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THE DISLOCATION STRESS FUNCTIONS FROM THE DOUBLE CURL T(3)-GAUGE EQUATION: LINEARITY AND A LOOK BEYOND

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ABSTRACT

$T(3)$-gauge model of defects based on the gauge Lagrangian quadratic in the gauge field strength is considered. The equilibrium equation of the medium is fulfilled by the double curl Kröner's ansatz for stresses. The problem of replication of the static edge dislocation along third axis is analysed under a special, though conventional, choice of this ansatz. The translational gauge equation is shown to constraint the functions parametrizing the ansatz (the stress functions) so that the resulting stress component σ_{33} is not that of the edge defect. Another translational gauge equation with the double curl differential operator is shown to reproduce both the stress functions, as well as the stress tensors, of the standard edge and screw dislocations. Non-linear extension of the newly proposed translational gauge equation is given to correct the linear defect solutions in next orders. New gauge Lagrangian is suggested in the Hilbert–Einstein form.

1. INTRODUCTION

Considerable attention has been paid in last years to various applications of the ISO (3) (ISO (3) is the group of rigid body $T(3) \otimes SO(3)$) gauge model of defects in (continuous) solids proposed in [1] and, as a more elaborated version, [2]. For instance, the gauge theory of continuum damage in solids [3], the gauge theory of elastic materials exhibiting a relaxation phenomenon [4], the gauge theory of "plastically incompressible" elastic–plastic non-dissipative medium [5] have been considered, as well as the problem of electronic states in solids containing isolated defects [6, 7]. All these applications [3–5, 7] have been mainly concerned with the translational sector of the model [1, 2], i.e., with the Lagrangian $\mathcal{L}_\phi$ quadratic in the gauge field strength postulated in [1, 2] to govern the translational gauge field.

The applications found look promising for the class of the $T(3)$-gauge models with the quadratic Lagrangians like $\mathcal{L}_\phi$. On the other hand, a conventional approach to the defects in solids, i.e., the defect theory [8] which is concerned with the Volterra-type solutions of incompatible elasticity theory, is widely acknowledged. The theory [8] deals with the singular dislocations and disclinations, as well as with their distributions, and it admits a number of reliable calculations in the field of mechanical problems of solid state physics [9] (for instance, and refs. therein). Besides, the use should be mentioned of the singular solutions [8] in the problem of sound propagation in elastic body with topological defects [10], and in such microscopic problem as particles diffusion in solid with randomly distributed defects [11]. For instance, the screw dislocation solution has been adapted both in [10] and [11].

Therefore it is seen that the conventional approach [8], as well as the non-conventional one [1, 2], are actively developed. However, with regard to the defect modelling in the framework of the Lagrangian $T(3)$-gauge approach, and namely with the quadratic $\mathcal{L}_\phi$, the following problem lacks a sufficient attention by now: whether some of the solutions considered in [8] can be reproduced ('replicated', the terminology of [12]) in the formalism [1, 2]? The Refs. [1, 2] themselves give a confirmative answer about this both for the screw and edge dislocations (one should also refer to the original paper [12]). Moreover, the last statement has stimulated the attempt [7] to calculate the second order corrections to the screw dislocation in the formalism [1, 2].

Conventional theory of defects [8] can be naturally considered as the Abelian gauge model [13–15] with the additive gauge group iso (3) $\approx \mathbb{R}^6$ [16] (iso (3) is the Lie algebra of ISO (3)). In this case the motor [17] of the disclination and dislocation loop densities plays the role of the 6-component Abelian gauge potential, and the motor of the defect densities that of the Abelian gauge field strength [18]. It can presumably be concluded from [18] that when ISO (3) is gauged, a successful replication (as soon as the last can be questioned) of the defects [8] should be valid both for the translational and rotational ones. Unfortunately, any attempt to obtain the standard wedge and twist disclinations [8] in the formalism [1, 2] is seemingly absent.

The present paper has originated as an attempt to understand, whether the Lagrangian $\mathcal{L}_\phi$ [1, 2] (i.e., practically, the translational model [12]) leads to an extension of the classical picture [8] for (singular) dislocations, or it provides a model of its own significance with the solutions which should be interpreted on their own rights.

Formally, the double curl Kröner's ansatz for stresses is used in [1, 2] to fulfil the equilibrium equation of the medium. In its turn, the translational gauge equation (the gauge equation, for brevity) determines the stress functions which parametrize this ansatz since can be re-expressed by means of the constitutive relation 'strain–stress'. As to the screw defect, the gauge equation has been reduced in [1, 2] to the homogeneous Helmholtz equation instead of the standard Poisson equation for the stress function. In the axisymmetric case ($\partial/\partial x_3 \equiv 0$) the Helmholtz equation has provided the modified Bessel function $K_0(\rho)$ as that solution, which coincides with the Prandtl stress function [19], i.e., behaves as $\log(1/\rho)+\text{const}$, at $\rho \ll 1$. Thus the screw defect is permissible in [1, 2].

In spite of the fact that the approach [12] to the screw dislocation is not contradictive, the situation with the edge one is more subtle. Generally speaking, the quadratic $\mathcal{L}_\phi$ becomes inappropriate as soon as the stress functions of the edge dislocation (along third axis) are obtained in the same way as for the screw one. Practically, the $T(3)$-gauge equation constraints the appropriate stress functions so that the resulting stress component σ_{33} is not of the edge defect.

An alternative form of the gauge equation is proposd in the present work, which eliminates the contradiction and, without any artificial effort, admits the linear solutions for the edge and screw dislocations. It is crucial that both the gauge equations, i.e., the newly proposed and that from [1, 2], being expressed through the general tensor of stress functions, pass each into other under the exchange $\nu \longleftrightarrow \nu^{-1}$, ν is the Poisson ratio. This is just the point needed to avoid the problem found for σ_{33}, thought to keep the situation with the screw dislocation unaffected.

The new gauge equation allows generalization, since can be written as an Einstein-type (non-linear) equation. This requires to modify the geometrical arena of the model [1, 2], and suggests $\mathcal{L}_\phi$ in the Hilbert–Einstein form. The stress tensor of non-linear elastic body plays the role of the source in the Einstein-type gauge equation. As a specialization of this source, the Murnaghan's constitutive relation of isotropic body is considered in the present paper. The new gauge equation should be appropriate to find out quadratic corrections to the linear solutions outside of the defect cores.

The paper is written in five sections. Sec. 1 is introductory, Sec. 2 is concerned with the difficulties of the replication of the static edge dislocation, and a modified linear model (i.e., another translational gauge equation, in fact) is proposed in Sec. 3. Further, Sec. 4 is devoted to the Lagrangian formulation of the corresponding non-linear model, and suggests $\mathcal{L}_\phi$ in the Hilbert–Einstein form, while Sec. 5 concludes the paper. In what follows, it is assumed that reader can be referred to [1, 2] for motives of the ISO (3)-gauging, as well as for certain details and comments on the original formulation. The consideration is time independent, Greek indices run from 0 to 3, Latin ones – from 1 to 3, and repeated up and down indices mean summation. Besides, the Latin letters a, b, c, $d, \ldots$ denote curvilinear indices of a deformed configuration, and i, j, k, $l, \ldots$ denote Cartesian frame of initial (undeformed) one.

2. T(3)-GAUGE EQUATION FROM THE QUADRATIC LAGRANGIAN AND RELATED DIFFICULTIES

Following [1, 2] let us briefly remind the basic elements of the theory of the translational gauge field ${\phi^a}_\alpha$ assuming that the gauge-rotational degrees of freedom of the whole ISO (3)-model are "frozen." The translational gauge field ${\phi^a}_\alpha$ plays "compensating" role as the Abelian group $T(3)$ acts by non-homogeneous shifts on the current configuration variable ξ^a, and the corresponding compensated derivative has the form ${B^a}_\alpha \equiv \partial_\alpha \xi^a + {\phi^a}_\alpha$, where ∂_α are the partial derivatives $\partial/\partial x_\alpha$, and x_0 implies time. The current configuration $\xi^a(\vec{x}, x_0)$ can be written in the form $\xi^a = \delta^a_i\, x^i + u^a(\vec{x}, x_0)$ with respect to a Cartesian coordinate system so that $\vec{x}$ implies an initial (undeformed) configuration, and $\vec{u}$ is displacement field. It should be noticed that when Abelian non-compact group like $T(3) \approx \mathbb{R}^3$ is gauged, appearance of ${B^a}_\alpha$ instead of the purely gradient quantity $\partial_\alpha \xi^a$ implies that the latter ceases to be adequate variable, because non-integrable contributions become essential.

As far as the problem of replication of the static edge dislocation is in focus of the present paper (see [8] for the standard results), it will be assumed that $\partial_0 \equiv 0$, ${\phi^a}_0 \equiv 0$ so that the governing equations are:

$$\partial_i\, {\sigma_a}^i = 0, \tag{1.1}$$

$$\partial_j\left(\partial^j \phi_a{}^i - \partial^i \phi_a{}^j\right) = \partial_j \partial^j \phi_a{}^i - \partial^i(\partial_j \phi_a{}^j) = (2s)^{-1} \sigma_a{}^i. \tag{1.2}$$

The source field $\sigma_a{}^i$ (1.2) is given by

$$\sigma_a{}^i = B_{aj}\left(\lambda\, \delta^{ij}\, E_k{}^k + 2\mu\, E^{ij}\right), \tag{2}$$

λ and μ are the Lamé constants, and the strain field E_{ij} is defined as $2E_{ij} = B^k{}_i\, B_{kj} - \delta_{ij}$ (see Sec.3 on the usage of up and down indices). Notice that the stress field σ^{ij} is just given by the brackets in the R.H.S. of (2). It is seen that the equilibrium Eq. (1.1) ensures integrability of (1.2). The parameter s in the R.H.S. of (1.2) (the "coupling" constant) is due to the following choice of the spatial part of the quadratic gauge-translational Lagrangian:

$$\mathcal{L}_\phi = (-\,2s)\,\partial_{[i}\phi^c{}_{j]}\,\partial^{[i}\phi_c{}^{j]}, \tag{3}$$

where the square brackets imply antisymmetrization. It has been stated in [1, 2] that the translational gauge equations (1.2) "are what replace the compatibility conditions of linear elasticity theory."

The Refs. [1, 2] propose the following approach to the dislocation problem. The purely integrable contribution to the state due to u_b is zero. Therefore, Eq. (2) can be linearized in ϕ_{ib}, since B_{ib} is $\delta_{ib} + \phi_{ib}$, as follows:

$$\sigma_{ij} = 2\mu\,\phi_{(ij)} + \lambda\delta_{ij}\phi_{kk}, \tag{4}$$

where the brackets mean symmetrization $\phi_{(ij)} = (1/2)(\phi_{ij} + \phi_{ji})$ (in linear case we do not distinguish up and down indices). Further, we impose the conditions $\phi_{12} = \phi_{21}$, $\phi_{13} = \phi_{31} = 0$, $\phi_{23} = \phi_{32} = 0$, and suppose $\partial_3 \equiv 0$ to adjust the axial orientation. In this case only a part of Eqs. (1.2) survives:

$$\begin{aligned}
\partial^2_{22}\phi_{11} - \partial^2_{12}\phi_{12} &= (2s)^{-1}\sigma_{11}\\
\partial^2_{11}\phi_{22} - \partial^2_{12}\phi_{12} &= (2s)^{-1}\sigma_{22},\\
\partial^2_{11}\phi_{12} - \partial^2_{12}\phi_{11} &= (2s)^{-1}\sigma_{12}\\
\partial^2_{22}\phi_{12} - \partial^2_{12}\phi_{22} &= (2s)^{-1}\sigma_{12},
\end{aligned} \tag{5}$$

$$\Delta\phi_{33} = (2s)^{-1}\sigma_{33}, \tag{6}$$

where $\Delta \equiv \partial^2_{11} + \partial^2_{22}$. It can be realized that the choice of ϕ_{ij} in the symmetric form $\phi_{ij} = \phi_{(ij)}$ should provide us the appropriate solution of (1.1), (5), (6) as a small elastic strain tensor, while Eq. (4) gets the status of the constitutive relation of isotropic body (the Hooke law).

The way to handle the system (1.1), (5), (6) proposed in [12] is as follows: to postulate the ansatz

$$\mu^{-1}\boldsymbol{\sigma} = \begin{pmatrix} \partial^2_{22}f & -\partial^2_{12}f & 0\\ -\partial^2_{12}f & \partial^2_{11}f & 0\\ 0 & 0 & p \end{pmatrix}, \tag{7}$$

which fulfils (1.1), while Eqs. (5), (6) enable to specify the unknown functions $f = f(x_1, x_2)$ and $p = p(x_1, x_2)$ provided the use of the linear law (4) is made.

As far as we are interested in the edge dislocation along the third axis Ox_3, the component ϕ_{33} (i.e., the strain (33)-component) is expected to become zero, at least as a limit of an "extended" solution. As $\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$, where $\nu = \lambda/2(\lambda + \mu)$ is the Poisson ratio, implies vanishing of the strain (33)-component, the constraint $\nu\Delta f = p$ has to be expected in that limit. The relation

(7) with $p = \nu\Delta f$ but, formally, with opposite sign at f is nothing but completely the standard ansatz of the theory of dislocations, where f is called Airy's stress function [19]. This function f has been found in [19] as the biharmonic potential $\partial_2(\rho^2 \log \rho)$ (or $\partial_1(\rho^2 \log \rho)$), $\rho^2 = x_1^2 + x_2^2$, and it enables to obtain via (7) all the stress tensor components of the edge defect.

Let us re-consider the solution [1, 2] of the problem of the edge dislocation. Clearly, Eq. (1.1) is respected by $\boldsymbol{\sigma}$ (7), whereas Eqs. (5) result in

$$(1-a)\Delta f - a\, p = \kappa^2 f, \tag{8}$$

and (6) leads to

$$(1-a)\Delta p - a\,\Delta\,\Delta f = \kappa^2 p, \tag{9}$$

where $a = \lambda(3\lambda + 2\mu)^{-1}$ and $\kappa^2 \equiv \mu/s$. Let us exclude p from (9) with the help of (8). We assume here the limit $\kappa \to 0$ proposed in [12] to restore the classical situation [8]. In this case $\Delta\Delta f = 0$ arises to define the limiting form of f, while (8) provides

$$p = \frac{1-a}{a}\,\Delta f \equiv \frac{1}{\nu}\,\Delta f \tag{10}$$

as the limiting form of p.

The point is that the fourth order differential equation, which appears to govern f at arbitrary κ, looks troublesome for analytical solution. It is why, in view of $\Delta\Delta f = 0$, the authors have been forced to "guess" that f can be found so that the needed Airy's function behaviour appears for it at $\kappa \to 0$. But since $\Delta\Delta f = 0$ is to indicate at the correct f, just the same reasons (i.e., linearity of the substitution of p) imply that Eq. (10) defines the limiting form of p. However, the Eq. (10) contradicts the constraint demonstrated above. Thus, in spite of the satisfactory equation for f, the correct stress component σ_{33} of the edge dislocation would not appear neither at finite κ nor at $\kappa \to 0$. Notice that in [1] it has been recognized only for finite κ that the new σ_{33} is different from the standard one.

Otherwise, let us try "rigidity" of (8), (9) against the constraint $p = \nu\Delta f$ by substituting it to them:

$$(1-\nu)\,\Delta f = \kappa^2 f,$$
$$0 = \kappa^2 \Delta f.$$

These equations can hardly be fulfilled with nontrivial f at finite κ, and, moreover, nontrivial solutions of $\Delta\Delta f = 0$ with the analytical behaviour we are interested in would not appear even at $\kappa \to 0$.

Therefore, the joint use of Eqs. (5), (6), and (7) to obtain the edge dislocation's stress function leads to the following conclusion. The resulting Eqs. (8) and (9) constraint the stress functions f and p so that it is impossible to get f as the biharmonic potential, and to fulfil at the same time $p = \nu\Delta f$. One gets either only σ_{33} is incorrect due to (10), or f is not the Airy function at all. Besides, there is a little hope that an improved f can be found at finite κ so that the additional contribution $(-1/a)\lim_{\kappa\to 0}\left(\kappa^2 f\right)$ from (8) to the R.H.S. of (10) could restore the desired $p = \nu\Delta f$.

With the purpose of illustration, let us obtain the limiting form of the matrix $\boldsymbol{\phi}$ by means of (4) inverted and (7). We shall substitute the known $f = (-C/2)\partial_2(\rho^2 \log \rho)$, where $C = (b/2\pi)(1-\nu)^{-1}$, but p will be evaluated accordingly to (10). Then we get:

$$\boldsymbol{\phi} = \frac{C}{2\,\rho^2}\begin{pmatrix} x_2\left(1 - 2\frac{x_1^2}{\rho^2}\right) & x_1\left(1 - 2\frac{x_2^2}{\rho^2}\right) & 0 \\ x_1\left(1 - 2\frac{x_2^2}{\rho^2}\right) & x_2\left(1 + 2\frac{x_1^2}{\rho^2}\right) & 0 \\ 0 & 0 & 2(1-\frac{1}{\nu})\,x_2 \end{pmatrix}. \tag{11}$$

Integrating the equations $\partial_1 u = \phi_{11}$, $\partial_2 v = \phi_{22}$, $\partial_2 u + \partial_1 v = 2\phi_{12}$ with respect to the new fields u, v, we obtain:

$$v + i\,u = v_d + i\,u_d + \frac{b}{2\pi}\log(x_1 - i\,x_2), \tag{12}$$

where $i = \sqrt{-1}$, and u_d, v_d are the edge dislocation's displacements in the plane perpendicular to Ox_3 (the Burgers vector is in x_1-direction) [8]. Equation (12) is to express that namely the logarithmic term responsible for the "closure failure" just characterizing the defect does not enter to u, v, because $v_d + iu_d$ contains $(-b/2\pi)\log(x_1 - ix_2)$. Besides, ϕ_{33} (11) looks embarrassantly to consider it as the corresponding strain component of the plane problem in question. In spite of the statement by [1, 2], it is plausible to conclude that Eq. (11), as the limiting form of possible solution to (5), (6), does not imply at all replication of the edge dislocation at $\kappa \to 0$.

It can occur that additional care with $\kappa \to 0$ is needed to make rigorous statement about absence (or presence) of the edge dislocation solution for the Eqs. (5), (6). The formal implications of the use of Eqs. (4), (5)–(7) have only been pointed out here. In the next section a modification of the situation, i.e., a way to keep the ansatz (7), to exchange the parameters a and $1-a$, and thus to avoid the unpleasant $p = \nu^{-1}\Delta f$, will be demonstrated which admits the standard dislocations in the formal limit $\kappa \to 0$.

3. THE DOUBLE CURL T(3)-GAUGE EQUATION

The equilibrium equation (1.1) can be satisfied identically provided the stress field $\boldsymbol{\sigma}$ is chosen as double curl of a twice differentiable symmetric tensor potential $\boldsymbol{\chi}$ which is called [19] the tensor field of second order stress functions:

$$\sigma_{ij} = (\mathrm{inc}\,\boldsymbol{\chi})_{ij} \equiv -\epsilon^{ikl}\,\epsilon^{jmn}\,\partial^2_{km}\chi_{ln}. \tag{13}$$

Both the particular ansatz proposed in [1, 2] to obtain the edge and screw dislocations appear as specializations of the general representation (13) (though (13) itself is not stressed in [1, 2]).

The so-called stress function method based on (13) has been developed in [20, 21] to approach to non-linear dislocation problems. For the internal stress problem (isotropic case) the representation (13) has also been discussed by Kröner in [19]. The concept of a torsion-free stress space and relationship to it of the Eq. (13) have been considered in [19]. In [22] a stress-strain duality has been presented which relates the double curl ansatz (13) to a similiarity in Riemannian descriptions of kinematics and statics of mechanical state of solid. It has been exploited in [22] that the tensor of secon order stress functions and the stress tensor could play the role of the metric and the Einstein tensors, accordingly, of a non-Euclidean stress space.

Therefore Eq. (7) looks more fundamentally than Eqs. (5), (6) with regard to the unsatisfactory implications of p (10). On the other hand, it is just the choice of the master equation (1.2) for the translational field which would require a justification. It turns out that the most direct way to get the function p consistent with the requirement $\phi_{33} = 0$, i.e., $\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$, at $\kappa \to 0$, is to choose the L.H.S. of Eq. (1.2) (linear consideration) as follows:

$$(\mathrm{inc}\,\boldsymbol{\phi}^{\mathrm{Sym}})_{ij} \equiv -\epsilon^{ikl}\,\epsilon^{jmn}\,\partial^2_{km}\phi_{(ln)} = (2s)^{-1}\sigma_{ij}, \tag{14}$$

where superscript $^{\mathrm{Sym}}$ implies tensor symmetrized, ϵ^{ikl} is the permutation symbol, and σ_{ij} is given by (4). When the R.H.S. of (14) is zero, it looks like a compatibility equation of linearized elasticity theory provided ϕ_{ij} is considered as distortion. Equation (14) replaces the compatibility condition in the sense that its R.H.S. is non-trivial while the L.H.S. looks conventionally. In the next section Eq. (14) will be viewed as the Einstein-type one.

Let us accept (14) instead of the linearized (1.2) in the $T(3)$-gauge model of defects. However, it is more appropriate to rewrite (14) through the unknown second order stress functions $\boldsymbol{\chi}$. To this end one should express $\boldsymbol{\phi}^{\rm Sym}$ using both the Hooke law (4) inverted and (13), and, after this, substitute it into the L.H.S. of (14). The stress field $\boldsymbol{\sigma}$ (13) takes its place in the R.H.S. of (14). Finally, the fourth order differential equation appears:

$$\Delta^{(3)}\Delta^{(3)}\chi_{ij} + a\,\mathrm{D}_{ij}\Delta^{(3)}\chi + \Big((1-a)\,\partial^2_{ij} + a\,\delta_{ij}\Delta^{(3)}\Big)\,\partial^2_{kl}\chi_{kl} - \Delta^{(3)}\left(\partial^2_{ik}\chi_{jk} + \partial^2_{jk}\chi_{ik}\right) =$$

$$= \kappa^2\left(\Delta^{(3)}\chi_{ij} + \mathrm{D}_{ij}\chi + \delta_{ij}\partial^2_{kl}\chi_{kl} - \partial^2_{ik}\chi_{jk} - \partial^2_{jk}\chi_{ik}\right), \tag{15}$$

where χ implies trace of $\boldsymbol{\chi}$, a is defined by Eq. (10), κ is defined in (8), (9), and we have denoted the differential operators $\Delta^{(3)} = \delta_{ij}\,\partial^2_{ij}$, and $\mathrm{D}_{ij} = \partial^2_{ij} - \delta_{ij}\,\Delta^{(3)}$. In its turn, the Eq. (15) can be considerably simplified if we replace $\boldsymbol{\chi}$ by another symmetric potential $\boldsymbol{\chi}'$ as follows:

$$\chi_{ij} = 2\mu\left(\chi'_{ij} + \frac{\nu}{1-\nu}\,\delta_{ij}\chi'\right),$$

where $\boldsymbol{\chi}'$ fulfils $\partial_i\chi'_{ij} = 0$. Thus we get:

$$\Delta^{(3)}\Delta^{(3)}\chi'_{ij} = \kappa^2\left(\Delta^{(3)}\chi'_{ij} + \frac{1}{1-\nu}\mathrm{D}_{ij}\chi'\right).$$

To clarify the situation let us repeat the procedure leading to (15) for the Eq. (1.2) as well, though assuming that the latter is written in terms of $\boldsymbol{\phi}^{\rm Sym}$. It is curious that thus resulting equation is completely similiar to (15) except only one thing: a and $1-a$ are just exchanged. In other terms, ν and ν^{-1} are exchanged. Therefore it can be guessed that the problem related to p (10) should disappear when (14) is used instead of (1.2).

It has to be noticed that linearizing Eq. (1.2) one gets another, apart from the problem of p, unpleasant thing: the source σ_{ij} (4) in its R.H.S. is symmetric in the indices, while the L.H.S. is obviously not. It is why in [1, 2] ϕ_{ij} has been additionally assumed symmetric to perform the concrete calculations. On the contrary, the L.H.S. in (14) is symmetric, and the antisymmetric part of the translational gauge field automatically drops out, but must be taken into account to restore, say, distortion from strain field.

In order to specialize (15) to the edge dislocation case, we introduce two functions p and f as follows:

$$f \equiv \chi_{33}, \qquad p \equiv -\partial^2_{22}\chi_{11} - \partial^2_{11}\chi_{22} + 2\partial^2_{12}\chi_{12},$$

while other χ_{ij} are zero. The plane problem is adjusted by $\partial_3 \equiv 0$, and thus $\Delta^{(3)}$ becomes Δ (see Sec. 2 for Δ). It can be verified that now six equations (15) are reduced to

$$a\Delta f + (1-a)\,p = \kappa^2 f, \tag{16}$$

$$(1-a)\Delta\Delta f + a\,\Delta p = \kappa^2 p. \tag{17}$$

More precisely, Eq. (16) appears instead of the three equations $\partial^2_{11}A = 0$, $\partial^2_{12}A = 0$, and $\partial^2_{22}A = 0$ where $A \equiv (1-a)p + a\Delta f - \kappa^2 f$. It is seen that changing f to $-f$ and $a \longleftrightarrow 1-a$ one obtains (8), (9) from (16), (17), accordingly.

Therefore Eq. (14) has provided us with the remarkable opportunity to convert the embarrassing ratio $(1-a)/a$ in (10) into $a/(1-a)$. Finally, the resulting function p (16) gets around the obstacle discussed in the previous section. Notice that choosing $f \equiv -\partial_1\chi_{23} + \partial_2\chi_{31}$ one can deduce from (15) a single equation

$$\Delta f = \kappa^2 f, \tag{18}$$

which is the same as in [1, 2] for the screw dislocation ansatz. It is clear that this coincidence is because (18) does not contain the Poisson ratio $\nu = a/(1-a)$. Solutions to (16)–(18) (and, generally, to (15)) should be called "modified stress functions" to distinguish them from the classical harmonic and biharmonic potentials.

As the Eq. (16) defines p, the second equation governing f gets the form:

$$\left(\Delta - \mathcal{M}^2\right)\left(\Delta + \mathcal{N}^2\right) f = 0, \qquad \mathcal{N}^2 \equiv \frac{\mu}{s}\frac{1}{1-2a}. \tag{19}$$

For correspondence with [1] we use $\mathcal{M}^2$ instead of our κ^2 when considering (19) and its solution. Equation (19) simply differs from the corresponding Eq. (4.6.27) in [1]: only $2a-1$ and $1-2a$ are interchanged under $a \longleftrightarrow 1-a$.

As ∂_2 commutes with Δ, one can be concerned with f in the form $\partial_2 h(\rho)$, $\rho = |\vec{x}|$, so that h also respects (19) with Δ reduced to

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right).$$

Equation (19) has the Bessel and Neumann functions $J_0(\mathcal{N}\rho)$ and $Y_0(\mathcal{N}\rho)$, and the modified Bessel functions $I_0(\mathcal{M}\rho)$, $K_0(\mathcal{M}\rho)$ as four angle-independent basic solutions [23]. These basic solutions can be combined so that their combination is decreased at infinity. Eventually, the following solution can be obtained:

$$\begin{gathered} f(\vec{x}) = \frac{bs}{2\pi}\,\partial_2\,\mathcal{F}(\rho), \\ \mathcal{F}(\rho) = \log\frac{\mathcal{N}}{\mathcal{M}}\,J_0(\mathcal{N}\rho) - \frac{\pi}{2}\,Y_0(\mathcal{N}\rho) - K_0(\mathcal{M}\rho). \end{gathered} \tag{20}$$

At $\mathcal{M}\rho, \mathcal{N}\rho \ll 1$, the solution f (20) results in the standard biharmonic potential, since $\mathcal{F} \simeq \mathcal{M}^2/(2(1-\nu))\rho^2\log\rho$. Notice, that the normalization of $\mathcal{F}(\rho)$ is chosen so that the stresses are $\sigma_{11} = -\partial^2_{22}f$, etc., but not, accordingly to (7), $\sigma_{11} = -\mu\,\partial^2_{22}f$, etc.. The limit $\mathcal{M}\rho, \mathcal{N}\rho \ll 1$ implies that ρ is finite but not zero while $\mu/s \to 0$. In the opposite case $\mathcal{M}\rho, \mathcal{N}\rho \gg 1$, the solution (20) becomes a linear combination of $x_2\,\rho^{-3/2}\sin(\mathcal{N}\rho)$ and $x_2\,\rho^{-3/2}\cos(\mathcal{N}\rho)$, i.e., it is $O\left((\mathcal{N}\rho)^{-1/2}\right)$. Therefore, we are expecting all the entries of the stress matrix $\boldsymbol{\sigma}$ at $\mathcal{M}\rho, \mathcal{N}\rho \ll 1$ to be those of the edge dislocation, because $p = \nu\Delta(-f)$ by $\kappa \to 0$ in (16). Recall that the solution to (18) K_0, which enables the screw dislocation, possesses fast and monotonic decreasing as $\rho^{-1/2}\exp(-\kappa\rho)$. In other words, this solution is really 'short-ranged' in comparison with the corresponding classical one.

Let us also mention the Ref. [39], where a "mass" of defects is discussed as an implication of the translational model [1] (specifically, as implication of the corresponding time-dependent $T(3)$-gauge equation). This is because the Klein–Gordon equation is possible with the mass κ^2 for the translational gauge field in the Lorentz-like gauge. The present section demonstrates, that such "mass" effect should be rather traced to Eq. (15) for the modified stress functions. The point is that namely (14), but not (1.2), seems to be related to the defects (at least, to the statics of the conventional ones). Moreover, it is just the Eq. (15), which implies the conventional, unboundly increasing, stress functions to become the limited modified ones just due to the "mass terms" κ^2 in (18) and (19).

However, in Sec. 4.3, and 5, it will be discussed, that only the replication demonstrated above (i.e., the regime $\kappa\rho \ll 1$) is of main interest in the linearized version of the Einstein–type consideration. Thus, the L.H.S. of (14) helps to avoid the problem with p (10). Both the sides of (14) admit non-linear generalizations, and therefore it is suggestive to put (14) into a more general form which can be derived in Lagrangian approach.

4. THE EINSTEIN-TYPE EQUATION AND ITS LAGRANGIAN DERIVATION

4.1. The geometric preliminaries

Before to proceed with the Lagrangian derivation of Eq. (14), let us briefly present the geometrical apparatus which will underly our main construction. To make the discussion reasonably compact, it will be assumed that one can be referred, say, to [24] for basics of geometry of the Riemann and Riemann–Cartan spaces in the form accomodated to describe defects. Other useful references can also be found in [25]. As to gauging of the important for us group ISO (3), a formally close gauging of the Poincaré group (which also is a semi-direct product of translation and pseudo-orthogonal rotation groups though of 4-dimensional Minkowskian space-time) have already been extensively developed in the realm of gravitational physics [26]. One should consult with [26, 27] for a similiar, though much more elaborated machinery.

Although we are restricted to the $T(3)$-gauging, it is more appropriate to admit, just for a moment, a more general framework of the ISO (3)-gauging. Here the couple of the Cartan structure equations

$$\begin{aligned} \mathcal{R}^a{}_{b,\,ij} &= \partial_i\, \omega^a{}_{b,\,j} - \partial_j\, \omega^a{}_{b,\,i} + \omega^a{}_{c,\,i}\, \omega^c{}_{b,\,j} - \omega^a{}_{c,\,j}\, \omega^c{}_{b,\,i}, \\ \mathcal{T}^a_{\,,ij} &= \partial_i B^a{}_j - \partial_j B^a{}_i + \omega^a{}_{c,\,i}\, B^c{}_j - \omega^a{}_{c,\,j}\, B^c{}_i. \end{aligned} \tag{21}$$

appears as one of the basic differential-geometric relations. Four tensors which enter the Eqs. (21) are components of the differential forms which define the geometry of the so-called Riemann–Cartan spaces: $\mathcal{R}^a{}_{b,\,ij}$ and $\mathcal{T}^a_{\,,ij}$, which are antisymmetric in i, j, determine the curvature and torsion 2-forms, accordingly $\mathcal{R}^a{}_b$ and $\mathcal{T}^a$. The coefficients $\omega^a{}_{b,\,i}$ and $B^a{}_i$ define the connection 1-form $\omega^a{}_b$ and the co-frame 1-form B^a, respectively. The couple of indices a, b demonstrate $\mathcal{R}^a{}_b$, $\omega^a{}_b$ as elements of the Lie algebra of the group $SO(3)$, while $\mathcal{T}^a$, B^a belong to the Lie algebra of the group $T(3)$. All the fields are considered in open domain of a 3-dimensional manifold with Euclidean signature.

It is known that gauging ISO (3) one gets the iso (3)-valued connection and curvature differential forms, both of which are split into "linear" and "translational" parts owing to the semi-direct sum structure of iso (3) [27, 28]. However, the "translational" curvature is transformed non-covariantly under the ISO (3) gauge transformation. In order to find out the quantity which is transformed (gauge-)covariantly, it is proposed to use the auxiliary field ξ^a [27–29]. As the result, the Cartan structure equations in the ISO (3) gauging acquire the form (21) provided the coefficients $B^a{}_i$ have the following structure:

$$B^a{}_i = \phi^a{}_i + \partial_i\, \xi^a + \omega^a{}_{b,\,i}\xi^b. \tag{22}$$

In (22) $\omega^a{}_{b,\,i}$ and $\phi^a{}_i$ are just the "linear" and "translational" parts of the iso (3)-valued connection 1-form. In other terminology, $\omega^a{}_{b,\,i}$ and $\phi^a{}_i$ are the gauge potentials which are due to non-homogeneous action of ISO (3). Going further, apart from $B^a{}_i$, it is also appropriate to define their reciprocals ${\mathcal B_b}^j$ as follows: ${B^a}_i\,{\mathcal B_b}^i = \delta^a_b$, ${B^a}_i\,{\mathcal B_a}^j = \delta^j_i$. The point is that the components ${\mathcal B_b}^j$ define an orthonormal triad $\mathcal B_b = {\mathcal B_b}^j \partial_j$ (more exactly, ${\mathcal B_b}^j$ are transitions between the coordinate basis in the tangent space $\{\partial_j\}$ and the orthonormal basis $\{\mathcal B_b\}$), whereas ${B^a}_i$ provide the dual basis of 1-forms $B^a = {B^a}_i dx^i$.

It is crucial that the auxiliary field ξ^a (22) has been identified in [1, 2] as the deformed configuration variable (see [28] for other problem-motivated interpretations of that special field) as follows:

$$\xi : x^i \longrightarrow \xi^a(x^i), \qquad \xi^{-1} : \xi^a \longrightarrow x^i(\xi^a). \tag{23}$$

This equally means that the group indices a, b, c in (21) get the "material" meaning since the sets $\{\xi^a\}$ label points in deformed configuration. The coordinate indices i, j, k in (21) get the status of the Cartesian ones in initial (undeformed) configuration. In particular, both ${B^a}_i$ and ${\mathcal B_a}^i$ are useful to pass from one set of indices to another.

In [18] the ISO (3) Cartan's equations (21) have been reduced to the conventional expression of the motor of dislocation and disclination densities through the motor of the defect loop densities. It is essential that the field ξ^a which is $x^a + u^a$ has enabled this truncation so that sequence of Schaefer's exterior differentiations inherent to [8] has naturally appeared. Besides, interpretation of the ISO (3)-connection in terms of the defect loop densities has become possible.

Regarding the material interpretation of $\{x^i\}$ and $\{\xi^a\}$, let us recall how the Green deformation tensor g_{ij} and the corresponding Lagrangian strain tensor E_{ij} appear in the conventional elasticity theory [30–32]. Indeed, the length element in a deformed configuration can be written in the Euclidean form $ds^2 = \eta_{ab} d\xi^a d\xi^b$ (η_{ab} is flat metric) as well. At the same time, the configuration $\{\xi^a\}$ can be considered in terms of the initial one by means of the maps (23). In this case the length element becomes expressed through the initial coordinates: $ds^2 = \eta_{ab}\partial_i\xi^a\partial_j\xi^b dx^i dx^j$. Thus we obtain the corresponding non-Euclidean metric (the Green deformation tensor) $g_{ij} = \eta_{ab}\,\partial_i\xi^a\partial_j\xi^b$, while the Lagrangian strain tensor is

$$2E_{ij} = g_{ij} - \eta_{ab}\delta^a_i\delta^b_j = g_{ij} - \eta_{ij}. \tag{24}$$

Following the close analogy, we define the Green deformation tensor as

$$g_{ij} = \eta_{ab}\,{B^a}_i\,{B^b}_j, \tag{25}$$

with ${B^a}_i$ (22), while the strain tensor is given by (24). The metric g_{ij} and its inverse can be used for raising and lowering the indices i, j, while η_{ab} – to handle analogously a, b, c.

Now it is time to stress that the present work is concerned with the Eq. (14) to avoid the problem discussed in the Sec. 2. The given section is to extend straightforwardly (14) as an Einstein-type equation in the context of torsion-free Riemannian geometry. Therefore, given the metric g_{ij} is (25), a unique linear connection without torsion Γ^k_{ij} can be associated with it so that g_{ij} is covariantly constant. The corresponding condition $\nabla_i\, g_{jk} = 0$ implies the conventional expression for such Γ^k_{ij}:

$$2\,\Gamma^k_{ij} = g^{kl}\,\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right).$$

The connection Γ^k_{ij} can be related with the SO(3)-connection ${\omega^a}_{b,\,i}$ by means of the relation

$${\omega^a}_{b,\,i} = {\mathcal B_b}^l\,(\Gamma^k_{il}\,{B^a}_k - \partial_i {B^a}_l),$$

which respects the covariant constance of g_{ij}, and, further, allows to obtain from ${\mathcal R^a}_{b,\,ij}$ (22) the Riemann–Christoffel curvature tensor ${\mathrm R^k}_{mij}$:

$${\mathcal R^a}_{b,\,ij} = {\mathcal B_b}^m\,{B^a}_k\,{\mathrm R^k}_{mij},$$

$$\mathrm{R}^k{}_{mij} = \partial_i \Gamma^k_{jm} - \partial_j \Gamma^k_{im} + \Gamma^k_{in}\,\Gamma^n_{jm} - \Gamma^k_{jn}\,\Gamma^n_{im}.$$

As the last step, we define the scalar curvature R, which is an important geometric invariant as follows: $\mathrm{R} = \mathrm{R}_i{}^i$, $\mathrm{R}_{ij} = \mathrm{R}^k{}_{ikj}$. The scalar R is just what we need to derive (14) in the Lagrangian approach. From now on, we "switch off" the gauge-rotational degree of freedom $\omega^{ab}_{\ ,i}$ in $B^a{}_i$.

The geometric presentation above is inevitably sketchy, since many appropriate things, like definition of principle bundle of affine (linear) frames, definition of associated vector bundle, gauge transformation rules, etc., are omitted. Necessary matter can be picked up from the literature cited, though more detailed and accurate geometric background of the non-Abelian model would require a separate presentation. However, as to the ISO (3) gauge model [1, 2], the idea to pass from the "triad" representation (21) to the purely "coordinate" one, in order to discuss replication of the conventional dislocations, belongs, seemingly, to this paper.

4.2. The Lagrangian derivation.

It is well known that the six compatibility equations of elasticity are, in fact, vanishing conditions of the six independent components of the 3-dimensional Riemann–Christoffel tensor R^i_{jkl}. Due to 3-dimensionality, one can equally use the second rank Einstein tensor G^{ij} instead of R^i_{jkl} as follows: $\mathrm{G}^{ij} = (1/4)e^{ikl}e^{jmn}\mathrm{R}_{klmn}$, where $e^{ijk} = \sqrt{g}\,\epsilon^{ijk}$ and g is $\det(g_{ij})$. Linearizing the Riemann–Christoffel tensor one gets the double curl expression like the L.H.S. of (14).

Therefore, the Eq. (14) can be considered as a weak field approximation of an Einstein-type equation. Indeed, the L.H.S. of (14) looks like linearization of G^{ij}, while the stress tensor in its R.H.S. can acquire higher powers of E_{ij}, as a non-linear constitutive relation 'stress-strain'. In other words, the situation is reminiscent to a gravitational equation which relates the Einstein tensor (though 4-dimensional) to a matter energy-momentum tensor as the source. Therefore both the sides of (14) can be extended, and the resulting equation admits a Lagrangian derivation.

Now we can proceed with the Lagrangian approach. First of all, we postulate the Hilbert–Einstein Lagrangian density which is responsible for the translational field:

$$\frac{1}{b}\,\mathcal{L}_\phi = s\,\mathrm{R},$$

where $b \equiv \det(B^a{}_i)$, and R is the scalar curvature. Variation of $\mathcal{L}_\phi$ takes the form (up to terms irrelevant by the Stokes theorem):

$$\frac{1}{b}\,\frac{\delta\mathcal{L}_\phi}{\delta B^a{}_i} = 2s\left(\mathrm{R}_a{}^i - \frac{\mathrm{R}}{2}\,\mathcal{B}_a{}^i\right) \equiv 2s\,\mathrm{G}_a{}^i, \tag{26}$$

where $\mathrm{G}_a{}^i$ is the Einstein tensor.

Let us obtain $\boldsymbol{\sigma}$ in the R.H.S. of the Einstein-type equation. The field ξ^a is an important constituent of more general ISO (3)-formalism, and it should be governed by an appropriate Lagrangian $\mathcal{L}_\xi$. Our consideration is static, and therefore we shall choose $(-1/b)\mathcal{L}_\xi$ in the form of potential energy W of isotropic non-linear elastic continuum (practically, in the so-called Murnaghan's form). Given the field $B^a{}_i$ has the $T(3)$-invariant form $\partial_i\xi^a + \phi^a{}_i$, variation of the Lagrangian $\mathcal{L}_\xi$ gets the form:

$$\frac{1}{b}\,\frac{\delta\mathcal{L}_\xi}{\delta B^a{}_i} = -\Sigma_a{}^i \equiv -\sigma_a{}^i - \mathcal{B}_a{}^i\,W. \tag{27}$$

In (27), by definition, the stress field σ^{ij} is given by $\delta W/\delta E_{ij} = \sigma^{ij}$, and the second term in $\Sigma_a{}^i$ is due to variation of b.

Putting together (26) and (27), we obtain the equation

$$\mathrm{G}_a{}^i = (2s)^{-1}\, \Sigma_a{}^i, \tag{28}$$

which generalizes (14) in the sense explained above. Rejecting in (28) higher terms and assuming coincidence, in the leading order, of its source with (4) (weak field approximation), we obtain (14). Tending s to infinity, we just recover the general compatibility equation $\mathrm{G}_{ij} = 0$. As far as the Einstein tensor is "covariantly conserved" by the appropriate Bianchi identity [24], the equation $\nabla_i \Sigma_a{}^i = 0$ appears to govern the source tensor. For the first time in the context of gauge dislocations, an equation similiar to (28) has been obtained by variational approach in [33], but without any further elaboration.

For definitness, let us specialize the potential energy W taking it in the form proposed by Murnaghan [34]:

$$W = \frac{\lambda}{2} I_1^2(\mathbb{E}) + \mu\, I_1(\mathbb{E}^2) + \frac{\nu_1}{6} I_1^3(\mathbb{E}) + \nu_2\, I_1(\mathbb{E})\, I_1(\mathbb{E}^2) + \frac{4}{3}\, \nu_3\, I_1(\mathbb{E}^3), \tag{29}$$

where $\mathbb{E}$ implies the gauge invariant strain tensor E_{ij} (24), invariant function $I_1(\dots)$ implies trace of the appropriate tensor argument, while λ and μ are the Lamé constants of second, and $\nu_{1,2,3}$ – of third orders. The stress tensor acquires the form

$$\sigma^{ij} = \delta^{ij} \left[\lambda\, I_1(\mathbb{E}) + \frac{\nu_1}{2}\, I_1^2(\mathbb{E}) + \nu_2\, I_1(\mathbb{E}^2) \right] + 2\, E^{ij} \left[\mu + \nu_2\, I_1(\mathbb{E}) \right] + 4\nu_3 E^{ik} E^j_k, \tag{30}$$

where $I_1(\mathbb{E}) = E_k{}^k$ and $I_1(\mathbb{E}^2) = E_{kl} E^{kl}$. When $\nu_{1,2,3}$ are zero, we obtain (2), while σ_{ij} coincides with (4) in the weak field limit.

The problem of corrections to the linear dislocation solutions has been actively investigated by various methods. For instance, the Refs. [20, 21, 35, 36], as well as a review of them in [31], are to be mentioned here. The Refs. [31, 32, 37] are also useful as the sources of information about others approaches to the non-linear dislocation problems. Moreover, in [38] second order corrections have been obtained for a wedge disclination solution. All the calculations in the refs. mentioned have been done assuming the constitutive relation in the Murnaghan's form to account for all the quadratic contributions properly. The Ref. [7] is concerned with the non-conventional approach to the screw dislocation [1, 2]. However the constants $\nu_{1,2,3}$ are taken zero in [7].

4.3. Vanishing torsion ?.

Before to conclude the paper, the Eq. (14) has been proposed in the Sec. 3 to avoid the contradiction between p (10) and $\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$ for the edge dislocation. In their turn, Secs. 4.2, 4.3, are to extend (14) to the Eq. (28), which is concerned with the Riemann–Christoffel picture. Therefore, it is appropriate to check, whether the solutions considered above are consistent with the requirement of zero torsion. Obviously, the torsion can not be asked to vanish everywhere, because thus defects will drop out. It seems sufficient, since our description should approximate a more adequate Riemann–Cartan situation, to ask about the torsion which is zero, at least, at $\kappa\rho \ll 1,\ 0 < R_{\text{core}} < \rho < R_{\text{exterior}} < \infty$.

First, it is appropriate to remind the situation [12] with the screw defect, which is more transparent. It is known how to calculate the torsion tensor [24] (see also (21)), and so we obtain from (4) and (7) for its single non-zero component:

$$\mathcal{T}^3{}_{,12} = \frac{1}{2}\, \left[2(\partial_1 \omega_2 + \partial_2 \omega_1) - \Delta f \right], \tag{31}$$

where the modified stress function f fulfills (18). There are two auxiliary functions ω_1 and ω_2 in (31) to account for the antisymmetric part of distortion:

$$\phi = \frac{1}{2}\begin{pmatrix} 0 & 0 & \partial_2 f - 2\omega_1 \\ 0 & 0 & -\partial_1 f + 2\omega_2 \\ \partial_2 f + 2\omega_1 & -\partial_1 f - 2\omega_2 & 0 \end{pmatrix}.$$

In the (anti-)plane problem we choose ω_1, ω_2 so that the distortion components ϕ_{31} and ϕ_{32} are zero (for the defect along Ox_3) [8]. Then (31) is reduced to

$$\mathcal{T}^3{}_{,12} = -\,\Delta f. \qquad (32)$$

In the axisymmetric case, the solution to (18) can be written as $f = (b/2\pi)K_0(\kappa\rho)$ [1, 2]. At $\kappa\rho \ll 1$, this solution results approximately in $(b/2\pi)\log(1/\rho) + \mathrm{const}$, i.e., in the Prandtl's stress function, while the corresponding value of $\mathcal{T}^3{}_{,12}$ (32) is zero, since we get $b\,\delta(x_1)\delta(x_2)$ at $\rho \neq 0$. At $\kappa\rho \gg 1$, the torsion $\mathcal{T}^3{}_{,12}$ approximately vanishes by (18), (32), since the Bessel function decays exponentially. Provided the entire f is used to consider the torsion component at $\kappa\rho \ll 1$, one gets:

$$\mathcal{T}^3{}_{,\,12} \simeq -\frac{b}{2\pi}\kappa^2\left(\mathrm{const} - \log(\kappa\rho)\right).$$

The latter still vanishes because ρ is limited $R_{\mathrm{core}} < \rho < R_{\mathrm{exterior}}$ and $\kappa \to 0$. However, $\mathcal{T}^3{}_{,12}$ (32) is not zero in the intermediate region $\kappa\rho \simeq 1$, thus violating our Riemannian interpretation.

Is it possible to make $\mathcal{T}^3{}_{,\,12}$ zero by an appropriate choice of ω_1, ω_2? Formally, it can be done relaxing the requirement $\phi_{31} = \phi_{32} = 0$. But therefore the framework of the (anti)-plane problem is left in favour of a 3-dimensional consideration. Notice, that the torsion components have not been considered in [1, 2] for the screw dislocation as we just did. This has influenced, for instance, the Burgers vector component b_3 found in [1] as $b/2$ instead of b owing to the neglection of $\phi_{31} = \phi_{32} = 0$ at $\kappa \to 0$.

Now let us turn to the edge dislocation which is more complicated. Here the torsion components are:

$$\begin{aligned} \mathcal{T}^1{}_{,12} &= \frac{1}{2}\left[2\,\partial_1\omega + (1-\nu)\partial_2\Delta f + \nu\,\kappa^2\partial_2 f\right], \\ \mathcal{T}^2{}_{,\,12} &= \frac{1}{2}\left[2\partial_2\omega - (1-\nu)\partial_1\Delta f - \nu\,\kappa^2\partial_1 f\right], \end{aligned} \qquad (33)$$

where ω is to account for the antisymmetric part in the distortion component ϕ_{12} (the normalization of f corresponds to (7)). The dependence on κ is present in (33), because p (16) has been used. When f, $\mathcal{T}^1{}_{,12}$, and $\mathcal{T}^2{}_{,12}$ are fixed, the function ω can be found from (33) provided, say, the integrability condition $(\partial_1\partial_2 - \partial_2\partial_1)\omega = 0$ is implemented, i.e., the equation

$$(1-\nu)\Delta\,\Delta f + \nu\,\kappa^2\,\Delta f = 2\left(\partial_2\mathcal{T}^1{}_{,\,12} - \partial_1\mathcal{T}^2{}_{,\,12}\right) \qquad (34)$$

holds.

In the formal limit $\kappa \to 0$ the choice

$$\omega = \frac{b}{2\pi}\,\frac{x_1}{\rho^2}$$

ensures that (33) can be fulfilled with

$$\mathcal{T}^1{}_{,\,12} = b\,\delta(x_1)\delta(x_2), \qquad \mathcal{T}^2{}_{,\,12} = 0, \qquad (35)$$

and $f = (b/4\pi)(1-\nu)^{-1}\partial_2(\rho^2 \log\rho)$. The torsion, Eq. (35), is given, the integrability Eq. (34) acquires the form at $\kappa \to 0$:

$$(1-\nu)\,\Delta\Delta f = 2b\,\delta(x_1)\partial_2\delta(x_2),$$

thus prescribing the limiting form of the modified stress function.

Taking into account the explicit solution f (20), and using (33), (34), we can also calculate the torsion components $\mathcal{T}^1{}_{,12}$, $\mathcal{T}^2{}_{,12}$ at arbitrary κ. Indeed, let us take $\mathcal{T}^2{}_{,12} = 0$ to express ω. Then we obtain $\mathcal{T}^1{}_{,12}$:

$$\mathcal{T}^1{}_{,12} = \frac{bs}{4\pi}\left((1-\nu)\,\Delta\Delta\mathcal{F} + \nu\kappa^2\Delta\mathcal{F}\right) = -\frac{b}{4\pi}\,\mu\,\nu\left(\Delta\mathcal{F} - \frac{\kappa^2}{a}\mathcal{F}\right),$$

where the Eq. (19) has been used to exclude $\Delta\Delta$. Using $\mathcal{F}(\rho)$ (20) explicitly, we obtain further:

$$\mathcal{T}^1{}_{,12} = \frac{b}{4\pi}\,\frac{\mu\kappa^2}{1-\nu}\left((1+\nu)\,\mathcal{F}(\rho) + 2\nu\,K_0(\mathcal{M}\rho)\right). \tag{36}$$

At $\mathcal{M}\rho, \mathcal{N}\rho \ll 1$, the R.H.S. of (36) is also vanishing due to the same reasons, as it happens for $\mathcal{T}^3{}_{,12}$ (32) in the analogous situation. At $\mathcal{M}\rho, \mathcal{N}\rho \gg 1$, $\mathcal{T}^1{}_{,12}$ decays as linear combination of $\rho^{-1/2}\sin(\mathcal{N}\rho)$ and $\rho^{-1/2}\cos(\mathcal{N}\rho)$, i.e., not so fast as $\mathcal{T}^3{}_{,12}$ (32) does.

Is it possible to make $\mathcal{T}^1{}_{,12}$, $\mathcal{T}^2{}_{,12}$ zero by adjusting ω? The integrability condition for ω can be written, without the use of (16), as follows:

$$(1-a)\Delta\,\Delta f + a\,\Delta p = 0$$

Then, formally, at finite s the parameter p, i.e., the stress component σ_{33}, is zero by (7), (17). Therefore the strain component E_{33} is

$$E_{33} = -\frac{a}{2\mu}(\sigma_{11} + \sigma_{22}) = \frac{a}{2}\,\Delta f.$$

If E_{33} is still zero (as at $\kappa \to 0$), the function f is also zero by (16). Otherwise the two-dimensional consideration is no longer sufficient.

Let us sum up this section. Here we have obtained the torsion components expressed through the stress functions. The results of the Sec. 3 have been used to establish that there exists a single limiting situation, i.e., the limit $R_{\rm core} < \rho < R_{\rm exterior}$, $\kappa \to 0$, when the modified stress functions can be replaced approximately by the conventional ones, and the torsion becomes zero. This would imply that the Einsteinian interpretation of the Eq. (14) is valid only in that limit, while the modified stress functions themselves for others κ, ρ should be discarded at the present stage.

5. DISCUSSION

Thus, in the present paper we have started with the demonstration of the fact that the conventional linear solution for the edge dislocation does not fit in the gauge model [1, 2] (specifically, its translational sector). The Kröner ansatz for stresses is used in [1, 2] to fulfil identically the equilibrium equations of the medium. It is essential that the corresponding parametrizing functions, i.e., the (modified) stress functions, are to be found from the translational gauge equation. In its turn, the form of the translational gauge equation [1, 2] is dictated by the Lagrangian quadratic in the $T(3)$-gauge field strength. As a special example, in [1, 2] the Kröner ansatz has been chosen in that form [19], which leads in conventional theory to the Airy's stress function of the edge dislocation along third axis. The present paper displays that in this case the gauge equation [1, 2] constraints the modified stress functions so that the resulting stress component σ_{33} arises as $\nu^{-1}(\sigma_{11}+\sigma_{22})$ instead of $\nu(\sigma_{11}+\sigma_{22})$ at $\kappa \to 0$, i.e., σ_{33} is not of the edge dislocation.

The ansatz itself looks more valuably for the problem in question, and therefore the contradiction found can be eliminated by another choice of the master translational gauge equation. Namely, the use of the double curl differential operator in its L.H.S. leads to the replacement $a \longleftrightarrow 1 - a$, or, equivalently, $\nu \longleftrightarrow \nu^{-1}$. More precisely, both the linear gauge equations in question (our double curl Eq. (14)and that from [1, 2]), provided they are written through the general tensor of stress functions, pass each into other just simply by inverting the Poisson ratio ν. Eventually, both the standard linear solutions for dislocations are possible with the new gauge equation, though in the formal limit $\kappa\rho \ll 1$ ($\kappa \to 0$, ρ is finite).

From a more formal point of view on Lagrangian field-theoretical modelling of conventional (and non-conventional) defects, the newly proposed Eq. (14) looks suitable for the replication of the dislocation solutions. Therefore, it is worth to be realized that the double curl in the L.H.S. of (14) can be considered not only formally, as a nice trick, but also as linearization of a three-dimensional Einstein tensor. Moreover, a non-linear constitutive relation 'stress-strain' can be thought of instead of the R.H.S. of (14). Thus, (14) can be treated as a weak field limit of a non-linear Einstein-type Eq. (29), which is the most straightforward extension of (14). The Eq. (29) uses the framework of torsion-free Riemannian geometry, and it can be derived by Lagrangian method.

The torsion components can also be expressed through the stress functions by means of the relation 'strain-stress' and the Kröner ansatz. The direct calculation in the linear approximation shows us that these components are not zero, though vanishing can be asked for a special choice of the stress functions. In our situation, Sec. 3, vanishing occurs in the particular limit $\kappa\rho \ll 1$, when the modified stress functions coincide approximately with the Airy's and Prandtl's ones. This simply means that the torsion-less interpretation behind (28) is valid for (14) just at $\kappa\rho \ll 1$. (Notice, that non-linear constitutive relation has to be used to re-express torsion in general situation).

Strictly speaking, the results concerning the torsion rather imply a necessity to extend the framework and to pass to the Riemann–Cartan formalism, which will allow for a non-trivial torsion. However, the Eq. (28) itself still looks appropriate in the following sense. As soon as the matter is concerned with the defects sufficiently separated each from other, the torsion can be required as localized inside tubes confining the defects. Outside such tubes (i.e., for $\rho > R_{\rm core} > 0$) but not very faraway from the cores ($\rho < R_{\rm exterior} < \infty$), the Eq. (28) can approximately be valid for the purely translational defects at $\kappa\rho \ll 1$. Indeed, the Eq. (14) with the solutions found capture properly the required conventional stress functions and can be considered as a first step of an iterative scheme at $\kappa \to 0$, provided (28) would dictate next corrections, valid outside $R_{\rm core}$.

Recall that the stress function approach [20, 21] postulates vanishing of the Einstein tensor, while the torsion (i.e., the density of dislocations) is presumed δ-like to obtain the first order approximation solutions. This picture corresponds to teleparallel geometry. On the contrary, in our case the source in the Einstein equation is present, but it is smeared. Hence, the first order solutions appear only in the special limit $\kappa \to 0$.

It is the Hilbert–Einstein Lagrangian which provides the Einstein tensor as the L.H.S. of the gauge equation. However it requires a modification of the geometric picture [1, 2]. Practically, the use of the "coordinate" representation seems to be appropriate. Besides, now it is more clear, how the matter Lagrangian $\mathcal{L}_\xi$ affects the linearized model, e.g., it leads to the limited modified stress functions instead of the increasing conventional ones. Although we have started with a rather special problem of modification of the σ_{33}-component, it seems, eventually, that the ideas discussed here should be valid for the whole ISO (3)-model also.

As the immediate implication of the present consideration, let us mention the following one. The second order corrections to the conventional screw dislocation, which are implemented by [1, 2], have been obtained in [7]. As far as $\mathcal{L}_\phi$ (3) is quadratic in derivatives of the gauge potential, the L.H.S. of the gauge equation is of first order in second derivatives. Therefore, it remains the same for both perturbative steps in [7]. But the Lagrangian $\mathcal{L}_\phi$ (3) is shown to be fairly inconsistent when replicating the edge dislocation, while the Hilbert–Einstein one removes the problem. Therefore usage of $\mathcal{L}_\phi$ suggested here could influence the results of [7], because the L.H.S. of the corresponding Einstein-type equation is non-linear. Moreover, the use in [7] of the constitutive relation without the Lam*é* constants of third order could be restrictive.

Generally speaking, both the Lagrangians, the Hilbert–Einstein and the quadratic, can occur in the gauge-translational models. The matter should be to specify a proper range of problems for each of them. The consideration presented is to show that the Hilbert–Einstein Lagrangian is rather appropriate when replicating the conventional dislocations, but further investigations are needed to go beyond the weak field. In its turn, the Lagrangian (3) can be more appropriate in such problems as [3–5]. Besides, the Refs. [40] should be mentioned where a Lagrangian quadratic in the translational field strength has been used in more general form to study a non-gravitational structure on space-time.

To summarize, a non-linear Lagrangian $T(3)$-gauge model is proposed which admits at $\kappa\rho \ll 1$, the conventional dislocation solutions of the linear isotropic (incompatible) elasticity. The calculation presented can be considered as a first order approximation. Incorporation of second order contributions to the gauge equation, and/or extension to the ISO (3)-case to include the torsion are needed to extend the present consideration, and to decide about further perspectives of the ISO (3)-gauging for defects modelling. With regard to the great attention to the $T(3)$-gauge models with the quadratic $\mathcal{L}_\phi$, it is hopeful that the work presented would serve to a more adequate understanding of the problem discussed.

A historical remark: more superficial versions of the critical content of the Sec. 2 and of the correction discussed in Sec. 3 can be found, respectively, in [41] and [42].

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