in (1.3) yield

$$\tilde{\gamma} = \begin{pmatrix} 1 & * & * & * \\ 0 & a & * & b \\ 0 & 0 & 1 & 0 \\ 0 & c & * & d \end{pmatrix}.$$

It is easy to see now that $\tilde{\gamma}$ has the decomposition $\tilde{\gamma} = \gamma' h_1(\lambda)$, where

$$\gamma' = \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{pmatrix} = n(*, *, 0, *) \in \Delta, \quad \lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

and h_1 is the monomorphism in (1.6). Then the formula $\tilde{\Delta}_\times = \Delta h_1(\Gamma_2(\mathfrak{q}))$ is obvious since Δ and $h_1(\Gamma_2(\mathfrak{q}))$ are subgroups of $\tilde{\Delta}_\times$. $\square$

Given $m \in \mathfrak{q}$, $n \in \mathfrak{o}$, $t \in \mathfrak{o}$, $t \neq 0$ with $n \equiv 1 \pmod 3$, $\gcd(m, n) = 1$ we choose the matrix $\lambda^{(t)}_{m,n} \in \Gamma(\mathfrak{q})$ in such a way that

$$\begin{aligned} &\text{(a) } \lambda^{(t)}_{m,n} = \begin{pmatrix} k & l \\ m & n \end{pmatrix} \text{ with } \gcd(l, t) = 1; \\ &\text{(b) if } m' \in \mathfrak{q} \text{ and } m' \equiv m \pmod n \text{ then } \lambda^{(t)}_{m',n} = \lambda^{(t)}_{m,n}. \end{aligned} \tag{2.2}$$

One can easily find that such choice is possible (but is not unique). We fix our choice of $\lambda^{(t)}_{m,n}$ once for all. For simplicity we will write $\lambda_{m,n}$ instead of $\lambda^{(1)}_{m,n}$.

Let $\tilde{c} = (m_2, n_2, m_3, n_3, m_4, n_4)$ and $\tilde{\Lambda}_\times$ be the set of all vectors $\tilde{c}$ that satisfy the conditions $m_i \in \mathfrak{q}$, $n_i \in \mathfrak{o}$, $n_i \equiv 1 \bmod 3$, $\gcd(m_i, n_i) = 1$, $i = 2, 3, 4$. For each pair (m_2, n_2), (m_3, n_3), (m_4, n_4) we choose the matrix λ_{m_2,n_2}, λ_{m_3,n_3}, $\lambda^{(n_2 n_3)}_{m_4,n_4}$ respectively. For $\tilde{c} \in \tilde{\Lambda}_\times$ we define $\gamma_{\tilde{c}}$ as the product $\gamma_{\tilde{c}} = \epsilon_2\epsilon_3\epsilon_4$ where

$$\epsilon_2 = h_2(\lambda_{m_2,n_2}), \quad \epsilon_3 = h_3(\lambda_{m_3,n_3}), \quad \epsilon_4 = h_4(\lambda^{(n_2 n_3)}_{m_4,n_4}) \tag{2.3}$$

and h_i are monomorphisms in (1.6).

Let us notice that:

$$\epsilon_2\epsilon_3 = \begin{pmatrix} k_2k_3 & 0 & k_2l_3 & l_2 \\ l_2m_3 & k_2 & l_2n_3 & 0 \\ n_2m_3 & m_2 & n_2n_3 & 0 \\ m_2k_3 & 0 & m_2l_3 & n_2 \end{pmatrix}, \ \epsilon_3\epsilon_4 = \begin{pmatrix} k_3k_4 & k_3l_4 & l_3n_4 & -l_3m_4 \\ m_4 & n_4 & 0 & 0 \\ m_3k_4 & m_3l_4 & n_3n_4 & -n_3m_4 \\ 0 & 0 & -l_4 & k_4 \end{pmatrix},$$

$$\epsilon_4^{-1}\epsilon_3^{-1} = \begin{pmatrix} n_3n_4 & -l_4 & -l_3n_4 & 0 \\ -n_3m_4 & k_4 & l_3m_4 & 0 \\ -m_3k_4 & 0 & k_3k_4 & m_4 \\ -m_3l_4 & 0 & k_3l_4 & n_4 \end{pmatrix}, \tag{2.4}$$

$$\epsilon_2\epsilon_3\epsilon_4 = \begin{pmatrix} k_2k_3k_4 & k_2k_3l_4 & k_2l_3n_4 - l_2l_4 & l_2k_4 - k_2l_3m_4 \\ l_2m_3k_4 + k_2m_4 & l_2m_3l_4 + k_2n_4 & l_2n_3n_4 & -l_2n_3m_4 \\ n_2m_3k_4 + m_2m_4 & n_2m_3l_4 + m_2n_4 & n_2n_3n_4 & -n_2n_3m_4 \\ m_2k_3k_4 & m_2k_3l_4 & m_2l_3n_4 - n_2l_4 & n_2k_4 - m_2l_3m_4 \end{pmatrix}.$$

Proposition 2.2. *Each coset in $\tilde{\Delta}_\times\backslash\Gamma(\mathfrak{q})$ contains some matrix $\gamma_{\tilde{c}}, \tilde{c} \in \tilde{\Lambda}_\times$.*

Proof. We only have to show for every $\gamma \in \Gamma(\mathfrak{q})$ that $\gamma = \delta\gamma_{\tilde{c}}$, where $\delta \in \tilde{\Delta}_\times$. Let

$$\gamma = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & d_1 & d_2 \\ * & * & * & * \end{pmatrix} \in \Gamma(\mathfrak{q})$$

and choose the matrix $\lambda = \begin{pmatrix} k_4 & l_4 \\ m_4 & n_4 \end{pmatrix} \in \Gamma_2(\mathfrak{q})$, where $m_4 = -\dfrac{d_2}{d}$, $n_4 = \dfrac{d_1}{d}$, $\gcd(l_4, d) = 1$, $d = \gcd(d_1, d_2)$. Obviously $(d_1, d_2) \cdot {}^t\lambda = (d, 0)$, thus we get $\gamma = \gamma' h_4(\lambda)$ and the vector $c_3(\gamma') = (*, *, d, 0)$ is the third row of the matrix γ'. Denote by m_2 the second entry of $c_3(\gamma')$ and by n_2 the greatest common divisor of the first and third entries of $c_3(\gamma')$. So, we have $c_3(\gamma') = (n_2m_3, m_2, n_2n_3, 0)$, where $m_i \in \mathfrak{q}$, $n_i \in \mathfrak{o}$, $n_i \equiv 1 \bmod 3$, $\gcd(m_i, n_i) = 1$, $i = 2, 3$. Clearly $d = n_2n_3$.

Next define matrices $\epsilon_2 = h_2(\lambda_{m_2,n_2})$, $\epsilon_3 = h_3(\lambda_{m_3,n_3})$, $\epsilon_4 = h_4(\lambda^{(n_2n_3)}_{m_4,n_4})$. Then $\gamma' = \tilde{\gamma}\epsilon_2\epsilon_3$ (see formulae in (2.4)) and the vector $c_3(\gamma') = (0, 0, 1, 0)$ is the third row of matrix $\tilde{\gamma} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \in \Gamma(\mathfrak{q})$. Now let us apply conditions in (1.3) to the matrix $\tilde{\gamma}$. Since $\mathcal{C}^t\mathcal{D} = \mathcal{D}^t\mathcal{C}$ bottom row of the matrix $\tilde{\gamma}$ is the vector $c_4(\tilde{\gamma}) = (0, *, *, *)$. Since ${}^t\mathcal{A}\mathcal{C} = {}^t\mathcal{C}\mathcal{A}$ either second row of $\tilde{\gamma}$ is equal to the vector $c_2(\tilde{\gamma}) = (0, *, *, *)$ or $\mathcal{C} = 0$. In the first case we have $\tilde{\gamma} \in \tilde{\Delta}_\times$. In the second case the condition ${}^t\mathcal{A}\mathcal{D} - {}^t\mathcal{C}\mathcal{B} = e_2$ leads to $\mathcal{A} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, $\mathcal{D} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$, that is $\tilde{\gamma} \in \tilde{\Delta}_\times$ too. Consequently $\gamma = \tilde{\gamma}\epsilon_2\epsilon_3\epsilon_4$ and each coset $\tilde{\Delta}_\times\backslash\Gamma(\mathfrak{q})$ contains some matrix $\gamma_{\tilde{c}}$. $\square$

Proposition 2.3. *Set $\{\gamma_{\tilde{c}} | \tilde{c} \in \tilde{\Lambda}_\times\}$ contains a unique representative of each coset in $\tilde{\Delta}_\times\backslash\Gamma(\mathfrak{q})$.*

Proof. The presence of the matrix $\gamma_{\tilde{c}}$ in each coset in $\tilde{\Delta}_\times\backslash\Gamma(\mathfrak{q})$ has been proved in the Proposition 2.2 and it is sufficient to prove uniqueness. Let $\tilde{c}, \tilde{c}' \in \tilde{\Lambda}_\times$ and $\gamma_{\tilde{c}}, \gamma_{\tilde{c}'}$ lie in one coset in $\tilde{\Delta}_\times\backslash\Gamma(\mathfrak{q})$. Then $\gamma_{\tilde{c}'} = \delta\gamma_{\tilde{c}}$, $\delta \in \tilde{\Delta}_\times$ and the third row of the matrix δ is the vector $c_3(\delta) = (0, 0, 1, 0)$. It yields that the third row of the matrix $\gamma_{\tilde{c}}$ and the third row of the matrix $\gamma_{\tilde{c}'}$ are same, hence $n_2n_3n_4 = n_2'n_3'n_4'$, $m_4n_2n_3 = m_4'n_2'n_3'$. Since $\gcd(m_4, n_4) = 1$ we get $\gcd(n_2n_3n_4, m_4n_2n_3) = n_2n_3$. Analogously $\gcd(n_2'n_3'n_4', m_4'n_2'n_3') = n_2'n_3'$, hence $n_2n_3 = n_2'n_3'$, $n_4 = n_4'$, $m_4 = m_4'$, consequently we have $\epsilon_4 = \epsilon_4', \ \epsilon_2\epsilon_3 = \delta\epsilon_2'\epsilon_3'$. Next put $\tilde{\epsilon}_3 = \epsilon_3{\epsilon'}_3^{-1}$ hence we get $\epsilon_2\tilde{\epsilon}_3 = \delta\epsilon_2'$. The third row of $\epsilon_2\tilde{\epsilon}_3$ is the vector $c_3(\epsilon_2\tilde{\epsilon}_3) = (n_2\tilde{m}_3, *, *, *)$, thus $n_2\tilde{m}_3 = 0$. But $n_2 \neq 0$ and it follows that $\tilde{m}_3 = 0, \ \tilde{n}_3 = 1, \ \tilde{\epsilon}_3 = e_4$ and $\epsilon_3 = \epsilon_3'$. Finally the condition $\epsilon_2 = \delta\epsilon_2'$ yields $\epsilon_2 = \epsilon_2'$ as claimed. □

Now we are going to describe a representative system for $\tilde{\Delta}_\times\backslash\Gamma(\mathfrak{q})/{}^t\Delta$ which we will use for the evaluation of Eisenstein series Fourier coefficients.

Let $\tilde{n} = (n_2, n_3, n_4)$ and $n_i \equiv 1 \bmod 3, \ \gcd(n_i, \mathfrak{q}) = 1$. Given $\tilde{n}$ we choose subset $\tilde{\mathfrak{M}}_4(\tilde{n})$ of $\mathfrak{q}$ which contains a unique representative of each class mod n_4 coprime n_4, subset $\tilde{\mathfrak{M}}_3(\tilde{n})$ of $\mathfrak{q}$ which contains a unique representative of each class mod n_3 coprime n_3 and subset $\tilde{\mathfrak{M}}_2(\tilde{n})$ of $\mathfrak{q}$ which contains a unique representative of each class mod n_2n_3 coprime n_2.

Proposition 2.4. *The family of matrices $\gamma = \epsilon_2\epsilon_3\epsilon_4, \ $ where $\epsilon_2 = h_2(\lambda_{m_2,n_2}), \ \epsilon_3 = h_3(\lambda_{m_3,n_3})$, $\epsilon_4 = h_4(\lambda^{(n_2n_3)}_{m_4,n_4}), \ \ m_i \in \tilde{\mathfrak{M}}_i(\tilde{n}), \ \ i = 2,3,4$ is the representative system of the double coset $\tilde{\Delta}_\times\backslash\Gamma(\mathfrak{q})/{}^t\Delta$.*

We will prove this proposition in following steps. Let us assume $(m_2, n_2, m_3, n_3, m_4, n_4) \in \tilde{\Lambda}_\times$.

Lemma 2.1. *Let $v = (0, v_2, v_3, v_4), \ z = n_3(-v_2n_4l_4 - v_3m_4k_4 + v_4(k_4n_4 + m_4l_4)), \ \ v_2n_4^2 + v_3m_4^2 - 2v_4n_4m_4 = 0$ and $v_i \in \mathfrak{q}$. If we put $\delta = \epsilon_3\epsilon_4{}^t n(v)\epsilon_4^{-1}\epsilon_3^{-1}$, where $n(v)$ and ϵ_i are defined in* (1.4) *and* (2.3) *respectively, then we have*

$$\delta = \begin{pmatrix} * & * & * & * \\ 0 & 1 & 0 & 0 \\ 0 & z & 1 & 0 \\ * & * & * & * \end{pmatrix}. \tag{2.5}$$

Proof. Obvious (but tedious) computation. □

The next lemma gives explicit expressions of variables v, z involved into the Lemma 2.1.

Lemma 2.2. *Let $v = (0, v_2, v_3, v_4), \ z = n_3(-v_2n_4l_4 - v_3m_4k_4 + v_4(k_4n_4 + m_4l_4))$ and $v_i \in \mathfrak{q}$. Then $v_2n_4^2 + v_3m_4^2 - 2v_4n_4m_4 = 0$ if and only if one has $v_4 = x, \ v_3 = -(2xl_4 + yn_4)n_4$, $v_2 = (2xk_4 + ym_4)m_4, \ z = n_3(x(k_4n_4 + m_4l_4) + yn_4m_4)$ for some $x, y \in \mathfrak{q}$.*

Proof. Since $\gcd(n_4, m_4) = 1$ the equation $v_2n_4^2 + v_3m_4^2 - 2v_4n_4m_4 = 0$ is equivalent to $v_2 = \tilde{v}_2m_4, \ v_3 = \tilde{v}_3n_4$ and $\tilde{v}_2n_4 + \tilde{v}_3m_4 = 2v_4$. From the equation $k_4n_4 - m_4l_4 = 1$ we obtain that $(\tilde{v}_2 - 2v_4k_4)n_4 + (\tilde{v}_3 + 2v_4l_4)m_4 = 0$. It yields $\tilde{v}_2 = 2v_4k_4 + ym_4, \ \tilde{v}_3 = -2v_4l_4 - yn_4$ for some $y \in \mathfrak{q}$. Substanding these expressions into the formula of z we get $z = n_3(-v_2n_4l_4 - v_3m_4k_4 + v_4(k_4n_4 + m_4l_4)) = n_3(x(k_4n_4 + m_4l_4) + yn_4m_4)$ as required. □

Proof of existence in the Proposition 2.4. Let $\mathcal{Z}$ be a double coset in $\tilde{\Delta}_\times\backslash\Gamma(\mathfrak{q})/{}^t\Delta$. According to Proposition 2.2 one exists some matrix $\nu_2\nu_3\nu_4 \in \mathcal{Z}$, where $\nu_2 = h_2(\lambda_{m_2,n_2}), \ \ \nu_3 = h_3(\lambda_{m_3,n_3}), \ \nu_4 = h_4(\lambda^{(n_2n_3)}_{m_4,n_4})$. Put

$$\epsilon_4 = \nu_4h_4\left(\begin{pmatrix} 1 & 0 \\ \xi_4 & 1 \end{pmatrix}\right) = h_4(\lambda^{(n_2n_3)}_{m_4',n_4}) \tag{2.6}$$

then $m_4' = m_4 + \xi_4n_4 \in \tilde{\mathfrak{M}}_4(\tilde{n})$ with suitably chosen ξ_4 and $\nu_2\nu_3\epsilon_4 \in \mathcal{Z}$. Let

$$\epsilon_3 = \nu_3h_3\left(\begin{pmatrix} 1 & 0 \\ \xi_3 & 1 \end{pmatrix}\right) = h_3(\lambda_{m_3',n_3}), \tag{2.7}$$

where $m_3' = m_3 + \xi_3 n_3 \in \tilde{\mathfrak{M}}_3(\tilde{n})$ with some ξ_3, hence we obtain $\nu_2\epsilon_3 h_3\left(\begin{pmatrix} 1 & 0 \\ -\xi_3 & 1\end{pmatrix}\right)\epsilon_4 \in \mathcal{Z}$. Since $h_3\left(\begin{pmatrix} 1 & 0 \\ -\xi_3 & 1\end{pmatrix}\right)\epsilon_4 = \epsilon_4\delta$ with

$$\delta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\xi_3 k_4^2 & -\xi_3 l_4 k_4 & 1 & 0 \\ -\xi_3 l_4 k_4 & -\xi_3 l_4^2 & 0 & 1 \end{pmatrix} \in {}^t\Delta, \tag{2.8}$$

we get $\nu_2\epsilon_3\epsilon_4 \in \mathcal{Z}$. Given v, z as in the Lemma 2.1 we have $\nu_2\epsilon_3\epsilon_4{}^t n(v) = \nu_2\delta\epsilon_3\epsilon_4$,

$$\nu_2\delta = \begin{pmatrix} k_2 & 0 & 0 & l_2 \\ 0 & k_2 & l_2 & 0 \\ 0 & m_2 & l_2 & 0 \\ m_2 & 0 & 0 & l_2 \end{pmatrix}\begin{pmatrix} * & * & * & * \\ 0 & 1 & 0 & 0 \\ 0 & z & 1 & 0 \\ * & * & * & * \end{pmatrix} = \begin{pmatrix} * & * & * & * \\ 0 & k_2' & l_2 & 0 \\ 0 & m_2' & n_2 & 0 \\ * & * & * & * \end{pmatrix} = \tilde{\delta}\epsilon_2. \tag{2.9}$$

Here $m_2' = m_2 + zn_2$, $\epsilon_2 = h_2(\lambda_{m_2', n_2})$, $\tilde{\delta} \in \tilde{\Delta}_\times$, hence $\epsilon_2\epsilon_3\epsilon_4 \in \mathcal{Z}$. Lemma 2.2 yields $z = n_3(x(k_4 n_4 + m_4 l_4) + y n_4 m_4)$ and since $\gcd(k_4 n_4 + m_4 l_4, n_4 m_4) = 1$ we obtain that $m_2' \in \tilde{\mathfrak{M}}_2(\tilde{n})$ for suitably chosen x and y. Thus existence in the Proposition 2.4 is proved. $\square$

Proof of uniqueness in the Proposition 2.4. Let $\gamma = \epsilon_2\epsilon_3\epsilon_4$ and $\gamma' = \epsilon_2'\epsilon_3'\epsilon_4'$ lies in one double coset $\tilde{\Delta}_\times \backslash \Gamma(\mathfrak{q}) / {}^t\Delta$ then $\gamma' = \lambda\gamma\delta$ where $\lambda \in \tilde{\Delta}_\times$, $\delta \in {}^t\Delta$, i.e.

$$\lambda = \begin{pmatrix} 1 & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ 0 & \lambda_{22} & \lambda_{23} & \lambda_{24} \\ 0 & 0 & 1 & 0 \\ 0 & \lambda_{42} & \lambda_{43} & \lambda_{44} \end{pmatrix}, \quad \delta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v & 1 & 0 & 1 \\ * & * & 1 & -v \\ * & * & 0 & 1 \end{pmatrix}. \tag{2.10}$$

Here $\epsilon_2 = h_2(\lambda_{m_2,n_2})$, $\epsilon_3 = h_3(\lambda_{m_3,n_3})$, $\epsilon_4 = h_4(\lambda^{(n_2n_3)}_{m_4,n_4})$ and $\epsilon_2' = h_2(\lambda_{m_2',n_2'})$, $\epsilon_3' = h_3(\lambda_{m_3',n_3'})$, $\epsilon_4' = h_4(\lambda^{(n_2'n_3')}_{m_4',n_4'})$. It follows from (2.4) and (2.10) that $n_2'n_3'n_4' = n_2n_3n_4$ and $n_2'n_3'm_4' = n_2n_3m_4 + vn_2n_3n_4$ where $n_i \neq 0$. Since $\gcd(n_4, m_4) = 1$ we get $n_2'n_3' | n_2n_3$; analogously $n_2n_3 | n_2'n_3'$. Thus $n_4' = n_4$, $m_4' = m_4 + vn_4$. But $m_4', m_4 \in \tilde{\mathfrak{M}}_4(\tilde{n})$, consequently $m_4 = m_4'$ and $\epsilon_4 = \epsilon_4'$.

Next, it is easy to see that $\epsilon_2'\epsilon_3' = \lambda\epsilon_2\epsilon_3\epsilon_4\delta\epsilon_4^{-1} = \lambda\epsilon_2\epsilon_3\delta'$ where

$$\delta' = \begin{pmatrix} \delta_{11} & \delta_{12} & 0 & 0 \\ \delta_{21} & \delta_{22} & 0 & 0 \\ * & * & \delta_{33} & \delta_{34} \\ * & * & \delta_{43} & \delta_{44} \end{pmatrix}, \quad \lambda\epsilon_2\epsilon_3\delta' = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & \delta_{34}n_2n_3 \\ * & * & * & * \end{pmatrix}. \tag{2.11}$$

These formulae yield $\delta_{34}n_2n_3 = 0$ and $\delta_{34} = 0$. But $\delta', \lambda\epsilon_2\epsilon_3\delta' \in \mathrm{Sp}_4(\mathbb{C})$ and we get

$$\delta' = \begin{pmatrix} 1 & v' & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \delta_{11}' & \delta_{12}' & 1 & 0 \\ \delta_{21}' & \delta_{22}' & -v' & 1 \end{pmatrix}, \quad \lambda\epsilon_2\epsilon_3\delta' = \begin{pmatrix} * & * & * & * \\ * & * & * & \lambda_{24}n_2 \\ * & * & * & 0 \\ * & * & * & * \end{pmatrix}. \tag{2.12}$$

Then it follows from formulae in (2.4) that $\lambda_{24} = 0$ and since $\lambda \in \mathrm{Sp}_4(\mathbb{C})$ we have

$$\lambda = \begin{pmatrix} 1 & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ 0 & 1 & \lambda_{23} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \lambda_{42} & \lambda_{43} & 1 \end{pmatrix}, \quad \lambda\epsilon_2\epsilon_3 = \begin{pmatrix} * & * & * & * \\ * & * & l_2n_3 + \lambda_{23}n_2n_3 & 0 \\ n_2m_3 & m_3 & n_2n_3 & 0 \\ * & * & * & n_2 \end{pmatrix}.$$

It yields $n_2'n_3' = n_2n_3,\ n_2' = n_2,\ n_2'm_3' = n_2m_3 + \delta_{11}'n_2n_3$. Consequently we get $n_3' = n_3,\ m_3' = m_3 + \delta_{11}'n_3$. But $m_3', m_3 \in \tilde{\mathfrak{M}}_3(\tilde{n})$, thus we have $m_3' = m_3,\ \epsilon_3' = \epsilon_3$.

It remainds to prove $m_2' = m_2$. We have obtained in (2.11) and (2.12) that

$$\delta' = \begin{pmatrix} \mathcal{A} & 0 \\ 0 & {}^t\!\mathcal{A}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ v & 1 & 0 & 0 \\ \delta_{11} & \delta_{12} & 1 & -v \\ \delta_{21} & \delta_{22} & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{A} & 0 \\ 0 & {}^t\!\mathcal{A}^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & v' & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \delta_{11}' & \delta_{12}' & 1 & 0 \\ \delta_{21}' & \delta_{22}' & -v' & 1 \end{pmatrix},$$

where $\mathcal{A} = \begin{pmatrix} k_4 & l_4 \\ m_4 & n_4 \end{pmatrix}$. But $\begin{pmatrix} k_4 & l_4 \\ m_4 & n_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \begin{pmatrix} k_4 & -m_4 \\ -l_4 & n_4 \end{pmatrix} = \begin{pmatrix} * & * \\ n_4^2 v & * \end{pmatrix}$, consequently $n_4^2 v = 0$ and $v = 0,\ v' = 0$. If we compare the second entries of the third rows of matrices $\epsilon_2'\epsilon_3$ and $\lambda\epsilon_2\epsilon_3\delta'$ we will obtain $m_2' = m_2 + n_2n_3\delta_{12}'$. Since $m_2', m_2 \in \tilde{\mathfrak{M}}_2(\tilde{n})$ we have $m_2' = m_2$. That is $\epsilon_i' = \epsilon_i, i = 2, 3, 4$ as required. $\square$

For $\gamma \in \Gamma(\mathfrak{q})$ we define the group $\tilde{\Delta}^\gamma$ by ${}^t\tilde{\Delta}^\gamma = \gamma^{-1}\Delta_\times\gamma \cap {}^t\Delta$. It is easy to check that $\tilde{\Delta}^\gamma = \{\delta \in \Delta | \gamma^t\delta\gamma^{-1} \in \tilde{\Delta}_\times\}$ and we will describe structure of this group in the following proposition.

Proposition 2.5. *Let* $\gamma = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$ *as in* (1.2) *and* $\tilde{\theta} = (d_1/\gcd(d_1,d_2))^2$ *then*

$$\tilde{\Delta}^\gamma = \left\{ n\left(0, \frac{d_2^2}{d_1^2}l_3, l_3, -\frac{d_2}{d_1}l_3\right) \middle| \, l_3 \in \tilde{\theta}\mathfrak{q} \right\}. \tag{2.13}$$

Proof. Let $\delta = n(l_1, l_2, l_3, l_4) \in \Delta$ and $\Omega = \begin{pmatrix} l_2 - l_1l_4 & l_4 - l_1l_3 \\ l_4 & l_3 \end{pmatrix}$, $L = \begin{pmatrix} 0 & 0 \\ l_1 & 0 \end{pmatrix}$, $M = \begin{pmatrix} 0 & -l_1 \\ 0 & 0 \end{pmatrix}$ are 2×2 matrices. Then we can express $\gamma^t\delta\gamma^{-1}$ as

$$\begin{aligned} \gamma^t\delta\gamma^{-1} &= \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \begin{pmatrix} L & 0 \\ \Omega & M \end{pmatrix} \begin{pmatrix} {}^t\mathcal{D} & -{}^t\mathcal{B} \\ -{}^t\mathcal{C} & {}^t\mathcal{A} \end{pmatrix} + e_4 = \\ &= \begin{pmatrix} \mathcal{A}L^t\mathcal{D} + \mathcal{B}\Omega^t\mathcal{D} - \mathcal{B}M^t\mathcal{C}, & -\mathcal{A}L\mathcal{B} - \mathcal{B}\Omega^t\mathcal{B} + \mathcal{B}M^t\mathcal{A} \\ \mathcal{C}L^t\mathcal{D} + \mathcal{D}\Omega^t\mathcal{D} - \mathcal{D}M^t\mathcal{C}, & -\mathcal{C}L^t\mathcal{B} - \mathcal{D}\Omega^t\mathcal{B} + \mathcal{D}M^t\mathcal{A} \end{pmatrix} + e_4. \end{aligned} \tag{2.14}$$

If $\gamma^t\delta\gamma^{-1} \in \tilde{\Delta}_\times$ then the matrix $\mathcal{C}L^t\mathcal{D} + \mathcal{D}\Omega^t\mathcal{D} - \mathcal{D}M^t\mathcal{C}$ is symmetric. Since $L = -{}^tM$ it yields $\mathcal{D}^t\Omega^t\mathcal{D} = \mathcal{D}\Omega^t\mathcal{D}$, but $\det\mathcal{D} \neq 0$ consequently $\Omega = {}^t\Omega$ and $l_1l_3 = 0$. Either $l_1 = 0$ or $l_3 = 0$ we deal with two cases:

1) Let $l_1 = 0$. Then $\gamma^t\delta\gamma^{-1} \in \tilde{\Delta}_\times$ if and only if $\mathcal{D}\Omega^t\mathcal{D} = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}$ (we recall that this matrix is symmetric). It is equivalent to the equation $\Omega^t\mathcal{D} = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}$, i.e.

$$\begin{pmatrix} l_2 & l_4 \\ l_4 & l_3 \end{pmatrix} \begin{pmatrix} d_1 & d_3 \\ d_2 & d_4 \end{pmatrix} = \begin{pmatrix} d_1l_2 + d_2l_4 & * \\ d_1l_4 + d_2l_3 & * \end{pmatrix} = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}. \tag{2.15}$$

If $d_2 = 0$ then $l_2 = l_4 = 0,\ l_3 \in \mathfrak{q}$ and $l_2 = d_2^2/d_1^2 \cdot l_3,\ l_4 = -d_2/d_1 \cdot l_3$ otherwise. Consequently we obtain

$$\tilde{\Delta}^\gamma \supset \left\{ n\left(0, \frac{d_2^2}{d_1^2}l_3, l_3, -\frac{d_2}{d_1}l_3\right) \middle| \, l_3 \in \tilde{\theta}\mathfrak{q} \right\}, \tag{2.16}$$

where $\tilde{\theta} = (d_1/\gcd(d_1, d_2))^2$.

2) In the case $l_3 = 0$ put $l_1 \neq 0$, then $\tilde{\Delta}^\gamma \supset n(l_1, *, 0, *)$. On the other hand the proof in the case $l_1 = 0$ leads to the conclusion that $\tilde{\Delta}^\gamma \supset n(0, *, l_3', *)$ for some $l_3' \neq 0$. But $\tilde{\Delta}^\gamma$ is group and we have $\tilde{\Delta}^\gamma \supset n(l_1, *, 0, *) \cdot n(0, *, l_3', *) = n(l_1, l_2', l_3', l_4')$. We have obtained the condition $\Omega = {}^t\Omega$, where $\Omega = \begin{pmatrix} l_2' - l_1 l_4' & l_4' - l_1 l_3' \\ l_4' & l_3' \end{pmatrix}$ and consequently $l_1 l_3' = 0$ for $l_1 \neq 0$, $l_3' \neq 0$. This contradiction shows that $l_1 = 0$ and we have described structure of $\tilde{\Delta}^\gamma$ entirely in the first part of the proof. Thus the Proposition 2.5 is proved. $\square$

We define for the ideal $\tilde{\theta}\mathfrak{q}$ a domain $\tilde{\mathcal{F}}_\gamma = \mathbb{C}^2 \times \mathbb{C}/\tilde{\theta}\mathfrak{q} \times \mathbb{C}$ in $\mathbb{C}^4$. Let a set $\tilde{\Omega} \subset \Gamma(\mathfrak{q})$ contains a unique representative of each double coset in $\tilde{\Delta}_\times \backslash \Gamma(\mathfrak{q}) / {}^t\Delta$, a set $\tilde{\Lambda}_\gamma$ contains a unique representative of each coset in ${}^t\tilde{\Delta}^\gamma / {}^t\Delta$ then we define a set

$$\tilde{\mathcal{P}} = \{(\gamma, h) | \gamma \in \tilde{\Omega}, h \in \tilde{\Lambda}_\gamma\}. \tag{2.17}$$

Proposition 2.6.

1) The mapping $\tilde{\mathcal{P}} \to \tilde{\Delta}_\times \backslash \Gamma(\mathfrak{q})$ defined by $(\gamma, h) \mapsto \tilde{\Delta}_\times \gamma h$ is bijective;

2) $\sigma^{-1} \tilde{\Lambda}_\gamma \sigma$ contains a unique representative of each coset in $\tilde{\Delta}^\gamma \backslash \Delta$;

3) $n(\tilde{\mathcal{F}}_\gamma) = \{n(l_1, l_2, l_3, l_4) | (l_1, l_2, l_3, l_4) \in \tilde{\mathcal{F}}_\gamma\}$ contains a unique representative of each coset in $\Delta^\gamma \backslash N$.

Proof.

1) To prove injectivity let $\overline{\gamma_1 h_1} = \overline{\gamma_2 h_2}$ then one exists some $\delta \in \tilde{\Delta}_\times$ such that $\gamma_1 h_1 = \delta \gamma_2 h_2$. Thus $\gamma_1 = \delta \gamma_2 h_2 h_1^{-1}$, $h_1, h_2 \in {}^t\Delta$, hence $\gamma_1 = \gamma_2 = \gamma$. Next $\gamma \delta^{-1} \gamma^{-1} = h_2 h_1^{-1} \in \gamma \tilde{\Delta}_\times \gamma^{-1}$ and it yields $h_2 h_1^{-1} \in {}^t\tilde{\Delta}^\gamma$. Thus we receive $h_1 = h_2$ as claimed. To prove surjectivity let $\alpha \in \tilde{\Delta}_\times \backslash \Gamma(\mathfrak{q})$ then $\underline{\alpha = \gamma\delta,}\ \gamma \in \tilde{\Omega}$, $\delta \in {}^t\Delta$ and $\delta = \omega h$, $\omega \in {}^t\tilde{\Delta}^\gamma$, $h \in \tilde{\Lambda}_\gamma$. Next $\gamma\omega = \mu\gamma$ where $\mu \in \tilde{\Delta}_\times$ and $\alpha = \overline{\gamma\omega h} = \overline{\gamma h}$;

2) This is obvious;

3) Let $\tilde{\mathcal{D}}_\gamma$ be the fundamental domain of the group $\tilde{\Delta}^\gamma \backslash N$. Obviously $\tilde{\mathcal{D}}_\gamma \subset \tilde{\mathcal{F}}_\gamma$. To prove that $\tilde{\mathcal{D}}_\gamma \supset \tilde{\mathcal{F}}_\gamma$ choose $z_1, z_2 \in \tilde{\mathcal{F}}_\gamma$ such that z_1 is equavalent to z_2 in $\tilde{\mathcal{D}}_\gamma$. Then $z_1 = z_2 + n\left(0, d_2^2/d_1^2 \cdot l_3,\ l_3, -d_2/d_1 \cdot l_3\right)$, $l_3 \in \tilde{\theta}\mathfrak{q}$, $l_3 \neq 0$ and $z_1^{(3)} = z_2^{(3)} + l_3$. But $z_1^{(3)}, z_2^{(3)} \in \mathbb{C}/\tilde{\theta}\mathfrak{q}$ consequently $l_3 = 0$ and $z_1 = z_2$ as required. $\square$

3. Eisenstein series Fourier coefficients.

We set $\mathbb{X} = \mathbb{C}^4 \times \mathbb{R}_+^{*2}$, considering the right-side as a product of real analytic manifolds. For $w = (z_1, z_2, z_3, z_4, u, v) \in \mathbb{X}$ we set

$$\begin{aligned} &n(w) = n(z_1, z_2, z_3, z_4), \quad a(w) = \mathrm{diag}(uv^{1/2}, v^{1/2}, u^{-1}v^{-1/2}, v^{-1/2}), \\ &U(w) = u, \quad V(w) = v, \quad Z_i = z_i, \quad Q(w) = (z_1, u), \quad \tilde{Q}(w) = (z_3, v). \end{aligned} \tag{3.1}$$

According to Iwasawa we have the real analytic manifolds isomorphism

$$\mathbb{X} \to \mathrm{Sp}_4(\mathbb{C})/\mathrm{Sp}_4, \quad w \mapsto n(w)a(w)\mathrm{Sp}_4 \tag{3.2}$$

which transfers on $\mathbb{X}$ the action by left multiplication of the group $\mathrm{Sp}_4(\mathbb{C})$ on $\mathrm{Sp}_4(\mathbb{C})/\mathrm{Sp}_4$. We will write γw for the image of the point $w \in \mathbb{X}$ under the action of $\gamma \in \mathrm{Sp}_4(\mathbb{C})$.

Let Γ be a discrete subgroup of $\mathrm{Sp}_4(\mathbb{C})$ and $\tilde{\psi}:\Gamma\to\mathbb{C}^*$ be an unitary character. We say $F:\mathbb{X}\to\mathbb{C}$ is an automorphic form under Γ with multiplier system $\tilde{\psi}$ if F is a $\mathbb{C}^\infty$–function satisfying the conditions

(a) $F(\gamma w)=\tilde{\psi}(\gamma)F(w)$ for all $\gamma\in\Gamma, w\in\mathbb{X}$;

(b) F is an eigenfunction of all the invariant differential operators on $\mathbb{X}$;

(c) There is $c\in\mathbb{R}$ such that $|F(w)|<\mathrm{Norm}(n(w)a(w))^c$ for all $w\in\mathbb{X}$.

In part (c), by $\mathrm{Norm}(\delta)$ for $\delta\in\mathrm{Sp}_4(\mathbb{C})$ we mean $\mathrm{trace}(\delta^t\bar{\delta})$.

Put $\Gamma=\Gamma(\mathfrak{q})$, $\mathfrak{q}\subset(3)$ and let Lebesgue measurable set $\mathcal{F}_\mathfrak{q}\subset\mathbb{C}^4$ be such that its image under the isomorphism of real analytic manifolds

$$(z_1,z_2,z_3,z_4)\mapsto n(z_1,z_2,z_3,z_4) \tag{3.3}$$

contains a unique representative of each coset in $\Delta\backslash N$. We write $\mathrm{vol}(\mathcal{F}_\mathfrak{q})$ for the volume of $\mathcal{F}_\mathfrak{q}$ and set $c=\mathrm{vol}(\mathcal{F}_\mathfrak{q})^{-1}$. It is obvious that $\mathrm{vol}(\mathcal{F}_\mathfrak{q})$ does not depend on the particular choice of $\mathcal{F}_\mathfrak{q}$. Now assume that Δ is contained in the kernel of the homomorphism $\tilde{\psi}$. Given $\mu,\nu\in\mathfrak{q}^*$, let

$$c_{\mu,\nu}(u,v;F)=c\iiiint\limits_{\mathcal{F}_\mathfrak{q}}F(w)\overline{e(\mu z_1+\nu z_3)}dz_1dz_2dz_3dz_4, \tag{3.4}$$

with $w\in\mathbb{X}$. This integral does not depend on the particular choice of $\mathcal{F}_\mathfrak{q}$ since Δ is contained in both the kernel of $\tilde{\psi}$ and the kernel of the homomorphism

$$N\to\mathbb{C}^*,\quad n(l_1,l_2,l_3,l_4)\mapsto e(\mu l_1+\nu l_3). \tag{3.5}$$

We call $c_{\mu,\nu}(u,v;F)$ the Fourier coefficients of the form F.

If F is an automorphic form under $\Gamma(\mathfrak{q})$ and $\lambda\in\mathrm{Sp}_4(\mathbb{C})$ is any matrix satisfying the conditions

$$\lambda\Gamma(\mathfrak{q})\lambda^{-1}=\Gamma(\mathfrak{q})\text{ and }\tilde{\psi}(\lambda\gamma\lambda^{-1})=\tilde{\psi}(\gamma)\text{ for all }\gamma\in\Gamma, \tag{3.6}$$

then the function $w\mapsto F(\lambda w)$, $w\in\mathbb{X}$ also is an automorphic form under the group $\Gamma(\mathfrak{q})$ with the same multiplier system $\tilde{\psi}$ as F is. This fact can be easily checked.

In a well known manner (see [2] for details) one can realize $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2$ as $\mathbb{H}=\mathbb{C}\times\mathbb{R}_+^*$. The action by left translations of the group $\mathrm{SL}_2(\mathbb{C})$ on $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2$ transfers on $\mathbb{H}$ and we will write γw for the image of w under γ. For this action one has

$$\gamma w=\left(\frac{(az+b)\overline{(cz+d)}+a\bar{c}v^2}{|cz+d|^2+|c|^2v^2},\frac{v}{|cz+d|^2+|c|^2v^2}\right), \tag{3.7}$$

where $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{C})$, $w=(z,v)\in\mathbb{H}$.

Let $f:\mathbb{H}\to\mathbb{C}$ be an automorphic form under the group $\Gamma_2(\mathfrak{q})$ with multiplier system $\tilde{k}(\gamma)=\tilde{\psi}(h_4(\gamma))$ and $\mathrm{D_{L-B}}f=r(2-r)f$, where $\gamma\in\Gamma_2(\mathfrak{q})$ and $\mathrm{D_{L-B}}$ is the Laplace-Beltrami operator. Then for f one has a Fourier expansion

$$f(\begin{pmatrix}0&-1\\1&0\end{pmatrix}(z,v))=c_0(v)+\sum_{\nu\in\mathfrak{q}^*\backslash\{0\}}\rho(\nu)vK_{r-1}(4\pi|\nu|v)e(\nu z),\ (z,v)\in\mathbb{H}. \tag{3.8}$$

Here K_m is the Bessel–MacDonald function with $z, m \in \mathbb{C},\ \Re(z) > 0,\ \rho(\nu) \in \mathbb{C}$, and, with some $\rho_+, \rho_- \in \mathbb{C}$, we have either $c_0(v) = \rho_+ v^r + \rho_- v^{2-r}$ or $c_0(v) = \rho_+ v \log v + \rho_- v$, according to $r \neq 1$ or $r = 1$ (see for example [5], part 0.3). In accordance with Selberg-Langlands theory one can attach to f and Jacobi maximal parabolic subgroup $\tilde{P}$ series

$$\tilde{E}(w, s; f) = \sum_{\gamma} \overline{\tilde{\psi}}(\gamma) U(\gamma w)^{2s} V(\gamma w)^s \bar{f}(\tilde{Q}(\gamma w)), \tag{3.9}$$

where $w \in \mathbb{X},\ s \in \mathbb{C}$ and the summation is carried out over a set of matrices $\gamma \in \Gamma(\mathfrak{q})$ that contains a unique representative of each coset in $\tilde{\Delta}_\times \backslash \Gamma(\mathfrak{q})$ (see [3]). In framework of the Selberg-Langlands theory this series is the maximal parabolic Eisensein series attached to the form f and the Jacobi maximal parabolic subgroup $\tilde{P}$. It is known that the maximal parabolic Eisensein series converges absolutely and locally uniformly in $\mathbb{X} \times \{s \in \mathbb{C} \mid \Re(s) > \tau\}$ for some τ and given $w \in \mathbb{X}$, the function $\tilde{E}(w, s; f)$ is regular on $\mathbb{X} \times \{s \in \mathbb{C} \mid \Re(s) > \tau\}$ and can be extended meromorphically to $\mathbb{C}$.

We will evaluate the Fourier coefficients $c_{\mu,\nu}(u, v, \tilde{E}(\sigma(\cdot), s; f))$, where $w \mapsto \tilde{E}(\sigma w, s; f),\ w \in \mathbb{X}$ is an automorphic form under the group $\Gamma(\mathfrak{q})$ with multiplier system $\tilde{\psi}$, as well as the Eisenstein series $\tilde{E}(\cdot, s; f)$ itself, because of we have $\sigma \Gamma(\mathfrak{q}) \sigma^{-1} = \Gamma(\mathfrak{q})$ and $\tilde{\psi}(\sigma\gamma\sigma^{-1}) = \tilde{\psi}(\gamma)$ for all $\gamma \in \Gamma(\mathfrak{q})$. Immediately from the definition we get

$$\begin{aligned} \tilde{c}_{\mu,\nu} &= c_{\mu,\nu}(u, v, \tilde{E}(\sigma(\cdot), s; f)) = \\ &= c \iiiint\limits_{\mathcal{F}_\mathfrak{q}} \tilde{E}(\sigma w, s; f) \overline{e(\mu z_1 + \nu z_3)} dz_1 dz_2 dz_3 dz_4 \\ &= c \sum_{\gamma} \overline{\tilde{\psi}}(\gamma) \iiiint\limits_{\mathcal{F}_\mathfrak{q}} \tilde{\triangledown}(\gamma \sigma w) \overline{e(\mu z_1 + \nu z_3)} dz_1 dz_2 dz_3 dz_4. \end{aligned} \tag{3.10}$$

The summation is carried out over some set of matrices γ that contains a unique representative of each coset in $\tilde{\Delta}_\times \backslash \Gamma(\mathfrak{q})$; $c = \mathrm{vol}(\mathcal{F}_\mathfrak{q})^{-1}$, $\mathcal{F}_\mathfrak{q}$ denotes $(\mathbb{C}/\mathfrak{q})^4$ and

$$\tilde{\triangledown}(w) = (U^2 V)(w)^s \bar{f}(\tilde{Q}(w)), \quad w = (z_1, z_2, z_3, z_4, u, v) \in \mathbb{X}. \tag{3.11}$$

Also we will use notations $m_+ = m\bar{\rho}_+,\ m_- = m\bar{\rho}_-,\ m = \mathrm{vol}(\mathbb{C}/\mathfrak{q})^{-3}$.

Theorem. *For the Eisenstein series Fourier coefficients following formulae hold:*

$$\tilde{c}_{\mu,0} = \begin{cases} m_+ \tilde{\mathcal{D}}_{\mu,0}(s, r) \tilde{\mathcal{J}}_{\mu,0}(u, v, s, r) + m_- \tilde{\mathcal{D}}_{\mu,0}(s, 2-r) \tilde{\mathcal{J}}_{\mu,0}(u, v, s, 2-r), \\ m_+ \left. \dfrac{\partial}{\partial t} \tilde{\mathcal{D}}_{\mu,0}(s, t) \tilde{\mathcal{J}}_{\mu,0}(u, v, s, t) \right|_{t=1} + m_- \tilde{\mathcal{D}}_{\mu,0}(s, 1) \tilde{\mathcal{J}}_{\mu,0}(u, v, s, 1), \end{cases} \tag{3.12}$$

according to $r \neq 0$ or $r = 0$ and

$$\tilde{c}_{\mu,\nu} = m \tilde{\mathcal{D}}_{\mu,\nu}(s) \tilde{\mathcal{J}}_{\mu,\nu}(u, v, s, r) \ \textit{for}\ \nu \neq 0\ \textit{and any}\ r, \tag{3.13}$$

where

$$\tilde{\mathcal{D}}_{\mu,0}(s, t) = \sum_{\gamma} \overline{\tilde{\psi}}(\gamma) e(\mu l_1') |t_1^2|^s |\tilde{\theta}|^2 \frac{1}{|t_2^2|^t}; \tag{3.14}$$

$$\tilde{\mathcal{D}}_{\mu,\nu}(s)=\sum_{\gamma}\overline{\tilde{\psi}}(\gamma)e(\mu l_1'+\nu l_3')\bar{\rho}(-\nu t_2^2)|t_1^2|^s|\tilde{\theta}|^2\frac{1}{|t_2^2|}; \tag{3.15}$$

$$\begin{aligned}&\tilde{\mathcal{J}}_{\mu,\nu}(u,v,s,t)=\\ &=\iiint\limits_{\mathbb{C}^3}\left(U^{2s}V^{s+t}\right)(\hat{\sigma}\tilde{w})\overline{e(\mu z_1)}dz_1dz_2dz_4 \text{ if } \nu=0 \text{ and}\\ &=\iiint\limits_{\mathbb{C}^3}\left(U^{2s}V^{s+1}\right)(\hat{\sigma}\tilde{w})K_{\bar{t}-1}(4\pi|\nu|V(\hat{\sigma}\tilde{w}))\overline{e(\mu z_1-\nu Z_3(\hat{\sigma}\tilde{w}))}dz_1dz_2dz_4\end{aligned} \tag{3.16}$$

otherwise. Here $\tilde{w}=(z_1,z_2,0,z_4,u,v)\in\mathbb{X}$,

$$\begin{gathered}\tilde{\theta}=\left(\frac{d_1}{\gcd(d_1,d_2)}\right)^2,\quad t_1=\frac{1}{d_1},\quad t_2=\frac{d_1}{d_1d_4-d_2d_3},\\ l_1'=\frac{d_2}{d_1},\quad l_3'=\frac{c_2d_3-c_4d_1}{d_1d_4-d_2d_3}\end{gathered} \tag{3.17}$$

and

$$\hat{\sigma}=\begin{pmatrix}0&0&-1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix}. \tag{3.18}$$

The summation in the series $\tilde{\mathcal{D}}_{\mu,0}(s,r)$ *and* $\tilde{\mathcal{D}}_{\mu,\nu}(s)$ *is over some set of matrices* $\gamma\in\Gamma(\mathfrak{q})$*, where* $\gamma=\begin{pmatrix}\mathcal{A}&\mathcal{B}\\\mathcal{C}&\mathcal{D}\end{pmatrix}$ *as defined in* (1.2). *This set contains a unique representative of each double coset in* $\tilde{\Delta}_\times\backslash\Gamma(\mathfrak{q})/{}^t\Delta$ *and it is chosen so that* $b_4d_1-b_3d_2=0$*. All series and integrals are absolutely convergent whenever* $\Re(s)$ *is sufficiently large and it is assumed that* $\rho(\lambda)=0$ *if* $\lambda\notin\mathfrak{q}^*$.

We will prove this theorem with the help of some auxiliary lemmas.

Lemma 3.1. *(a) If* $\gamma=n(l),\ l=(l_1,l_2,l_3,l_4)\in\mathbb{C}^4$ *then*

$$Z_3(\gamma w)=z_3+l_3,\quad U(\gamma w)=u,\quad V(\gamma w)=v,\quad \tilde{Q}(\gamma w)=(z_3+l_3,v); \tag{3.19}$$

(b) If $\gamma=\operatorname{diag}(t_1,t_2,t_1^{-1},t_2^{-1}),\ t_1,t_2\in\mathbb{C}^*$ *then*

$$Z_3(\gamma w)=z_3t_2^2,\ U(\gamma w)=|t_1t_2^{-1}|u,\ V(\gamma w)=|t_2^2|v,\ \tilde{Q}(\gamma w)=(t_2^2z_3,|t_2^2|v). \tag{3.20}$$

Proof. See ([5] prop. 2.3.1). □

Lemma 3.2. *Let* $\gamma\in h_1(\alpha),\ \alpha=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{C})$*. Then we have*

$$\left(U^2V\right)(\gamma w)=u^2v,\quad \tilde{Q}(\gamma w)=\begin{pmatrix}a&b\\c&d\end{pmatrix}\tilde{Q}(w). \tag{3.21}$$

Proof. See ([5] prop.2.3.4). □

Lemma 3.3. *Let* $\tilde{w} = (z_1, z_2, 0, z_4, u, v)$*,* $a \in \mathbb{C}^*$*. Then* $(U^2V)(\sigma w) = (U^2V)(\hat{\sigma}\tilde{w})$*,*

$$(a^2 Z_3(\sigma w), |a^2| V(\sigma w)) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(\frac{1}{a^2}(z_3 + Z_3(\hat{\sigma}\tilde{w})), \frac{1}{|a^2|} V(\hat{\sigma}\tilde{w}) \right). \tag{3.22}$$

Proof. If we put $\tilde{p} = h_1(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})$, $\tilde{\delta} = n(0, 0, z_3, 0)$, $\tilde{\delta}' = h_1(\begin{pmatrix} 1 & z_3 \\ 0 & 1 \end{pmatrix})$ then we get $\sigma = \tilde{p}\hat{\sigma}$, $\hat{\sigma}\tilde{\delta} = \tilde{\delta}'\hat{\sigma}$. Hence one has $\hat{\sigma}n(w)a(w) = \tilde{\delta}'\hat{\sigma}n(\tilde{w})a(\tilde{w})$ and $\hat{\sigma}w = \tilde{\delta}'\hat{\sigma}\tilde{w}$. Thus by using Lemma 3.2 we obtain

$$(U^2V)(\sigma w) = (U^2V)(\tilde{p}\hat{\sigma}w) = (U^2V)(\hat{\sigma}w) = (U^2V)(\tilde{\delta}'\hat{\sigma}\tilde{w}) = (U^2V)(\hat{\sigma}\tilde{w}), \tag{3.23}$$

$$\begin{aligned} \tilde{Q}(\sigma w) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tilde{Q}(\hat{\sigma}w) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tilde{Q}(\tilde{\delta}'\hat{\sigma}\tilde{w}) = \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & z_3 \\ 0 & 1 \end{pmatrix} \tilde{Q}(\hat{\sigma}\tilde{w}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (z_3 + Z_3(\hat{\sigma}\tilde{w}), V(\hat{\sigma}\tilde{w})). \end{aligned} \tag{3.24}$$

Next, in the equation

$$(a^2 Z_3(\sigma w), |a^2| V(\sigma w)) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(\frac{1}{a^2}(z_3 + Z_3(\hat{\sigma}\tilde{w})), \frac{1}{|a^2|} V(\hat{\sigma}\tilde{w}) \right) \tag{3.25}$$

the left side is nothing but else $\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} (Z_3(\sigma w), V(\sigma w))$, the right side is equal to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/a & 0 \\ 0 & a \end{pmatrix} (z_3 + Z_3(\hat{\sigma}\tilde{w}), V(\hat{\sigma}\tilde{w}))$. This concludes the lemma. $\square$

Proof of Theorem. Immediately from the definition we get

$$\tilde{c}_{\mu,\nu} = c \sum_{\gamma} \overline{\tilde{\psi}}(\gamma) \sum_{h} \iiiint\limits_{\tilde{\mathcal{F}}_{\mathfrak{q}}} \tilde{\bigtriangledown}(\gamma\sigma h w) \overline{e(\mu z_1 + \nu z_3)} dz_1 dz_2 dz_3 dz_4, \tag{3.26}$$

where the summation is over $\gamma \in \tilde{\Omega}$, $h \in \sigma^{-1}\tilde{\Lambda}_\gamma \sigma$, consequently

$$\tilde{c}_{\mu,\nu} = c \sum_{\gamma} \overline{\tilde{\psi}}(\gamma) \iiiint\limits_{\tilde{\mathcal{F}}_{\gamma}} \tilde{\bigtriangledown}(\gamma\sigma w) \overline{e(\mu z_1 + \nu z_3)} dz_1 dz_2 dz_3 dz_4. \tag{3.27}$$

We can express $\gamma\sigma w$ as $n(l)\mathrm{diag}(t_1, t_2, t_1^{-1}, t_2^{-1})\sigma n(l')w = n\tilde{a}\sigma\tilde{n}w$ by formulae in the Proposition 1.1. This proposition yields $l_3 = (b_4 d_1 - b_3 d_2)/(d_1 d_4 - d_2 d_3) = 0$, thus by using Lemma 3.1 we obtain $(U^2V)(\gamma\sigma w) = |t_1^2|(U^2V)(\sigma\tilde{n}w)$ and

$$\tilde{Q}(\gamma\sigma w) = (Z_3(\tilde{a}\sigma\tilde{n}w), V(\tilde{a}\sigma\tilde{n}w)) = (t_2^2 Z_3(\sigma\tilde{n}w), |t_2^2| V(\sigma\tilde{n}w)). \tag{3.28}$$

Hence we get $\tilde{\bigtriangledown}(\gamma\sigma w) = |t_1^2|^s (U^2V)(\sigma\tilde{n}w)^s \bar{f}(t_2^2 Z_3(\sigma\tilde{n}w), |t_2^2| V(\sigma\tilde{n}w))$.

Now let us make a change of variables by the formula $\tilde{n}w = w$. Since

$$n(l_1, l_2, l_3, l_4)\ n(l_1', l_2', l_3', l_4') = n(t_1, t_2, t_3, t_4)$$

with $t_1 = l_1 + l_1',\ t_2 = l_2 + l_2' + l_1^2 l_3' + 2l_1 l_4',\ t_3 = l_3 + l_3',\ t_4 = l_4 + l_4' + l_1 l_3'$, we have

$$\begin{aligned}
\tilde{c}_{\mu,\nu} &= c \sum_{\gamma} \overline{\tilde{\psi}}(\gamma) e(\mu l_1' + \nu l_3') |t_1^2|^s \tilde{I}, \\
\tilde{I} &= \iiiint\limits_{\tilde{\mathcal{F}}_\gamma} (U^2 V)(\sigma w)^s \bar{f}(t_2^2 Z_3(\sigma w), |t_2^2| V(\sigma w)) \overline{e(\mu z_1 + \nu z_3)} dz_1 dz_2 dz_3 dz_4.
\end{aligned} \tag{3.29}$$

Next, by using Lemma 3.3 we get

$$\begin{aligned}
\tilde{I} &= \iiint\limits_{\mathbb{C}^3} (U^2 V)(\hat{\sigma}\tilde{w})^s \overline{e(\mu z_1)} \tilde{I}_3 dz_1 dz_2 dz_4, \\
\tilde{I}_3 &= \int\limits_{\mathbb{C}/\tilde{\theta}\mathfrak{q}} \bar{f}(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(\frac{1}{t_2^2}(z_3 + Z_3(\hat{\sigma}\tilde{w})), \frac{1}{|t_2^2|} V(\hat{\sigma}\tilde{w}) \right)) \overline{e(\nu z_3)} dz_3.
\end{aligned} \tag{3.30}$$

To evaluate the integral $\tilde{I}_3$ one would notice two facts. In the first place, formulae in (3.17) yield $\tilde{\theta}/t_2^2 \in \mathfrak{o}$. And secondly, one can check that

$$\int\limits_{\mathbb{C}/\tilde{\theta}\mathfrak{q}} e(\lambda z) dz = \begin{cases} \operatorname{vol}(\mathbb{C}/\mathfrak{q}) |\tilde{\theta}|^2, \text{ if } \lambda = 0, \\ 0, \text{ if } \lambda\tilde{\theta} \in \mathfrak{q}^*,\ \lambda \neq 0. \end{cases} \tag{3.31}$$

It follows from the Fourier expansion of f in (3.7) that

$$\begin{aligned}
& f(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(\frac{1}{t_2^2}(z_3 + Z_3(\hat{\sigma}\tilde{w})), \frac{1}{|t_2^2|} V(\hat{\sigma}\tilde{w}) \right)) = c_0 \left(\frac{1}{|t_2^2|} V(\hat{\sigma}\tilde{w}) \right) + \\
&+ \sum_{\kappa \in \mathfrak{q}^* \setminus \{0\}} \rho(\kappa) \frac{1}{|t_2^2|} V(\hat{\sigma}\tilde{w}) K_{r-1} \left(4\pi |\kappa| \frac{1}{|t_2^2|} V(\hat{\sigma}\tilde{w}) \right) e \left(\kappa \frac{1}{t_2^2} (z_3 + Z_3(\hat{\sigma}\tilde{w})) \right)
\end{aligned} \tag{3.32}$$

and if we substitute (3.32) into (3.30) then integrating term–by–term we have

$$\begin{aligned}
\overline{\tilde{I}_3} =& \operatorname{vol}(\mathbb{C}/\mathfrak{q}) |\tilde{\theta}|^2 \cdot \\
& \cdot \begin{cases} \rho_+ \frac{1}{|t_2^2|^r} V(\hat{\sigma}\tilde{w})^r + \rho_- \frac{1}{|t_2^2|^{2-r}} V(\hat{\sigma}\tilde{w})^{2-r},\ \nu = 0,\ r \neq 1 \\ \rho_+ \frac{1}{|t_2^2|} V(\hat{\sigma}\tilde{w}) \log \frac{1}{|t_2^2|} V(\hat{\sigma}\tilde{w}) + \rho_- \frac{1}{|t_2^2|} V(\hat{\sigma}\tilde{w}),\ \nu = 0,\ r = 1 \\ \rho(-\nu t_2^2) \frac{1}{|t_2^2|} V(\hat{\sigma}\tilde{w}) K_{r-1}(4\pi|\nu| V(\hat{\sigma}\tilde{w})) e(-\nu Z_3(\hat{\sigma}\tilde{w})),\ \nu \neq 0 \end{cases}
\end{aligned} \tag{3.33}$$

Then the final result can be obtained from this formula by simple computation. $\square$

4. On the series $\tilde{\mathcal{D}}_{\mu,\nu}$.

We have described in the Proposition 2.4 the representative system of the coset $\tilde{\Delta}_\times \backslash \Gamma(\mathfrak{q})/\Delta$ and we can apply this fact to give more manageable form for the series $\tilde{\mathcal{D}}_{\mu,\nu}$ that arose in the

Theorem. Indeed, by means of the Proposition 2.4 and Theorem sums in (3.14) and (3.15) can be rewritten as the sums over $n_2, n_3, n_4 \equiv 1 \bmod 3$, $\gcd(n_2n_3n_4, \mathfrak{q}) = 1$ and $m_2 \in \tilde{\mathfrak{M}}_2(\tilde{n}),\ m_3 \in \tilde{\mathfrak{M}}_3(\tilde{n}),\ m_4 \in \tilde{\mathfrak{M}}_4(\tilde{n}),\ \tilde{n} = (n_2, n_3, n_4)$. Also we can rewrite formulae in (3.17) as

$$\begin{gathered} \tilde{\theta} = n_4, \quad t_1 = \frac{1}{n_2n_3n_4}, \quad t_2 = \frac{n_4}{n_2}, \quad l_1' = -\frac{m_4}{n_4}, \\ l_3' = \frac{m_2^2 l_3 n_4^2}{n_2^2 n_3} - \frac{m_3 l_4^2}{n_3} - \frac{2m_2n_4l_4}{n_2n_3}. \end{gathered} \tag{4.1}$$

Let us now restrict our attention by the case when $\tilde{\psi}$ be the so-called Bass–Milnor–Serre's homomorphism of the group $\Gamma(3)$ and κ_2 be Kubota's homomorphism of the group $\Gamma_2(3)$ (see [1] and [5]). As it was stated by Proskurin (see [5], prop.2.2.9) $\tilde{\psi}(h_i(\lambda)) = \bar{\kappa}_2(\lambda),\ i = 1, 3,\ \tilde{\psi}(h_i(\lambda)) = \kappa_2(\lambda),\ i = 2, 4$. Thus we have:

$$\tilde{\mathcal{D}}_{\mu,0}(s,t) = \sum_{\substack{n_2,n_3,n_4 \equiv 1 \bmod 3 \\ \gcd(n_2n_3n_4,\mathfrak{q})=1}} \frac{\tilde{\mathcal{E}}(\mu, 0, \tilde{n})}{\|n_2\|^{s-t}\|n_3\|^{s}\|n_4\|^{s+t-1}}, \tag{4.2}$$

$$\tilde{\mathcal{D}}_{\mu,\nu}(s) = \sum_{\substack{n_2,n_3,n_4 \equiv 1 \bmod 3 \\ \gcd(n_2n_3n_4,\mathfrak{q})=1, \nu n_4^2/n_2^2 \in \mathfrak{q}^*}} \frac{\bar{\rho}(-\nu n_4^2/n_2^2)\tilde{\mathcal{E}}(\mu, \nu, \tilde{n})}{\|n_2\|^{s-1}\|n_3\|^{s}\|n_4\|^{s}}, \tag{4.3}$$

where $\|\cdot\|$ is a norm in the field $\mathbb{Q}(\sqrt{-3})$, $\rho(\lambda) = 0$ if $\lambda \notin \mathfrak{q}^*$,

$$\begin{aligned} \tilde{\mathcal{E}}(\mu, \nu, \tilde{n}) = \sum_{m_i \in \tilde{\mathfrak{M}}_i(\tilde{n})} \left(\frac{m_2}{n_2}\right) \overline{\left(\frac{m_3}{n_3}\right)} \left(\frac{m_4}{n_4}\right) \times \\ \times e\left(\nu\left(\frac{m_2^2 l_3 n_4^2}{n_2^2 n_3} - \frac{m_3 l_4^2}{n_3} - \frac{2m_2n_4l_4}{n_2n_3}\right) - \mu\frac{m_4}{n_4}\right) \end{aligned} \tag{4.4}$$

and $\left(\frac{\cdot}{\cdot}\right)$ is a cubic residue symbol. For $\lambda \in \mathfrak{q}^*, \mathfrak{q} \subset (3), c, c' \in 1 + \mathfrak{q}$ we set

$$S(\lambda, c, c') = \sum_{k \in \mathfrak{c}(c,c')} \left(\frac{k}{c}\right) \left(\frac{k}{c'}\right)_e (\lambda k/c) \quad \text{(Gauss sum)}, \tag{4.5}$$

where $e(z) = \exp(2\pi i(z+\bar{z}))$ and $\mathfrak{c}(c, c')$ is a reduced residue system $\bmod\ c$ composed of numbers coprime with c'. Let us assume that $\nu n_4^2/n_2^2 \in \mathfrak{q}^*$ therefore $\nu n_4/n_2 \in \mathfrak{q}^*$ too. Then we can prove following proposition.

Proposition 4.1. *The series $\tilde{\mathcal{E}}(\mu, \nu, \tilde{n})$ in (4.4) can be factored as*

$$\tilde{\mathcal{E}}(\mu, \nu, \tilde{n}) = S(\mu, n_4, n_2n_3) \cdot \mathcal{P}(\nu, \tilde{n}), \ \textit{where} \tag{4.6}$$

$$\mathcal{P}(\nu, \tilde{n}) = \sum_{m_2 \in \tilde{\mathfrak{M}}_2(\tilde{n})} \left(\frac{m_2}{n_2}\right) \overline{S\left(\nu\left(\frac{m_2^2 n_4^2}{n_2^2} + 2\frac{m_2n_4}{n_2} - 1\right), n_3, n_2^2\right)}. \tag{4.7}$$

Proof. Let us choose a set $\tilde{\mathfrak{M}}_3(\tilde{n})$ in such a way that for all $m_3 \in \tilde{\mathfrak{M}}_3(\tilde{n})$, one has $\gcd(m_3, n_2) = 1$. Also we can choose the set $\tilde{\mathfrak{M}}_4(\tilde{n})$ so that for all matrices $\lambda^{(n_2n_3)}_{m_4,n_4}$ we have $m_4l_4 \equiv -1$

(mod $n_2n_3n_4$) and $\gcd(m_4, n_2n_3) = 1$. Now let us make a change of variables $m_3 \equiv m_3' m_4^2$ (mod n_3), $m_2 \equiv m_2' m_4$ (mod n_2n_3). Then we have $l_3 \equiv l_3' \hat{m}_4^2$ (mod n_3) where $m_4\hat{m}_4 \equiv 1$ (mod n_2n_3). Therefore we obtain:

$$\begin{aligned}\tilde{\mathcal{E}}(\mu,\nu,\tilde{n}) = \sum_{m_2', m_3', m_4} \left(\frac{m_2'}{n_2}\right)\left(\frac{m_4}{n_2}\right)\overline{\left(\frac{m_3'}{n_3}\right)}\left(\frac{m_4}{n_3}\right)\left(\frac{m_4}{n_4}\right)\times \\ \times e\left(-\mu\frac{m_4}{n_4} + \nu\left(-\frac{m_3'}{n_3} + \frac{m_2'^2 l_3' n_4^2}{n_2^2 n_3} - 2\frac{m_2' n_4}{n_2n_3}\right)\right) = \\ = S(\mu, n_4, n_2n_3)\sum_{m_2, m_3}\left(\frac{m_2}{n_2}\right)\overline{\left(\frac{m_3}{n_3}\right)} e\left(\nu\left(-\frac{m_3}{n_3} + \frac{m_2^2 l_3 n_4^2}{n_2^2 n_3} - 2\frac{m_2 n_4}{n_2 n_3}\right)\right).\end{aligned} \tag{4.8}$$

Finally if we change variables $m_2 \equiv m_2' m_3$ (mod n_2n_3) then we get proposition. □

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