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BITOPOLOGICAL REPRESENTATIONS

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ABSTRACT

A class $\mathcal{B}$ of bitopological structures is a bitopological representation of a class $\mathcal{C}$ of continuous mappings of topological spaces if $\mathcal{C}$ coinsides with the class of all bicontinuous mappings $f : (X, \beta) \longrightarrow (X', \beta')$ for all $\beta, \beta' \in \mathcal{B}$. The paper contains a short survey of old author's results and some new results on bitopological representations of classes of metric space mappings.

§1. Preliminaries

1. Some classes of functions. Let $X = \mathbb{R}$ be the real line, τ_u be its usual topological structure, τ_r be the topological structure consisting of $\mathbb{R}$, $\emptyset$ and all sets $G_a = \{x | x > a\}$, $a \in \mathbb{R}$. Fix on $\mathbb{R}$ two topological structures τ_u and τ_r, i.e., consider bitopological space (X, τ_u, τ_r). Then we can consider the set of all bicontinuous mappings of (X, τ_u, τ_r) into itself. It is easy to prove that this set coinsides with the set of all non-decreasing continuous functions.

Now, consider another topological structure $\tau_a = \{\mathbb{R}, \emptyset, \{a\}\}$. It is easy to prove that the set of all bicontinuous mappings of (X, τ_u, τ_a) into itself coinsides with the set of all continuous functions identical in a and constant functions.

The following construction is rather exotic. We shall consider here so-called rational points in $\mathbb{R}^2$ (points (x, y) with rational co-ordinates). Introduce on $\mathbb{R}^2$ topological structure β_e which consists of subsets of the set of all rational points not belonging to Δ and the sets $(\mathbb{R}^2 \setminus \Delta) \cup U$, where $U \in \tau_u \times \tau_u | \Delta$. It can be proved that the set of all bicontinuous mappings of $(\mathbb{R}, \beta_e)$ into itself coinsides with the set all constant functions and all continuous functions such that their invers images of any rational point consist of rational points only.

And it seems really quite exotic the following construction. Consider the set $Z(\Delta)$ of all rational points of Δ. This set is a countable set and so we can introduce one-to-one correspondence $n : Z(\Delta) \longrightarrow \mathbb{N}$. Draw $n(a)$ non-diagonal lines through each rational point a on Δ. Now we can introduce on $\mathbb{R}^2$ a new topological structure β_l. This topological structure is defined by all sequences converging in the usual sense to points of Δ, which are contained either in Δ or in one of lines drawn in above described construction through some rational point. It is easy to prove that identical mapping is the only bihomeomorphism of $(\mathbb{R}, \beta_l)$ onto itself.

2. Classes of continuous mappings. In almost every topological theory some classes of continuous mappings are considered. These classes are defined often not in pure topological terms even if we deal, in reality, with a topological theory. But in some cases it proves to be that we can convert a considered theory in pure topological theory replacing the initial definitions by other ones.

We shall consider here different classes $\mathcal{C}$ of continuous mappings of topological spaces and their descriptions. There are various methods of such descriptions. We shall be interesting here in a very special kind of descriptions, namely in so called *bitopological representations* based on the notions of *bitopological structure, bitopological space* and *bicontinuous mapping.* Let us remind of the corresponding definitions.

Let X be a set then topological structure β given on the set $X \times X$ is called *bitopological structure on* X, the paire (X, β) is called *bitopological space*, any mapping $f : X \longrightarrow X'$ is called *bicontinuous mapping* of (X, β) into (X', β') iff

$$f \times f : (X \times X, \beta) \longrightarrow (X' \times X', \beta'),$$

$$(f \times f)(x, y) = (f(x), f(y)),$$

is continuous mapping.

If $\beta = \tau_1 \times \tau_2$ then one can write (X, τ_1, τ_2) instead of (X, β).

We shall say that a class $\mathcal{C}$ of continuous mappings of topolological spaces has a *bitopological representation by a class* $\mathcal{B}$ *of bitopological structures* iff $\mathcal{C}$ coincides with the class of all bicontinuous mappings

$$f : (X, \beta) \longrightarrow (X', \beta'),$$

which are considered for all $\beta,\ \beta' \in \mathcal{B}$.

It is possible that some class $\mathcal{C}$ of continuous mappings introduced in non topological terms can not be described really in terms of topological structures of considered spaces but, nevertheless, this class has a bitopological representation. Since any bitopological structure on X is, in the end, a topological structure on another set (on $X \times X$), one can consider the given bitopological representation as a topological type description of the class under consideration. It should be noticed that bitopological representations of classes of continuous mappings have the other, sometimes extremely important, significance. In many cases they are prove to be suitable instrument of investigation of the corresponding classes.

3. The history of the problem. The idea of replacement of non-topological description of one or another class of continuous mappings of topological spaces by some topological type description have been stated distinctly by the author as long ago as 1967 at 5-th All-Union topological conference. Later on the author repeatedly returned to this subject, especially in connection with piece-linear and differential mappings, at several conferences (Moscow, Tiraspol, Herzeg-Novy, Warszawa). The first more or less detailed account of the corresponding author's results was contained in the paper [1]. Different author's results on bitopological representations can be found in the papers published in Zap. Nauchn. Semin. LOMI (subsequently POMI): vol. 122 (1982), vol. 143 (1985), vol. 167 (1988), vol. 208 (1993), vol. 231 (1995), vol. 242 (1997).

This paper contains a short exposition of some old author's results on this theory and detailed account of new ones.

§2. A survey of old results

1. The bitopological space of graphs. Let X be a set, τ be a topological structure on X, φ be a continuius mapping of the topological space (X, τ) into itself, Γ_φ be the graph of φ. Let us define the topological structure β_φ on $X \times X$ setting the following:
– β_φ is discrete topology on $(X \times X) \setminus (\Gamma_\varphi \cup \Delta)$,
– β_φ coincides with $\tau \times \tau$ on $\Gamma_\varphi \cup \Delta$,
– $\Gamma_\varphi \cup \Delta$ is open and closed set in $(X \times X, \beta_\varphi)$ simultaneously.

We shall say that *β_φ is the bitopological structure of graph* (for φ) on X and (X, β_φ) is the *bitopological space of graph.* The following theorem has been proved in [1].

Theorem. *Let (X, τ) be a connected topological space and $\varphi : (X, \tau) \longrightarrow (X, \tau)$ be a continuous mapping having no fixed point. Then the set of all continuous mappings of (X, τ) into itself, which commute with φ, has a bitopological representation in the class of all one-to-one mappings.*

The condition of connectedness in this theorem is essential.

Remark. The class $\mathcal{B}$ consists here of one point β_φ only.

We can consider more general situation when arbitrary fixed set Φ of continuous mappings of (X, τ) into itself is considered. In the case we can defined on X the bitopological structure $\beta_\Phi = \{G | G \cap \Gamma_\varphi \in \beta_\varphi \text{ for all } \varphi \in \Phi\}$. We shall say that *$\beta_\Phi$ is the bitopological structure of graphs* (for Φ) on X and (X, β_Φ) is the *bitopological space of graphs.* The following theorem has been proved in [1].

Theorem. *Let Φ be a bitopologically complete system of continuous mappings $\varphi : (X, \tau) \longrightarrow (X, \tau)$. Then any homeomorphism $\psi : (X, \tau) \longrightarrow (X, \tau)$ is bicontinuous mapping of (X, β_Φ) on itself iff $\psi\varphi\psi^{-1} \in \Phi$ for all $\varphi \in \Phi$.*

Bitopological completness means here the following:

A system Φ of continuous mappings of a topological space (X, τ) into itself is called *bitopologically complete* if any continuous mapping $\psi : (X, \tau) \longrightarrow (X, \tau)$ such that mapping $1 \times \psi : (X, \tau) \longrightarrow (X \times X, \beta_\Phi)$ is a continuous mapping will belong to Φ.

Remark. The class $\mathcal{B}$ consists here of one point β_Φ only.

2. Special bitopologies on $\mathbb{R}^n$. We consider now three special bitopologies on $X = \mathbb{R}^n$, which show possibility of bitopological approach to fix different classes of continuous mappings.

Example 1. Let β be a topological structure on $X \times X$ defined as follows:
– on spherical cylinder $S_1 \times \Delta = \{(x, y)|d(x, y) = 1\}$ the usual topological structure is defined and each subset of $\mathbb{R}^n$ having no common with $S_1 \times \Delta$ belongs to β.

Example 2. Let β be a topological structure on $X \times X$ defined as follows:
– on cylinders $S_a \times \Delta = \{(x, y)|d(x, y) = a\}$, $a \geq 0$, the usual topological structures are defined and every of these cylinders is a closed and open set of the space $(X \times X, \beta)$ simultaneosly.

Example 3. Let β be a topological structure on $X \times X$ defined as follows:
– on hyperplanes which are parallel to Δ in $\mathbb{R}^n \times \mathbb{R}^n$ a topological structure is the usual one and every such set is a closed and open set of the space $(X \times X, \beta)$ simultaneously.

3. Piece-linear mappings. Valuable results have been received concerning bitopological representations of piece-linear mappings. Here we mention the following results [1].

Theorem. *Let β_n be a topological structure on $\mathbb{R}^{2n}$ which is the weakening of the usual structure by means of the system of n-dimensional hyperplanes $y - a = \lambda(x - a)$, where $a \in \mathbb{R}^n$, $\lambda > 0$. Then any mapping*

$$f : (\mathbb{R}^n, \beta_n) \longrightarrow (\mathbb{R}^m, \beta_m)$$

is bicontinuous one iff f is piece-linear mapping $\mathbb{R}^n$ into $\mathbb{R}^m$.

Notice, that condition $\lambda > 0$ can not be replaced by the condition $\lambda \geq 0$.

Theorem. *Let β_n be a topological structure on $\mathbb{R}^{2n}$ which is the weakening of the usual structure by means of the system of n-dimensional hyperplanes $y - a = \lambda(x - a)$, where $a \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$. Then any mapping*

$$f : (\mathbb{R}^n, \beta_n) \longrightarrow (\mathbb{R}^m, \beta_m)$$

is bicontinuous one iff f is piece-linear mapping $\mathbb{R}^n$ into $\mathbb{R}^m$.

§3. Bitopological representations of metric spaces mappings

We turn, now, to new results, which concerns of metric spaces.

1. Keeping contact mappings. Let d be a metric, which defines on X a topology τ, and $C \subset X \times X$. It is known that τ will be sequential topological structure, i.e., τ is defined by convergence of sequencies of points. We shall say that a sequence $((x_k, x'_k)|k \in \mathbb{N})$ *contacts with* C if there exists a point $(x_0, x'_0) \in C$ and a converging to 0 sequence $(\epsilon_k|k \in \mathbb{N})$ of positive numbers such that $(x_k|k \in \mathbb{N}) \longrightarrow x_0$, $(x'_k|k \in \mathbb{N}) \longrightarrow x'_0$ in (X, τ) and $(d \times d)((x_k, x'_k), C) \leq \epsilon_k(d \times d)((x_k, x'_k), (x_0, x'_0))$ for all $k \in \mathbb{N}$.

Consider some continuous mapping $f : (X, \tau) \longrightarrow (X', \tau')$ and sets $C \subset X \times X$, $C' \subset X' \times X'$. We shall say that f *keeps contacts with* C and C' if $f \times f$ converts any contacting with C sequence to a contacting with C' sequence. Later on we shall suppose that $(f \times f)(C) \subset C'$. One can see that it will be the fact if C' is a closed set in $(X' \times X', \tau' \times \tau')$. Indeed, let $(x, y) \in C$ then the sequence $((x_k, y_k)|k \in \mathbb{N})$, where $x_k = x$, $y_k = y$ for all k, contacts with C and so the sequence $((f(x_k), f(y_k))|k \in \mathbb{N})$ contacts with C'. But it means that $(d \times d)((f(x), f(y)), C') = 0$ and so $(f(x), f(y)) \in C$.

The cases when $C = \Delta$, $C' = \Delta'$ or else C and $C\prime$ are horizontal or vertical sets, present a special interest. It is clear that in the first case the inclusion $f(C) \subset C$ takes place for any mapping f. On the other hand the condition "f keeps contacts with Δ and Δ'" can be formulated in other form, namely as follows:

Since

$$(d \times d)((x_k, y_k), \Delta) = \frac{|x_k - y_k|}{\sqrt{2}}$$

$$(d \times d)((x_k, y_k), (x_0, x_0)) = \sqrt{((x_k - x_0)^2 + (y_k - x_0)^2}$$

so the condition

$$(d \times d)((x_k, x'_k), C) \leq \epsilon_k (d \times d)((x_k, x'_k), (x_0, x'_0)) \text{ for all } k \in \mathbb{N}$$

turns into the condition

$$d(x_k, x'_k) \leq \delta_k d(x_k, x_0),\ \ \delta_k \longrightarrow 0.$$

We shall say that two sequencies $(x_k | k \in \mathbb{N})$ and $(x'_k) | k \in \mathbb{N})$ converging to a point x_0 *contact with each other* if $|x_k - x'_k| \leq \delta_k |x_k - x_0|$, for all $k \in \mathbb{N}$, $\delta_k \longrightarrow 0$. Then we can assert in the situation $C = \Delta$, $C' = \Delta'$, that *mapping f keeps contacts iff f converts contacting with each other sequencies to the contacting with each other sequencies.*

We can consider also more complex case, when many sets C are involved simultaneously, for example all horizontal sets $X \times \{y\}$, $y \in X$ or all vertical sets $\{x\} \times X$, $x \in X$ or both ones, but it will be investigated some later.

The concept of contact with some fixed set C in $X \times X$ (C-*contact*) defined for sequencies by a metric d on X or even a system of such sets can be naturally generalized. We can simply fix arbitrary set of sequencies converging to points of C, which satisfies corresponding conditions.

2. Contact bitopology. It is easy to see, that any C-contact structure is naturally equivalent to the corresponding bitopology on X, which we shall designate by β_C. This bitopology can be introduced by the following convergence:

If point (x, y) does not belong to C then only becoming stable sequencies converge to this point and if $(x, y) \in C$ then only contact with C sequencies converge to (x, y). Notice, that on C the initial topology, which is defined by the metric $d \times d$, is preserved.

Such bitopology will be called C-*contact bitopology* according to the given subset C of $X \times X$. It is easy to see that C-contact bitopologies on topological spaces give us a bitopological representation of the corresponding set of mappings, which keep contacts.

Example 4. Let $X = \mathbb{R}$ be the real line with the usual topology, d be the usual metric on $\mathbb{R}$, $d(x, x') = |x' - x|$, $C = \Delta$ be the diagonal of $\mathbb{R}^2$, β_Δ be the Δ-contact bitopology on $\mathbb{R}$, which is defined by $d \times d$ and Δ. Bitopological structure β_Δ induces on Δ usual topological structure of real line therefore it will be a bitopological representation of the corresponding set of continuous real functions.

3. Properties of keeping contact mappings. Consider some properties of keeping contact mappings, not necessary all ones. We need, first of all, in concept of so-called locally double-sided lipschitz mappings. A continuous mapping $f : (X, d) \longrightarrow (X', d')$ is called *locally double-sided lipschitz mapping* if for any point $x \in X$ there exist its neighbourhood U_x and some numbers m_x, M_x such that

$$m_x d(x, x_1) \leq d'(f(x), f(x_1)),$$

$$d'(f(x_1), f(x_2)) \leq M_x d(x_1, x_2).$$

where $x_1, x_2 \in U_x$.

Theorem. *Every locally double-sided lipschitz mapping is a keeping contact mapping.*

Proof. Let (x_k) and (y_k) be contacting with each other sequences in U_x, converging to x, i.e., $\frac{d(y_k,x_k)}{d(x_k,x)} \longrightarrow 0$ for $k \longrightarrow \infty$. Since

$$m_x d(x, x_k) \leq d'(f(x), f(x_k)),$$

$$d'(f(x_k), f(y_k)) \leq M_x d(x_k, y_k),$$

so

$$\frac{d(f(y_k), f(x_k))}{d(f(x_k), f(x))} \leq \frac{M_x}{m_x} \frac{d(y_k, x_k)}{d(x_k, x)} \longrightarrow 0$$

for $k \longrightarrow \infty$, i.e., sequences $(f(x_k))$ and $(f(y_k))$ have to contact to each other. But it just means that the mapping f is a keeping contact mapping. The theorem is proved.

To formulate the next theorem we need in the following condition of strong local connectedness.

A topological space (X, τ) is called *strongly locally connected* if for any $x \in X$ and any neighbourhood U of x in (X, τ) there exists a neighbourhood $V \subset U$ such that $V \setminus \{x\}$ is a connected set (V is a strongly connected neighbourhood).

Theorem. *Let (X, d) be a strongly locally connected metric space and $f : (X, d) \longrightarrow (X', d')$ be a keeping contact continuous mapping. Then for any point $x \in X$ there exists either a neighbourhood U_x such that $U_x \subset f^{-1}(f(x))$ or a neighbourhood U_x such that $U_x \cap f^{-1}(f(x)) = \{x\}$.*

Proof. Let V be a strongly connected neighbourhood of a point x. Consider the set $F = f^{-1}(f(x)) \cap U$. Suppose that there exists a sequence $(x_k | k \in \mathbb{N},\ x_k \neq x)$ in F converging to x, which consists of limit points of the set $V \setminus F$. Fix some sequence (ϵ_k) converging to zero. Then we can form a sequence $(y_k | k \in \mathbb{N},\ y_k \in V \setminus F)$ such that $\frac{d(y_k,x_k)}{d(x_k,x)} < \epsilon_k$. It means that the sequences (x_k) and (y_k) contact to each other, what is impossible since $f(x_k) = x$ for all $k \in \mathbb{N}$ and $f(y_k) \neq x$. It implies that for some strongly connected neighbourhood $U_x \subset V$ the set $F \cap U_x \setminus \{x\}$ contains no limit point of the set $U_x \setminus F$, i.e., $F \setminus \{x\}$ is a closed and open set in $U_x \setminus \{x\}$. Since $U_x \setminus \{x\}$ is connected set we have only two variant: $F = \{x\}$ and $F = U_x$. The theorem is proved.

4. Another Δ-contact bitopology. Now let us consider another bitopological representation of the class of keeping contact mappings. Let (X, d) be a metric space and β_Δ be the contact bitopology on X, i.e., a topology on $X \times X$. This topology is determined by sequences contacting with the diagonal $\Delta \subset X \times X$. Join to these sequences all vertical converging sequences and all horizontal converging sequences, i.e., sequences of the type $((x, y_k)|k \in \mathbb{N})$ and of the type $((x_k, y)|k \in \mathbb{N})$, where $(y_k|k \in \mathbb{N})$ and $(x_k|k \in \mathbb{N})$ are converging in (X, d) sequences. All such sequences just determine the corresponding topological structure β_{hv} on $X \times X$, which is near to contact bitopology on (X, d) but doesn't coincide with it. Nevertheless, it can be proved under a special conditions that β_{hv} is really a contact bitopology. We shall say that a mapping $f : (X, d) \longrightarrow (X', d')$ is *an almost keeping contact mapping* if the mapping $(f \times f) : (X, \beta_{hv}) \longrightarrow (X', \beta'_{hv})$ is a bicontinuous mapping, in other words, if $f \times f$ converts contacting diagonal sequence either in contacting diagonal sequences or in vertical sentence or else in horizontal sequence.

Theorem. *If (X, d) is a strongly locally connected metric space then any almost keeping contact mapping is a keeping contact mapping.*

Proof. Let (X, d) be a strongly locally connected metric space and $f : (X, d) \longrightarrow (X', d')$ be an almost keeping contact mapping. We shall consider different neighbourhoods of point in (X, d). There are three types of neighbourhoods U_x of a point x, namely the following ones:
– $U_x \setminus \{x\}$ consists of points of $f^{-1}(f(x))$ only,
– $(U_x \setminus \{x\}) \cap f^{-1}(f(x)) = \emptyset$
– $U_x \setminus \{x\}$ consists both of points of $f^{-1}(f(x))$ and of points which doesn't belong to $f^{-1}(f(x))$.

It is easy to see that this theorem will be proved if we prove that any point x has a neighbourhood of the first or second type. Suppose that it is not the fact, i.e., there exists a point x such that any its neighbourhood is a neighbourhood of the third type. Let U be a neighbourhood of x, consider the sets $F = f^{-1}(f)) \cap U$ and $U \setminus F$, which are not empty sets according to our supposition. We can think that U is a strongly connected neighbourhood, i.e., $U \setminus \{x\}$ is a connected set. All this implies that there exists in F a sequence (x_k) converging to x, which consists of limit points of the set $U \setminus F$. Now, we can find a sequence (y_k) in $U \setminus F$ such that $\frac{d(y_k, x_k)}{d(x_k, x)} < \epsilon_k$, $\epsilon_k \longrightarrow 0$, i.e., the sequences (x_k) and (y_k) are contact with each other. We have by construction $f(x_k) = f(x)$, $f(y_k) \neq f(x)$ for all $k \in \mathbb{N}$. Now, we can fix all y_k and move all x_k into x'_k such that $f(x'_k) \neq f(x)$ and $\frac{d(f(y_k), f(x_k))}{d(f(x_k), f(x))} > 1$ for all $k \in \mathbb{N}$. It means that $f \times f$ converts the sequence $((x_k, y_k)|k \in \mathbb{N})$ contacting Δ neither in a sequence contacting Δ nor in vertical (horizontal) sequence, what contradicts the supposition relative to f. The theorem is proved.

Remark. The following example demonstrate that the condition of strongly locally connectedness is an essential one.

Example 5. Let (X, d) be a set $\{x_k, x_k + x_k^2 | x_k \in \mathbb{R}, k \in \mathbb{N}\}$, where $x_k \longrightarrow 0$ according to the usual metric on the set of real numbers. Suppose that $x_{k+1} + {x_{k+1}}^2 < x_k$ for all k. It is easy to see that the sequences (x_k) and $(x_k + x_k^2)$ are contact with each other. Consider the mapping $f : (X, d) \longrightarrow (X, d)$ such that $f(x_k) = 0$, $f(x_k + x_k^2) = x_k + x_k^2$. This mapping will be an almost keeping contact, but not keeping contact, continuous mapping.

5. Once again on contact bitopology. We shall see a little later that almost contact bitopology has some shortcoming. In this connection we shall consider one more variant of this conception, which can be determined by subsequent coarcening of the initial topological structure.

Let (X, d) be an arbitrary metric space, consider the following convergence of sequences (x_k, y_k) in $X \times X$. If $x \neq y$ then the convergence of sequences to (x, y) is the usual convergence in $(X \times X, d \times d)$, but the convergence to points (x, x) of the diagonal coincides with the almost contact convergence. In that way introduced convergence determines the bitopology β_w, which is coarser one than β_{hv} and, moreover, than β_C.

It is easy to see that *the theorem of previous section proved for bitopologies β_{hv} remains true for bitopologies β_w*. Thus, *the class of bitopologies of the type β_w gives us a bitopological representation of the class of keeping contact continuous mappings of strongly locally connected metric spaces.*

6. Function spaces conjugated to contact bitopologies. In [2] the conception of duality between products of spaces and ratios of spaces (i.e., function spaces) was introduced. A partial case of such duality can be introduced as follows:

Let (X, τ) be a topological space, (X, β) be a bitopological space such that β induces on each vertical set $\{x\} \times X$ the topological structure canonically isomorf to the structure τ, i.e., imbedding $j : X \longrightarrow X \times X$, $j(x) = (x, x)$, is a topological imbedding (X, τ) into $(X \times X, \beta)$.

Consider mappings (see [3])

$$F : (X \times X, \beta) \longrightarrow X,$$

$$G : X \longrightarrow (X/X, \beta'),$$

where X/X is the set of all continuous mappings $f : (X, \tau) \longrightarrow (X, \tau)$ and $F(x, y) = (G(x))(y)$ (F and G *correspond to each other*).

We shall say that a topological structure β' is a *β-splitting topological structure* if the continuity of any mapping $F : (X \times X, \beta) \longrightarrow X$ implies the continuity of the corresponding mapping $G : X \longrightarrow (X/X, \beta')$.

We shall say that a topological structure β is a *β'-jointly continuous topological structure* if the continuity of any mapping $G : X \longrightarrow (X/X, \beta')$ implies the continuity of the corresponding mapping $F : (X \times X, \beta) \longrightarrow X$.

We shall say that the topological structures β and β^* fixed on sets $X \times X$ and X/X, accordingly, *are dual to each other* if each $F : (X \times X, \beta) \longrightarrow (X, \tau)$ is a continuous mapping iff the corresponding mapping $G : X \longrightarrow (X/X, \beta^*)$ is a continuous mapping. In that case the structure β^* will be a β-splitting topology and the structure β will be a β^*-jointly continuous topology.

For given topological structure β on $X \times X$ there exists on X/X the maximal β-splitting topological structure $\beta'(\beta)$, which we shall call *conjugated to β bitopological structure*, and for given topological structure β' on X/X there exists on $X \times X$ the minimal β'-jointly continuous topological structure $\beta(\beta')$, which we shall call *conjugated to β' bitopological structure.*

If $\beta(\beta'(\beta)) = \beta$ or $\beta'(\beta(\beta')) = \beta'$ then topological structures β and β' are dual to each other.

Let us consider, at first, the case when (X, d) is a metric space and β is a Δ-contact bitopology on X. In this case we can not construct conjugated to β topological structure on X/X, with the exception the case when (X, d) is discrete space, since in the situation considered here β induced on all vertical sets in $X \times X$ the discrete topological structure.

Now, consider the case when (X, d) is a metric space and β_{hv} is an almost contact bitopology on X. In this case we can construct conjugated to β topological structure β'_{hv} on X/X as follows: Fix arbitrary continuous mapping $F : (X \times X, \beta) \longrightarrow X$ and a sequence (x_k) converging in (X, d) to a point x. Then we can consider the points $G(x_k)$ for all $k \in \mathbb{N}$ and the point $G(x)$ in X/X, where G corresponds to F. Now, let us introduce a topological structure β'_{hv} on X/X determined by convergences $(G(x_k)|k \in \mathbb{N}) \longrightarrow G(x)$, which are considered for all continuous mappings $F : (X \times X, \beta) \longrightarrow (X, \tau)$ and all convergences $(x_k|k \in \mathbb{N}) \longrightarrow x$. It will be the maximal topological structure on X/X with the given convergences.

Remark. This construction is a partial case of a construction of conjugated topological structure for an arbitrary topological space (X, τ) and an admissable bitopological structure β on X.

The structure β' on X/X consists here of all sets U such that $G^{-1}(U) \in \tau$ for all $G : X \longrightarrow X/X$ corresponding to continuous mappings $F : (X \times X, \beta) \longrightarrow X$.

Consider, at last, bitopological structure β_w. This structure considered as a topological structure on $X \times X$ coincides with the topological structure of usual product of $(X, \tau) \times (X, \tau)$ at all points with the exception of diagonal points. We can consider the structure β'_w. This topological structure on X/X has rather good properties, but, in general case, β_w and β'_w are not dual to each other, as we shall see some later.

7. The structures β_Δ, β_{hv}, β_w. The bitopological structures β_Δ, β_{hv}, β_w on $\mathbb{R}$ are different ones according to their definitions. Let us consider also the bitopological structure β_u and β_{st} on $\mathbb{R}$, which were introduced in [2]. The bitopology $\beta_u = \tau_u \times \tau_u$ is the usual topology of the product $(\mathbb{R}, \tau_u) \times (\mathbb{R}, \tau_u)$ and the bitopology β_{st} is defined by the convergence of sequences

$((x_k, y_k)|k \in \mathbb{N})$ stationary at least by one of co-ordinates. All these five structures are different and we have the following inclusions:

$$\beta_u \subset \beta_w \subset \beta_{hv} \subset \beta_{st}$$

$$\beta_\Delta \subset \beta_{st}$$

Since there is no sence a structure conjugated to the β_Δ we shall consider the structures, which are members of the first sequence of inclusions only, and conjugated to them sructures on $\mathbb{R}/\mathbb{R}$

$$\beta'_{st} \subset \beta'_{hv} \subset \beta'_w \subset \beta'_u.$$

The first and the last members of this sequence of inclusions are known. The first member is the topology of point-wise convergence of real functions, the last member is the compact-open topology. It was proved in [2] that the topological structure β_{st} and β_u on $\mathbb{R} \times \mathbb{R}$ are dual to the topological structures β'_{st} and β'_u on $\mathbb{R}/\mathbb{R}$ accordingly. Connections between the topological structures β_w, β_{hv} and, correspondingly, between the topological structures β'_w, β'_{hv} we shall consider in the next section.

8. The structure β'_{hv}. According to the definition, the stucture β'_{hv} is defined by the set of continuous mappings $F : (X \times X, \beta_{hv}) \longrightarrow (X, \tau)$, where τ corresponds to the given metric d. Notice, first of all, that this set differs from the set of continuous mappings $F : (X \times X, \beta_{st}) \longrightarrow (X, \tau)$, namely the transition $\beta_{hv} \mapsto \beta_{st}$ extends the set of continuous mapping F, as the following example shows.

Example 6. Let $F : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a real function defined as follows:

$$F(x,y) = \begin{cases} 0, & \text{for } x \leq 0 \\ 0, & \text{for } x \geq 0,\ y \leq x \text{ or } y \geq x + 2x^2 \\ \frac{y-x}{x^2}, & \text{for } x \geq 0,\ x \leq y \leq x + x^2 \\ 1 - \frac{y-x-x^2}{x^2}, & \text{for } x \geq 0,\ x + x^2 \leq y \leq x + 2x^2. \end{cases}$$

It is clear that this mapping is continuous one with respect to the structure β_{st}.On the other hand, let us consider the sequence $((\frac{1}{n}, \frac{1}{n} + \frac{1}{n^2})|n \in \mathbb{N})$. This sequence converges to the point $(0,0)$ with respect to the structure β_{hv} since it contacts with the diagonal at the point $(0,0)$, but $F(\frac{1}{n}, \frac{1}{n} + \frac{1}{n^2}) = 1$ for all $n \in \mathbb{N}$ and $F(0,0) = 0$, i. e., $(F(\frac{1}{n}, \frac{1}{n} + \frac{1}{n^2})|n \in \mathbb{N}) \not\longrightarrow F(0,0)$. It means that F is not continuous mapping with respect to the structure β_{hv}.

Now, we need in the following definition.

Let f_n, which are defined for all $n \in \mathbb{N}$, and f be mappings $\mathbb{R}$ into itself. We shall say that there is a *β_{hv}-tangential convergence $(f_n|n \in \mathbb{N}) \longrightarrow f$ along a convergence $(x_n|n \in \mathbb{N}) \longrightarrow x$* in (X,d) if $(f_n|n \in \mathbb{N})$ point-wise converges to f and $(f_n(y_n)|n \in \mathbb{N}) \longrightarrow f(x)$ for any contacting with diagonal sequence $(x_n, y_n)|n \in \mathbb{N})$.

Theorem. *The structure β'_{hv} on $\mathbb{R} \times \mathbb{R}$ is defined by the set of all tangential convergences.*

Proof. Since each convergence $(G(x_n)|n \in \mathbb{N}) \longrightarrow G(x)$ is a tangential convergence it is enough to prove that for any tangential convergence $(f_n|n \in \mathbb{N}) \longrightarrow f$ along a convergence $(x_n|n \in \mathbb{N}) \longrightarrow x_0$ in (X,d) there exists a continuous mapping $F : (X \times X, \beta_{hv}) \longrightarrow (X, \tau)$ such that $f_n = G(x_n)$ for all $n \in \mathbb{N}$ and $f = G(x_0)$, where G is the corresponding to F mapping. It is clear that we can consider only the case when $(x_n|n \in \mathbb{N})$ is an increasing sequence. In the case one can introduce the following mapping $F: X \times X \longrightarrow X$:

$$F(x,y) = \begin{cases} f_1(y + x_1 - x), & \text{for } x \leq x_1 \\ (1-\lambda)f_{n-1}(y + x_{n-1} - x) + \lambda f_n(y + x_n - x), & \text{for } x = x_{n-1} + \lambda(x_n - x_{n-1}), \\ & 0 \leq \lambda \leq 1 \\ f(y - x + x_0), & \text{for } x_0 \leq x. \end{cases}$$

We have to prove that F is a continuous mapping of $(X \times X, \beta_{hv})$ into (X, τ), i.e., F is continuous mapping on every vertical line, on every horizontal line and on every contacting with the diagonal sequence.

Fix, at first, some $u \in \mathbb{R}$ and consider the function f_u such that $f_u(y) = F(u,y)$, which maps the vertical line with abscissa u into $\mathbb{R}$. It is necessary to mark out the following cases:

$$u \leq x_1, \;\; x_{n-1} \leq u \leq x_n \text{ for each } n \in \mathbb{N} \text{ in turn}, \;\; x \leq u$$

Then we shall have the following:

$$f_u(y) = \begin{cases} f_1(y + x_1 - u), & \text{for } u \leq x_1 \\ (1-\lambda)f_{n-1}(y + x_{n-1} - u) + \lambda f_n(y + x_n - u), & \text{for } u = x_{n-1} + \lambda(x_n - x_{n-1}), \\ & 0 \leq \lambda \leq 1 \\ f(y - u + x_0), & \text{for } x_0 \leq u. \end{cases}$$

In every of these cases the function f_u is continuous one.

Fix, now, some $v \in \mathbb{R}$ and consider the function g_v such that $g_v(x) = F(x,v)$, which maps the horizontal line with ordinate v into $\mathbb{R}$. We shall have in this situation the following.

$$g_v(x) = \begin{cases} f_1(v + x_1 - x), & \text{for } x \leq x_1 \\ (1-\lambda)f_{n-1}(v + x_{n-1} - x) + \lambda f_n(v + x_n - x), & \text{for } x = x_{n-1} + \lambda(x_n - x_{n-1}), \\ & 0 \leq \lambda \leq 1 \\ f(v - x + x_0), & \text{for } x_0 \leq x. \end{cases}$$

In every of these intervals the function g_v is continuous one and so g_v is continuous function.

Fix, at last, a contacting with the diagonal sequence $((u_n, v_n)|n \in \mathbb{N})$, which convergences to a point (w,w), where $(u_n|n \in \mathbb{N})$ is an increasing sequence. We have to consider the following cases:

$$w < x_1, \;\; x_{k-1} \leq w \leq x_k, \;\; w = x_0, \quad x_0 < w.$$

In the first case we have

$$F(u_n, v_n) = f_1(v_n + x_1 - u_n), \;\; F(w,w) = f_1(x_1)$$

and so

$$(F(u_n, v_n)|n \in \mathbb{N}) \longrightarrow F(w,w) \text{ since } (f_1(v_n + x_1 - u_n)|n \in \mathbb{N}) \longrightarrow f_1(x_1).$$

In the case $x_{k-1} \leq w \leq x_k$ we have

$$F(u_n, v_n) = (1-\lambda_n)f_{k-1}(v_n + x_{k-1} - u_n) + \lambda_n f_k(v_n + x_k - u_n), u_n = x_{k-1} + \lambda_n(x_k - x_{k-1}),$$

$$F(w,w) = (1-\lambda)f_{k-1}(x_{k-1}) + \lambda f_k(x_k), w = x_{k-1} + \lambda(x_k - x_{k-1}),$$

and so

$$(F(u_n, v_n)|n \in \mathbb{N}) \longrightarrow F(w,w) \text{ since } (\lambda_n|n \in \mathbb{N}) \longrightarrow \lambda$$

The most complex case is the case $w = x_0$. With no loss of generality we can think in that case that $x_{n-1} \leq u_n \leq x_n$ and so $u_n = x_{n-1} + \lambda_n(x_n - x_{n-1}), 0 \leq \lambda_n \leq 1$ for all $n \in \mathbb{N}$. Then we shall have

$$F(u_n, v_n) = (1-\lambda)f_{n-1}(v_n + x_{n-1} - u_n) + \lambda_n f_n(v_n + x_n - u_n), \ F(w,w) = f(x_0).$$

Notice, now, that $((x_{n-1}, v_n + x_{n-1} - u_n)|n \in \mathbb{N})$ and $((x_n, v_n + x_n - u_n)|n \in \mathbb{N})$ are contacting with the diagonal sequences and so

$$(f_{n-1}(v_n + x_{n-1} - u_n)|n \in \mathbb{N}) \longrightarrow f(x_0), \ (f_n(v_n + x_n - u_n)|n \in \mathbb{N}) \longrightarrow f(x_0),$$

thus

$$(F(u_n, v_n)|n \in \mathbb{N}) = ((1-\lambda)f_{n-1}(v_n + x_{n-1} - u_n) + \lambda_n f_n(v_n + x_n - u_n)|n \in \mathbb{N}) \longrightarrow f(x_0)$$

and so

$$(F(u_n, v_n)|n \in \mathbb{N}) \longrightarrow F(w,w) = f(x_0).$$

At last,in the case $x < w$ we can think that $x < u_n$ for all $n \in \mathbb{N}$. Then we shall have

$$F(u_n, v_n) = f(v_n - u_n + x_0), \ F(w,w) = f(x_0),$$

and so

$$(F(u_n, v_n)|n \in \mathbb{N}) \longrightarrow F(w,w)$$

since

$$(v_n - u_n + x_0|n \in \mathbb{N}) \longrightarrow x_0.$$

Thus we proved that all tangential convergences are β'_{hv}-convergences. The theorem is proved.

Remark. It is easy to see that the structures β_{hv} and β'_{hv} are not dual to each other. *Moreover, we can prove that* $\beta(\beta'_{hv}) = \beta_{st}$. To prove it we consider a mapping $F : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$F(x,y) = \begin{cases} 0, & \text{for } x \leq 0 \text{ or } y \leq 0 \\ \frac{2y}{x}, & \text{for } 0 < y \leq \frac{x}{2} \\ -\frac{2y}{x} + 2, & \text{for } 0 < \frac{x}{2} \leq y \leq x. \end{cases}$$

It is easy to see that $F : (\mathbb{R} \times \mathbb{R}, \beta_{hv}) \longrightarrow \mathbb{R}$ is a continuous mapping.

With no loss of generality it is enough to consider only the case of contacting with the diagonal sequence $((x_k, y_k)|k \in \mathbb{N})$ such that $(x_k) \longrightarrow 0$ being a decreasing sequence and $y_k < x_k$ for all $k \in \mathbb{N}$. In this case there exists a continuous mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x_k) = 2y_k$ therefore the mapping $Gf : \mathbb{R} \longrightarrow (\mathbb{R}/\mathbb{R}, \beta')$, where G corresponds to F, will be a continuous mapping too. Now, we can consider the mapping $F(f \times i)$, where i is identical mapping. This mapping corresponds to the mapping Gf, and so the mapping $F(f \times i) : (\mathbb{R} \times \mathbb{R}, \beta(\beta'_{hv})) \longrightarrow \mathbb{R}$ must be a continuous mapping. But $F(f \times i)(x_k, y_k) = F(2y_k, y_k) = 1$ for all $k \in \mathbb{N}$ while $F(f \times i)(0,0) = 0$ and so $((x_k, y_k)|k \in \mathbb{N}) \not\longrightarrow (0,0)$ in $(\mathbb{R} \times \mathbb{R}, \beta(\beta'_{hv}))$. The statement we need is proved.

9. The problem of invariantness. In the previous section it was shown that β_{hv} on $\mathbb{R}$ has no dual structure. This fact has the simple reason that this structure is not invariant relative to all mappings $f \times i : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}$, where f is a continuous mapping $\mathbb{R}$ into itself. In this connection it is interesting to consider the whole class of invariant in this sense *admissible* bitopological structures on $\mathbb{R}$, i.e., bitopological structures, which induce the usual topology on all vertical and all horizontal lines.

Theorem. *There exists one-to-one correspondence between the set of all sequential invariant admissible bitopological structures on $\mathbb{R}$ and the set of all sequential topological structures on $\mathbb{R}$ finer then the usual topological structure.*

Proof. Let β be a sequential invariant admissible bitopological structure on $\mathbb{R}$, $H_y = \{(x, y)|x \in \mathbb{R}\}$, $y \in \mathbb{R}$, be a horizontal set in $\mathbb{R} \times \mathbb{R}$, $(y_k|k \in \mathbb{N})$ be a sequence converging to y, $(x_k|k \in N)$ and $(x'_k|k \in N)$ be non becoming stable sequences converging to x and x', accordingly. We need to prove that $((x_k, y_k)|k \in N)$ and $((x'_k, y_k)|k \in N)$ will converge or will not converge simultaneously. If they converge we shall include $(y_k|k \in \mathbb{N})$ in the corresponding convergence system on $\mathbb{R}$.

With no loss of generality we can suppose that both $(x_k|k \in N)$ and $(x'_k|k \in N)$ are strictly monotonous ones. In this case there are continuous mappings f and g of $\mathbb{R}$ into itself such that $f(x_k) = x'_k$ and $g(x'_k) = x_k$. Then the mappings $f \times i$ and $g \times i$ will be the continuous mappings of $(\mathbb{R} \times \mathbb{R}, \beta)$ into itself and so convergence of $((x_k, y_k)|k \in N)$ implies convergence of $((x'_k, y_k)|k \in N)$ and vice versa. The theorem is proved.

Remark. It is easy to see that β_{st}, which is defined by the convergence of horizontal and vertical sequences, corresponds to the discrete topological structure on $\mathbb{R}$ and β_u, which is the usual topological structure on the plane $(\mathbb{R} \times \mathbb{R}, \beta)$, corresponds to the usual topological structure on $\mathbb{R}$. Since any stable sequence in $\mathbb{R}$ converges so any sequential invariant admissible bitopological structures on $\mathbb{R}$ induces on all horizontal lines the usual topology. Thus any sequential invariant admissible bitopological structures on $\mathbb{R}$ containes β_{st} and it is contained in β_u.

10. Topologies dual to sequential invariant admissible bitopological structures on $\mathbb{R}$. Let us introduce some definitions and notations. Any system of convergent sequences in $\mathbb{R}$, which includes all stable sequences, we shall designate by $\mathcal{S}$, its elements we shall designate by s. Any such system corresponds to a sequential bitopological structure $\beta_{\mathcal{S}}$ on $\mathbb{R}$, which is defined by all vertical convergences and convergences $((x_k, y_k)|k \in \mathbb{N}) \longrightarrow (x_0, y_0)$, where $(y_k|k \in \mathbb{N}) \in \mathcal{S}$. The system $\mathcal{S}$ defines also a topological structure $\beta^*_{\mathcal{S}}$ on $\mathbb{R}/\mathbb{R}$, which is defined by the neighbourhoods $U(V, s) = \{f|f(s) \subset G\}$, where V is an open in $\mathbb{R}$ with the usual topology and s is a set of points of the corresponding element of $\mathcal{S}$ including its limite point (here s is understood in two different meanings but it will not entail misunderstanding).

Theorem. *Structures $\beta_{\mathcal{S}}$ and $\beta^*_{\mathcal{S}}$ are dual to each other.*

Proof. We shall consider here some corresponding to each other mappings $F : (\mathbb{R} \times \mathbb{R}, \beta_{\mathcal{S}}) \longrightarrow \mathbb{R}$ and $G : \mathbb{R} \longrightarrow (\mathbb{R}/\mathbb{R}, \beta^*_{\mathcal{S}})$, i.e., mappings for which the relation $F(x, y) = G(x)(y), x, y \in \mathbb{R}$, holds.

Let F is a continuous mapping, prove that G is a continuous mapping too. Considering a set $G^{-1}(U(V, s))$, we need to prove that it is an open set in $\mathbb{R}$ (with usual topology). Notice, first of all, that

$$G^{-1}(U(V, s)) = \{x|F(\{x\} \times s) \subset V\}.$$

It is clear that with no loss of generality we can confine ourselves by consideration of sets $G^{-1}(U(V, s_n))$, where $s_n = \{y_k, y_0|k \geq n\}$, instead of the set $s = \{y_k, y_0|k \in \mathbb{N}\}$. Let $x_0 \in G^{-1}(U(V, s))$ and $F(x, y_0) \in V$ for $|x - x_0| \leq \epsilon$. Since each set $[x_0 - \epsilon, x_0 + \epsilon] \times s_n$ is a compact set according to $\beta_{\mathcal{S}}$ for all $n \in \mathbb{N}$ (notice that in general case the point (x_0, y_0) can have no compact neighbourhood according to $\beta_{\mathcal{S}}$) and F-image of their intersection is contained in V so there exists $n \in \mathbb{N}$ such that $F([x_0 - \epsilon, x_0 + \epsilon] \times s_n) \subset V$. We proved that x_0 is contained in $G^{-1}(U(V, s))$ with some its neighbourhood and so this set is open set in $\mathbb{R}$. Continuity of G is proved.

Let, now, G is continuous mapping, prove that F is continuous mapping too. Considering a set $F^{-1}(V)$, where V is an open set in $\mathbb{R}$ (with usual topology), we need to prove that this set is an open set in $(\mathbb{R} \times \mathbb{R}, \beta_{\mathcal{S}})$. To prove it we consider any point $(x_0, y_0) \in F^{-1}(V)$ and any sequence $((x_k, y_k)|k \in \mathbb{N})$ convergent to (x_0, y_0). Then $s = ((x_k, y_k)|k \in \mathbb{N}) \in \mathcal{S}$ and the set $U(V, s)$ is an open set in $(\mathbb{R}/\mathbb{R}, \beta^*_{\mathcal{S}})$. It implies, because of continuity G, that the set $G^{-1}(U(V, s))$ is open set in $\mathbb{R}$ and so there exists $n \in \mathbb{N}$ such that $x_k \in G^{-1}(U(V, s))$ for $k \geq n$. It means that $F(\{x_k\} \times s) \subset V$ for all $k \geq n$ and so $F(x_k, y_k) \in V$ for all $k \geq n$, i.e., $(x_k, y_k) \in F^{-1}(V)$ for all $k \geq n$. The continuity of F is proved.

Remark. We found a new class of topological structures on the set $\mathbb{R}/\mathbb{R}$ of continuous real functions having dual structures on the plane $\mathbb{R} \times \mathbb{R}$. The extreme cases of these structures are the topology of point-wise convergence and the compact-open topology.

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