PDMI PREPRINT — 24/1998

Liouville central charge in quantum Teichmüller theory

R. M. Kashaev
Russian Academy of Sciences
Steklov Mathematical Institute
St. Petersburg Department
Fontanka 27, St. Petersburg 191011, Russia
E-mail:kashaev@pdmi.ras.ru

November 1998

ABSTRACT

In the quantum Teichmüller theory, based on Penner coordinates, the mapping class groups of punctured surfaces are represented projectively. The case of a genus three surface with one puncture is worked out explicitly. The projective factor is calculated. It is given by the exponential of the Liouville central charge.

1 Introduction

In [5] L.D. Faddeev suggested to use as a building block for the evolution operator in the quantum Liouville model on discrete space-time [6] the following non-compact version of the quantum dilogarithm function:

$$\psi(x) \equiv \exp\left(-\frac{1}{4}\int_{-\infty}^{+\infty} \frac{\exp(-\mathrm{i}xz)\,dz}{\sinh(\pi\lambda z)\sinh(\pi\lambda^{-1}z)z}\right), \tag{1}$$

where the singularity at $z = 0$ is put below of the contour of integration, and λ is a real parameter. Among other things this function satisfies an inversion relation

$$\psi(x)\psi(-x) = \exp\left(-\frac{\mathrm{i}\pi}{12}(\lambda^2 + \lambda^{-2}) - \frac{\mathrm{i}x^2}{4\pi}\right). \tag{2}$$

Faddeev conjectured that the inversion factor in this equation (the constant part of the r.h.s) is related to the Liouville central charge [7].

The purpose of this paper is to prove Faddeev's conjecture. Our approach is based on the quantum theory of Teichmüller spaces of punctured surfaces suggested recently in [13] and independently by Fock, Chekhov and Frolov [8, 9]. It is known [17] that the Teichmüller spaces appear as the classical phase spaces of Chern–Simons theory with gauge group $SL(2,\mathbb{R})$. This theory is related to $2+1$ dimensional gravity with cosmological constant [1, 18, 17], which in turn induces the Liouville theory at spatial infinity [2, 3].

The quantum theory of the Teichmúller spaces, described in [13], is based on Penner's coordinates [16]. The mapping class (or modular) group of a punctured surface is projectively represented in the corresponding Hilbert space of physical states in terms of the non-compact quantum dilogarithm. We work out explicitly the case of a genus three surface with one puncture, and calculate the corresponding projective factor which, being essentially given by the inversion factor of the non-compact quantum dilogarithm, appears to be the exponential of the quantum Liouville central charge. This is in agreement with the interpretation of the Hilbert space of quantum states as Virasoro conformal blocks [17], since the projective factors in the representations of the mapping class groups, coming from two-dimensional conformal field theories, are known to be given by the exponential of the corresponding Virasoro central charges [10, 15].

2 The non-compact quantum dilogarithm

2.1 Notation

For any natural m define embeddings

$$\iota_i \colon \mathrm{End}\, L^2(\mathbb{R}) \ni a \mapsto a_i = 1 \otimes \cdots \otimes a \otimes \cdots \otimes 1 \in \mathrm{End}\, L^2(\mathbb{R}^m), \quad 1 \le i \le m,$$

where a stands in the i-th position, which mean that the operator a_i acts on the i-th argument as a, and identically on the others:

$$a_i f(x_1, \ldots, x_m) = a f(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_m)|_{x=x_i}.$$

If $b \in \mathrm{End}\, L^2(\mathbb{R}^k)$ for some $0 < k \le m$ and $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, m\}$, we write

$$b_{i_1 i_2 \ldots i_2} \equiv \iota_{i_1} \otimes \iota_{i_2} \otimes \cdots \otimes \iota_{i_k}(b).$$

The permutation group $\mathbb{S}_m$ is represented faithfully in $\mathrm{End}\, L^2(\mathbb{R}^m)$:

$$\mathsf{P}_\sigma f(\ldots, x_i, \ldots) \equiv f(\ldots, x_{\sigma(i)}, \ldots), \quad \forall \sigma \in \mathbb{S}_m. \tag{3}$$

2.2 Some algebraic relations

Let $\mathsf{p}, \mathsf{q} \in \mathrm{End}\, L^2(\mathbb{R})$ be defined by

$$\mathsf{p} f(x) \equiv -2\pi \mathsf{i} \partial f(x)/\partial x, \quad \mathsf{q} f(x) \equiv x f(x).$$

They satisfy the Heisenberg commutation relation

$$[\mathsf{q}, \mathsf{p}] = 2\pi \mathsf{i}.$$

The non-compact quantum dilogarithm, defined by eqn (1), solves a pair of functional equations

$$\psi(x + \mathsf{i}\pi\lambda^{\pm 1}) = (1 + \exp(x\lambda^{\pm 1}))\psi(x - \mathsf{i}\pi\lambda^{\pm 1}),$$

and, as was argued in [5], satisfies the following five-term operator equation:

$$\psi(\mathsf{q})\psi(\mathsf{p}) = \psi(\mathsf{p})\psi(\mathsf{p} + \mathsf{q})\psi(\mathsf{q}). \tag{4}$$

This fact can be proved rigorously [19].

We consider a unitary operator $\mathsf{T} \in \mathrm{End}\, L^2(\mathbb{R}^2)$, defined by

$$\mathsf{T} \equiv \exp\left(\frac{\mathsf{i}}{2\pi} \mathsf{p}_1 \mathsf{q}_2\right) \psi(\mathsf{q}_1 + \mathsf{p}_2 - \mathsf{q}_2). \tag{5}$$

Eqn (4) is equivalent to

$$\mathsf{T}_{12}\mathsf{T}_{13}\mathsf{T}_{23} = \mathsf{T}_{23}\mathsf{T}_{12}, \tag{6}$$

while the inversion relation (2) leads to the equation

$$\mathsf{T}_{12}\mathsf{R}_1\mathsf{T}_{21}\mathsf{R}_1^{-1} = \zeta\mathsf{R}_1\mathsf{P}_{(12)}, \tag{7}$$

where unitary operator $\mathsf{R} \in \mathrm{End}\, L^2(\mathbb{R})$ is defined by

$$\mathsf{R}f(x) \equiv \exp\left(\frac{\mathsf{i}x^2}{4\pi} - \frac{\mathsf{i}\pi}{12}\right)\int_{-\infty}^{+\infty} f(y)\exp\left(\frac{\mathsf{i}xy}{2\pi}\right)\frac{dy}{2\pi}, \tag{8}$$

and

$$\zeta = \exp\left(-\frac{\mathsf{i}\pi}{12}(\lambda + \lambda^{-1})^2\right). \tag{9}$$

Besides these we have a symmetry property

$$\mathsf{T}_{12} = \mathsf{R}_1^{-1}\mathsf{R}_2\mathsf{T}_{21}\mathsf{R}_1\mathsf{R}_2^{-1} \tag{10}$$

with the same operator R defined by eqn (8). Note that R satisfies the following equations:

$$\mathsf{R}\mathsf{q}\mathsf{R}^{-1} = \mathsf{p} - \mathsf{q}, \quad \mathsf{R}\mathsf{p}\mathsf{R}^{-1} = -\mathsf{q}, \quad \mathsf{R}^3 = 1.$$

It can also be written in an operator form

$$\mathsf{R} = \exp\left(-\frac{\mathsf{i}\pi}{3}\right)\exp\left(\frac{\mathsf{i}}{2\pi}\mathsf{q}^2\right)\exp\left(\frac{\mathsf{i}}{4\pi}\mathsf{p}^2\right)\exp\left(\frac{\mathsf{i}}{4\pi}\mathsf{q}^2\right).$$

3 Projective representations of mapping class groups

We call a two-cell in CW complex *triangle* if exactly three boundary points of the corresponding two-disk are mapped to the zero-skeleton. We shall also call zero-cells and one-cells *vertices* and *edges*, respectively.

Let $\Sigma = \Sigma_{g,r}$ be a closed oriented Riemann surface of genus g with a set V of $r > 0$ marked points such that $M \equiv 2g - 2 + r > 0$. By *ideal triangulation (i.t.)* τ of Σ we mean the isotopy class of a representation of Σ as a CW complex with V as the set of vertices, and all two-cells being triangles. It is easy to count that there are $2M$ triangles and $3M$ edges in any i.t.. We shall denote by $E(\tau)$ and $T(\tau)$ the sets of edges and triangles respectively in i.t. τ.

An i.t. τ is called *decorated (d.i.t.)* if each triangle is provided by a marked corner and a bijective *ordering* mapping

$$\bar{\tau}\colon \{1,\ldots,2M\} \ni j \mapsto \bar{\tau}_j \in T(\tau)$$

is fixed. Denote the set of all d.i.t. as $\Delta(\Sigma)$. The permutation group $\mathbb{S}_{2M}$ naturally acts in $\Delta(\Sigma)$ from the right:

$$\Delta(\Sigma) \times \mathbb{S}_{2M} \ni (\tau,\sigma) \mapsto \tau\sigma \in \Delta(\Sigma),$$

where $\tau\sigma$ differs from τ only in ordering:

$$\overline{\tau\sigma} = \bar{\tau} \circ \sigma.$$

We also define two elementary transformations

$$\rho^t, \omega_e\colon \Delta(\Sigma) \to \Delta(\Sigma)$$

as follows.

Let t be a triangle in d.i.t. τ. We define d.i.t. $\tau^t \equiv \rho^t(\tau)$ by changing the marked corner of triangle t as is shown in Figure 1 and leaving the ordering mapping unchanged: $\bar{\tau} = \bar{\tau}^t$. Let now e be an edge in d.i.t. τ shared by

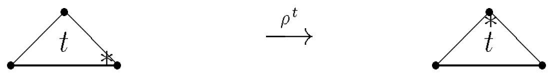

Figure 1: Elementary transformation ρ^t consisting of changing the marked corner of triangle t.

two distinct triangles s, t with marked corners as is shown in the l.h.s. of Figure 2. Then d.i.t. $\tau_e \equiv \omega_e(\tau)$ is obtained from τ by replacing the cells

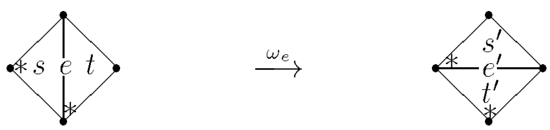

Figure 2: Elementary transformation ω_e consisting of replacing edge e by the complementary edge e'.

e, s, t by new cells e', s', t' as is shown in the r.h.s. of Figure 2. The ordering mapping $\bar{\tau}_e$ is defined by

$$\bar{\tau}_e^{-1}\big|_{T(\tau_e)-\{s',t'\}} = \bar{\tau}^{-1}\big|_{T(\tau)-\{s,t\}}, \quad \bar{\tau}_e^{-1}(s') = \bar{\tau}^{-1}(s).$$

For each d.i.t. τ and for each $1 \le i \le 2M$ define

$$\mathsf{F}(\tau, \tau^{\bar{\tau}(i)}) \equiv \mathsf{R}_i \in \operatorname{End} L^2(\mathbb{R}^{2M}).$$

Let $i \neq j$ be such that triangles $s \equiv \bar{\tau}(i)$ and $t \equiv \bar{\tau}(j)$ share an edge e as in the l.h.s. of Figure 2. Then we define

$$\mathsf{F}(\tau, \tau_e) \equiv \mathsf{T}_{ij} \in \operatorname{End} L^2(\mathbb{R}^{2M}). \tag{11}$$

Finally, for any permutation $\sigma \in \mathbb{S}_{2M}$ define

$$\mathsf{F}(\tau, \tau\sigma) \equiv \mathsf{P}_\sigma,$$

where P_σ is defined by eqn (3). We claim that due to relations (6), (10) and (7), F can be extended to a unitary operator valued function $\mathsf{F}(\tau, \tau')$ on $\Delta(\Sigma) \times \Delta(\Sigma)$ such that for any d.i.t. τ, τ', τ''

$$\mathsf{F}(\tau, \tau')\mathsf{F}(\tau', \tau'')\mathsf{F}(\tau'', \tau) \in \mathbb{C}.$$

The mapping class or modular group M_Σ of Σ naturally acts in $\Delta(\Sigma)$. By construction we have the following invariance property of the function F:

$$\mathsf{F}(f(\tau), f(\tau')) = \mathsf{F}(\tau, \tau'), \qquad \forall f \in M_\Sigma.$$

This enables us to construct a unitary projective representation of M_Σ:

$$M_\Sigma \ni f \mapsto \mathsf{F}(\tau, f(\tau)) \in \operatorname{End} L^2(\mathbb{R}^{2M}).$$

Indeed,

$$\mathsf{F}(\tau, f(\tau))\mathsf{F}(\tau, h(\tau)) = \mathsf{F}(\tau, f(\tau))\mathsf{F}(f(\tau), f(h(\tau))) \simeq \mathsf{F}(\tau, fh(\tau)),$$

where we denote by $\simeq$ an equality up to a numerical factor. In the next section we calculate the projective factor in the case of the surface $\Sigma_{3,1}$ of genus three and with one puncture.

4 The case of $\Sigma_{3,1}$

To find the central charge we shall need a suitable presentation of the mapping class group. First, following [14] introduce some notation.

Let $\mathcal{S} = \mathcal{S}(\Sigma)$ be the set of isotopy classes of simple closed curves on the surface Σ. We denote by D_α, $\alpha \in \mathcal{S}$, the Dehn twist along α. Define

$$I(\alpha, \beta) = \min\{|a \cap b| : a \in \alpha, b \in \beta\}, \quad \forall \alpha, \beta \in \mathcal{S}.$$

We shall use $\alpha \cap \beta = \emptyset$ to denote $I(\alpha, \beta) = 0$; $\alpha \perp \beta$ to denote $I(\alpha, \beta) = 1$; and $\alpha \perp_0 \beta$ to denote $I(\alpha, \beta) = 2$ so that their algebraic intersection number is zero. If a, b are two arcs intersecting transversely at a point p, then the *resolution of $a \cup b$ at p from a to b* is defined as follows. Fix any orientation on a and use the orientation on the surface to determine an orientation on b. Then resolve the intersection according to the orientations, see Figure 3.

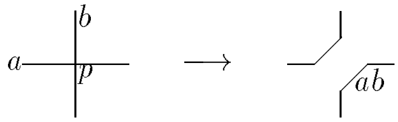

Figure 3: Resolution from a to b at p.

The resolution is independent on the choice of the orientations on a. If $\alpha \perp \beta$ or $\alpha \perp_0 \beta$, take $a \in \alpha, b \in \beta$ so that $|a \cap b| = I(\alpha, \beta)$. Then the curve obtained by resolving all intersection points in $a \cap b$ from a to b is again simple closed curve. Denote by $\alpha\beta$ its' isotopy class. It follows that when $\alpha \perp \beta$ then $\alpha\beta \perp \alpha, \beta$ and when $\alpha \perp_0 \beta$ then $\alpha\beta \perp_0 \alpha, \beta$. Also the Dehn twist of β along α is given by $D_\alpha(\beta) = \alpha\beta$ if $\alpha \perp \beta$, and $D_\alpha(\beta) = \alpha(\alpha\beta)$ if $\alpha \perp_0 \beta$. Let $N(a)$ and $N(b)$ be small regular neighborhoods of a and b. Then $N(a \cup b) = N(a) \cup N(b)$ is homeomorphic to a torus with one boundary component when $\alpha \perp \beta$, and to a sphere with four boundary components when $\alpha \perp_0 \beta$. Denote by $\partial(\alpha, \beta)$ the isotopy class of the curve system $\partial N(a \cup b)$.

One has a *lantern* relation [4, 12]:

$$D_\alpha D_\beta D_{\alpha\beta} = D_{\partial(\alpha,\beta)} \text{ if } \alpha \perp_0 \beta, \tag{12}$$

and a *chain* relation [4]:

$$(D_\alpha D_\beta D_\gamma)^4 = D_{\epsilon_1} D_{\epsilon_2} \text{ if } \alpha, \beta, \gamma, \epsilon_i \text{ are as shown in Figure 4.} \tag{13}$$

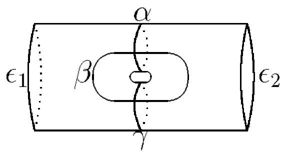

Figure 4: The Dehn twists along these curves satisfy the chain relation.

Theorem 1 (Gervais [11]) *For any compact oriented surface Σ of genus $g > 1$, the mapping class group M_Σ has the following presentation:*

$$\begin{array}{rrl} \textit{generators}: & & \{D_\alpha\colon \textit{ non-separating } \alpha \in \mathcal{S}(\Sigma)\}; \\ \textit{relations}: & (i) & D_\alpha D_\beta = D_\beta D_\alpha \textit{ if } \alpha \cap \beta = \emptyset, \\ & (ii) & D_{\alpha\beta} = D_\alpha D_\beta D_\alpha^{-1} \textit{ if } \alpha \perp \beta, \\ & (iii) & \textit{one lantern relation (12)}, \\ & (iv) & \textit{one chain relation (13)}. \end{array}$$

If $\alpha \perp \beta$, relations (ii) together with $\alpha\beta \perp \alpha$, $(\alpha\beta)\alpha = \beta$ imply that

$$D_\alpha D_\beta D_\alpha = D_\beta D_\alpha D_\beta \text{ if } \alpha \perp \beta. \tag{14}$$

4.1 Realization of the Dehn twists

To begin with[1], we describe the simplest case of the Dehn twist of an annulus along the only simple non-contractible curve denoted in Figure 5 as α. From

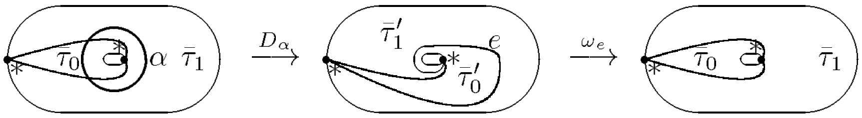

Figure 5: The Dehn twist $\tau' \equiv D_\alpha(\tau)$ of d.i.t. τ of an annulus along contour α is mapped by the elementary transformation ω_e back to $\tau = \omega_e(\tau')$.

the same Figure 5 it follows that $\omega_e \circ D_\alpha = \mathrm{id}$. So, using definition (11), we obtain

$$\mathsf{F}(\tau, D_\alpha(\tau)) = \mathsf{F}(D_\alpha^{-1}(\tau), \tau) = \mathsf{F}(\omega_e(\tau), \tau) \simeq \mathsf{T}_{01}^{-1}, \tag{15}$$

where the normalization remains to be fixed.

Now we consider the curves on $\Sigma_{3,1}$, shown in Figure 6. We choose the d.i.t. τ, obtained by cutting the surface into three handles and a triangle as shown in Figures 7, 8. The corresponding Dehn twists can be calculated by choosing appropriate d.i.t. where the annular neighborhoods of the curves look like as in Figure 5, and using formula (15). The result of such calculations reads:

$$\left.\begin{aligned} \mathsf{F}_{\alpha_i} &= \zeta^{-6}\,\mathrm{Ad}(\mathsf{T}_{3i-1,3i-2})\mathsf{T}^{-1}_{3i,\widehat{3i-2}}, \\ \mathsf{F}_{\beta_i} &= \zeta^{-6}\,\mathrm{Ad}(\mathsf{T}_{3i,\widehat{3i-2}})\mathsf{T}^{-1}_{\widehat{3i-1},3i}, \end{aligned}\right. \qquad i = 1, 2, 3;$$

$$\mathsf{F}_{\gamma_1} = \zeta^{-6}\,\mathrm{Ad}(\mathsf{T}_{0\check{7}}\mathsf{T}_{87}\mathsf{T}_{10}\mathsf{T}_{1\check{9}}\mathsf{T}_{21}\mathsf{T}_{\check{3}1}\mathsf{T}_{27}\mathsf{T}_{30})\mathsf{T}^{-1}_{\check{0}\hat{7}},$$

[1]In the rest of the paper we numerate triangles from 0 thru $2M - 1$ rather than from 1 thru $2M$.

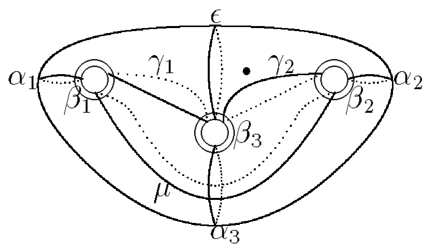

Figure 6: The Dehn twists along these curves generate the mapping class group of the surface $\Sigma_{3,1}$

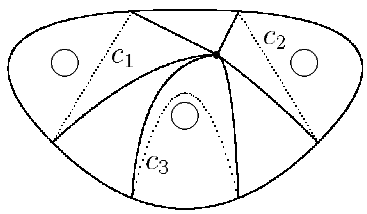

Figure 7: The surface $\Sigma_{3,1}$ is cut into a triangle and three handles

$$\mathsf{F}_{\gamma_2} = \zeta^{-6}\,\mathrm{Ad}(\mathsf{T}_{87}\mathsf{T}_{8\check{0}}\mathsf{T}_{8\check{4}}\mathsf{T}_{\hat{8}9}\mathsf{T}_{69}\mathsf{T}_{\check{9}7}\mathsf{T}_{54}\mathsf{T}_{\check{7}8})\mathsf{T}^{-1}_{\check{8}\hat{4}},$$
$$\mathsf{F}_{\mu} = \zeta^{-6}\,\mathrm{Ad}(\mathsf{T}_{0\check{7}}\mathsf{T}_{87}\mathsf{T}_{10}\mathsf{T}_{1\check{9}}\mathsf{T}_{\hat{8}9}\mathsf{T}_{21}\mathsf{T}_{49}\mathsf{T}_{69}\mathsf{T}_{54}\mathsf{T}_{5\check{8}}\mathsf{T}_{08}\mathsf{T}_{30}\mathsf{T}_{3\check{4}}\mathsf{T}_{\hat{4}9}\mathsf{T}_{\hat{1}0})\mathsf{T}^{-1}_{\check{1}\hat{9}},$$
$$\mathsf{F}_{\alpha_3\mu} = \zeta^{-6}\,\mathrm{Ad}(\mathsf{T}_{0\check{7}}\mathsf{T}_{87}\mathsf{T}_{10}\mathsf{T}_{\hat{8}9}\mathsf{T}_{18}\mathsf{T}_{21}\mathsf{T}_{49}\mathsf{T}_{69}\mathsf{T}_{30}\mathsf{T}_{\check{9}7}\mathsf{T}_{4\check{2}}\mathsf{T}_{54}\mathsf{T}_{\hat{4}7}\mathsf{T}_{\hat{1}0}\mathsf{T}_{50})\mathsf{T}^{-1}_{\check{0}\hat{7}},$$
$$\mathsf{F}_{\epsilon} = \zeta^{-6}\,\mathrm{Ad}(\mathsf{T}_{0\check{7}}\mathsf{T}_{87}\mathsf{T}_{10}\mathsf{T}_{30}\mathsf{T}_{\check{2}0}\mathsf{T}_{\hat{3}0}\mathsf{T}_{21}\mathsf{T}_{\hat{3}2}\mathsf{T}_{93}\mathsf{T}_{\check{3}1}\mathsf{T}_{\check{1}0})\mathsf{T}^{-1}_{\check{0}\hat{7}},$$

where

$$\mathsf{F}_{\alpha} \simeq \mathsf{F}(\tau, D_{\alpha}(\tau)),$$
$$\mathrm{Ad}(a)b \equiv aba^{-1},$$
$$a_{\hat{k}} \equiv \mathrm{Ad}(\mathsf{R}_k)a_k, \quad a_{\check{k}} \equiv \mathrm{Ad}(\mathsf{R}_k^{-1})a_k,$$

see eqn (8) for the definition of the operator R. Note that the mutual normalization of these operators is fixed by the relations of the type (14):

$$\mathsf{F}_{\alpha_1}\mathsf{F}_{\beta_1}\mathsf{F}_{\alpha_1} = \mathsf{F}_{\beta_1}\mathsf{F}_{\alpha_1}\mathsf{F}_{\beta_1}, \quad \ldots$$

while the overall normalization is fixed by the lantern relation:

$$\mathsf{F}_{\alpha_3}\mathsf{F}_{\mu}\mathsf{F}_{\alpha_3\mu} = \mathsf{F}_{\alpha_1}\mathsf{F}_{\alpha_2}\mathsf{F}_{\gamma_1}\mathsf{F}_{\gamma_2}.$$

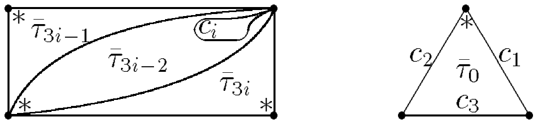

Figure 8: D.i.t. τ of the surface $\Sigma_{3,1}$. Index i takes three values $1, 2, 3$. The rectangles with identified opposite sides represent the three handles, which are glued along the boundary loops c_i to the corresponding sides of the triangle $\bar\tau_0$.

Thus, we do not have any other freedom in normalization. Checking the chain relation we recover the projective factor:

$$(\mathsf{F}_{\alpha_1}\mathsf{F}_{\beta_1}\mathsf{F}_{\gamma_1})^4 = \xi_{\mathsf{F}}\mathsf{F}_\epsilon\mathsf{F}_{\alpha_3}, \quad \xi_{\mathsf{F}} = \zeta^{-72}. \tag{16}$$

4.2 Relation to the Liouville central charge

In the projective representations of the mapping class groups, associated with quantum CFT, the projective factors have the form $\exp(2\pi\mathrm{i}nc/24)$, where n is an integer and c is the Virasoro central charge [10, 15]. According to the result of [13] on quantization of the Teichmüller spaces of punctured surfaces, the representation given by the operators F_α is not quite the representation, corresponding to the quantum Teichmüller space of $\Sigma_{3,1}$. There are additional degrees of freedom, given by the constraints associated with homologies of the surface. Explicit form of six basis constraints, corresponding to oriented contours α_i, β_i, is as follows:

$$\begin{aligned} \mathsf{z}_{\alpha_i} &= \mathsf{q}_{3i-2} - \mathsf{p}_{3i-2} + \mathsf{p}_{3i-1} - \mathsf{p}_{3i}, \\ \mathsf{z}_{\beta_i} &= \mathsf{p}_{3i-2} - \mathsf{q}_{3i-2} - \mathsf{q}_{3i-1} + \mathsf{q}_{3i}, \end{aligned} \quad i = 1, 2, 3.$$

All the constraints, corresponding to other contours, are linear combinations of the basis ones:

$$\mathsf{z}_{\gamma_i} = \mathsf{z}_{\alpha_i} - \mathsf{z}_{\alpha_3}, \quad i = 1, 2,$$

$$\mathsf{z}_\mu = \mathsf{z}_{\alpha_1} - \mathsf{z}_{\alpha_2}, \quad \mathsf{z}_{\alpha_3\mu} = \mathsf{z}_{\alpha_3} - \mathsf{z}_{\alpha_1} - \mathsf{z}_{\alpha_2}, \quad \mathsf{z}_\epsilon = \mathsf{z}_{\alpha_3}.$$

To get rid of these degrees of freedom, we note the following action of the Dehn twists on the constraints:

$$\mathrm{Ad}(\mathsf{F}_\alpha)\mathsf{z}_\beta = \mathsf{z}_\beta + \alpha \circ \beta \mathsf{z}_\alpha, \tag{17}$$

for oriented contours α, β, $\alpha \circ \beta$ being their algebraic intersection index. The constraints themselves satisfy the following commutation relations:

$$[\mathsf{z}_\alpha, \mathsf{z}_\beta] = 4\pi\mathrm{i}\alpha \circ \beta. \tag{18}$$

From eqns (17), (18) it follows that the combinations

$$\mathsf{D}_\alpha \equiv \exp(\mathsf{i}\mathsf{z}_\alpha^2/8\pi)\mathsf{F}_\alpha$$

commute with the constraints and satisfy the defining relations for the Dehn twists with the projective factor $\xi_{\mathsf{D}} = -\xi_{\mathsf{F}}$ in eqn (16). Taking into account eqn (9) we obtain

$$\xi_{\mathsf{D}} = -\xi_{\mathsf{F}} = -\zeta^{-72} = \exp(\mathrm{i}\pi c_L), \tag{19}$$

where

$$c_L = 1 + 6(\lambda + \lambda^{-1})^2 \pmod 2 \tag{20}$$

is the Virasoro central charge in quantum Liouville theory. This is in agreement with interpretation of physical states of quantum Teichmüller theory as Virasoro conformal blocks.

5 Summary

In the quantum Teichmüller theory of punctured surfaces, based on Penner coordinates, the mapping class groups are represented projectively in terms of non-compact quantum dilogarithm. Algebraically the representations are based on three operator equations (6),(7), and (10).

We have calculated the projective factor for the case of genus three surface with one puncture. The reason for considering the genus three surface is due to the fact that for lower genus the projective factor can be absorbed into a redefinition of generators of the mapping class group. Only starting from genus three one has simultaneously the lantern (12) and chain (13) relations for the Dehn twists along non-separating curves [2], which are not homogeneous in generators.

The result, given by eqns (19), (20), is the exponentiated Liouville central charge. This is in agreement with the connection of the $SL(2,\mathbb{R})$ Chern–Simons and Liouville theories on the classical level, as well as with the expected interpretation of the Hilbert space of states in quantum Teichmüller theory as the space of Virasoro conformal blocks. Note that the right answer has been obtained only after elimination of the non-physical (Gaussian) degrees of freedom associated with the homologies of the surface.

[2] the relative normalization of Dehn twists along non-separating curves is fixed by the braid type relations (14)

6 Acknowledgements

It is a pleasure to thank L.D. Faddeev for encouragement and helpful suggestions at various stages of the work. I would also like to thank V. Fock, N. Reshetikhin, S. Shatashvili, V. Tarasov, O. Tirkkonen, S.L. Woronowicz, P. Zograf for discussions. The work is supported in part by grant RFFI-96-01-00851.

References

[1] Achúcarro, A., Townsend, P.K.: Chern–Simons actions for three dimensional anti-De Sitter supergravity theories. *Phys. Lett.* B **180** (1986), 89

[2] Carlip, S.: Inducing Liouville theory from topologically massive gravity. *Nuclear Phys.* B**362** (1991), 111–24

[3] Coussaert, O., Henneaux, M., van Driel,P.: The asymptotic dynamics of three-dimensional Einstein gravity with a negative cosmological constant. *Class. Quant. Grav.* **12** (1995), 2961–6, gr-qc/9506019

[4] Dehn, M.: Papers on group theory and topology. J.Stillwell (eds.). Springer–Verlag, Berlin–New York, 1987.

[5] Faddeev, L.D.: Discrete Heisenberg–Weyl group and modular group. *Lett. Math. Phys.*, **34** (1995), 249–54, hep-th/9504111

[6] Faddeev, L.D., Volkov, A.Yu.: Hirota equation as an example of integrable symplectic map. *Lett. Math. Phys.*, **32** (1994), 125–36, hep-th/9405087

[7] Faddeev, L.D.: Private communication, 1995

[8] Fock, V.: Dual Teichmüller spaces. ITEP-TH-5-96, dg-ga/9702018

[9] Fock, V., Chekhov, L., Frolov, S.: To be published.

[10] Friedan, D., Shenker, S.: The analytic geometry of two-dimensional conformal field theory.*Nuclear Phys.* B **281** (1987), 509–45.

[11] Gervais, S.: Presentation and central extensions of mapping class groups. *Trans. Amer. Math. Soc.* **348**(8) (1996), 3097–132.

[12] Johnson, D.: Homeomorphisms of a surface which act trivially on homology. *Proc. of Amer. Math. Soc.* **75** (1979), 119–25.

[13] Kashaev, R.M.: Quantization of Teichmüller spaces and the quantum dilogarithm. *Lett. Math. Phys.* **43**(2) (1998), 105–15, q-alg/9705021

[14] Luo, F.: A presentation of the mapping class groups. math.GT/9801025

[15] Moore,G., Seiberg, N.: Classical and quantum conformal field theory. *Commun. Math. Phys.* **123** (1989), 177–254.

[16] Penner, R.C.: The decorated Teichmüller space of punctured surfaces. *Commun. Math. Phys.* **113** (1987), 299–339.

[17] Verlinde, H.: Conformal field theory, two-dimensional quantum gravity and quantization of Teichmüller space.*Nuclear Phys.* B **337** (1990), 652–80.

[18] Witten, E.: $2+1$-dimensional gravity as an exactly soluble system. *Nuclear Phys.* B **311** (1988/89), 46–78.

[19] Woronowicz, S.L.: Private communication, 1998.