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ON A GROTHENDIECK CONJECTURE ABOUT PRINCIPAL HOMOGENEOUS SPACES FOR SPECIAL UNITARY GROUP

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ABSTRACT

The goal of this paper is to introduce some applications and consequences of the injectivity of the map $\mathfrak{F}(R) \to \mathfrak{F}(K)$ induced by the canonical inclusion of a local regular ring R of geometric type to its field of fractions $R \hookrightarrow K$, where $\mathfrak{F}$ is a homotopy invariant functor with transfers. In particular, we get an original proof of a Grothendieck Conjecture about principal homogeneous spaces for Special Unitary group.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

§1. Introduction

The goal of the present work is to consider some interesting examples of functors by which we can apply the theory developed before in [Z$_1$]. As a consequences, we will get two cases of Grothendieck Conjecture on principal homogeneous spaces [G] — for the Special Linear and for the Special Unitary group.

Let's remind that the case of Special Linear group was originally considered by I.Panin and A.Suslin in [PS]. But the method they have offered don't work for other interesting examples like Special Unitary group.

In our discussion we will often refer to the paper of I.Panin and M.Ojanguren [PO] which was a starting point of our investigations.

We keep all notations and agreements used in [Z$_1$]. Let A be a smooth algebra over an infinite field k and R be its local regular ring at some maximal ideal $\mathfrak{m}$, i.e. $R = A_{\mathfrak{m}}$. Let K be the field of fractions of R. Let $\mathfrak{F}$ be some specially chosen functor from the category of A-algebras to abelian groups.

The point is to show that this functor satisfies to the axioms **C**, **TE**, **TA**, **TB** and **E** from [Z$_1$]. Then we can apply our main theorem

Theorem (Non-Constant Case)**.** *Let $\mathfrak{F}$ be the functor on the category of A-algebras and let $\mathfrak{F}_R$ be its restriction to the category of R-algebras. If $\mathfrak{F}$ and $\mathfrak{F}_R$ satisfy to the axioms* **C**, **TE**, **TA**, **TB** *and* **E** *then the homomorphism $\mathfrak{F}(R) \to \mathfrak{F}(K)$ induced by a canonical inclusion is injective.*

and get the injectivity of the map $\mathfrak{F}(R) \to \mathfrak{F}(K)$ (see the (a) statement of the theorems below). Finally, playing with a few long exact cohomology sequences and using the injectivity above we get injectivity on the first cohomology level (see the (b) statement of the theorems below).

To make a content clear we are going to formulate all the axioms excluding **E** (see [Z$_1$]):

For a given functors $\mathfrak{F}$ and $\mathfrak{F}_R$ we must have:

C. (continuity) For any A-algebra S smooth over k and for any localization $M^{-1}S$ of S via the multiplicative system M functor $\mathfrak{F}$ commutes with direct limit

$$\varinjlim_{g \in M} \mathfrak{F}(S_g) = \mathfrak{F}(M^{-1}S)$$

taken over all filtered multiplicative systems induced by $g \in M$

Axioms for the transfer map.

For any finitely generated and projective R-algebra T (since R is local, T is a free R-module)

TE. (existance) There exists homomorphism $\mathrm{Tr}_R^T : \mathfrak{F}_R(T) \to \mathfrak{F}_R(R)$ called transfer map;

TA. (additivity) For every $x \in \mathfrak{F}_R(R \times T)$, $x_R = \mathrm{pr}_R^*(x) \in \mathfrak{F}_R(R)$ and $x_T = \mathrm{pr}_T^*(x) \in \mathfrak{F}_R(T)$ the equality $\mathrm{Tr}_R^{R\times T}(x) = x_R + \mathrm{Tr}_R^T(x_T)$ holds in $\mathfrak{F}_R(R)$, where pr_R^* and pr_T^* are induced by projections;

TB. (base changing and homotopy invariance) For any finitely generated projective $R[t]$-algebra S (thus, $S/(t)$ and $S/(t-1)$ are finitely generated projective R-algebras) the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{F}_R(S) & \xrightarrow{\mathrm{can}_0^*} & \mathfrak{F}_R(S/(t)) \\ {\scriptstyle \mathrm{can}_1^*}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}_0} \\ \mathfrak{F}_R(S/(t-1)) & \xrightarrow[\mathrm{Tr}_1]{} & \mathfrak{F}_R(R) \end{array},$$

where can_0^*, can_1^* are induced by canonical projections and Tr_0, Tr_1 denote the corresponding transfer maps $\mathrm{Tr}_R^{S/(t)}$ and $\mathrm{Tr}_R^{S/(t-1)}$.

Now let describe our special functors and corresponding results we are interested in: (detailed consideration of each case separately one can find in the text below)

Linear Case. Let $\mathcal{A}$ be some Azumaya algebra over the given local regular ring R. Let $\mathrm{Nrd} : \mathcal{A}^* \to R^*$ denotes the reduced norm homomorphism. For any R-algebra S let $\mathcal{A}_S$ be an extended Azumaya algebra over S.

Define the group scheme $\mathrm{SL}_{1,\mathcal{A}}$ related to an Azumaya R-algebra $\mathcal{A}$ as

$$\mathrm{SL}_{1,\mathcal{A}} : S \mapsto \mathrm{SL}(\mathcal{A}_S) = \{a \in \mathcal{A}_S \mid \mathrm{Nrd}(a) = 1\}.$$

Our aim is to show:

Theorem. *Let $\mathcal{A}$ be an Azumaya algebra over a local regular ring R of geometric type. Then*

(a) *the homomorphism $R^*/\mathrm{Nrd}(\mathcal{A}^*) \longrightarrow K^*/\mathrm{Nrd}(\mathcal{A}_K^*)$ is injective;*

(b) *the canonical map $H^1_{\text{ét}}(R, \mathrm{SL}_{1,\mathcal{A}}) \longrightarrow H^1_{\text{ét}}(K, \mathrm{SL}_{1,\mathcal{A}_K})$ on the first cohomology groups induced by inclusion is injective.*

Note that the statement (a) of the theorem corresponds to the case when the functor $\mathfrak{F}$ is defined as $\mathfrak{F} : S \mapsto S^*/\mathrm{Nrd}(\mathcal{A}_S^*)$.

We recall that the conjecture of Grothendieck [G] states the triviality of the kernel of the canonical map

$$H^1_{\text{ét}}(R, G) \longrightarrow H^1_{\text{ét}}(K, G_K)$$

for any reductive flat group scheme G over a regular semilocal ring R. Thus, the statement (b) of the theorem above proves the following assertion.

Corollary (Special Linear Group)**.** *The Grothendieck conjecture is true for the group scheme $G = \mathrm{SL}_{1,\mathcal{A}}$ related to an Azumaya R-algebra $\mathcal{A}$ over a local regular ring of geometric type.*

In the same notation assume that there is the additional structure on our Azumaya algebra: Let $(\mathcal{A}, \sigma)$ be an Azumaya algebra with involution over R.

We know that there can be a three different types of involution: ortogonal, sympletic and unitary. It turns out that in case of ortogonal and sympletic involution we can prove the same results as above by using a well-known facts about quadratic forms. So the only interesting case for us is the unitary case:

Unitary Case. Let $(\mathcal{A}, \sigma)$ be an Azumaya algebra with unitary involution over R. It means that there is a tower $\mathcal{A}/C/R$, where C is the center of $\mathcal{A}$ and C/R is an etale quadratic extension over R with restricted involution σ. Therefore, C_K/K is a separable quadratic extension of the corresponding fields of fractions.

Let $\mathrm{U}(\mathcal{A}_S) = \{a \in \mathcal{A}_S \mid aa^\sigma = 1\}$ be the unitary group of an algebra $(\mathcal{A}_S, \sigma)$ for any R-algebra S. We define the group scheme $\mathrm{SU}_{1,\mathcal{A}}$ related to an Azumaya R-algebra $\mathcal{A}$ with unitary involution σ as

$$\mathrm{SU}_{1,\mathcal{A}} : S \mapsto \mathrm{SU}(\mathcal{A}_S) = \{a \in \mathcal{A}_S \mid aa^\sigma = 1,\ \mathrm{Nrd}(a) = 1\}$$

Our goal will be to show:

Theorem. *Let $(\mathcal{A}, \sigma)$ be an Azumaya algebra with unitary involution over a local regular ring R of geometric type. Then*

(a) *the homomorphism $\mathrm{U}(C)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A})) \longrightarrow \mathrm{U}(C_K)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A}_K))$ is injective;*

(b) *the canonical map* $H^1_{\text{ét}}(R, \mathrm{SU}_{1,\mathcal{A}}) \longrightarrow H^1_{\text{ét}}(K, \mathrm{SU}_{1,\mathcal{A}_K})$ *on the first cohomology groups induced by inclusion is injective.*

Clearly, the statement (a) corresponds to the functor $\mathfrak{F} : S \mapsto \mathrm{U}(C_S)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A}_S))$, where $C_S = C \otimes_R S$ is the center of the extended Azumaya algebra $\mathcal{A}_S$.

As before, there is a consequence of (b):

Corollary (Special Unitary Group)**.** *The Grothendieck conjecture is true for the group scheme* $G = \mathrm{SU}_{1,\mathcal{A}}$ *related to an Azumaya* R*-algebra* $\mathcal{A}$ *with unitary involution over a local regular ring of geometric type.*

The next two theorems represent independent interest. One can look on it as on modification of corresponding Linear and Unitary Case.

For the functor $\mathfrak{F} : S \mapsto S^*/\mathrm{Nrd}(\mathcal{A}^*_S)(S^*)^d$, $d \in \mathbb{N}$, we have

Theorem (Linear Torsion Case)**.** *Let* $\mathcal{A}$ *be an Azumaya algebra over a local regular ring* R *of geometric type and* d *be some natural number. Then the homomorphism*

$$R^*/\mathrm{Nrd}(\mathcal{A}^*)(R^*)^d \longrightarrow K^*/\mathrm{Nrd}(\mathcal{A}^*_K)(K^*)^d$$

is injective.

And for the functor $\mathfrak{F} : S \mapsto \mathrm{U}(C_S)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A}_S))\mathrm{U}(C_S)^d$, $d \in \mathbb{N}$, we have

Theorem (Unitary Torsion Case)**.** *Let* $(\mathcal{A}, \sigma)$ *be an Azumaya algebra with unitary involution over a local regular ring* R *of geometric type and* d *be some natural number. Then the homomorphism*

$$\mathrm{U}(C)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A}))\mathrm{U}(C)^d \longrightarrow \mathrm{U}(C_K)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A}_K))\mathrm{U}(C_K)^d$$

is injective.

§2. The Properties of an Azumaya algebras

Let us formulate some properties of an Azumaya algebras which we are going to use. Mostly, one can find them in [K] and [PO].

We recall that an Azumaya R-algebra over a commutative ring R is an R-algebra $\mathcal{A}$ that has the following two properties: it is a finitely generated projective R-module, and the canonical R-algebra homomorphism $\mathcal{A} \otimes_R \mathcal{A}^{op} \to \mathrm{End}_R(\mathcal{A})$ is an isomorphism. Clearly, for each R-algebra S, the scalar extension $\mathcal{A}_S = \mathcal{A} \otimes_R S$ of the algebra $\mathcal{A}$ is an Azumaya algebra over S.

We will use the reduced norm homomorphism $\mathrm{Nrd}_R : \mathcal{A}^* \to R^*$. This homomorphism respects the scalar extensions. In the case where R is a field and S is its algebraic closure, the homomorphism Nrd_S can be identified with the usual determinant $\det : \mathrm{GL}_d(S) \to S^*$; this identification is induced by an S-algebra isomorphism between $\mathcal{A}_S$ and the matrix algebra $M_d(S)$ mentioned above.

By an Azumaya algebra with an involution over a commutative ring R we mean a pair $(\mathcal{A}, \sigma)$ consisting of an R-algebra $\mathcal{A}$ and an R-linear involution σ on $\mathcal{A}$ such that: i) $\mathcal{A}$ is an Azumaya algebra over its center C; ii) C is either R or an etale quadratic extension of R; and iii) $C^\sigma = R$. Note that an involution commutes with reduced norm map, i.e. $\mathrm{Nrd}_C(a^\sigma) = \mathrm{Nrd}_C(a)^\sigma$ for any $a \in \mathcal{A}^*$.

When $C = R$ we have an involution of the first kind. More precise, involution σ of the first kind is called ortogonal (sympletic) if the induced involution on the matrix algebra $\mathcal{A} \otimes \mathcal{A}^{op}$ is given by a symmetric (alternating) bilinear form.

In the case when C is quadratic etale over R we have an unitary involution (involution of the second kind).

Now we fix a local regular ring $R = A_{\mathfrak{m}}$ of a smooth k-algebra A and consider an Azumaya R-algebra $\mathcal{A}$. Then we have the following properties:

I. Since an R-algebra $\mathcal{A}$ is given by a finite number of equations we can find a localization A_g, $g \in A \setminus \mathfrak{m}$, such that an algebra $\mathcal{A}$ came from some Azumaya A_g-algebra $\mathcal{A}_g$, i.e. $\mathcal{A} = \mathcal{A}_g \otimes_{A_g} R$. On geometric language it means that we can extend an Azumaya algebra given at a point to some neighbourhood of this point.

By the same reasons as before we can state that if an Azumaya R-algebras are isomorphic then there exists a localization A_g, $g \in A \setminus \mathfrak{m}$, such that the induced Azumaya A_g-algebras are isomorphic, too.

These arguments also work in the case when our Azumaya algebra has the involution structure.

II. We also will need in the following reformulation of the Proposition 7.1 of [PO]: Let $\mathcal{O}$ be some semilocal regular ring such that there is the inclusion $R \overset{i}{\hookrightarrow} \mathcal{O}$ with section ε. Let $(\mathcal{A}_{\mathcal{O}}, \sigma)$ and $(\mathcal{B}_{\mathcal{O}}, \tau)$ be two Azumaya with involution over $\mathcal{O}$, of the same rank. Assume that there exists an isomorphism $\phi : (\mathcal{A}_R, \sigma) \to (\mathcal{B}_R, \tau)$. Then there exists a finite etale extension $e : \mathcal{O} \to \tilde{\mathcal{O}}$, a section $\tilde{\varepsilon} : \tilde{\mathcal{O}} \to R$ of e over R and an isomorphism $\Phi : (\mathcal{A}_{\tilde{\mathcal{O}}}, \tilde{\sigma}) \to (\mathcal{B}_{\tilde{\mathcal{O}}}, \tilde{\tau})$ of extended Azumaya algebras over $\tilde{\mathcal{O}}$ such that the restriction of Φ via $\tilde{\varepsilon}$ on Azumaya R-algebras $\Phi \otimes \mathrm{id}_R : (\mathcal{A}_{\tilde{\mathcal{O}}}, \tilde{\sigma}) \otimes_{\tilde{\mathcal{O}}} R \to (\mathcal{B}_{\tilde{\mathcal{O}}}, \tilde{\tau}) \otimes_{\tilde{\mathcal{O}}} R$ coinsides with ϕ.

Remark. *We have to add that all our properties will be right for the case when R is a semilocal regular ring, i.e. R is a localization at some finite number of maximal ideals of A.*

§3. Linear Case

Take $R = A_{\mathfrak{m}}$ to be a local regular ring of a smooth k-algebra A and let $\mathcal{A}$ be an Azumaya algebra over R.

Let $\mathfrak{F}$ be the functor defined on the category of R-algebras as:

$$\mathfrak{F} : S \mapsto S^*/\mathrm{Nrd}(\mathcal{A}_S^*).$$

First of all, because of the property **I**, we can assume that our functor is defined on the category of A_g-algebras, for some $g \in A \setminus \mathfrak{m}$. Since A_g is also a smooth k-algebra and R is its local regular ring we can write A instead of A_g and nothing will change. Hence, we will write $\mathcal{A}_R$ instead of $\mathcal{A}$ meaning that $\mathcal{A}_R$ is an extension of scalars of $\mathcal{A}$ via the canonical inclusion $A \overset{i_R}{\hookrightarrow} R$. Thus, we can say that our functor $\mathfrak{F}$ is defined on the category of A-algebras.

We would like to show that $\mathfrak{F}$ and $\mathfrak{F}_R$ satisfy to the axioms **C**, **TE**, **TA**, **TB** and **E**:

C. Note that the functor $G_m : S \mapsto S^*$ sending any A-algebra S to its group of units is continuous. Since $\mathcal{A}$ is a finitely generated projective A-algebra, the functor of extension of scalars $S \mapsto \mathcal{A}_S$ is continuous. Thus the functor $S \mapsto \mathrm{Nrd}(G_m(\mathcal{A}_S))$ is continuous, too, and we get the required.

E. This axiom is a combination of the properties **I** and **II**:

By conditions of **E** let $R \overset{i}{\to} S$ be some R-algebra smooth over the field k. Let there exists the inclusion $i_S : A \to S$ and the section (augmentation) $\varepsilon : S \to R$ of i, such that the following

diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{i_S} & S \\ \| & & \downarrow \varepsilon \\ A & \xrightarrow[i_R]{} & R \end{array}$$

Let $\{\mathfrak{m}_i\}_{i\in\mathcal{I}}$ be some finite set of maximal ideals in S with the property $\varepsilon^{-1}(\mathfrak{m}) \subset \cup_{i\in\mathcal{I}}\mathfrak{m}_i$. Let M be the multiplicative system $S \setminus \cup_{i\in\mathcal{I}}\mathfrak{m}_i$.

Now look on two functors given on the category of S-algebras:

$$\mathfrak{F} : T \mapsto T^*/\mathrm{Nrd}(\mathcal{A}_T^*) \qquad \text{and} \qquad \mathfrak{F}_R : T \mapsto T^*/\mathrm{Nrd}(\mathcal{B}_T^*),$$

where for any S-algebra T, $\mathcal{A}_T$ is an extension of scalars of $\mathcal{A}$ via the inclusion $A \xrightarrow{i_S} S \to T$ and $\mathcal{B}_T$ is an extension of scalars of $\mathcal{A}$ via $A \xrightarrow{i_R} R \xrightarrow{i} S \to T$. In general, $\mathcal{A}_T$ and $\mathcal{B}_T$ are not isomorphic.

The geometric idea here is to produce a finite etale extension at some fix points of S by using the second property such that the functors $\mathfrak{F}$ and $\mathfrak{F}_R$ become there equivalent. Then by the first property we can continue this extension to some neighbourhood of these points keeping the equivalence between $\mathfrak{F}$ and $\mathfrak{F}_R$.

We introduce a semilocal ring $\mathcal{O}$ as the localization $M^{-1}S$ of S at the maximal ideals $\{\mathfrak{m}_i\}_{i\in\mathcal{I}}$. By Remark 3 of [Z$_1$] we have an augmentation map $\varepsilon : \mathcal{O} \to R$ for the inclusion $R \xrightarrow{i} S \xrightarrow{i_{\mathcal{O}}} \mathcal{O}$ compatible with augmentation on S. Hence, there is the commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow{i_S} & S & \xrightarrow{i_{\mathcal{O}}} & \mathcal{O} \\ \| & & \varepsilon\downarrow & & \downarrow\varepsilon \\ A & \xrightarrow[i_R]{} & R & = & R \end{array}$$

Consider an extended Azumaya $\mathcal{O}$-algebras $\mathcal{A}_{\mathcal{O}}$ and $\mathcal{B}_{\mathcal{O}}$. Since its restrictions via ε coinside, i.e. there is the identity $\mathcal{A}_R = \mathcal{B}_R$, we are in conditions of the second property (our ϕ is the identity).

So there exists a finite etale extension $e : \mathcal{O} \to \tilde{\mathcal{O}}$, a section $\tilde{\varepsilon} : \tilde{\mathcal{O}} \to R$ for the inclusion $R \xrightarrow{i} S \xrightarrow{i_{\mathcal{O}}} \mathcal{O} \xrightarrow{e} \tilde{\mathcal{O}}$ compatible with ε and an isomorphism $\Phi : \mathcal{A}_{\tilde{\mathcal{O}}} \to \mathcal{B}_{\tilde{\mathcal{O}}}$ of extended Azumaya algebras which restricted via $\tilde{\varepsilon}$ to an Azumaya R-algebras coincides with the identity.

By the properties of a finite etale extensions there exists the localization S_h, $h \in M$, a finite etale extension $e : S_h \to \tilde{S}$ such that $\tilde{\mathcal{O}} = \mathcal{O} \otimes_{S_h} \tilde{S}$. By the well-known properties of a finite extensions for any direct system of localizations $\varinjlim_{S_h \subset S_g} S_g = \mathcal{O}$ we have an induced direct system $\{\tilde{S}_g\}_{\tilde{S}\subset\tilde{S}_g}$, where $\tilde{S}_g = S_g \otimes_{S_h} \tilde{S}$, and the canonical map $\varinjlim_{\tilde{S}\subset\tilde{S}_g} \tilde{S}_g \to \tilde{\mathcal{O}}$ is an isomorphism.

Take the composition $\tilde{S} \to \tilde{\mathcal{O}} \xrightarrow{\tilde{\varepsilon}} R$ to be the augmentation on $\tilde{S}$. Clearly, it is compatible with ε.

We know that our Azumaya $\tilde{\mathcal{O}}$-algebras $\mathcal{A}_{\tilde{\mathcal{O}}}$ and $\mathcal{B}_{\tilde{\mathcal{O}}}$ are isomorphic so by the first property there exists a localization S_g, $g \in M$, containing S_h, such that an induced Azumaya $\tilde{S}_g$-algebras

$\mathcal{A}_{\tilde{S}_g}$ and $\mathcal{B}_{\tilde{S}_g}$ are isomorphic also, where $\tilde{S}_g = S_g \otimes_{S_h} \tilde{S}$. Thus, one have the diagram:

$$\begin{array}{ccccc} & & \tilde{S} & \longrightarrow & \tilde{S}_g & \longrightarrow & \tilde{\mathcal{O}} \\ & & {\scriptstyle e}\uparrow & & {\scriptstyle e}\uparrow & & \uparrow{\scriptstyle e} \\ S & \longrightarrow & S_h & \longrightarrow & S_g & \longrightarrow & \mathcal{O} \end{array}$$

To simplify the notations we replace $\tilde{S}_g$ by $\tilde{S}$.

We already have checked properties **E.(a)** and **E.(b)**, so the last step is to show **E.(c)**:

Since we have an isomorphism $\Phi : \mathcal{A}_{\tilde{S}} \to \mathcal{B}_{\tilde{S}}$ on Azumaya $\tilde{S}$-algebras and by definition $\mathfrak{F}(\tilde{S}) = \tilde{S}^*/\mathrm{Nrd}(\mathcal{A}^*_{\tilde{S}})$, $\mathfrak{F}_R(\tilde{S}) = \tilde{S}^*/\mathrm{Nrd}(\mathcal{B}^*_{\tilde{S}})$, we have an isomorphism on the corresponding Azumaya algebras got by extensions of scalars and, therefore, we have a natural equivalence of the functors $\Phi : \mathfrak{F} \to \mathfrak{F}_R$ on the category of $\tilde{S}$-algebras.

Moreover, our restricted Azumaya algebras via the augmenation $\tilde{\varepsilon}$ coincide, i.e. $\mathcal{A}_R = \mathcal{B}_R$. Hence, we have the commutative diagram:

$$\begin{array}{ccc} \mathfrak{F}(\tilde{S}) = \tilde{S}^*/\mathrm{Nrd}(\mathcal{A}^*_{\tilde{S}}) & \xrightarrow{\Phi(\tilde{S})} & \tilde{S}^*/\mathrm{Nrd}(\mathcal{B}^*_{\tilde{S}}) = \mathfrak{F}_R(\tilde{S}) \\ {\scriptstyle \tilde{\varepsilon}^*}\downarrow & & \downarrow{\scriptstyle \tilde{\varepsilon}^*} \\ \mathfrak{F}(R) = R^*/\mathrm{Nrd}(\mathcal{A}^*_R) & = & R^*/\mathrm{Nrd}(\mathcal{B}^*_R) = \mathfrak{F}_R(R) \end{array} .$$

And we have finished.

TE. Let T be some finitely generated projective R-algebra. Thus, T is a semilocal ring. We define the transfer map

$$\mathrm{Tr}^T_R : T^*/\mathrm{Nrd}(\mathcal{A}^*_T) \longrightarrow R^*/\mathrm{Nrd}(\mathcal{A}^*_R)$$

to be the usual norm map $\mathrm{N}^T_R : T^* \to R^*$ on the quotients.

To see that it is a well-defined we need in:

Lemma. *Let $\mathcal{A}$ be an Azumaya algebra over a semilocal ring R. Let T/R be a finitely generated projective extension and let $\mathrm{N}^T_R : T^* \to R^*$ be its norm map. Then we claim that:*

$$\mathrm{N}^T_R(\mathrm{Nrd}_T(\mathcal{A}^*_T)) \subset \mathrm{Nrd}_R(\mathcal{A}^*).$$

Proof: To prove this inclusion we are passing our reduced norms through K_1-groups. By results of [B] there is the commutative diagram:

$$\begin{array}{ccccccc} \mathcal{A}^*_T & \xrightarrow{i_T} & K_1(\mathcal{A}_T) & \xrightarrow{\mathrm{N}^*} & K_1(\mathcal{A}) & \xleftarrow{i} & \mathcal{A}^* \\ {\scriptstyle \mathrm{Nrd}_T}\downarrow & & {\scriptstyle \mathrm{Nrd}^*_T}\downarrow & & \downarrow{\scriptstyle \mathrm{Nrd}^*_R} & & \downarrow{\scriptstyle \mathrm{Nrd}_R} \\ T^* & = & T^* & \xrightarrow[\mathrm{N}]{} & R^* & = & R^* \end{array}$$

where N^* is the norm map on K_1-groups; i^* is surjective homomorphism induced by inclusion $\mathcal{A}^* = \mathrm{GL}_1(\mathcal{A}) \hookrightarrow K_1(\mathcal{A})$; and Nrd^* is the reduced norm on K_1 of an Azumaya algebra. If $\mathcal{A}$ splits, i.e. $\mathcal{A} = M_d(R)$, we can write our reduced norm as the composition:

$$\mathrm{Nrd}^* : K_1(M_d(R)) \xrightarrow{\text{Morita equivalence}} K_1(R) \xrightarrow{\det} R^*.$$

We get the required inclusion since i is surjective. $\square$

TA. The additivity follows from the corresponding property of the norm map: If T_1 and T_2 are finitely generated projective over R then for any $(t_1, t_2) \in T_1 \times T_2$ we have $\mathrm{N}_R^{T_1 \times T_2}(t_1, t_2) = \mathrm{N}_R^{T_1}(t_1)\mathrm{N}_R^{T_2}(t_2)$. Since $\mathfrak{F}_R(T_1 \times T_2) = \mathfrak{F}_R(T_1) \times \mathfrak{F}_R(T_2)$ we get the required.

TB. Let S be any finitely generated projective $R[t]$-algebra. Observe that the functor $G_m : T \mapsto T^*$ with norm homomorphism as the transfer map satisfies homotopy invariance **H** and base changing **TB'** properties, thus, it satisfies **TB**.

To see the last property let look on the diagram induced by **TB** applying to G_m:

$$(*) \qquad \begin{array}{ccc} G_m(S) & \xrightarrow{\mathrm{can}_0^*} & G_m(S/(t)) \\ {\scriptstyle \mathrm{can}_1^*}\downarrow & & \downarrow{\scriptstyle \mathrm{N}_0} \\ G_m(S/(t-1)) & \xrightarrow[\mathrm{N}_1]{} & G_m(R) \end{array}$$

where can_i^*, $i = 0, 1$ are induced by projections, and N_i denote the corresponding norm map.

Four diagrams induced by projection $\mathrm{pr} : G_m(T) \to G_m(T)/\mathrm{Nrd}(\mathcal{A}_T^*) = \mathfrak{F}_R(T)$ are commutes as well:

$$\begin{array}{ccc} G_m(S) & \xrightarrow{\mathrm{can}_i^*} & G_m(S/(t-i)) \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ \mathfrak{F}_R(S) & \xrightarrow[\mathrm{can}_i^*]{} & \mathfrak{F}_R(S/(t-i)) \end{array} \qquad \begin{array}{ccc} G_m(S/(t-i)) & \xrightarrow{\mathrm{N}_i} & G_m(R) \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ \mathfrak{F}_R(S/(t-i)) & \xrightarrow[\mathrm{Tr}_i]{} & \mathfrak{F}_R(R) \end{array}$$

where Tr_i, $i = 0, 1$ denote the corresponding transfer map.

Thus, since the projection pr is surjective, we get the required diagram commutes:

$$\begin{array}{ccc} \mathfrak{F}_R(S) & \xrightarrow{\mathrm{can}_0^*} & \mathfrak{F}_R(S/(t)) \\ {\scriptstyle \mathrm{can}_1^*}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}_0} \\ \mathfrak{F}_R(S/(t-1)) & \xrightarrow[\mathrm{Tr}_1]{} & \mathfrak{F}_R(R) \end{array}$$

Applying now our theory we get the statement (a) of the Linear Case:

Theorem (a)**.** *The map* $R^*/\mathrm{Nrd}(\mathcal{A}^*) \longrightarrow K^*/\mathrm{Nrd}(\mathcal{A}_K^*)$ *induced by a canonical inclusion is injective.*

To prove the statement (b) let look on the short exact sequence of a smooth group schemes:

$$1 \longrightarrow \mathrm{SL}_{1,\mathcal{A}} \longrightarrow \mathrm{GL}_{1,\mathcal{A}} \xrightarrow{\mathrm{Nrd}} G_m \to 1,$$

where the $\mathrm{GL}_{1,\mathcal{A}} : S \mapsto \mathcal{A}_S^*$.

It induces the long exact cohomology sequence:

$$1 \longrightarrow \mathrm{SL}(\mathcal{A}) \longrightarrow \mathcal{A}^* \xrightarrow{\mathrm{Nrd}} R^* \longrightarrow H^1_{\text{ét}}(R, \mathrm{SL}_{1,\mathcal{A}}) \longrightarrow H^1_{\text{ét}}(R, \mathrm{GL}_{1,\mathcal{A}}) \longrightarrow \cdots$$

Since R is local, the group $H^1_{\text{ét}}(R, \mathrm{GL}_{1,\mathcal{A}})$ is trivial (see [K]).

The inclusion $R \hookrightarrow K$ induces the natural map on our long cohomology sequence, so that the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{SL}(\mathcal{A}) & \longrightarrow & \mathcal{A}^* & \xrightarrow{\mathrm{Nrd}} & R^* & \longrightarrow & H^1_{\text{ét}}(R, \mathrm{SL}_{1,\mathcal{A}}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{SL}(\mathcal{A}_K) & \longrightarrow & \mathcal{A}_K^* & \xrightarrow[\mathrm{Nrd}_K]{} & K^* & \longrightarrow & H^1_{\text{ét}}(K, \mathrm{SL}_{1,\mathcal{A}_K}) & \longrightarrow & 1 \end{array}$$

commutes.

Taking the cokernels we get:

$$\begin{array}{ccc} R^*/\mathrm{Nrd}(\mathcal{A}^*) & \xrightarrow{\cong} & H^1_{\text{ét}}(R, \mathrm{SL}_{1,\mathcal{A}}) \\ \downarrow & & \downarrow \\ K^*/\mathrm{Nrd}(\mathcal{A}_K^*) & \xrightarrow[\cong]{} & H^1_{\text{ét}}(K, \mathrm{SL}_{1,\mathcal{A}_K}) \end{array} .$$

Since the left arrow is injective by (a) we get the statement (b):

Theorem (b)**.** *The map $H^1_{\text{ét}}(R, \mathrm{SL}_{1,\mathcal{A}}) \longrightarrow H^1_{\text{ét}}(K, \mathrm{SL}_{1,\mathcal{A}_K})$ induced by a canonical inclusion is injective.*

§4. Unitary Case

We keep all notations and definitions used in the Linear Case. The points of our discussion here mostly base on the previous case. Only the change is that we have an additional structure of unitary involution on our Azumaya algebras.

Thus, let $(\mathcal{A}, \sigma)$ be an Azumaya algebra with unitary involution over R.

Let $\mathfrak{F}$ be the functor defined on the category of R-algebras as:

$$\mathfrak{F} : S \mapsto \mathrm{U}(C_S)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A}_S)),$$

where $\mathrm{U}(C_S) = \{c \in C_S \mid cc^\sigma = 1\}$ is the unitary group of the center of $\mathcal{A}_S$.

By the same arguments as before, we can assume that our functor is given on the category of A-algebras.

Now the problem is to check the axioms **C**, **TE**, **TA**, **TB** and **E**. But it turns out that the first two axioms **C** and **E** follows immediately from the previous discussion because all used facts are carried out at presence of additional structure of unitary involution.

The main difficulty is to show the existance of the transfer map for some finitely generated projective extension T/R. Indeed, we would like to see the norm homomorphism in the role of the transfer map again but there is no way to show the inclusion

$$\mathrm{N}_C^{C_T}(\mathrm{Nrd}_{C_T}(\mathrm{U}(\mathcal{A}_T))) \subset \mathrm{Nrd}_C(\mathrm{U}(\mathcal{A}))$$

by using arguments with K_1 (there is no well-defined norm map for the unitary K_1).

The following important theorem [Z_2] gives us another possibility to do it.

Theorem (Norm Principle for the Unitary Group)**.** *Let $(\mathcal{A}_T, \sigma)$ be an Azumaya algebra with unitary involution σ over a semilocal ring T. Let C_T be the center of $\mathcal{A}_T$, so C_T/T is an etale quadratic extension with restricted involution σ. Then the following equality holds:*

$$\mathrm{Nrd}_{C_T}(\mathrm{U}(\mathcal{A}_T)) = \mathrm{Nrd}_{C_T}(\mathcal{A}_T^*)^{1-\sigma},$$

where $c^{1-\sigma} = c(c^{\sigma})^{-1}$, *for any* $c \in C_T^*$.

Let's notice that the paper [Z_2] contains the proof only for the case when T is a local ring, but one can easily extend it to the semilocal case.

Since the norm commutes with an involution we get that

$$N_C^{C_T}(\mathrm{Nrd}_{C_T}(\mathrm{U}(\mathcal{A}_T))) = N_C^{C_T}(\mathrm{Nrd}_{C_T}(\mathcal{A}_T^*)^{1-\sigma}) =$$

$$N_C^{C_T}(\mathrm{Nrd}_{C_T}(\mathcal{A}_T^*))^{1-\sigma} \subset \mathrm{Nrd}_C(\mathcal{A}^*)^{1-\sigma} = \mathrm{Nrd}_C(\mathrm{U}(\mathcal{A})),$$

where the inclusion follows from the Lemma of §3 applied to an Azumaya algebra $\mathcal{A}$ over the semilocal ring C and extension C_T/C.

Hence, we can take the norm homomorphism as the transfer map.

Since we again have the identity $\mathfrak{F}_R(T_1 \times T_2) = \mathfrak{F}_R(T_1) \times \mathfrak{F}_R(T_2)$, an additivity axiom holds.

To see **TB** we have an analogous situation but here the diagram $(*)$ must be induced by the functor $\mathcal{R}^1_{C/R}(G_m) : S \mapsto \mathrm{U}(C_S)$ coming from the short exact sequence of group schemes:

$$1 \longrightarrow \mathcal{R}^1_{C/R}(G_m) \longrightarrow \mathcal{R}_{C/R}(G_m) \xrightarrow{\mathrm{N}_R^C} G_m \longrightarrow 1,$$

where $\mathcal{R}_{C/R}$ denotes Weil restriction.

Indeed, we know that the functor $\mathcal{R}_{C/R}(G_m) : S \mapsto C_S^*$ satisfies **TB** and the involution is compatible with norm map, hence, the kernel $\mathcal{R}^1_{C/R}(G_m)$ satisfies the axiom **TB**, too.

Thus, we have proved the statement (a) of the Unitary Case:

Theorem (a)**.** *The map* $\mathrm{U}(C)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A})) \longrightarrow \mathrm{U}(C_K)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A}_K))$ *induced by a canonical inclusion is injective.*

To show the statement (b) let look on the short exact sequence of a smooth group schemes:

$$1 \longrightarrow \mathrm{SU}_{1,\mathcal{A}} \longrightarrow \mathrm{U}_{1,\mathcal{A}} \xrightarrow{\mathrm{Nrd}} \mathcal{R}^1_{C/R}(G_m) \longrightarrow 1,$$

where the $\mathrm{U}_{1,\mathcal{A}} : S \mapsto \mathrm{U}(A_S)$.

It induces the long exact cohomology sequence:

$$1 \longrightarrow \mathrm{SU}(\mathcal{A}) \longrightarrow \mathrm{U}(\mathcal{A}) \xrightarrow{\mathrm{Nrd}} \mathrm{U}(C) \longrightarrow H^1_{\text{ét}}(R, \mathrm{SU}_{1,\mathcal{A}}) \longrightarrow H^1_{\text{ét}}(R, \mathrm{U}_{1,\mathcal{A}}) \longrightarrow \cdots$$

Observe that here the group $H^1_{\text{ét}}(R, \mathrm{U}_{1,\mathcal{A}})$ is not trivial.

The inclusion $R \hookrightarrow K$ induces the natural map on our long cohomology sequence, so that the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{SU}(\mathcal{A}) & \longrightarrow & \mathrm{U}(\mathcal{A}) & \xrightarrow{\mathrm{Nrd}} & \mathrm{U}(C) & \longrightarrow & H^1_{\text{ét}}(R, \mathrm{SU}_{1,\mathcal{A}}) \longrightarrow H^1_{\text{ét}}(R, \mathrm{U}_{1,\mathcal{A}}) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \qquad\qquad \downarrow \\
1 & \longrightarrow & \mathrm{SU}(\mathcal{A}_K) & \longrightarrow & \mathrm{U}(\mathcal{A}_K) & \xrightarrow[\mathrm{Nrd}_K]{} & \mathrm{U}(C_K) & \longrightarrow & H^1_{\text{ét}}(K, \mathrm{SU}_{1,\mathcal{A}_K}) \longrightarrow H^1_{\text{ét}}(K, \mathrm{U}_{1,\mathcal{A}_K})
\end{array}$$

commutes.

Taking the cokernels we get:

$$\begin{array}{ccccccc}
1 & \longrightarrow & \mathrm{U}(C)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A})) & \longrightarrow & H^1_{\text{ét}}(R, \mathrm{SU}_{1,\mathcal{A}}) & \longrightarrow & H^1_{\text{ét}}(R, \mathrm{U}_{1,\mathcal{A}}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \mathrm{U}(C_K)/\mathrm{Nrd}(\mathrm{U}(\mathcal{A}_K)) & \longrightarrow & H^1_{\text{ét}}(K, \mathrm{SU}_{1,\mathcal{A}_K}) & \longrightarrow & H^1_{\text{ét}}(K, \mathrm{U}_{1,\mathcal{A}_K})
\end{array} .$$

Since the left vertical arrow is injective by (a) and the right vertical arrow is injective by the result of [PO], we get the injectivity of the middle vertical arrow, i.e we have:

Theorem (b). *The map* $H^1_{\text{ét}}(R, \mathrm{SU}_{1,\mathcal{A}}) \longrightarrow H^1_{\text{ét}}(K, \mathrm{SU}_{1,\mathcal{A}_K})$ *induced by a canonical inclusion is injective.*

Torsion Cases. The last two theorems in the introduction one can prove easily by following the proof of the corresponding Linear and Unitary case. Nothing will change, since the functor of rising up to the power d has as good behaviour as the functor G_m does.

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