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The Theorem on Three Spheres for Harmonic Differential Forms

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ABSTRACT

We study a generalization of the Hadamard theorem on three circles to harmonic differential forms. An inequality for the L^2-norms of a harmonic form over concentric spheres is proved. Also, we obtain an estimate for the L^∞-norms.

1. Introduction

Let u be a harmonic differential form on the *spherical shell* $\{r_1 \le |x| \le r_2\} \subset \mathbf{R}^n$. By $\|u\|_R$ we denote the L^2-norm of the form u over the sphere $\{|x| = R\}$, i.e., $\|u\|_R^2 = R^{1-n} \int_S |u(Rx')|^2 ds(x')$. In the first part of the paper we prove that

$$\|u\|_r \le \|u\|_{r_1}^{\alpha} \|u\|_{r_2}^{1-\alpha}, \tag{1}$$

where $r_1 < r < r_2$ and $\alpha = \log(r_2/r)/\log(r_2/r_1)$, i.e., $r = r_1^{\alpha} r_2^{1-\alpha}$. In order to obtain this inequality, we represent the norm $\|u\|_r^2$ as a series $\sum_{-\infty}^{\infty} c_k r^k$ with nonnegative coefficients c_k. Then we give another proof of (1), using direct calculations of the derivatives of $\|u\|_r$ with respect to r. In the second part of the paper we employ (1) to obtain a rather rough estimate for the L^{∞}-norms.

Many results of this type have been proved for solutions of second order elliptic equations. We mention here the recent paper by Brummelhuis [B], the paper concerning harmonic functions by Korevaar and Meyers [KM], and the paper by Garofalo and Lin [GL]. In all these papers, inequalities similar to (1) and the corresponding inequalities for the L^{∞}-norms were considered for the functions satisfying a certain elliptic equation near the entire ball $\{|x| \le r_2\}$, while the classical Hadamard theorem is true for the functions analytic on the closed annulus $\{r_1 \le |x| \le r_2\}$. Note that the harmonic differential forms present a natural generalization of analytic functions. For a detailed discussion of this issue, see the paper [PH] by Presa Sagué and Havin.

Finally, we mention the paper [JP] by Janson and Peetre and the later paper [PS] by Peetre and Sjölin, where the three-line theorem was proved in the more general setting of the Clifford analysis.

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We use the usual notation d, $*$, δ, and Δ for the standard operators in $\mathbf{R}^n$. We denote by d_S, $*_S$, δ_S, and Δ_S the corresponding operators on the unit sphere $\mathbf{S}^{n-1}$. A differential form with C^1-smooth coefficients is said to be *harmonic* on an open set $O \subset \mathbf{R}^n$ if $du = 0$ and $\delta u = 0$ on O. It is easy to check that the coefficients of a harmonic form are harmonic functions, so that they are C^{∞}-smooth.

For $x \in \mathbf{R}^n$, we write $x = rx'$, where $r = \sqrt{x_1^2 + \ldots + x_n^2}$, $x' \in \mathbf{S}^{n-1}$. By $S(r)$ we denote the sphere $\{|x| = r\}$. A p-form u defined on a spherical shell can be uniquely decomposed into a sum $u(x) = u_t(x) + u_n(x) \wedge dr$, where $(u_t)_r(x') = u_t(rx')$ and $(u_n)_r(x') = u_n(rx')$ are forms on the unit sphere of degree p and $p-1$, respectively. We say that u_t is the *tangent* part of u and u_n is the *normal* part of u.

2. L^2-inequality

Let q be a positive integer, $0 < q < n-1$. We denote by L_q^2 the space of square integrable q-forms on the unit sphere $\mathbf{S}^{n-1}$. Consider the Laplace–Beltrami operator Δ_S on L_q^2. By the Hodge theorem, we have $L_q^2 = H_1^q + H_2^q$, where H_1^q (H_2^q) is the closure of the space of exact (coexact) forms in L_q^2. Observe that the spaces H_1^q and H_2^q are invariant under the action of the operator Δ_S. We shall use the following facts about the eigenvalues and eigenforms of the operator Δ_S (see [IK] and [P]). The eigenvalues of the Laplace–Beltrami operator Δ_S on the space of q-forms on the unit sphere are $\lambda_k = -(k+q)(n-q+k)$, $k = 0, 1, \ldots$, and $\mu_k = -(k+q+1)(n-q+k-1)$, $k = 0, 1, \ldots$, each eigenvalue being of finite multiplicity. The eigenforms corresponding to λ_k are closed, and the eigenforms corresponding to μ_k are coclosed. In L_q^2 we can choose a basis consisting of eigenforms of Δ_S. If $\{\beta_{kl}\}$, $k = 0, 1, \ldots$, $1 \le l \le N_k$, is an orthonormal basis of the space H_2^q such that $\Delta_S \beta_{kl} = \mu_k \beta_{kl}$, then $\{|\mu_k|^{-1/2} d_S \beta_{kl}\}$, $\quad k = 0, 1, \dots,\quad 1 \le l \le N_k$, is an orthonormal basis of the space H_1^{q+1}, and $\Delta_S(d_S\beta_{kl}) = \mu_k(d_S\beta_{kl})$ (here N_k is the multiplicity of the eigenvalue μ_k).

Let u be a harmonic p-form on O, where O is an open subset of $\mathbf{R}^n$ containing the spherical shell $K = \{r_1 \le |x| \le r_2\}$. Consider the tangent and the normal part of u, $u = u_t + u_n \wedge dr$. Suppose that $\{\beta_{kl}\}$ is a basis of the space H_2^{p-1} such that $\Delta_S \beta_{kl} = -(k+p)(n+k-p)\beta_{kl}$; then the forms u_t and u_n have the following decompositions in L_p^2 and in L_{p-1}^2, respectively.

Lemma 2.1. *There exist real numbers a_{kl} and b_{kl}, $k = 0, 1, \dots,\ 1 \le l \le N_k$, such that*

$$u_t(x) = (-1)^{p+1} \sum_{k,l} (a_{kl} r^{k+p} + b_{kl} r^{-n-k+p}) d_S \beta_{kl}(x'), \tag{2}$$

$$u_n(x) = \sum_{k,l} ((k+p) a_{kl} r^{k+p-1} - (n+k-p) b_{kl} r^{-n-k+p-1}) \beta_{kl}(x'). \tag{3}$$

The series on the right-hand sides are convergent in $L_p^2\,(S(r))$ and $L_{p-1}^2\,(S(r))$, respectively, for $r_1 \le r \le r_2$.

Proof. We rewrite the equations $du = 0$ and $\delta u = 0$ in terms of the forms u_t and u_n. Evidently, the equation $du = 0$ is equivalent to the system

$$0 = (du)_t = d_S u_t, \qquad 0 = (du)_n = d_S u_n + (-1)^p \partial u_t / \partial r. \tag{4}$$

Direct calculation shows that for any q-form ω we have $(*\omega)_n = r^{n-2q-1} *_S \omega_t$ and $(*\omega)_t = (-1)^{n-q} r^{n-2q+1} *_S \omega_n$. Hence, the equation $\delta u = 0$ can be written as follows:

$$0 = (\delta u)_t = r^{-2} \delta_S u_t + (-1)^p (n - 2p + 1) r^{-1} u_n + (-1)^p \partial u_n / \partial r, \tag{5}$$

$$0 = (\delta u)_n = r^{-2} \delta_S u_n. \tag{6}$$

Combined with (6), the first identity in (4) implies that $(u_t)_r \in H_1^p$ and $(u_n)_r \in H_2^{p-1}$. Thus, we can decompose the forms u_t and u_n into the series

$$u_t(x) = \sum_{k,l} f_{kl}(r) d_S \beta_{kl}(x'), \quad u_n(x) = \sum_{k,l} g_{kl}(r) \beta_{kl}(x'). \tag{7}$$

Obviously, the functions f_{kl} and g_{kl} are C^∞-smooth on $[r_1, r_2]$. We use the following notation for the scalar product in L_q^2. If $\phi, \psi \in L_q^2$, then

$$\langle \phi, \psi \rangle = \int_S \phi \wedge *\psi.$$

Clearly, $\langle \phi, \psi \rangle = \langle \psi, \phi \rangle$. The definition of the operator δ_S shows that $\langle d_S w, v \rangle = \langle w, \delta_S v \rangle$. Multiplying in L_p^2 the second identity in (4) by $d_S \beta_{kl}$, we get

$$0 = \langle d_S u_n, d_S \beta_{kl} \rangle + \langle (-1)^p \partial u_t / \partial r, d_S \beta_{kl} \rangle = \langle u_n, \delta_S d_S \beta_{kl} \rangle + (-1)^p \partial / \partial r \langle u_t, d_S \beta_{kl} \rangle.$$

By (7), this yields

$$g_{kl}(r) = (-1)^{p+1} f'_{kl}(r). \tag{8}$$

Next, we multiply (5) by β_{kl}, obtaining

$$0 = \langle \delta_S u_t, \beta_{kl} \rangle + (-1)^p (n-2p+1) r \langle u_n, \beta_{kl} \rangle + (-1)^p r^2 \langle \partial u_n \partial r, \beta_{kl} \rangle =$$

$$= (k+p)(n+k-p) f_{kl}(r) + (-1)^p \left((n-2p+1) r g_{kl} + r^2 g'_{kl} \right).$$

Substituting $(-1)^{p+1} f'_{kl}$ for g_{kl} in the above formula, we conclude that the function $f_{kl}(r)$ satisfies the differential equation

$$(k+p)(n+k-p) f(r) - (n-2p+1) r f'(r) - r^2 f''(r) = 0.$$

Then

$$f_{kl}(r) = (-1)^{p+1} (a_{kl} r^{k+p} + b_{kl} r^{-n-k+p}), \quad \text{where} \quad a_{kl}, b_{kl} \in \mathbf{R}. \tag{9}$$

Finally, (7), (8), and (9) imply (2).

Theorem 2.2. *Suppose that u is a p-form continuous on $\{r_1 \le |x| \le r_2\}$ and harmonic in $\{r_1 < |x| < r_2\}$; then inequality* (1) *is fulfilled.*

Proof. #1. The norm $\|u\|_r$ is continuous with respect to r on $[r_1, r_2]$. Therefore, in the proof of (1) we may assume that u is harmonic near the set $\{r_1 \le |x| \le r_2\}$. Then we can use Lemma 2.1. In order to compute the L^2-norm of the form u, we separate the normal and the tangent parts of u. Namely,

$$r^{1+2p-n} |u(rx')|^2 = r^{1+2p-n} * (u(rx') \wedge *u(rx')) =$$

$$= *(u_t(rx') \wedge *_S u_t(rx') \wedge dr + r^2 u_n(rx') \wedge *_S u_n(rx') \wedge dr).$$

Let ω be a differential form on the unit sphere. We put $\|\omega\|^2 = \int_S \omega \wedge *_S \omega$. We recall that $\|u\|_R^2 = R^{1-n} \int_S |u(Rx')|^2 ds(x')$. Thus, we have

$$\|u\|_r^2 = r^{-2p} \|(u_t)_r\|^2 + r^{-2p+2} \|(u_n)_r\|^2.$$

Since the $d_S \beta_{kl}$ are mutually orthogonal in L_p^2, the β_{kl} are mutually orthogonal in L_{p-1}^2, and $\|d_S \beta_{kl}\|^2 = (k+p)(n+k-p)$, $\|\beta_{kl}\|^2 = 1$, from (2) we deduce that

$$\|(u_t)_r\|_S^2 = \sum_{k,l} (k+p)(n+k-p)(a_{kl} r^{k+p} + b_{kl} r^{-n-k+p})^2,$$

$$\|(u_n)_r\|_S^2 = \sum_{k,l} \left((k+p) a_{kl} r^{k+p-1} - (n+k-p) b_{kl} r^{-n-k+p-1} \right)^2.$$

Consequently,

$$\|u\|_r^2 = \sum_{k,l} ((k+p)(n+2k) a_{kl}^2 r^{2k} + (n+k-p)(n+2k) b_{kl}^2 r^{-2n-2k}).$$

Now, it suffices to observe that the right-hand side of the above identity is a sum of logarithmically convex positive functions of the form $c_m r^m$. Hence, $\|u\|_r^2$ is also logarithmically convex, which proves inequality (1).

Proof. #2. Put $q(r) = r^{2p} \|u\|_r^2$. Inequality (1) means that q satisfies the following differential inequality:

$$r q''(r) q(r) + q'(r) q(r) - r (q'(r))^2 \ge 0 \tag{10}$$

((10) expresses the logarithmic convexity of q). Now we calculate the left-hand side of (10) with the help of (4) and (5). First, we note that

$$q(r) = \|(u_t)_r\|^2 + r^2\|(u_n)_r\|^2 = \langle u_t, u_t\rangle + r^2\langle u_n, u_n\rangle.$$

We omit the index r, having in mind that all forms actually depend on r, and we use f' to denote the differentiation of f with respect to r. We have

$$\begin{aligned} q'(r) = 2\langle u_t', u_t\rangle + 2r\langle u_n, u_n\rangle + 2\langle u_n', u_n\rangle = (-1)^{p+1}2\langle d_S u_n, u_t\rangle + \\ +2r\langle u_n, u_n\rangle + (-1)^{p+1}2\langle \delta_S u_t, u_n\rangle - 2(n-2p+1)r\langle u_n, u_n\rangle = \\ = (-1)^{p+1}4\langle d_S u_n, u_t\rangle - 2(n-2p)r\langle u_n, u_n\rangle. \end{aligned} \tag{11}$$

In the same way we calculate the second derivative:

$$\begin{aligned} q''(r) =&(-1)^{p+1}4\langle u_n', \delta_S u_t\rangle + (-1)^{p+1}4\langle d_S u_n, u_t'\rangle - 2(n-2p)\langle u_n, u_n\rangle + \\ &+4(n-2p)r\langle u_n', u_n\rangle = 4\langle \delta_S u_t, \delta_S u_t\rangle + (-1)^p 4(n-2p+1)\langle u_n, \delta_S u_t\rangle + \\ &+4\langle d_S u_n, d_S u_n\rangle + (-1)^p 4(n-2p)\langle \delta_S u_t, u_n\rangle - 2(n-2p)\langle u_n, u_n\rangle + \\ &+4(n-2p)(n-2p+1)\langle u_n, u_n\rangle = 4\langle \delta_S u_t, \delta_S u_t\rangle + 4\langle d_S u_n, d_S u_n\rangle + \\ &+(-1)^p 4(2n-4p+1)\langle \delta_S u_t, u_n\rangle + 2(n-2p)(2n-4p+1)\langle u_n, u_n\rangle. \end{aligned}$$

After trivial calculations, we get

$$\begin{aligned} A :=&\frac{r}{4}\left(rq''(r)q(r) + q'(r)q(r) - r(q'(r))^2\right) = r^2(n-2p)^2\|u_n\|^2\|u_t\|^2 + \\ &+2(-1)^p(n-2p)r\left(\|u_t\|^2 - r^2\|u_n\|^2\right)\langle \delta_S u_t, u_n\rangle + r^2\|d_S u_n\|^2\|u_t\|^2 + \\ &+r^2\|\delta_S u_t\|^2\|u_n\|^2 + r^4\| \, d_S u_n\|^2\|u_n\|^2 + \|\delta_S u_t\|^2\|u_t\|^2 - 4r\langle \delta_S u_t, u_n\rangle^2. \end{aligned}$$

Now we rearrange the terms of the sum above to show that A is positive:

$$\begin{aligned} A = &\left(r(n-2p)\|u_t\| \; \|u_n\| + (-1)^p\frac{\|u_t\|^2 - r^2\|u_n\|^2}{\|u_t\| \; \|u_n\|}\langle d_S u_n, u_t\rangle\right)^2 + \\ &+\frac{\| \, u_t\|^2}{\|u_n\|^2}\left(\|\delta_S u_t\|^2\|u_n\|^2 - \langle \delta_S u_t, u_n\rangle^2\right) + \\ &+r^4\frac{\|u_n\|^2}{\|u_t\|^2}\left(\|d_S u_n\|^2\|u_t\|^2 - \langle d_S u_n, u_t\rangle\right)^2 + \\ &+r^2\left(\|d_S u_n\|^2\|u_t\|^2 + \|\delta_S u_t\|^2\|u_n\|^2 - 2\langle \delta_S u_t, u_n\rangle\langle d_S u_n, u_t\rangle\right). \end{aligned} \tag{12}$$

Each summand on the right in (12) is indeed positive. The first is a perfect square, the second and the third are positive thanks to the Cauchy inequality, namely,

$$\langle \delta_S u_t, u_n\rangle \le \|\delta_S u_t\| \; \|u_n\| \qquad \text{and} \qquad \langle d_S u_n, u_t\rangle \le \|d_S u_n\| \; \|u_t\|.$$

To show that the last term in (12) is also positive, we can use the Cauchy inequality twice:

$$\|d_S u_n\|^2\|u_t\|^2 + \|\delta_S u_t\|^2\|u_n\|^2 \ge 2 \; \|d_S u_n\| \; \|u_t\| \; \|\delta_S u_t\| \; \|u_n\| \ge 2\langle \, d_S u_n, u_t\rangle\langle \delta_S u_t, u_n\rangle.$$

3. An estimate for the L^∞-norms

Using the L^2-inequality proved in the preceding section, we obtain an estimate for the L^∞-norms. This may be regarded as an illustration for "propagation of smallness", but the inequality we prove is rather rough. Our proof is similar to that in [KM]. Also, we wish to emphasize that in the case of an analytic function u the function $\log|u|$ is subharmonic, and the maximum principle implies the theorem on three circles for the sup-norms. However, no explicit form of a differential equation for $\log|u|$ is known for harmonic forms in $\mathbf{R}^n$, $n > 2$, although such an equation is known to exist (for the case of harmonic vector fields, see, e.g., [GH]).

For a form u defined near the sphere $S(r)$, we put $\|u\|_{r,\infty} = \sup_{|x|=r} |u(x)|$.

Theorem 3.1. *Let r be a fixed number in (r_1, r_2). Then for any p-form u continuous on $\{r_1 \le |x| \le r_2\}$ and harmonic on $\{r_1 < |x| < r_2\}$ we have*

$$\|u\|_{r,\infty} \le \max\{ \ \|u\|_{r_1,\infty}^{\beta} \|u\|_{r_2,\infty}^{1-\beta}, \ \|u\|_{r_1,\infty}^{\gamma} \|u\|_{r_2,\infty}^{1-\gamma} \ \} \tag{13}$$

with some constants $\beta = \beta(r_1, r, r_2)$ and $\gamma = \gamma(r_1, r, r_2)$, $0 < \beta < \gamma < 1$,.

Proof. Fixing a point $z \in S(r)$, we put $f(y) := *(u(y) \wedge *u(z))$. Then f is a harmonic function, $|f(y)| \le |u(y)||u(z)|$ for any $y \in \{r_1 \le |x| \le r_2\}$, and $|f(z)| = |u(z)|^2$. We choose $r_3 \in (r, r_2)$ and represent $f(z)$ by the Poisson integral of its boundary values on the spheres $S(r_1)$ and $S(r_3)$. Let $p(x, y)$ denote the Poisson kernel for the Laplace operator on the spherical shell $K_1 = \{r_1 \le |x| \le r_3\}$. Then

$$f(z) = \int_{S(r_1)} p(z,y) f(y) ds(y) + \int_{S(r_3)} p(z,y) f(y) ds(y). \tag{14}$$

The first integral can easily be estimated. Obviously, $p(z, y)$ is positive, and $\int_{S(r_1)} p(z, y)\, ds(y) \le \int_{\partial K_1} p(z, y) ds(y) = 1$. Hence,

$$\int_{S(r_1)} p(x,y) f(y) ds(y) \le sup_{|y|=r_1} |f(y)| \le |u(z)| \ \|u\|_{r_1,\infty}. \tag{15}$$

To estimate the second integral in (14) we apply the Cauchy–Schwartz inequality:

$$\int_{S(r_3)} p(z,y) f(y) ds(y) \le \left(\int_{S(r_3)} f(y)^2 ds(y) \int_{S(r_3)} p(z,y)^2 ds(y) \right)^{1/2} \le$$

$$\le r_3^{(n-1)/2} \|u\|_{r_3} |u(z)| \left(\max_{|y|=r_3} p(z,y) \right)^{1/2}.$$

We put $C_1 = r_3^{(n-1)/2} \left(\max_{|y|=r_3} p(z,y) \right)^{1/2}$ and $\beta_1 = \log(r_2/r_3)/\log(r_2/r_1)$. Now we use (1), having in mind that $\|u\|_\rho \le \|u\|_{\rho,\infty}$. This yields

$$\int_{S(r_3)} p(z,y) f(y) ds(y) \le C_1 |u(z)| \|u\|_{r_1,\infty}^{\beta_1} \|u\|_{r_2,\infty}^{1-\beta_1}. \tag{16}$$

Since $f(z) = |u(z)|^2$, from (14), (15), and (16) we get

$$|u(z)| \le \|u\|_{r_1,\infty} + C_1 \|u\|_{r_1,\infty}^{\beta_1} \|u\|_{r_2,\infty}^{1-\beta_1}. \tag{17}$$

The same argument shows that

$$|u(z)| \leq \|u\|_{r_2,\infty} + C_2 \|u\|^{\gamma_1}_{r_1,\infty} \|u\|^{1-\gamma_1}_{r_2,\infty}, \tag{18}$$

where $0 < \beta_1 < \alpha < \gamma_1 < 1$, and $\alpha = \log(r_2/r)/\log(r_2/r_1)$. The maximum principle applied to the harmonic function f implies yet another inequality for $|u(z)|$, namely,

$$f(z) \leq \frac{r^{2-n} - r_2^{2-n}}{r_1^{2-n} - r_2^{2-n}} \|f\|_{r_1,\infty} + \frac{r_1^{2-n} - r^{2-n}}{r_1^{2-n} - r_2^{2-n}} \|f\|_{r_2,\infty}.$$

Dividing both sides by $|u(z)|$ and denoting the first ratio on the right by c, we obtain

$$|u(z)| \leq c\|u\|_{r_1,\infty} + (1-c)\|u\|_{r_2,\infty}. \tag{19}$$

First, we assume that $\|u\|_{r_1,\infty} \leq \|u\|_{r_2,\infty}$ and compare estimates (17) and (19). Let t_1 be the smallest positive solution of the equation $t + C_1 t^{\beta_1} = ct + 1 - c$. We choose β such that $t_1^{\beta} = ct_1 + 1 - c$. It can easily be verified that $t^{\beta} \geq \min\{t + C_1 t^{\beta_1}, ct + 1 - c\}$ for $0 \leq t \leq 1$. Thus, we have

$$|u(z)| \leq \|u\|^{\beta}_{r_1,\infty} \|u\|^{1-\beta}_{r_2,\infty} \quad \text{whenever} \quad \|u\|_{r_1,\infty} \leq \|u\|_{r_2,\infty}.$$

Now, suppose that $\|u\|_{r_1,\infty} \geq \|u\|_{r_2,\infty}$. In this case we use (18) and (19). Let t_2 be the greatest solution of the equation $1 + C_2 t^{\gamma_1} = ct + 1 - c$. Then we introduce γ by $t_2^{\gamma} = ct_2 + 1 - c$, and claim that $t^{\gamma} \geq \min\{1 + C_2 t^{\gamma_1}, ct + 1 - c\}$ for $t > 1$. Consequently,

$$|u(z)| \leq \|u\|^{\gamma}_{r_1,\infty} \|u\|^{1-\gamma}_{r_2,\infty} \quad \text{whenever} \quad \|u\|_{r_1,\infty} \geq \|u\|_{r_2,\infty}.$$

This completes the proof.

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