PDMI PREPRINT — 22/1998

ON A RATIONAL INJECTIVITY FOR HOMOTOPY INVARIANT FUNCTORS WITH TRANSFERS

K. Zainoulline

Russian Academy of Sciences
Steklov Mathematical Institute
St. Petersburg Department
Fontanka 27, St. Petersburg 191011, Russia

November 11, 1998

ABSTRACT

In the present paper we investigate the question about injectivity of the map $\mathfrak{F}(R) \to \mathfrak{F}(K)$ induced by the canonical inclusion of a local regular ring of geometric type R to its field of fractions K for a homotopy invariant functor $\mathfrak{F}$ with transfers.

Let R be some local regular ring of geometric type, i.e. R is a local ring at some closed point of a smooth affine variety over the field k. Let K be its field of fractions. Let $\mathfrak{F}$ be a functor from the category of R-algebras to the category of abelian groups.

Assume that $\mathfrak{F}$ is a homotopy invariant functor with transfers. Our main goal in the present article is to prove an injectivity of the map $\mathfrak{F}(R) \to \mathfrak{F}(K)$ induced by the canonical inclusion $R \hookrightarrow K$.

As a consequences of this result we get the Special Linear Case and the Special Unitary Case of Grothendieck's conjecture about principal homogeneous spaces [G] and some other important applications which we will describe precisely in another paper [Z].

The Special Linear Case was originally considered in the work of I. Panin and A. Suslin [PS] but the method they have used there doesn't work for the other interesting examples of a smooth group schemes like Special Unitary group.

Our work was motivated by the work of V. Voevodsky [V] and, mostly, by the paper of I. Panin and M. Ojanguren on one Grothendieck's conjecture for hermitian spaces [PO]. The point was to offer a good axiomatization for the theory they were introducing.

Acknowledgements. I personally grateful to Prof. Ivan Panin for his attention and help. I thank to University Franche-Comte Besancon and, espesially, to prof. Eva Bayer-Fluckiger for hospitality and useful discussions promoted to a writing this article.

Agreements

All rings are assumed to be commutative with identity. By k we will always mean an infinite field of characteristic different from 2.

Let A and S be any rings. By an A-algebra S we will mean the pair (S, i), where $i : A \to S$ is a ring homomorphism. Sometimes, we will write just S, instead of the pair (S, i), keeping in mind the homomorphism i. We will denote an A-algebra S by $A \xrightarrow{i} S$.

By a morphism $f : (S, i) \to (S', i')$ between two A-algebras we will mean the commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{i} & S \\ \| & & \downarrow f \\ A & \xrightarrow[i']{} & S' \end{array}$$

Let $\mathfrak{F}$ be a functor on the category of A-algebras to the category of abelian groups. Let $A \xrightarrow{i} R$ be some A-algebra. By restriction of $\mathfrak{F}$ to the category of R-algebras along i we will call the functor denoted by $\mathfrak{F}_R$ and given by the identity $\mathfrak{F}_R(R \xrightarrow{t} T) = \mathfrak{F}(A \xrightarrow{i} R \xrightarrow{t} T)$ on objects and the identity

$$\mathrm{Mor}(\mathfrak{F}_R(R \xrightarrow{t_1} T_1), \mathfrak{F}_R(R \xrightarrow{t_2} T_2)) = \mathrm{Mor}(\mathfrak{F}(A \xrightarrow{i} R \xrightarrow{t_1} T_1), \mathfrak{F}(A \xrightarrow{i} R \xrightarrow{t_2} T_2)),$$

on morphisms for any R-algebra T, T_1, T_2.

Further in the paper we will use the result of Grothendieck [E] which says that if we have a finite extension $A \to B$ of smooth k-algebras then B is a finitely generated projective as the A-module.

§1. The Constant Case

Let A be a smooth k-algebra. Let $R = A_{\mathfrak{m}}$ be a local ring at some fixed maximal ideal $\mathfrak{m}$. By $\mathfrak{m}_R$ we will denote the corresponding maximal ideal of R. Clearly, $A \overset{i_R}{\hookrightarrow} R$ is an A-algebra, where i_R is a canonical inclusion.

Let $\mathfrak{F}$ be a functor from the category of k-algebras to the category of abelian groups. By $\mathfrak{F}_R$ we denote its restriction to the category of R-algebras along $k \hookrightarrow A \overset{i_R}{\hookrightarrow} R$. Let functors $\mathfrak{F}$ and $\mathfrak{F}_R$ satisfy to the following list of axioms:

C. **(continuity)** For any A-algebra S smooth over k and for any localization $M^{-1}S$ of S via the multiplicative system M functor $\mathfrak{F}$ commutes with direct limit

$$\varinjlim_{g \in M} \mathfrak{F}(S_g) = \mathfrak{F}(M^{-1}S)$$

taken over all filtered multiplicative systems induced by $g \in M$;

Axioms for the transfer map. For any finitely generated and projective R-algebra T (since R is local, T is a free R-module)

TE. **(existance)** There exists homomorphism $\mathrm{Tr}_R^T : \mathfrak{F}_R(T) \to \mathfrak{F}_R(R)$ called transfer map;

TA. **(additivity)** For every $x \in \mathfrak{F}_R(R \times T)$, $x_R = \mathrm{pr}_R^*(x) \in \mathfrak{F}_R(R)$ and $x_T = \mathrm{pr}_T^*(x) \in \mathfrak{F}_R(T)$ the equality $\mathrm{Tr}_R^{R\times T}(x) = x_R + \mathrm{Tr}_R^T(x_T)$ holds in $\mathfrak{F}_R(R)$, where pr_R^* and pr_T^* are induced by projections;

TB. **(base changing and homotopy invariance)** For any finitely generated projective $R[t]$-algebra S (thus, $S/(t)$ and $S/(t-1)$ are finitely generated projective R-algebras) the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{F}_R(S) & \xrightarrow{\mathrm{can}_0^*} & \mathfrak{F}_R(S/(t)) \\ {\scriptstyle \mathrm{can}_1^*}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}_0} \\ \mathfrak{F}_R(S/(t-1)) & \xrightarrow[\mathrm{Tr}_1]{} & \mathfrak{F}_R(R) \end{array},$$

where can_0^*, can_1^* are induced by canonical projections and Tr_0, Tr_1 denote the corresponding transfer maps $\mathrm{Tr}_R^{S/(t)}$ and $\mathrm{Tr}_R^{S/(t-1)}$.

Let K be a field of fractions of the ring A. Then our aim is to prove:

Theorem (Constant case)**.** *Let $\mathfrak{F}$ be the functor on the category of k-algebras and let $\mathfrak{F}_R$ be its restriction to the category of R-algebras. If $\mathfrak{F}$ and $\mathfrak{F}_R$ satisfy to the axioms* **C**, **TE**, **TA**, **TB** *then the homomorphism $\mathfrak{F}(R) \to \mathfrak{F}(K)$ induced by the canonical inclusion is injective.*

Remark 1. To get a better feeling for the axioms above note that **TE**, **TA**, **TB** are the consequences of the following more strong conditions:

For any finitely generated projective S-algebras T_1, T_2 and T, where S is an R-algebra smooth over k

H. **(homotopy invariance)** There is the identity $\mathfrak{F}_R(S) = \mathfrak{F}_R(S[t])$ induced by inclusion;

TE′. **(existance)** There exists a homomorphism $\mathrm{Tr}_S^T : \mathfrak{F}_R(T) \to \mathfrak{F}_R(S)$ called transfer map;

TA′. **(additivity)** Let $T = T_1 \times T_2$. For every $x \in \mathfrak{F}_R(T)$, $x_1 = \mathrm{pr}_1^*(x) \in \mathfrak{F}_R(T_1)$ and $x_2 = \mathrm{pr}_2^*(x) \in \mathfrak{F}_R(T_2)$ the equality $\mathrm{Tr}_S^T(x) = \mathrm{Tr}_S^{T_1}(x_1) + \mathrm{Tr}_S^{T_2}(x_2)$ holds in $\mathfrak{F}_R(S)$, where $\mathrm{pr}_i^* : \mathfrak{F}_R(T_1 \times T_2) \to \mathfrak{F}_R(T_i)$ are induced by projections;

TB′. (base changing) For any extension U/S of an R-algebras the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{F}_R(T) & \longrightarrow & \mathfrak{F}_R(U \otimes_S T) \\ {\scriptstyle \mathrm{Tr}_S^T}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Tr}_U^{U\otimes_S T}} . \\ \mathfrak{F}_R(S) & \longrightarrow & \mathfrak{F}_R(U) \end{array}$$

Indeed, to show that the axiom **TB** is the consequence of the axioms **H** and **TB′** let look on two commutative diagrams arising when we are applying **TB′** to the extension $S/R[t]$ and evaluations $i_0 : R[t] \overset{t\mapsto 0}{\to} R$, $i_1 : R[t] \overset{t\mapsto 1}{\to} R$ at 0 and 1 correspondingly:

$$\begin{array}{ccc} \mathfrak{F}_R(S) & \xrightarrow{\mathrm{can}_0^*} & \mathfrak{F}_R(S/(t)) \\ {\scriptstyle \mathrm{Tr}_{R[t]}^S}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Tr}_0} \\ \mathfrak{F}_R(R[t]) & \xrightarrow[i_0^*]{} & \mathfrak{F}_R(R) \end{array} \qquad \begin{array}{ccc} \mathfrak{F}_R(S) & \xrightarrow{\mathrm{Tr}_{R[t]}^S} & \mathfrak{F}_R(R[t]) \\ {\scriptstyle \mathrm{can}_1^*}\big\downarrow & & \big\downarrow{\scriptstyle i_1^*} \\ \mathfrak{F}_R(S/(t-1)) & \xrightarrow[\mathrm{Tr}_1]{} & \mathfrak{F}_R(R) \end{array} .$$

By axiom **H** there is the identity $\mathfrak{F}_R(R) \overset{\mathrm{id}}{\to} \mathfrak{F}_R(R[t])$ induced by inclusion. Since the compositions $R \hookrightarrow R[t] \overset{i_0}{\to} R$ and $R \hookrightarrow R[t] \overset{i_1}{\to} R$ are identities, it is clear that the evaluations i_0^* and i_1^* coinside with identity, too.

Hence, the left-down part of the first diagram coinside with the upper-right part of the second diagram and we can glue it together to get the required one. □

This section is organized as follows: 1.1 contains the proof of Specialization Lemma, 1.2 devoted to the version of the Quillen's Trick, and the last subsection 1.3 contains the proof of our theorem.

1.1. Specialization Lemma

Let R be a local regular ring of a smooth k-algebra A and let $R \overset{i}{\to} S$ be a given R-algebra with properties:

- **S1. (finitness conditions)** a) S is finite over $R[t]$ for some specially chosen $t \in S$; b) The quotient $S/(f)$ is finite over R;
- **S2. (existance of augmentation)** There is the augmentation map $\varepsilon : S \to R$ such that the composition $R \overset{i}{\to} S \overset{\varepsilon}{\to} R$ is the identity;
- **S3. (smoothness)** a) S is smooth over the field k; b) $S/\mathfrak{m}S$ is smooth over the residue field $R/\mathfrak{m}_R$ at the maximal ideal $\varepsilon^{-1}(\mathfrak{m}_R)/\mathfrak{m}S$.

Remark 2. To see that **S3.(b)** is described correctly we have an obvious inclusion $\varepsilon(\mathfrak{m}S) \subset \mathfrak{m}_R$, therefore $\varepsilon^{-1}(\mathfrak{m}_R) \supset \mathfrak{m}S$. Since R is a local we conclude that the ideal $\varepsilon^{-1}(\mathfrak{m}_R)$ is an unique maximal ideal lying over the kernel of augmentation $I = \ker(\varepsilon)$.

Now we are going to prove:

Theorem (Specialization Lemma)**.** *Let S be an R-algebra above satisfying conditions* **S1**, **S2** *and* **S3**. *Let $\mathfrak{F}_R$ be the functor from the category of R-algebras to the category of abelian groups satisfying the axioms* **TE**, **TA** *and* **TB**.

If the image in $\mathfrak{F}_R(S_f)$ of an element $\alpha_S \in \mathfrak{F}_R(S)$ is trivial then it is trivial under augmentation, i.e. $\varepsilon^*(\alpha_S) = 0$ in $\mathfrak{F}_R(R)$.

Proof. To simplify the notation, for any element $\alpha_S \in \mathfrak{F}_R(S)$, by α_{S_f} we will denote its image under the map induced by inclusion $S \hookrightarrow S_f$ and by α_0, α_1 and α_I – its image under the map induced by the canonical projection can_0^*, can_1^* and can_I^* for the ideal I, correspondingly.

By "Geometric Presentation Lemma" ([PO], Lemma 10.1) for a given $f \in A$ we can find an element $t' \in S$ such that:
G1. S is finite over $R[t']$;
G2. There exists an ideal J coprime with I and $I \cap J = (t')$;
G3. (f) is coprime with J and with $(t'-1)$.

I. The last property **G3** means that we can pass the canonical projections $S \overset{\mathrm{can}_J}{\to} S/J$ and $S \overset{\mathrm{can}_1}{\to} S/(t'-1)$ through the localization S_f.

Hence, if $\alpha_{S_f} = 0$ in $\mathfrak{F}_R(S_f)$ for some $\alpha_S \in \mathfrak{F}_R(S)$ then $\alpha_J = \mathrm{can}_J^*(\alpha_S) = 0$ and $\alpha_1 = \mathrm{can}_1^*(\alpha_S) = 0$.

II. Now look on the commutative diagram coming from **G1**:

$$\begin{tikzcd}
\mathfrak{F}_R(S) \arrow[r, "\mathrm{can}_0^*"] \arrow[d, "\mathrm{can}_1^*"'] & \mathfrak{F}_R(S/(t')) \arrow[d, "\mathrm{Tr}_0"] \\
\mathfrak{F}_R(S/(t'-1)) \arrow[r, "\mathrm{Tr}_1"'] & \mathfrak{F}_R(R)
\end{tikzcd} .$$

It turns out that $\mathrm{Tr}_0(\alpha_0) = \mathrm{Tr}_1(\alpha_1) = 0$.

III. By **G2**, there is the decomposition $S/(t') \cong R \times S/J$ via the projections $S/(t') \overset{\mathrm{can}_{0I}}{\to} S/I \overset{\bar{\varepsilon}}{\to} R$ and $S/(t') \overset{\mathrm{can}_{0J}}{\to} S/J$.

Since $S/(t')$ is finitely generated projective over R, the quotient S/J is finitely generated projective over R, too. So by **TE** we have well-defined transfer map $\mathrm{Tr}_J : \mathfrak{F}_R(S/J) \to \mathfrak{F}(R)$.

By additivity of the transfer **TA**, we can write:

$$0 = \mathrm{Tr}_0(\alpha_0) = \bar{\varepsilon}^*(\mathrm{can}_{0I}^*(\alpha_0)) + \mathrm{Tr}_J(\mathrm{can}_{0J}^*(\alpha_0)).$$

Observe that ε decomposes as $\varepsilon : S \overset{\mathrm{can}_I}{\to} S/I \overset{\bar{\varepsilon}}{\to} R$, where $\bar{\varepsilon}$ is an isomorphism.

Since $\mathrm{can}_{0I}^*(\alpha_0) = \alpha_I$ and $\mathrm{can}_{0J}^*(\alpha_0) = \alpha_J$, we can rewrite the equality above as:

$$0 = \bar{\varepsilon}^*(\alpha_I) + \mathrm{Tr}_J(\alpha_J) = \varepsilon^*(\alpha_S),$$

And we have finished. □

1.2. A version of the Quillen's Trick

Let we have a triple (A, R, f), where A is a smooth d-dimensional k-algebra, R is a local regular ring at the maximal ideal $\mathfrak{m}$ of A and $f \in \mathfrak{m}$ is some fixed regular element.

We would like to produce some extension S of the ring R, such that all properties **S1**, **S2**, **S3** of Specialization Lemma are satisfied.

We shall need the following lemma of Quillen (see [Q], Lemma 5.12):

Theorem (Quillen's Lemma)**.** *Let A be a smooth finite type algebra of dimension d over a field k, let f be a regular element of A, and let $\mathfrak{I}$ be a finite subset of* Spec$A$. *Then in A there exist elements $x_1, \dots, x_d$ algebraically independed over k and such that if $P = k[x_1, \dots, x_{d-1}] \overset{q}{\hookrightarrow} A$, then* **i)** *$A/(f)$ is finite over P;* **ii)** *A is smooth over P at points of $\mathfrak{I}$; and* **iii)** *the inclusion q factors as $q : P \hookrightarrow P[x_d] \overset{q_1}{\to} A$, where q_1 is finite.*

Let put $\mathfrak{I} = \{\mathfrak{m}\}$ and apply Quillen's Lemma to the given pair (A, f).

Look on the canonical tensor product diagram (we keep all notation coming from Quillen's Lemma):

$$\begin{tikzcd} A \arrow[r, "i_S"] & A \otimes_P R \\ P \arrow[u, "q"] \arrow[r, "r"'] & R \arrow[u, "i"'] \end{tikzcd},$$

where the map r is the composition $r : P \overset{q}{\hookrightarrow} A \overset{i_R}{\hookrightarrow} R$ and the maps $i_S : a \mapsto a \otimes 1$, $i : r \mapsto 1 \otimes r$ are canonical inclusions.

We are going to show that $S = A \otimes_P R$ is our required extension:

S1. a) Since A is a finite over $P[x_d]$ via q_1, we get that $A \otimes_P R$ is finite over $P[x_d] \otimes_P R = R[x_d]$; b) Let $f' = f \otimes 1 \in S$ be the image of our element $f \in A$ under inclusion i_S. Obviously, $S/(f') = A/(f) \otimes_P R$ is finite over R. Further we will identify f with f' assuming that it lies in A or S correspondingly.

S2. Take multiplication map $a \otimes r \mapsto ar$ as augmentation ε.

S3. a) The fact that A is smooth over the P at $\mathfrak{m}$ is equivalent to the smoothness of the map $r : P \overset{q}{\hookrightarrow} A \overset{i_R}{\hookrightarrow} A_{\mathfrak{m}}$. Hence, since r is smooth, i_S is smooth also and since A is smooth over k, so is S; b) By the property of a smooth extensions, since A is smooth over the P at $\mathfrak{m}$, we have that S must be smooth over R at all primes lying above $\mathfrak{m}S$. But we have seen already that $\varepsilon^{-1}(\mathfrak{m}_R)$ lies over $\mathfrak{m}S$, hence, S is smooth over R at $\varepsilon^{-1}(\mathfrak{m}_R)$.

1.3. Proof of the Constant Case

Let $\mathfrak{F}$ be a functor on the category of k-algebras and let $\mathfrak{F}_R$ be its restriction to the category of R-algebras. Let $\mathfrak{F}$ and $\mathfrak{F}_R$ satisfy to the axioms **C**, **TE**, **TA** and **TB**. We would like to show that the map $\mathfrak{F}(R) \to \mathfrak{F}(K)$ induced by the canonical inclusion is injective.

Let $\alpha' \in \mathfrak{F}(R)$ be such that its image α'_K in $\mathfrak{F}(K)$ is trivial.

Since $\mathfrak{F}$ is continuous (**C**), we can assume that α' came from some element α in $\mathfrak{F}(A_g)$, where A_g is localization of A at some $g \in A \setminus \mathfrak{m}$, and the image of α in $\mathfrak{F}(K)$ is trivial. Thus, we can write $\alpha' = i_R^*(\alpha)$, where $i_R : A_g \hookrightarrow R$ is the canonical inclusion.

Clearly, A_g is again a smooth k-algebra and R is its local regular ring, therefore, we can replace A by A_g and think of our element α as lying in $\mathfrak{F}(A)$.

Using **C** again we get that there must exist an element $f \in \mathfrak{m}$ such that the image α_f in $\mathfrak{F}(A_f)$ of our α is trivial.

Hence, our problem has reduced to the following one:

Theorem. *For a given $\alpha \in \mathfrak{F}(A)$ and $f \in \mathfrak{m}$ such that the image α_f is trivial in $\mathfrak{F}(A_f)$ the element $\alpha' = i_R^*(\alpha)$ is trivial in $\mathfrak{F}(R)$.*

Proof. **I.** Let S is built up by using Quillen's Trick for any triple (A, R, f). So there is the commutative diagram:

$$\begin{tikzcd} A \arrow[r, "i_S"] \arrow[d, equal] & S \arrow[d, "\varepsilon"] \\ A \arrow[r, "i_R"'] & R \end{tikzcd} \qquad (*)$$

where $\varepsilon : S \to R$ is the section for the inclusion $i : R \to S$.

By construction, S/R satisfies all conditions of Specialization Lemma.

II. Let $\mathfrak{F}_R$ be the restriction of the functor $\mathfrak{F}$ to the category of R-algebras.

By commutativity of the diagram

$$\begin{tikzcd} k \arrow[r] \arrow[d, equal] & A \arrow[r, "i_S"] & S \\ k \arrow[r] & A \arrow[r, "i_R"'] & R \arrow[u, "i"'] \end{tikzcd}$$

we have the identities $\mathfrak{F}(S) = \mathfrak{F}(k \hookrightarrow A \xrightarrow{i_S} S) = \mathfrak{F}(k \hookrightarrow A \overset{i_R}{\hookrightarrow} R \xrightarrow{i} S) = \mathfrak{F}_R(R \xrightarrow{i} S) = \mathfrak{F}_R(S)$, $\mathfrak{F}(S_f) = \mathfrak{F}_R(S_f)$ and $\mathfrak{F}(R) = \mathfrak{F}_R(R)$. Moreover, there is a natural identity between functors $\mathfrak{F}$ and $\mathfrak{F}_R$ restricted to the category of S-algebras and the following diagram commutes:

$$\begin{tikzcd} \mathfrak{F}(S) \arrow[r, equal] \arrow[d, "\varepsilon^*"'] & \mathfrak{F}_R(S) \arrow[d, "\varepsilon^*"] \\ \mathfrak{F}(R) \arrow[r, equal] & \mathfrak{F}_R(R) \end{tikzcd} . \qquad (**)$$

III. Now let $\alpha \in \mathfrak{F}(A)$ and f be an element from $\mathfrak{m}$ such that $\alpha_f = 0$.

Consider the commutative diagram built up in steps **I–II**:

$$\begin{tikzcd} \alpha_f \in \mathfrak{F}(A_f) \arrow[r] & \mathfrak{F}(S_f) \arrow[r, equal] & \mathfrak{F}_R(S_f) \ni \alpha_{S_f} \\ \alpha \in \mathfrak{F}(A) \arrow[r, "i_S^*"] \arrow[u] \arrow[d, equal] & \mathfrak{F}(S) \arrow[r, equal] \arrow[u] \arrow[d, "\varepsilon^*"'] & \mathfrak{F}_R(S) \ni \alpha_S \arrow[u] \arrow[d, "\varepsilon^*"] \\ \mathfrak{F}(A) \arrow[r, "i_R^*"'] & \mathfrak{F}(R) \arrow[r, equal] & \mathfrak{F}_R(R) \ni \alpha' \end{tikzcd} , \qquad (***)$$

where the left-down square is induced by $(*)$ and the right-down square coinsides with $(**)$.

Let $\alpha_S = i_S^*(\alpha)$ be the image in $\mathfrak{F}_R(S)$ of our element α. Since $\alpha_f = 0$ in $\mathfrak{F}(A_f)$ and upper squares of the diagram commute, we get that the image α_{S_f} in $\mathfrak{F}_R(S_f)$ of the element α_S is trivial.

To finish our proof we would like to apply Specialization Lemma (see 1.1) to the right column of our diagram $(***)$.

By construction S/R satisfies all properties **S1**, **S2**, **S3**. And by the very assumption the functor $\mathfrak{F}_R$ satisfies to the axioms **TE**, **TA**, **TB**. So we are in conditions of Specialization Lemma.

Using Specialization Lemma we conclude that the image of α_S under augmentation map ε^* is trivial and by commutativity of lower squares it coincides with α'. And we have finished. $\square$

§2. The Non-Constant Case

Let we have more general situation, namely, our functor $\mathfrak{F}$ is defined on the category of A-algebras and we still want to prove an injectivity of the homomorphism $\mathfrak{F}(R) \to \mathfrak{F}(K)$, where $R = A_{\mathfrak{m}}$ is a local regular ring and K is its field of fractions.

Let as before $\mathfrak{F}_R$ denote the restriction of $\mathfrak{F}$. In this case arguments used in the last part of the previous proof don't work because our functor is not given on k-algebras. Moreover, the objects $\mathfrak{F}(S)$ and $\mathfrak{F}_R(S)$ are not isomorphic in general.

To evoid this problem we have to assume that our functor satisfies to the additional axiom:

For a given functor $\mathfrak{F}$ on the category of A-algebras and its restriction $\mathfrak{F}_R$

E. (extension property) Let $R \xrightarrow{i} S$ be some R-algebra smooth over the field k.

Let there exists the inclusion $i_S : A \to S$ and the section (augmentation) $\varepsilon : S \to R$ of i, such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "i_S"] \arrow[d, equal] & S \arrow[d, "\varepsilon"] \\ A \arrow[r, "i_R"'] & R \end{tikzcd}$$

Let $\{\mathfrak{m}_i\}_{i \in \mathfrak{I}}$ be some finite set of maximal ideals in S with the property $\varepsilon^{-1}(\mathfrak{m}_R) \subset \cup_{i \in \mathfrak{I}} \mathfrak{m}_i$. Denote by M the multiplicative system $S \setminus \cup_{i \in \mathfrak{I}} \mathfrak{m}_i$.

Then there exists

a) The localization S_g, $g \in M$, with a finite etale extension $e : S_g \to \tilde{S}$;

b) The augmentation $\tilde{\varepsilon} : \tilde{S} \to R$ for the inclusion $R \xrightarrow{i} S_g \xrightarrow{e} \tilde{S}$ such that the following diagram commutes:

$$\begin{tikzcd} S_g \arrow[r, "e"] \arrow[d, "\varepsilon"'] & \tilde{S} \arrow[d, "\tilde{\varepsilon}"] \\ R \arrow[r, equal] & R \end{tikzcd}$$

c) The natural transformation $\Phi : \mathfrak{F} \to \mathfrak{F}_R$ between two functors $\mathfrak{F}$ and $\mathfrak{F}_R$ restricted to the category of $\tilde{S}$-algebras along $A \xrightarrow{i_S} S_g \xrightarrow{e} \tilde{S}$ and $R \xrightarrow{i} S_g \xrightarrow{e} \tilde{S}$ correspondingly, such that the following diagram commutes:

$$\begin{tikzcd} \mathfrak{F}(\tilde{S}) \arrow[r, "\Phi(\tilde{S})"] \arrow[d, "\tilde{\varepsilon}^*"'] & \mathfrak{F}_R(\tilde{S}) \arrow[d, "\tilde{\varepsilon}^*"] \\ \mathfrak{F}(R) \arrow[r, equal] & \mathfrak{F}_R(R) \end{tikzcd} .$$

Remark 3. Since $\varepsilon^{-1}(\mathfrak{m}_R) \cap M = \varnothing$, we can extend our augmentation ε given on S to the augmentation given on the localization S_g, for any $g \in M$, i.e., we have the commutative diagram:

$$\begin{tikzcd} S \arrow[r, "i_g"] \arrow[d, "\varepsilon"'] & S_g \arrow[d, "\varepsilon"] \\ R \arrow[r, equal] & R \end{tikzcd}$$

with canonical inclusion $S \overset{i_g}{\hookrightarrow} S_g$. Indeed, since R is a local with unique maximal ideal $\mathfrak{m}_R$, the image $\varepsilon(g)$ of any $g \in M$ is invertible in R.

Then our main result can be stated as follows:

Theorem (Non-Constant Case)**.** *Let $\mathfrak{F}$ be the functor on the category of A-algebras and let $\mathfrak{F}_R$ be its restriction to the category of R-algebras. If $\mathfrak{F}$ and $\mathfrak{F}_R$ satisfy to the axioms* **C**, **TE**, **TA**, **TB** *and* **E** *then the homomorphism $\mathfrak{F}(R) \to \mathfrak{F}(K)$ induced by a canonical inclusion is injective.*

Proof. **I.** Let $R \xrightarrow{i} S$ be the extension built up by using Quillen's Trick for any triple (A, R, f). In particular, it means that there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{i_S} & S \\ \Vert & & \downarrow{\scriptstyle \varepsilon} \\ A & \xrightarrow[i_R]{} & R \end{array} \qquad (*)$$

where ε is the augmentation, S is finite over $R[t]$ for specially chosen t in S and $S/(f)$ is finite over R.

Let $I = \ker(\varepsilon)$, $J = (f) \cap I$ and $J' = (J \cap R[t])S$ (we identify $R[t]$ with its embedding in S). Observe, that $J' \subset J \subset I$.

Lemma 1. *The quotients S/J and S/J' are finite over R.*

Proof. Since $J = (f) \cap I$, there is the inclusion $S/J \hookrightarrow S/(f) \times S/I$ induced by projections. It is easy to see now that S/J is finite over R.

Denote $J_t = J \cap R[t]$ the ideal in $R[t]$. Then $S/J' = S/J_tS = S \otimes_{R[t]} (R[t]/J_t)$ and, since S is finite over $R[t]$, our quotient S/J_tS is finite over $R[t]/J_t$. But there is the inclusion $R[t]/J_t \hookrightarrow S/J$ induced by the given one $R[t] \hookrightarrow S$ and we have just proved that S/J is finite over R. Hence, $R[t]/J_t$ is finite over R and we have done. $\square$

Let $\{\mathfrak{m}_i\}_{i\in\mathfrak{I}}$ be the system of all maximal ideals lying over J'. It is finite because of the lemma. Let M be the corresponding multiplicative system. We see that $\varepsilon^{-1}(\mathfrak{m}_R)$ lies above $I \supset J'$, so it is one of the $\mathfrak{m}_i$, and $\varepsilon^{-1}(\mathfrak{m}_R) \subset \cup_{i\in\mathfrak{I}}\mathfrak{m}_i$.

II. Now we are interested in the properties of localizations taken via the multiplicative system M.

Lemma 2. *For any localization S_g, $g \in M$,* **i)** *the quotient $S_g/(f)$ is a finite over R;* **ii)** *there exists an extension S_h of S_g, $h \in M$, such that S_h has a structure of a finite algebra over $R[t']$ for some specially chosen $t' \in S_h$.*

Proof. **i)** Since $(f) \supset J$, it is enough to show that S_g/JS_g is finite over R. By construction of M we know that $g \in M$ is invertible in S/J, therefore, $S_g/JS_g = (S/J)_g = S/J$ is finite over R.

ii) Indeed, it is the reformulation of Lemma 8.2 [PO]: $\square$

Let show that the extension S_h/R satisfies all properties **S1**, **S2** and **S3** of Specialization Lemma.

The property **S1** follows from **i)** and **ii)**. The existance of augmentation **S2** can be conclude from Remark 3. The property **S3** is the consequence of the fact that the localization of a smooth algebra is again smooth and the fact that the element h is coprime with the kernel of augmentation I.

III. Apply now axiom **E** to our situation $(*)$ with multiplicative system M.

We get that there exists a localization S_g, $g \in M$, a finite etale extension $e : S_g \to \tilde{S}$, the augmentation $\tilde{\varepsilon} : \tilde{S} \to R$ and the natural transformation $\Phi : \mathfrak{F} \to \mathfrak{F}_R$ between two functors $\mathfrak{F}$ and $\mathfrak{F}_R$ restricted to the category of $\tilde{S}$-algebras.

By the previous step, we can find an extension S_h, $h \in M$, of S_g with the property that S_h/R satisfies all conditions of Specialization Lemma.

Consider now the canonical tensor product diagram:

$$\begin{tikzcd} S_h \arrow[r, "i_1"] & S_h \otimes_{S_g} \tilde{S} \\ S_g \arrow[u] \arrow[r, "e"'] & \tilde{S} \arrow[u, "i_2"'] \end{tikzcd},$$

where $i_1 : s_h \mapsto s_h \otimes 1$ and $i_2 : \tilde{s} \mapsto 1 \otimes \tilde{s}$ are the canonical inclusions.

Let $\tilde{S}_h = S_h \otimes_{S_g} \tilde{S}$. Define the augmentation map as $\tilde{\varepsilon}_h : s_h \otimes \tilde{s} \mapsto \varepsilon(s_h)\tilde{\varepsilon}(\tilde{s})$. Clearly, we have the commutative diagram:

$$\begin{tikzcd} \tilde{S} \arrow[r, "i_2"] \arrow[d, "\tilde{\varepsilon}"'] & \tilde{S}_h \arrow[d, "\tilde{\varepsilon}_h"] \\ R \arrow[r, equal] & R \end{tikzcd} \tag{$*'$}$$

Since Φ is a natural equivalence by **E.(c)**, the diagram

$$\begin{tikzcd} \mathfrak{F}(\tilde{S}_h) \arrow[r, "\Phi(\tilde{S}_h)"] \arrow[d, "\tilde{\varepsilon}_h^*"'] & \mathfrak{F}_R(\tilde{S}_h) \arrow[d, "\tilde{\varepsilon}_h^*"] \\ \mathfrak{F}(R) \arrow[r, equal] & \mathfrak{F}_R(R) \end{tikzcd} \tag{$**'$}$$

will commute, too.

We claim that the extension $\tilde{S}_h/R$ satisfies to the conditions of Specialization Lemma.

Indeed, we know that the extension S_h/R do so. The augmentation $\tilde{\varepsilon}_h$ has been defined above, thus we have **S2**. It is clear that the properties **S1** and **S3** are invariant under taking a finite etale extensions. Since $\tilde{S}_h/S_h$ is a finite etale, we have done.

IV. Repeat now the arguments used in the proof of the Constant Case:

As before, by using continuity **C**, we can reduce the problem about injectivity of the map $\mathfrak{F}(R) \to \mathfrak{F}(K)$ to the following one:

Theorem. *For a given $\alpha \in \mathfrak{F}(A)$ and $f \in \mathfrak{m}$ such that the image α_f is trivial in $\mathfrak{F}(A_f)$ the element $i_R^*(\alpha)$ is trivial in $\mathfrak{F}(R)$.*

Let $\alpha \in \mathfrak{F}(A)$ and f be an element from $\mathfrak{m}$ such that $\alpha_f = 0$.

Consider the modification of the diagram $(***)$ built up in steps **I–III** for the given triple (A, R, f):

$$\begin{tikzcd} \mathfrak{F}(A_f) \arrow[r] & \mathfrak{F}(S_{gf}) \arrow[r] & \mathfrak{F}(\tilde{S}_f) \arrow[r] & \mathfrak{F}(\tilde{S}_{hf}) \arrow[r, "\Phi(\tilde{S}_{hf})"] & \mathfrak{F}_R(\tilde{S}_{hf}) \\ \mathfrak{F}(A) \arrow[u] \arrow[r, "i_S^* \circ i_g^*"] \arrow[d, equal] & \mathfrak{F}(S_g) \arrow[u] \arrow[r, "e^*"] \arrow[d, "\varepsilon^*"'] & \mathfrak{F}(\tilde{S}) \arrow[u] \arrow[r, "i_2^*"] \arrow[d, "\tilde{\varepsilon}^*"'] & \mathfrak{F}(\tilde{S}_h) \arrow[u] \arrow[r, "\Phi(\tilde{S}_h)"] \arrow[d, "\tilde{\varepsilon}_h^*"'] & \mathfrak{F}_R(\tilde{S}_h) \arrow[u] \arrow[d, "\tilde{\varepsilon}_h^*"] \\ \mathfrak{F}(A) \arrow[r, "i_R^*"'] & \mathfrak{F}(R) \arrow[r, equal] & \mathfrak{F}(R) \arrow[r, equal] & \mathfrak{F}(R) \arrow[r, equal] & \mathfrak{F}_R(R) \end{tikzcd}$$

where the left side is induced by $(*)$ as in $(***)$ and by Remark 3; the next squares are induced by **E.(b)**, $(*')$ and $(**')$, correspondingly. Since all these squares commute, our diagram commutes, too.

Take $\alpha \in \mathfrak{F}(A)$ and denote by $\alpha_{\tilde{S}_h}$ its image in $\mathfrak{F}_R(\tilde{S}_h)$ under the composition $i_S^* \circ i_g^* \circ e^* \circ i_2^* \circ \Phi(\tilde{S}_h)$. Since the image in $\mathfrak{F}(A_f)$ of α is trivial, then by commutativity of upper squares the image in $\mathfrak{F}_R(\tilde{S}_{hf})$ of the element $\alpha_{\tilde{S}_h}$ is trivial, too.

Hence, we can apply Specialization Lemma to the right column of our diagram and we get $i_R^*(\alpha) = \tilde{\varepsilon}_h^*(\alpha_{\tilde{S}_h}) = 0$. Thus, we have finished. $\square$

References

[E] D. Eisenbud, *Commutative Algebra*, Graduate Texts in Mathematics 150, Springer, 1994.

[G] A. Grothendieck, *Le groupe de Brauer. II*, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam; Masson, Paris, 1968, pp. 67–87.

[PO] I. Panin and M. Ojanguren, *Rationally trivial hermitian spaces are locally trivial*, Preprint, May 1998.

[PS] I. Panin and A. Suslin, *On a Grothendieck conjecture for Azumaya algebras*, St. Petersburg Math. J., **9**, N.4 (1998).

[Q] D. Quillen, *Higher algebraic K-theory I*, in Algebraic K-theory I, Lecture Notes in Math., 341, Springer-Verlag, Berlin 1973, pp. 85-147.

[V] V. Voevodsky, *Homology of Schemes II*, Preprint, April 1993.

[Z] K. Zainoulline, *On a Grothendieck Conjecture about principal homogeneous spaces for Special Unitary group*, Preprint LOMI, November 1998.