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Stationary measures on the space of virtual permutations for an action of the infinite symmetric group

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ABSTRACT

We describe stationary central measures on the space of virtual permutations (that is a projective limit of finite symmetric groups) for an action of the infinite symmetric group. The most important class of central distributions consists of measures μ such that the sum of normalized cycle lengths is equal to 1 for almost all with respect to μ virtual permutations. In this class, the only stationary distribution is the Ewens measure with parameter 1, that is the projective limit of the Haar measures on symmetric groups, and this distribution is invariant.

Equivalent setting of the problem is to describe invariant measures for a family of Markovian operators on the simplex of infinite monotone sequences with sum at most 1. The ergodic invariant measures are homothetic images of the famous Poisson–Dirichlet distribution PD(1) with parameter 1. In particular, we obtain a new characterization of PD(1) as the only invariant distribution on the simplex of sequences with sum 1.

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1. Introduction.

The space of virtual permutations $\mathfrak{S}^\infty$, that is the projective limit of finite symmetric groups, was introduced in [2]. The group $G = \mathfrak{S}_\infty \times \mathfrak{S}_\infty$, where $\mathfrak{S}_\infty$ is the group of all finite permutations of $\mathbb{N}$, acts on this compact space, and the related unitary representations of the infinite symmetric group $\mathfrak{S}_\infty$ were considered in [2]. In a number of subsequent papers ([1, 3, 7]) the properties of this space were studied. The study of *central* distributions on the space of virtual permutations, i.e. distributions that are invariant under the diagonal subgroup H of G, is parallel to Kingman's theory of partition structures ([4, 5, 6]). In particular, ergodic H-invariant measures are indexed with the points of the infinite-dimensional simplex. This allows one to translate to the simplex the action of the infinite symmetric group $\mathfrak{S}_\infty$ on the space of virtual permutations $\mathfrak{S}^\infty$. The result is a family of Markovian operators, and the problem proposed by A. M. Vershik was to find all measures that are invariant under this family. In this paper, we give the solution of this problem (Theorem 4.3). It turns out that ergodic invariant measures are homothetic images of the famous *Poisson–Dirichlet distribution* with parameter 1.

The result of this paper admits another interpretation suggested by A. M. Vershik. Given a central measure μ on $\mathfrak{S}^\infty$, denote by μ^g its image under $g \in \mathfrak{S}_\infty$. Then μ^g is not a central measure. However, one can define a projection P onto the space of central measures, and the question is to find all *stationary* central measures, i.e. satisfying the condition

$$P\mu^g = \mu \qquad \forall g \in \mathfrak{S}_\infty.$$

Theorem 4.11 gives the description of these measures. If we confine ourselves to the class of 1-*measures* (see Section 3), then the only stationary distribution is the *Ewens measure* with parameter 1 (the projective limit of Haar measures on symmetric groups), which is obviously $\mathfrak{S}_\infty$-invariant. Thus for this class our result takes the form "each stationary measure is invariant".

The paper is organized as follows. Sections 2, 3 contain necessary background concerning the space of virtual permutations and central measures. In Section 4 we formulate the Main Theorem, and its proof is given in Sections 5–7.

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2. The space of virtual permutations.

Let $\mathfrak{S}_n$ be the symmetric group of order n, i.e. the group of all permutations of the set $[n] = \{1, 2, \ldots, n\}$. Denote by $\mathfrak{S}_\infty = \cup_{n=1}^\infty \mathfrak{S}_n$ the infinite symmetric group (the group of all finite permutations of $\mathbb{N}$).

(2.1) Definition. Given a subset $A \subset [n]$ and a permutation $w \in \mathfrak{S}_n$, we denote by $\pi_{n,A} w$ the permutation of the set A obtained by deleting from the cycles of w all elements that do not belong to A. The permutation $\pi_{n,A} w$ is said to be the *induced permutation* of w on A.

The notation $\pi_{n,m} w$ stands for the induced permutation on the subset $A = [m]$. We usually omit the index n when it is clear from the context. For example, if $w = (6\,3\,5\,1)(8\,2)(7\,4)$, then $\pi_4 w = (3\,1)(2)(4)$.

Each permutation $w \in \mathfrak{S}_n$ has a unique decomposition into a product of transpositions

$$w = (1, i_1)(2, i_2) \ldots (n, i_n),$$

where $i_k \in [k]$. (We assume that in a product of permutations the leftmost permutation acts the first.) In this notation, the induced permutation $w' = \pi_{n-1} w$ equals $(1, i_1) \ldots (n-1, i_{n-1})$, i.e. $w = w'(n, k)$ for some $k \in [n]$. It is clear that $k = w(n)$. If $k \le n$, this means that n immediately precedes k in its cycle. If $k = n$, then n is a fixed point of w.

(2.2) Definition. The space of virtual permutations $\mathfrak{S}^\infty$ is the projective limit $\mathfrak{S}^\infty = \varprojlim \mathfrak{S}_n$ of finite symmetric groups $\mathfrak{S}_n$ with respect to the projections $\pi_{n+1,n}$.

Thus a virtual permutation is a sequence

$$\omega = (w_1, w_2, \dots, w_n, \dots) \in \mathfrak{S}_1 \times \mathfrak{S}_2 \times \dots,$$

where w_n is the induced permutation of w_{n+1} for all $n = 1, 2, \dots$

Note that π_n commutes with the shifts by elements of $\mathfrak{S}_n$, i.e.

$$\pi_n(g_1^{-1} h g_2) = g_1^{-1} \pi_n(h) g_2 \qquad \forall N > n \; \forall h \in \mathfrak{S}_N \; \forall g_1, g_2 \in \mathfrak{S}_n.$$

Thus we have an action of the group $G = \mathfrak{S}_\infty \times \mathfrak{S}_\infty$ on the space of virtual permutations $\mathfrak{S}^\infty$ defined by the formula

$$(g_1, g_2)\omega = g_1^{-1} \omega g_2 = g_1^{-1}(w_1, w_2, \dots) g_2 = (w_1', w_2', \dots),$$

where $\omega \in \mathfrak{S}^\infty$, $(g_1, g_2) \in \mathfrak{S}_n \times \mathfrak{S}_n$, and

$$w_i' = \begin{cases} g_1^{-1} w_i g_2 & \text{if } i \geq n; \\ \pi_i(g_1^{-1} w_n g_2) & \text{if } i < n. \end{cases}$$

We say that $\mathfrak{S}_\infty$ acts on $\mathfrak{S}^\infty$ by *shifts*, but this terminology is conditional since $\mathfrak{S}^\infty$ is not a group.

3. Central measures on the space of virtual permutations.

A sequence $\{\mu_n\}$ of measures on finite symmetric groups $\mathfrak{S}_n$ is said to be *coherent* if it accords with the projections $\pi_{n+1,n}$:

$$\pi_{n+1,n} \mu_{n+1} = \mu_n, \qquad n = 1, 2, \dots .$$

Each coherent sequence defines a measure $\mu = \varprojlim \mu_n$ on the space of virtual permutations. If all measures μ_n are central, i.e. invariant under inner automorphisms, then μ is invariant under the action of the diagonal subgroup $H = \{(g_1, g_2) \in G : \; g_1 = g_2\}$. Such measures on $\mathfrak{S}^\infty$ are called *central* (they correspond to central measures on the branching graph of conjugacy classes of symmetric groups). We denote by $M^H(\mathfrak{S}^\infty)$ the set of all central probability measures on the space of virtual permutations.

(3.1) Example. Let m_n be the Haar measure on the symmetric group $\mathfrak{S}_n$. The sequence $\{m_n\}$ is coherent, and the measure $m = \varprojlim m_n$ on the space of virtual permutations is called the *Ewens measure with parameter* 1. It is clear that this measure is central, and moreover, it is invariant under the whole group G.

The problem of describing all central measures on the space of virtual permutations is parallel to Kingman's theory of *partition structures* (see [4, 5, 6]). In fact, Kingman considered coherent sequences of random partitions defined on an abstract probability space (using Kolmogorov's theorem). The space of virtual permutations gives a very natural and convenient realization of this abstract space. In this paper we do not discuss in details this relationship, we just reformulate the results in our terms.

The key point of Kingman's theory is the following theorem. Applied to our situation, it describes ergodic central measures on the space of virtual permutations.

(3.2) Theorem. *The set of all ergodic H-invariant measures on the space of virtual permutations is indexed by the points of the simplex*

$$\Sigma = \left\{ x = (x_1, x_2, \ldots) : \; x_1 \geq x_2 \geq \ldots \geq 0, \; \sum_{i=1}^{\infty} x_i \leq 1 \right\}.$$

There is an explicit formula for the ergodic measure P^x corresponding to $x \in \Sigma$. First, let $x \in \Sigma_0$, where $\Sigma_0 = \{x \in \Sigma, \sum_{i=1}^{\infty} x_i = 1\}$. Then the projection P_n^x of P^x on the symmetric group $\mathfrak{S}_n$ is defined by the formula

$$P_n^x(h) = \prod_{j \geq 1} \frac{r_j!}{((j-1)!)^{r_j}} \sum x_1^{k_1} x_2^{k_2} \ldots , \tag{3.3}$$

where r_j is the number of cycles of length j in the permutation $h \in \mathfrak{S}_n$, and the sum extends over all sequences $(k_1, k_2, \ldots)$ of non-negative integers such that exactly r_j of the k_i are equal to j.

For arbitrary non-zero $x \in \Sigma$, let $x_0 = 1 - x_1 - x_2 - \ldots$ and $y = (\frac{x_1}{1-x_0}, \frac{x_2}{1-x_0}, \ldots)$. Then $y \in \Sigma_0$, and the ergodic measure P^x is given by

$$P_n^x(h) = \sum_{k=0}^{s} \binom{n}{k} x_0^k (1-x_0)^{n-k} P_{n-k}^y(h^k), \tag{3.4}$$

where s is the number of 1-cycles (fixed points) of the permutation $h \in \mathfrak{S}_n$, and h^k is a permutation of $n-k$ objects obtained by deleting k fixed points from h (this definition is correct, since the distribution P^y is central, i.e. the measure of the permutation depends only on its cycle structure).

Finally, if $x = (0, 0, \ldots)$, the corresponding ergodic measure is the δ-measure at $e = (e_1, e_2, \ldots)$, where $e_n \in \mathfrak{S}_n$ is the identity permutation, i.e. $P_n^0(e_n) = 1$ for all $n \in \mathbb{N}$.

(3.5) Corollary. *A general central measure on the space of virtual permutations is given by the formula*

$$\mu = \int_{\Sigma} P^x d\nu(x), \tag{3.6}$$

where ν is an arbitrary distribution on Σ.

Denote by ρ the operator $\nu \mapsto \rho\nu = \int P^x d\nu(x)$.

(3.7) Example. The projection of P^x on the symmetric group $\mathfrak{S}_2$ is given by

$$P_2^x((1)(2)) = 2 \sum_{1 \leq i < j < \infty} x_i x_j + x_0^2 + 2x_0(1-x_0),$$

$$P_2^x((12)) = \sum_{i=1}^{\infty} x_i^2.$$

The next theorem shows that the representing measure ν is in fact the distribution of a quite natural functional on the space of virtual permutations, namely, it is the distribution of the normalized cycle lengths.

Given a finite permutation $w \in \mathfrak{S}_n$, let $l_i(w)$ denote the length of the ith largest cycle of w. We assume that $l_i(w) = 0$ if i is greater than the number of cycles of w.

(3.8) Theorem. *Let μ be a central measure on the space of virtual permutations. The limits*

$$X_i(\omega) = \lim_{n\to\infty} \frac{l_i(w_n)}{n}, \qquad i = 1, 2, \dots$$

exist for almost all with respect to μ virtual permutations $\omega = (w_1, w_2, \dots) \in \mathfrak{S}^\infty$. The distribution of the sequence $(X_1(\omega), X_2(\omega), \dots)$ on the simplex Σ coincides with the representing measure ν in (3.6).

The limit $X_i(\omega)$ is called the *normalized length of the ith largest cycle* of ω.

Thus we have a mapping $\phi : \mathfrak{S}^\infty \to \Sigma$, where $\phi(\omega) = (X_1(\omega), X_2(\omega), \dots)$, which is defined a.e. with respect to μ. The ergodic measure P^x is just the conditional distribution of μ, given $\phi(\omega) = x$.

Note that though the sum of the normalized cycle lengths of a finite permutation equals 1, this does not hold for virtual permutations. The simplest example is the virtual permutation $e = (e_1, e_2, \dots)$; it is clear that all its cycle lengths are zero.

(3.9) Definition. A central measure μ on the space of virtual permutations is said to be 1-*measure* if the sum of the normalized cycle lengths is equal to 1 a.e. with respect to μ. A coherent family $\{\mu_n\}$ of central measures on symmetric groups is called a 1-*family* if the corresponding limiting measure $\mu = \varprojlim \mu_n$ on the space of virtual permutations is a 1-measure.

Thus μ is a 1-measure if and only if its representing distribution ν is concentrated on the simplex Σ_0 of monotone sequences with sum 1. We denote by $M_1^H(\mathfrak{S}^\infty)$ the set of all probability 1-measures.

The class of 1-measures plays a fundamental role in the theory as well as in applications. In fact, a problem concerning general central measures can be reduced in most cases to that for 1-measures, due to the fact that the ergodic measure corresponding to a point $x \in \Sigma$ can be expressed in terms of its "defect" $x_0 = 1 - x_1 - x_2 - \dots$ and its projection $y = \frac{x}{1-x_0}$ on Σ_0 (see (3.4)). Given $a \in [0, 1]$, denote by Σ_a the simplex of monotone sequences with sum $1 - a$:

$$\Sigma_a = \{x \in \Sigma : \sum_{i=1}^{\infty} x_i = 1 - a\}.$$

If $a \neq 1$, we identify the standard simplex Σ_0 with Σ_a by the homothetic transformation $\Gamma_a : x \mapsto x(1-a)$. We write Γ_1 for the constant mapping from Σ_0 to $\Sigma_1 = \{0\}$.

Given a distribution ν on Σ, denote by ν^a the conditional measure of ν on Σ_a, and let $\bar\nu^a = \Gamma_a^{-1}\nu^a$ be the image of ν^a on Σ_0. Denote by τ the image of ν under the mapping $x \mapsto x_0 = 1 - x_1 - x_2 - \dots$, thus we have $\nu = \int_0^1 \nu^{x_0} d\tau(x_0)$. Then the central measure $\mu = \rho\nu$ on the space of virtual permutations is given by the finite-dimensional projections

$$\mu_n(h) = \int_0^1 \sum_{k=0}^{s} \binom{n}{k} x_0^k (1-x_0)^{n-k} \mu_{n-k}^{x_0}(h^k) d\tau(x_0), \quad h \in \mathfrak{S}_n, \tag{3.10}$$

where $\mu^{x_0} = \int_{\Sigma_0} P^y d\bar\nu^{x_0}(y)$ is the central measure corresponding to $\bar\nu^{x_0}$, s is the number of fixed points of h, and h^k is a permutation obtained from h by deleting k fixed points.

(3.11) Example. The distribution of the cycle lengths of virtual permutations with the Ewens distribution with parameter 1 is the famous *Poisson–Dirichlet distribution* with parameter 1 denoted by $PD(1)$ ([4]). This measure is concentrated on Σ_0, thus $m \in M_1^H(\mathfrak{S}^\infty)$.

Explicit formulae for ergodic measures seem rather complicated, but there is a simple stochastic procedure that allows one to generate random virtual permutations with distribution P^x. It suffices to describe this construction in case $x \in \Sigma_0$.

(3.12) Construction. We put the elements $1, 2, \dots$ successively into cycles and associate random labels with these cycles according to the following rule:

(1) on the first step, the element 1 forms a 1-cycle, and we attach the label j to this cycle with probability x_j ($j = 1, 2, \dots$);

(2) if the elements $1, \dots, m$ are already placed, and they form k cycles with labels $i_1, \dots, i_k$, then with probability x_{i_j} ($j = 1, \dots, k$) we put the element $m+1$ into one of the possible positions in the jth cycle, or with probability x_i ($i \neq i_1, \dots, i_k$) we let it form a new 1-cycle with label i.

After the nth step of this procedure we obtain a random permutation $w_n \in \mathfrak{S}_n$. The construction guarantees that the sequence $\{w_n\}$ is coherent, i.e. $\pi_n w_{n+1} = w_n$, hence it defines a random virtual permutation. One can check that its distribution is described by (3.3).

4. Main Theorem.

As was mentioned in Section 2, the infinite symmetric group $\mathfrak{S}_\infty$ acts on the space of virtual permutations $\mathfrak{S}^\infty$. Denote by $R_g : \mathfrak{S}^\infty \to \mathfrak{S}^\infty$ the right shift by $g \in \mathfrak{S}_n$. We use the same notation for the corresponding operator that acts from the space $M^H(\mathfrak{S}^\infty)$ of central probability measures on $\mathfrak{S}^\infty$ to the space $M(\mathfrak{S}^\infty)$ of arbitrary probability measures on $\mathfrak{S}^\infty$. It is clear that the unique central measure invariant under $\{R_g\}_{g \in \mathfrak{S}_\infty}$ is the Ewens measure with parameter 1.

One can easily check that the image $\mu^g = R_g \mu$ of a central measure μ enjoys the property that the normalized cycle lengths exist for almost all with respect to μ^g virtual permutations. Hence we may define the projection of R_g on Σ, which is already a Markovian operator. It sends a point $x \in \Sigma$ to $\phi(\omega g)$, where ω is a random virtual permutation with distribution P^x. Thus the corresponding operator $\hat{T}_g$ on the space $M(\Sigma)$ of probability measures on Σ completes the commutative diagram

$$
\begin{array}{ccc}
M^H(\mathfrak{S}^\infty) & \xrightarrow{R_g} & M(\mathfrak{S}^\infty) \\
\rho \big\uparrow & & \phi \big\downarrow \\
M(\Sigma) & \xrightarrow{\hat{T}_g} & M(\Sigma).
\end{array}
\tag{4.1}
$$

One can easily see that the operator $\hat{T}_g$ depends in fact only on the conjugacy class of the permutation g.

(4.2) Example. It is obvious that $\hat{T}_e$ is the identity operator. The simplest non-trivial case is a transposition g. It follows from Construction 3.12 that the restriction $T_{(1,2)}$ of $\hat{T}_{(1,2)}$ on Σ_0 is given by the following formula. Denote by V the operator that arranges the elements of a sequence in non-increasing order. Then

$$
T_{(1,2)}x = \begin{cases}
V(x_i + x_j, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots), \quad i < j \\
\qquad\qquad\qquad\qquad \text{with probability } 2x_i x_j; \\
V(t, x_i - t, x_1, \dots, x_{i-1}, x_{i+1}, \dots), \quad i = 1, 2, \dots \\
\qquad\qquad\qquad\qquad \text{with probability } dt,\ t \in [0, x_i].
\end{cases}
$$

It is clear that if μ is R_g-invariant, then its representing measure $\nu = \phi\mu$ is $\hat{T}_g$-invariant. Hence the Poisson–Dirichlet measure $PD(1)$ is $\hat{T}_g$-invariant for all $g \in \mathfrak{S}_\infty$. The problem is to describe *all* such distributions.

One can easily check that the set Σ_a is $\hat{T}_g$-invariant for all $a \in [0,1]$ (it follows from the fact that the shift by a finite permutation does not change the sum of the normalized cycle lengths of a virtual permutation). It turns out that these sets are ergodic components, and the ergodic measure concentrated on Σ_a is the image of $PD(1)$ under the homothetic transformation $\Gamma_a : \Sigma_0 \to \Sigma_a$.

(4.3) Main Theorem. *Ergodic measures for the family of operators $\{\hat{T}_g\}_{g\in\mathfrak{S}_\infty}$ are indexed with the points of the interval $[0,1]$. The ergodic distribution corresponding to $a \in [0,1]$ is concentrated on Σ_a, and equals $PD_a(1) = \Gamma_a(PD(1))$.*

(4.4) Corollary. *A general probability measure on Σ invariant under $\{\hat{T}_g\}_{g\in\mathfrak{S}_\infty}$ is given by the formula*

$$\nu = \int_0^1 PD_{x_0}(1) d\tau(x_0), \tag{4.5}$$

where τ is an arbitrary probability distribution on $[0,1]$.

Now we consider another setting of our problem. Given a distribution ν on Σ, consider the central measure $\rho(\hat{T}_g\nu)$ associated with $\hat{T}_g\nu$. By definition of $\hat{T}_g$, we have

$$\rho(\hat{T}_g\nu) = \rho(\phi R_g \mu) = \rho(\phi\mu^g),$$

where μ is the central measure on $\mathfrak{S}^\infty$ corresponding to ν, and $\mu^g = R_g\mu$. It follows that the central measure $\rho(\hat{T}_g\nu)$ has the same distribution of the cycle lengths as μ^g. Thus $\rho(\hat{T}_g\nu)$ can be obtained from μ^g by taking $\phi\mu^g$ as representing measure and restoring *regular* conditional distributions P^x, i.e.

$$\rho(\hat{T}_g\nu) = \int_\Sigma P^x d(\phi\mu^g)(x).$$

Hence ν is $\hat{T}_g$-invariant if and only if μ satisfies the condition

$$\mu = \int_\Sigma P^x d(\phi\mu^g)(x). \tag{4.6}$$

Note that the right-hand side of (4.6) defines a projection $P\mu^g$ of μ^g onto the space $M^H(\mathfrak{S}^\infty)$ of H-invariant measures. Namely, $P\mu^g$ is the only central measure that has the same distribution of cycle lengths as μ^g. The group H isomorphic to $\mathfrak{S}_\infty$ is a locally finite group, and the very fruitful idea due to A. M. Vershik was to use the *ergodic method* (see [8]) to define this projection. Namely, the following Lemma holds.

(4.7) Lemma. *Let η be a distribution on the space of virtual permutations such that the normalized cycle lengths exist a.e. with respect to η. Then the projection of η on $M^H(\mathfrak{S}^\infty)$ is given by the formula*

$$P\eta = \lim_{N\to\infty} \frac{1}{N!} \sum_{u\in S_N} u^{-1}\eta u. \tag{4.8}$$

Proof. Denote $\eta^N = \frac{1}{N!}\sum_{u\in S_N} u^{-1}\eta u$. Let $\{\eta^{N_k}\}$ be a converging subsequence, and $\eta_0 = \lim_{k\to\infty}\eta^{N_k}$. Since the conjugation by u does not change the cycle lengths, we have $\phi(u^{-1}\eta u) = \phi(\eta)$ for every $u \in \mathfrak{S}_\infty$, hence $\phi(\eta_0) = \phi(\eta)$.

Given $n \in \mathbb{N}$, denote $H_n = H \cap (\mathfrak{S}_n \times \mathfrak{S}_n) = \{(h,h) : h \in \mathfrak{S}_n\}$, then $H = \cup H_n$. It is clear that η^N is H_n-invariant for $N \geq n$. It follows that η_0 is H_n-invariant for every $n \in \mathbb{N}$, i.e. $\eta_0 \in M^H(\mathfrak{S}^\infty)$.

Thus η_0 is a central measure that has the same distribution of the cycle lengths as η, i.e. $\eta_0 = P\eta$. We obtain that each converging subsequence of $\{\eta^N\}$ converges to $P\eta$, hence the whole sequence $\{\eta^N\}$ converges to $P\eta$, and Lemma follows.

(4.9) Remark. By Lemma 4.7, our problem is to find all central measures satisfying

$$\mu = \lim_{N\to\infty} \frac{1}{N!} \sum_{u \in S_N} u^{-1} \mu^g u \qquad \forall g \in \mathfrak{S}_\infty. \tag{4.10}$$

It is very natural to call such measures *stationary*. Recall that this term was introduced by H. Furstenberg in the following situation. Let a compact group $\mathfrak{G}$ with a measure κ act on a measurable space X. Then a distribution η on X is said to be stationary if it satisfies the condition

$$\eta = \kappa * \eta = \int_{\mathfrak{G}} \eta^a d\kappa(a),$$

where η^a is the image of η under $a \in \mathfrak{G}$. The formula (4.10) is a quite natural generalization of this definition for a locally finite group.

Thus the original problem may be reformulated as follows: find all central stationary measures on the space of virtual permutations. The following Theorem is a straightforward consequence of Theorem 4.3 and (3.10).

(4.11) Theorem. *A general central stationary probability measure on the space of virtual permutations is given by the finite-dimensional projections*

$$\mu_n(h) = \sum_{k=0}^{s} \frac{1}{(n-k)!} \binom{n}{k} \int_0^1 x_0^k (1-x_0)^{n-k} d\tau(x_0),$$

where s is the number of fixed points of $h \in \mathfrak{S}_n$, and τ is an arbitrary probability distribution on $[0,1]$.

If we confine ourselves to the class of 1-distributions, then the only stationary measure is the Ewens measure with parameter 1, which is invariant. Thus in this class there are no non-invariant stationary measures.

(4.12) Remark Since $\{\hat{T}_g\}$ is not a group action of $\mathfrak{S}_\infty$, it is quite natural to ask whether it suffices to consider only operators associated with transpositions (i.e. only $\hat{T}_{(1,2)}$ since $\hat{T}_g$ depends only on the class of permutation). This question is yet more intriguing because of a very simple form of this operator (see (4.2)). The conjecture is that there are no $\hat{T}_{(1,2)}$-invariant measures that are not invariant under the whole family $\{\hat{T}_g\}_{g\in\mathfrak{S}_\infty}$.

(4.13) Conjecture. *A general $\hat{T}_{(1,2)}$–invariant probability measure on Σ is given by the formula*

$$\nu = \int_0^1 PD_{x_0}(1) d\tau(x_0),$$

where τ is an arbitrary probability distribution on $[0,1]$.

To prove this conjecture, one could use the same argument as in the Main Theorem. The only difficulty is Lemma 6.6. Namely, the key point of the proof is that taking average over the whole group $\mathfrak{S}^k$ guarantees good uniform properties of the resulting random permutation. Now we may take average only over involutions in $\mathfrak{S}^k$, and the argument does not work. However, we hope that finer arguments and approximation tools will overcome this difficulty.

5. Reduction to finite-dimensional conditions.

The first part of the proof consists of reformulating the original infinite-dimensional problem in terms of distributions on finite symmetric groups.

(5.1) Lemma. *Let μ be a central measure on the space of virtual permutations, $\{\mu_n\}$ be the corresponding coherent family of distributions on symmetric groups, $g \in \mathfrak{S}_k$. Then the finite-dimensional projections of $P\mu^g$ are given by the formula*

$$(P\mu^g)_n(u) = \sum_{\substack{w \in \mathfrak{S}_{n+k} \\ \pi_n(w)=u}} \mu_{n+k}(wg^n), \qquad u \in \mathfrak{S}_n,$$

where $g^n \in \mathfrak{S}_{n+k}$ is a permutation such that $g^n(i) = i$ for $i = 1, \dots, n$, and $\pi_{\{n+1,\dots,n+k\}}g^n$ has the same cycle structure as g.

Proof. Denote $\eta = P\mu^g$, and let $u \in \mathfrak{S}_n$. Then we have

$$\eta_n(u) = \lim_{N\to\infty} \frac{1}{N!} \sum_{h \in \mathfrak{S}_N} (h^{-1}\mu^g h)_n(u) = \lim_{N\to\infty} \frac{1}{N!} \sum_{h \in \mathfrak{S}_N} \sum_{\substack{w \in \mathfrak{S}_N \\ \pi_n(w)=u}} \mu_N(h^{-1}whg).$$

The conjugacy classes of the symmetric group $\mathfrak{S}_n$ are indexed with partitions of the number n. Denote by Π_n the set of all partitions of n. Given $\lambda \in \Pi_n$, denote by $[\lambda]$ the conjugacy class associated with λ, i.e. the set of permutations whose cycle lengths form the partition λ, and let $|\lambda|$ be the cardinality of this class. We also denote by $[w]$ the class of the permutation w. Let

$$\bar\eta_n(\lambda) = \sum_{v \in [\lambda]} \eta_n(v) = |\lambda|\, \eta_n(u)$$

for an arbitrary representative u of the class $[\lambda]$. We have

$$\bar\eta_n(\lambda) = \lim_{N\to\infty} \frac{1}{N!} \sum_{h \in \mathfrak{S}_N} \sum_{\substack{w \in \mathfrak{S}_N \\ \pi_n(w)\in[\lambda]}} \mu_N(h^{-1}whg) = \lim_{N\to\infty} \sigma_N.$$

Note that σ_N can be rewritten as

$$\sum_{v \in \mathfrak{S}_N} \mu_N(v) \frac{1}{N!} \#\left\{h \in \mathfrak{S}_N :\ \pi_n(hvg^{-1}h^{-1}) \in [\lambda]\right\} = \sum_{v \in \mathfrak{S}_N} \mu_N(v) q(v,\lambda),$$

where $q(v,\lambda)$ are the transition probabilities of a Markovian operator that acts from $\mathfrak{S}_N$ to Π_n as follows. Given a permutation $v \in \mathfrak{S}_N$, it picks w uniformly distributed on the class $[vg^{-1}]$ and sends v to $[\pi_n(w)]$. But instead of picking a random representative of the class $[vg^{-1}]$ and then taking its induced permutation on $[n]$, we may consider the induced permutation of vg^{-1} itself but on a *random* subset $J \subset [N]$ of cardinality n, i.e.

$$q(v,\lambda) = \frac{1}{\binom{N}{n}} \#\left\{J \subset [N] :\ \#J = n,\ \pi_J(vg^{-1}) \in [\lambda]\right\}.$$

It follows that

$$\sum_{v \in \mathfrak{S}_N} \mu_N(v) q(v,\lambda) = \frac{1}{\binom{N}{n}} \sum_{\substack{J \subset [N] \\ \#J=n}} \sum_{v \in \mathfrak{S}_N} \mu_N(v) \delta_\lambda\big(\pi_J(vg^{-1})\big),$$

where
$$\delta_\lambda(u)=\begin{cases}1 & \text{if } u\in[\lambda];\\ 0 & \text{otherwise.}\end{cases}$$
Let $N\geq n+k$. Since $P\mu^g$ depends only on the cycle structure of g, we may assume that $g\in\mathfrak{S}[n+1,\dots,n+k]$. Let $J=\{j_1,\dots,j_n\}\cap\{n+1,\dots,n+k\}=\emptyset$. The probability that we choose such a subset equals

$$\frac{\binom{N-k}{n}}{\binom{N}{n}}\to 1 \qquad \text{as } N\to\infty. \tag{5.2}$$

Consider a permutation $h\in\mathfrak{S}_N$ such that $h(j_1)=1,\dots,h(j_n)=n$, and $h(i)=i$ if $i\notin J$. It is clear that h commutes with g. Then, since μ_N is central,
$$\sum_{v\in\mathfrak{S}_N}\mu_N(v)\delta_\lambda\big(\pi_J(vg^{-1})\big)=\sum_{v\in\mathfrak{S}_N}\mu_N(v)\delta_\lambda\big(\pi_J(h^{-1}vhg^{-1})\big)=$$
$$=\sum_{v\in\mathfrak{S}_N}\mu_N(v)\delta_\lambda\big(\pi_J(h^{-1}vg^{-1}h)\big)=\sum_{v\in\mathfrak{S}_N}\mu_N(v)\delta_\lambda\big(\pi_n(vg^{-1})\big)$$
does not depend on J. Moreover, since $g\in\mathfrak{S}[n+1,...,n+k]$, the last sum equals
$$\sum_{v\in\mathfrak{S}_{n+k}}\mu_{n+k}(v)\delta_\lambda\big(\pi_n(vg^{-1})\big).$$
It follows that
$$\sigma_N=\frac{\binom{N-k}{n}}{\binom{N}{n}}\sum_{v\in\mathfrak{S}_{n+k}}\mu_{n+k}(v)\delta_\lambda\big(\pi_n(vg^{-1})\big)+R_N,$$
where $|R_N|\leq 1-\binom{N-k}{n}/\binom{N}{n}$. In view of (5.2), we obtain that
$$\bar\eta_n(\lambda)=\sum_{v\in\mathfrak{S}_{n+k}}\mu_{n+k}(v)\delta_\lambda\big(\pi_n(vg^{-1})\big),$$
and Lemma follows.

(5.3) Corollary. *A central measure μ on the space of virtual permutations is stationary if and only if the corresponding coherent family of distributions μ_n on finite symmetric groups satisfies the conditions*
$$\sum_{\substack{w\in\mathfrak{S}_{n+k}\\ \pi_n(w)=u}}\mu_{n+k}(wg)=\mu_n(u)\qquad \forall n,k\in\mathbb{N},\ \forall g\in\mathfrak{S}^k,\ \forall u\in\mathfrak{S}_n,$$
where $\mathfrak{S}^k=\mathfrak{S}[n+1,\dots,n+k]\subset\mathfrak{S}_{n+k}$.

6. Main Lemma.

As was mentioned above, statements concerning general central measures on the space of virtual permutations is usually reducible to the case of 1-measures. In accordance with this idea, the main part of the proof of Theorem 4.3 is the following Proposition 6.3. The corresponding reduction is given in Section 7.

Denote by T_g the restriction of $\hat T_g$ on the invariant subset Σ_0, i.e. consider the commutative diagram

$$\begin{array}{ccc} M_1^H(\mathfrak{S}^\infty) & \xrightarrow{R_g} & M(\mathfrak{S}^\infty)\\ \rho\big\uparrow & & \big\downarrow\phi\\ M(\Sigma_0) & \xrightarrow{T_g} & M(\Sigma_0). \end{array} \tag{6.1}$$

(6.2) Proposition. *The Poisson–Dirichlet distribution $PD(1)$ is the unique probability measure on Σ_0 invariant under $\{T_g\}_{g\in\mathfrak{S}_\infty}$.*

In fact, we need a stronger statement.

(6.3) Proposition. *Denote by T_n $(n = 1, 2, \ldots)$ the average of operators T_g over the symmetric group $\mathfrak{S}_n$,*

$$T_n = \frac{1}{n!} \sum_{g\in\mathfrak{S}_n} T_g.$$

The Poisson–Dirichlet distribution $PD(1)$ is the unique probability measure on Σ_0 that is invariant under $\{T_n\}_{n\in\mathbb{N}}$.

In view of Lemma 5.1, Proposition 6.3 is a corollary of the following lemma.

(6.4) Main Lemma. *Let $\{\mu_n\}$ be a 1-family of central probability distributions on symmetric groups $\mathfrak{S}_n$ satisfying the conditions*

$$\sum_{\substack{w\in\mathfrak{S}_{n+k}\\ \pi_n(w)=u}} \frac{1}{k!} \sum_{g\in\mathfrak{S}^k} \mu_{n+k}(wg) = \mu_n(u) \quad \forall n, k \in \mathbb{N},\ \forall u \in \mathfrak{S}_n, \tag{6.5}$$

where $\mathfrak{S}^k = \mathfrak{S}[n+1, \ldots, n+k] \subset \mathfrak{S}_{n+k}$. Then μ_n coincides with the Haar measure m_n on $\mathfrak{S}_n$ for all $n \in \mathbb{N}$.

Proof. Rewrite (6.5) in the form

$$\mu_n(u) = \sum_{w\in\mathfrak{S}_{n+k}} \mu_{n+k}(w) \cdot \frac{1}{k!}\#\{g \in \mathfrak{S}^k :\ \pi_n(wg) = u\} = \sum_{w\in\mathfrak{S}_{n+k}} \mu_{n+k}(w)P(w,u),$$

where

$$P(w,u) = \frac{1}{k!}\#\{g \in \mathfrak{S}^k :\ \pi_n(wg) = u\}.$$

Note that $P(w,u)$ are the transition probabilities of a Markovian operator that acts from $\mathfrak{S}_{n+k}$ to $\mathfrak{S}_n$ and sends a permutation $w \in \mathfrak{S}_{n+k}$ to $\pi_n(wg)$, where g is uniformly distributed on $\mathfrak{S}^k$. Considering the distribution $\bar{\mu}_{n+k}$ on the space of partitions Π_{n+k} corresponding to μ_{n+k}, we obtain

$$\mu_n(u) = \sum_{w\in\mathfrak{S}_{n+k}} \mu_{n+k}(w)P(w,u) = \sum_{\lambda\in\Pi_{n+k}} \bar{\mu}_{n+k}(\lambda)\bar{P}(\lambda,u),$$

where

$$\bar{P}(\lambda,u) = \frac{1}{|\lambda|} \sum_{w\in[\lambda]} P(w,u).$$

The following lemma shows that if n is fixed, $N = n+k$ is great enough, and $\lambda \in \Pi_N$ has a small number of 1-summands, then $\bar{P}(\lambda,u)$ is close to the Haar measure $m_n(u) = 1/n!$

(6.6) Lemma. *Let $\lambda \in \Pi_N$. Denote by $r_1(\lambda)$ the number of 1-summands in λ. Then for each permutation $u \in \mathfrak{S}_n$,*

$$\bar{P}(\lambda,u) = m_n(u) + R(\lambda,u),$$

where

$$|R(\lambda,u)| \le 1 - \frac{(N-r_1(\lambda))(N-r_1(\lambda)-2)\ldots(N-r_1(\lambda)-2(n-1))}{N(N-1)\ldots(N-(n-1))}.$$

Proof. Let us call *low* the elements $1, \ldots, n$ of the set $[N]$, and the rest elements $n+1, \ldots, N$ are called *high*. Consider the set D of all permutations from $\mathfrak{S}_N$ such that the inverse images of low elements are high elements:

$$D = \left\{ w \in \mathfrak{S}_N : \ w^{-1}(i) \in \{n+1, \ldots, n+k\}, \ i = 1, \ldots, n \right\}.$$

Let $w \in D$. Denote $j_1 = w^{-1}(1), \ldots, j_n = w^{-1}(n)$. We fix an isomorphism of $\mathfrak{S}_n$ and $\mathfrak{S}[j_1, \ldots, j_n]$ that identifies i with j_i. It is clear that it identifies $\pi_n(w)$ with $\pi_{\{j_1, \ldots, j_n\}}(w)$. Moreover, for each $g \in \mathfrak{S}^k$, we have $(wg)^{-1}(i) = w^{-1}(i)$ for all $i = 1, \ldots, n$, since the low elements $1, \ldots, n$ are fixed points of g. Hence $\pi_n(wg)$ is identified with $\pi_{\{j_1, \ldots, j_n\}}(wg)$ for all $g \in \mathfrak{S}^k$. Thus the distribution of $\pi_n(wg)$ on $\mathfrak{S}_n$ coincides in sense of our isomorphism with the distribution of $\pi_{\{j_1, \ldots, j_n\}}(wg)$ on $\mathfrak{S}[j_1, \ldots, j_n]$. But

$$\pi_{\{j_1, \ldots, j_n\}}(wg) = \pi_{\{j_1, \ldots, j_n\}} \pi_{\{n+1, \ldots, n+k\}}(wg),$$

and if g is uniformly distributed on $\mathfrak{S}^k$, then $\pi_{\{n+1, \ldots, n+k\}}(wg)$ is uniformly distributed on $\mathfrak{S}^k$ for all $w \in \mathfrak{S}_{n+k}$, hence the induced permutation $\pi_{\{j_1, \ldots, j_n\}}(wg)$ also has the uniform distribution on $\mathfrak{S}[j_1, \ldots, j_n]$.

This means that $P(w, u) = m_n(u)$ if $w \in D$. It follows that

$$\bar{P}(\lambda, u) = \frac{1}{|\lambda|} \sum_{w \in [\lambda]} P(w, u) = \frac{\#(D \cap |\lambda|)}{|\lambda|} m_n(u) + r(\lambda, u),$$

where

$$r(\lambda, u) = \frac{1}{|\lambda|} \sum_{w \in [\lambda] \setminus D} P(w, u) \leq 1 - \frac{\#(D \cap |\lambda|)}{|\lambda|},$$

hence

$$|R(\lambda, u)| \leq \left| r(\lambda, u) - m_n(u) \left(1 - \frac{\#(D \cap |\lambda|)}{|\lambda|} \right) \right| \leq 1 - \frac{\#(D \cap |\lambda|)}{|\lambda|}.$$

Thus we only have to estimate $\frac{\#(D \cap |\lambda|)}{|\lambda|}$, i.e. the probability that a random permutation $w \in [\lambda]$ belongs to D.

To obtain a uniformly distributed permutation $w \in [\lambda]$, we apply the following procedure. We consider the Young diagram of the partition λ and put the elements $1, \ldots, n$ successively into the squares of this diagram assuming that at each step all empty squares have equal probabilities. Then we regard the rows of the filled diagram as cycles, and thus obtain a random permutation w which is uniformly distributed on $[\lambda]$.

At the first step of this procedure, we call a square *bad* if it forms a row of length 1. Recall that the number of such squares is denoted by $r_1(\lambda)$, thus the element 1 falls into a good square with probability $(N - r_1(\lambda))/N$. At the second step, we add one more bad square, namely, we call bad the left neighbour of the square occupied by 1 (recall that we regard rows as cycles, thus the left neighbour of the leftmost square in a row is the rightmost square of the same row). Hence at the second step we have $(N-1)$ empty squares and at most $(r_1(\lambda) + 1)$ bad squares. It follows that both 1 and 2 fall into good squares with probability at least

$$\frac{N - r_1(\lambda)}{N} \cdot \frac{N - r_1(\lambda) - 2}{N - 1}.$$

We continue this procedure and at each step add the left neighbour of the occupied square to the set of bad squares. Since each time the number of bad squares increases at most by one, the probability that all low elements $1, \dots, n$ fall into good squares is at least

$$\frac{N - r_1(\lambda)}{N} \cdot \frac{N - r_1(\lambda) - 2}{N - 1} \cdot \frac{N - r_1(\lambda) - 2(n-1)}{N - (n-1)}.$$

But in this case the elements $1, \dots, n$ do not belong to 1-rows (the squares of 1-rows are bad a priori), and besides, the inverse images of these elements lie in the set $\{n+1, \dots, n+k\}$ (the left neighbours of the squares containing low elements are bad, hence they are occupied by high elements). This means that the corresponding permutation belongs to the set D, therefore

$$\frac{\#(D \cap |\lambda|)}{|\lambda|} \geq \frac{N - r_1(\lambda)}{N} \cdot \frac{N - r_1(\lambda) - 2}{N - 1} \cdot \frac{N - r_1(\lambda) - 2(n-1)}{N - (n-1)},$$

and Lemma 6.6 follows.

To make use of Lemma 6.6, we show that in case of 1-measures, the probability that a random permutation has many fixed points is asymptotically zero.

(6.7) Lemma. *Denote*

$$A_N(\delta) = \left\{ \lambda \in \Pi_N : \frac{r_1(\lambda)}{N} \leq \delta \right\}, \qquad 0 < \delta < 1.$$

If $\{\mu_n\}$ is a 1-family, then for all $\delta \in (0,1)$

$$\bar{\mu}_N(A_N(\delta)) \to 1 \quad \text{as} \quad N \to \infty.$$

Proof. Let us fix $\delta \in (0,1)$ and denote $A_n = A_n(\delta)$. Let

$$x^N(w) = \left(\frac{l_1(w)}{N}, \frac{l_2(w)}{N}, \dots \right) \in \Sigma$$

be the vector of normalized cycle lengths of the permutation $w \in \mathfrak{S}_N$ in decreasing order. Let ν_N be the x^N-image of μ_N. Then the sequence ν_N weakly converges to the measure ν that describes the distribution of the cycle lengths of virtual permutations with respect to the limiting measure $\mu = \varprojlim \mu_N$. By definition of 1-measures, ν is concentrated on the simplex Σ_0 of sequences with sum 1.

Denote by $F_N : \Sigma \to [0,1]$ the function

$$F_N(x) = \sum_{i:\, x_i \geq \frac{2}{N}} x_i.$$

It is clear that $F_N(x) \nearrow \sum_{i=1}^{\infty} x_i$ as $N \to \infty$, hence $F_N(x) \to 1$ a.e. with respect to ν. Let $B_N = \{x \in \Sigma : F_N(x) \geq 1 - \delta\}$. Since $B_N \nearrow B = \{x : \sum x_i \geq 1 - \delta\}$, we have $\nu B_N \nearrow 1$.

One can easily see that $\lambda \in A_N$ if and only if $x^N(w) \in B_N$ for each permutation $w \in [\lambda]$, hence $\bar{\mu}_N A_N = \nu_N B_N$. But since the distributions ν_N weakly converge to ν, the sequence of sets B_N is increasing, and $\nu B_N \nearrow 1$, we have $\nu_N B_N \to 1$, and Lemma 6.7 follows.

Let us fix $\varepsilon > 0$. For $\lambda \in \Pi_N$, denote

$$f(\lambda) = \frac{(N - r_1(\lambda))(N - r_1(\lambda) - 2)\dots(N - r_1(\lambda) - 2(n-1))}{N(N-1)\dots(N-(n-1))}.$$

It is clear that $f(\lambda) \to 1$ as $N \to \infty$ and $r_1(\lambda)/N \to 0$. Hence there exist $N_1 \in \mathbb{N}$ and $\delta > 0$ such that $1 - \varepsilon \le f(\lambda) \le 1$ when $N \ge N_1$ and $r_1(\lambda)/N < \delta$. By Lemma 6.7, there exists $N \ge N_1$ such that

$$\bar{\mu}_N(A_N(\delta)) \ge 1 - \varepsilon.$$

By Lemma 6.6, if $\lambda \in A_N(\delta)$, then $P(\lambda, u) = m_n(u) + R(\lambda, u)$, where $|R(\lambda, u)| \le \varepsilon$. It follows that

$$\mu_n(u) = \sum_{\lambda \in A_N(\delta)} (m_n(u) + R(\lambda, u))\bar{\mu}_N(\lambda) + \sum_{\lambda \in \Pi_N \setminus A_n(\delta)} \bar{\mu}_N(\lambda)\bar{P}(\lambda, u) =$$
$$= m_n(u) + R_1 + R_2 - R_3,$$

where

$$R_1 = \sum_{\lambda \in A_N(\delta)} \bar{R}(\lambda, u)\bar{\mu}_N(\lambda) \le \varepsilon \bar{\mu}_N(A_N(\delta)) \le \varepsilon,$$
$$R_2 = \sum_{\lambda \in \Pi_N \setminus A_n(\delta)} \bar{\mu}_N(\lambda)\bar{P}(\lambda, u) \le \bar{\mu}_N(\Pi_n \setminus A_N(\delta)) \le \varepsilon,$$
$$R_3 = m_n(u)(1 - \bar{\mu}_N(A_N(\delta))) \le \varepsilon.$$

Hence

$$|\mu_n(u) - m_n(u)| \le 2\varepsilon,$$

and the Main Lemma follows.

7. Reduction to the case of 1-measures.

To complete the proof of the Main Theorem, we reduce it to Proposition 6.3.

As was mentioned in Section 4, the sets Σ_a are $\hat{T}_g$-invariant for all $g \in \mathfrak{S}_\infty$. The case $a = 1$ is trivial since Σ_1 consists of one point. We assume in the sequel that $a \ne 1$.

It follows from definition of $\hat{T}_g$ and (3.4) that if we identify Σ_a with Σ_0, then the restriction of $\hat{T}_g$, $g \in \mathfrak{S}_n$ on Σ_a takes the form

$$T_g^a = \Gamma_a^{-1}\left(\hat{T}_g\Big|_{\Sigma_a}\right) = \sum_{k=0}^{n} a^{n-k}(1-a)^k \sum_{\substack{J \subset [n] \\ |J| = k}} T_{\pi_J(g)}.$$

Taking the average over $g \in \mathfrak{S}_n$, we obtain

$$T_n^a = \frac{1}{n!}\sum_{g \in \mathfrak{S}_n} T_g^a = \sum_{k=0}^{n} a^{n-k}(1-a)^k \sum_{\substack{J \subset [n] \\ |J| = k}} \frac{1}{n!}\sum_{g \in \mathfrak{S}_n} T_{\pi_J(g)} =$$
$$= \sum_{k=0}^{n} a^{n-k}(1-a)^k \sum_{\substack{J \subset [n] \\ |J| = k}} \frac{1}{k!}\sum_{h \in \mathfrak{S}[J]} T_h.$$

Since T_g depends only on the class of g, we have

$$\frac{1}{k!}\sum_{h \in \mathfrak{S}[J]} T_h = T_k.$$

Hence

$$T_n^a = \sum_{k=0}^{n} a^{n-k}(1-a)^k \sum_{\substack{J\subset[n]\\ |J|=k}} T_k = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}(1-a)^k T_k. \tag{7.1}$$

Let ν be a $\{\hat{T}_g\}$-invariant measure, and ν^a be its conditional distribution on Σ_a. Note that $T_0^a = T_1^a = E$, where E is the identity operator on Σ_0. By (7.1),

$$T_2^a = (1-a)^2 T_2 + (1-(1-a)^2)E.$$

Since $\bar{\nu}^a = \Gamma_a^{-1}\nu^a$ is T_2^a-invariant, we obtain that it is T_2-invariant. It follows by induction that $\bar{\nu}^a$ is invariant under $\{T_n\}_{n=1}^{\infty}$. Then, by Proposition 6.3, $\bar{\nu}^a = PD(1)$ for all $a \in [0,1]$, and the Main Theorem follows.

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