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SOME EXTREMAL PROBLEMS ON THE HYPERBOLIC POLYGONS

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ABSTRACT

We study some isoperimetric problems for plane polygons. In particular we show that among all hyperbolic n-gons with a fixed number of sides the regular one has the maximal value of the ratio *"conformal radius perimeter"*. For n-gon admitting a full n-sides reflection by the *amplification coefficient* we mean the ratio of the conformal radii of given and reflected polygons. We prove that the amplification coefficient takes the minimal value only for the regular n-gons, that confirms the conjecture posed by J. Hersch [6].

1. Introduction

The present paper deals with extremal problems on hyperbolic polygons. In particular we shall give solutions of two problems of this kind posed by J. Hersch in [6].

By a (topological) n-gon D_n, $n \geq 2$, we mean a simply connected domain D on a Riemann surface $\mathcal{R}$ with n distinguished boundary points $a_1, a_2, \ldots, a_n$ ordered in the positive direction of ∂D if $n \geq 3$. The points $a_1, \ldots, a_n$ and the boundary arcs $a_j a_{j+1}$, $j = 1, \ldots, n$, are called vertices and sides of D_n, respectively. From now on, $a_{n+1} = a_1$ if we consider a polygon with n sides. In this paper we deal only with Riemann surfaces over the sphere $\overline{\mathbb{C}}$.

Any image of an n-gon D_n under a conformal mapping of D continued to the correspondence of vertices a_j will be called a conformal realization of D_n. Let $z_0 \in D$ and let $f(z)$, $f(z_0) = 0$, be a Riemann mapping function of D. The disk $U = \{z : |z| < 1\}$ with distinguished boundary points $A_j = f(a_j)$ gives a "natural realization" of D_n. Let us recall that the quantity

$$R(D, z_0) = |f'(z_0)|^{-1}$$

is called the conformal radius of D at the point z_0;

$$R(D) = \max_{z \in D} R(D, z).$$

A polygon D_n is called regularly centered at $z_0 \in D_n$ if its natural realization (Riemann mapping function normalized by the conditions $f(z_0) = 0$, $f(a_1) = 1$) is the unit disk U with vertices

$$A_j^* = e^{2\pi i (j-1)/n}, \quad j = 1, \ldots, n.$$

As shown in Lemma 2 below each n-gon D_n can be mapped conformally onto a hyperbolic polygon, i.e., a polygon in U having its sides on circles orthogonal to the absolute $\mathbb{T} = \{z : |z| = 1\}$ with vertices on the circle $\mathbb{T}$. Such a realization of D_n used by J. Hersch [6] will be called a hyperbolic or H-realization. It seems of interest in view of the importance of hyperbolic polygons in the theory of automorphic functions and Fuchsian groups. By D_n^* we shall denote the regular hyperbolic n-gon with vertices A_j^*, $j = 1, \ldots, n$.

One of the problems considered in [6] is the problem on the maximum of the conformal radius among all hyperbolic polygons D_n with n sides. In the case of triangles this problem was solved by R. Kühnau [8]. For arbitrary n its solution was found in [13], it is given in the following theorem.

Theorem A. *Let D_n be a hyperbolic n-gon, $n \geq 3$. Then*

$$R(D_n) \leq R(D_n^*) = \frac{\Gamma(1 - \frac{1}{n})\Gamma(\frac{1}{2} + \frac{1}{n})}{\Gamma(1 + \frac{1}{n})\Gamma(\frac{1}{2} - \frac{1}{n})}. \tag{1.1}$$

Here and below $\Gamma(z)$ denotes Euler's gamma function.

Equality holds in (1.1) *only for the regular n-gon centered at the origin.*

Let again D_n be a hyperbolic n-gon and $z_0 \in D_n$ and let $D_n{}'$ be the image of D_n under the Möbius mapping

$$\tau(z) = e^{i\alpha} \frac{z - z_0}{1 - \overline{z}_0 z}, \quad \alpha \in \mathbb{R}.$$

Theorem A together with the assertion on variation of the conformal radius under conformal mapping implies

$$R(D_n, z_0) = (1 - |z_0|^2) R(D'_n, 0) \leq (1 - |z_0|^2) R(D_n^*, 0), \tag{1.2}$$

equality here holds only for regular n-gons centered at z_0. (1.2) shows that the maximum of the conformal radius of D_n^* is attained only in its center. This presents some analogy fact to the well-known property of convex n-fold symmetric domains (cf. [10, p. 161]) but it seems to have never been remarked in the literature.

The regular hyperbolic polygon D_n^* extremizes many geometric functionals, for example, it has the maximal Euclidian perimeter among all hyperbolic polygons with given number of sides, i.e.,

$$\operatorname{length}(\partial D_n) \leq \pi(n-2)\tan(\pi/n). \tag{1.3}$$

Thus both the numerator and denominator of the ratio $R(D_n, 0)/\operatorname{length}(\partial D_n)$ take their maximal value in the case of regular polygons. Therefore Theorem A follows from the isoperimetric inequality of Theorem 1 below, which is one of the main results of the paper.

Theorem 1. *Let D_n be a hyperbolic n-gon with vertices on $\mathbb{T}$, $0 \in D_n$. Then*

$$\frac{R(D_n, 0)}{\operatorname{length}(\partial D_n)} \leq \frac{n\Gamma^2(1-\frac{1}{n})}{\pi(n-2)\Gamma^2(\frac{1}{2}-\frac{1}{n})}. \tag{1.4}$$

Equality is attained in (1.4) *only for the regular hypobolic n-gon centered at the origin.*

Let D_n be a polygon on a Riemann surface $\mathcal{R}$ with vertices a_j, $j = 1, \dots, n$, and let $\zeta = f(z)$ be a Riemann mapping function of D_n, $A_j = f(a_j)$.

We say that D_n admits the reflection $D_{n,j}$ with respect to its side $a_j a_{j+1}$ if the inverse Riemann mapping $f^{-1}(\zeta)$ can be continued conformally into the domain $U' = \overline{\mathbb{C}} \setminus \overline{U}$ (the extended mapping might not be univalent) across the arc $A_j A_{j+1}$ of $\mathbb{T}$ so that $f^{-1}(U') = D_{n,j}$. If D_n admits reflection with respect to each of its sides, then $\widehat{D}_n$ will denote the full n-sides reflection, i.e., a Riemann surface obtained from D_n by "sticking" along each side $a_j a_{j+1}$ the reflected $D_{n,j}$. Following J. Hersch [6], the quantity

$$\kappa(D_n, z_0) = \frac{R(\widehat{D}_n, z_0)}{R(D_n, z_0)}$$

will be called the "*amplification coefficient*" of the conformal radius of D_n at the point $z_0 \in D_n$.

Let

$$\kappa(D_n) = \inf_{z \in D_n} \kappa(D_n, z). \tag{1.5}$$

By an "*amplification center*" we mean a point $z_0 \in D_n$ such that $\kappa(D_n) = \kappa(D_n, z_0)$. Clearly, $\kappa(D_n, z_0)$ and $\kappa(D_n)$ are conformal invariants.

If D_n is a hyperbolic n-gon, then $\widehat{D}_n$ is also a hyperbolic polygon having $n(n-1)$ sides. Due to this fact J. Hersch [6] was able to consider further reflections of appeared polygons. When studying the conformal mapping onto such polygons he posed the problem of finding of the minimal value of the amplification coefficient among all the hyperbolic n-gons having vertices on $\mathbb{T}$. The following theorem solves this problem.

Theorem 2. *Let D_n be an arbitrary n-gon, $n \geq 2$, admitting the full n-sides reflection $\widehat{D}_n$, $z_0 \in D_n$. Then*

$$\kappa(D_n, z_0) \geq \frac{n+1}{n-1}. \tag{1.6}$$

Equality in (1.6) *is attained only in the case when D_n is a regular n-gon centered at z_0.*

Here we should remark that the equality statement of Theorem 2 confirms the conjecture on the amplification coefficient of the conformal radius of the regular n-gon, formulated initially by

J. Hersch in [5]. Soon after that he proved it and now his proof is being published in [6]. Our proof is based on the c oncept of the reduced module of a topological triangle, it differs from the proof in [6] based on the construction of a mapping onto the corresponding covering surface.

Corollary. *Under the hypotheses of Theorem* 2

$$\kappa(D_2) = 3, \qquad \kappa(D_3) = 2, \tag{1.7}$$

and

$$\kappa(D_n) \geq \frac{n+1}{n-1} \tag{1.8}$$

for all $n \geq 4$.

Equality in (1.8) *is attained only for the regular* n*-gons.*

There is an essential difference in the methods of proving Theorems 1 and 2. The proof of Theorem 1 given in Sec. 3 is *essentially analitic.* One of its main points is the solution of an extremum problem for a weighted sum of the reduced moduli of triangles as it has been done in [11] to solve an isoperimetric problem by Polya and Szegö. On the contrary the proof of Theorem 2 in Sec. 4 is *purely geometric.* It is based on the application of a special dissymetrization trasformation of a Riemann surface. The concept of dissymetrization in the case of plane domains and condensers was first introduced by V. N. Dubinin in [1].

Necessary results concernig the methods used in the paper are given in Sec. 2.

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2. Preliminaries

2.1. Reduced module of a triangle. Let $D = D(e_0, e_1, e_2)$ be a triangle on a Riemann surface $\mathcal{R}$ over $\overline{\mathbb{C}}$ with vertices e_0, e_1, e_2, where e_0 is a single point boundary element. Let the sides e_0e_1 and e_0e_2 form an inner angle $\varphi > 0$ at the vertex e_0. We will denote by z_P an affix of a point $P \in \mathcal{R}$ to underline the difference between these notions if necessary. Let $\varepsilon > 0$ be small enough and let $D'(\varepsilon)$ be the connected component of the set $D \setminus \{P : |z_P - z_{e_0}| = \varepsilon\}$ having e_0 on its boundary. Let $D(\varepsilon)$ denote the domain $D \setminus \overline{D'(\varepsilon)}$ considered as a quadrangle with vetices e_1, e_2, e_2', e_1'; where e_k' is the point of intersection of the set $\{P : |z_P - z_{e_0}| = \varepsilon\}$ and the side e_0e_k, $k = 1, 2$.

We will write $\mathrm{mod}(D(\varepsilon))$ for the module of $D(\varepsilon)$ with respect to the family of curves joining its sides e_1e_1' and e_2e_2', for the definitions see [7].

By the reduced module of the triangle $D = D(e_0, e_1, e_2)$ with respect to the vertex e_0 we mean the limit

$$m(D; e_0|e_1, e_2) = \lim_{\varepsilon \to 0}\{\mathrm{mod}(D(\varepsilon)) + (1/\varphi)\log\varepsilon\}, \tag{2.1}$$

if it exists.

The notion of the reduced module of a plane triangle was introduced in [11]. It is very close to the concept of the reduced module $m(D, z_0)$ of a simply connected domain D with respect to the point $z_0 \in D$ introduced by O. Teichmüller in [14]. The well-known relation (cf. [7, p. 24])

$$m(D, z_0) = \frac{1}{2\pi}\log R(D, z_0) \tag{2.2}$$

connects the reduced module and the conformal radius of a simply connected domain. An analogous equality

$$m(D; e_0|e_1, e_2) = \frac{1}{\varphi}\log R(D; e_0|e_1, e_2) \tag{2.3}$$

connects the reduced module of a triangle with the angular conformal radius introduced earlier by J. Hersch [4].

The existence of the limit (2.1) depends on the behaviour of ∂D near the point e_0. In [11] the conditions of the existence of the limit are given in terms of a conformal mapping of D onto a canonical domain, in [12] some geometric sufficient conditions for the existence of (2.1) are obtained, they are easy to verify under symmetrization trasformations.

Let the function $Q = f(P)$ map conformally the triangle D onto a triangle Δ, $f(e_k) = q_k$, $k = 0, 1, 2$, and let

$$f(P) = \zeta_0 + a(P - e_0)^\alpha + o(|P - e_0|^\alpha)$$

in a neibourhood of e_0. Here $\alpha > 0$ and $a \in \mathbb{C}$. The following formula gives the change in the reduced module under conformal mapping, it is used to calculate the reduced module of concrete triangles (cf. [11])

$$m(\Delta; q_0|q_1, q_2) = m(D; e_0|e_1, e_2) + \frac{1}{\alpha\varphi}\log|a|. \tag{2.4}$$

Let us consider an isosceles circular triangle $\Delta = \Delta(\alpha, \beta, r)$ bounded by the segments $[0, r]$, $[0, re^{2i\alpha\pi}]$ and by the circular arc joining vertices r and $re^{2i\alpha\pi}$ and forming the angle $\beta\pi$ with the segment $[0, r]$, $r > 0$, $0 < \alpha \leq 1$, $0 \leq \beta \leq 3/2$ (the domain Δ contains the point at infinity if $\beta > \frac{3}{2} - \alpha$).

Applying (2.4) to the conformal mapping of the upper half-plane $H_+ = \{z : \Im > 0\}$ onto Δ, we shall find (following [13]) that

$$m(\Delta; 0|r, re^{2i\alpha\pi}) = \frac{1}{2\alpha\pi}\log\frac{r\Gamma(1-\alpha)\Gamma(\frac{1}{2}+\alpha+\beta)}{\Gamma(1+\alpha)\Gamma(\frac{1}{2}-\alpha+\beta)}. \tag{2.5}$$

One of the key points in solving the extremal problems of the concedered type is using the inequality (2.6) of Theorem B [11] formulated below. Let D be a simply connected domain, $z_0 \in D$, and let $\Delta_j = \Delta_j(z_0, e_1^{(j)}, e_2^{(j)})$, $j = 1, 2, \ldots$, be a system of nonoverlapping triangles in D such that Δ_j has the angle $2\alpha_j\pi$ at its vertex z_0 and has the opposite side $e_1^{(j)}e_2^{(j)}$ on ∂D.

Theorem B. *Let* $\sum_j \alpha_j = 1, \alpha_j \geq 0$. *Then*

$$m(D, z_0) \leq \sum_j \alpha_j^2 m(\Delta_j; 0|e_1^{(j)}, e_2^{(j)}). \tag{2.6}$$

Equality holds in (2.6) if and only if the triangles Δ_j correspond to the circular sectors of opening angles $2\alpha_j\pi$ under the conformal mapping $f(z)$ of D onto the disk U, $f(z_0) = 0$.

The idea to apply some trasformations connected with the decomposition of geometric objects into parts is due to the researches of J. Steiner, R. Courant and to the stips' method of H. Grötzsch. G. Polya [9] applied such a transformation and called it the "alternative symmetrization." Resently such transformations obtained many applications due to the researches of V. N. Dubinin (cf.[2]) who suggest the term "piecewise symmetrization."

2.2. Capacity and the inner radius. A condenser on $\overline{\mathcal{R}}$ is an ordered pair $C = (E_0, E_1)$ of disjoint nonempty closed sets E_0 and E_1, called the plates. The capacity of C is defined by

$$\operatorname{cap} C = \operatorname{cap}(E_0, E_1) = \inf \int\int_{\mathcal{R}} |\nabla v|^2 \, dA_z, \tag{2.7}$$

where the infimum is taken over all functions $v : \overline{\mathcal{R}} \mapsto \mathbb{R}$ that are continuous on $\overline{\mathcal{R}}$, Lipschitz' in $\mathcal{R} \setminus E_0 \setminus E_1$ and equal to zero on E_0, and to one on E_1. We shall consider only condersers having a potential function, i.e., the function ω harmonic in $\mathcal{R} \setminus E_0 \setminus E_1$ and satisfying the conditions mentioned above. It is well known that ω minimizes the integral in (2.7).

Let D be a domain in $\mathcal{R}$ and let $g(z, z_0)$ denote the generalized Green's function of D with pole at $z_0 \in D$, (such a Green's function might be not equal to 0 at irregular boundary points). By the inner radius of D at z_0 we mean the quantity $r(D, z_0)$ satisfying the condition

$$\log r(D, z_0) = \lim_{z \longrightarrow z_0} \{g(z, z_0) + \log |z - z_0|\},$$

if the limit exists. It is well-known that

$$r(D, z_0) = R(D, z_0)$$

for every simply connected domain. For our purposes it is sometimes more convenient to deal with the inner radius rather then with the conformal radius because the property of simple connectivity is not invariant under geometrical transformations considered below.

2.3. Dissymmetrization. Following V. N. Dubinin [1,2] we will define the dissymmetrization of sets and functions. Let n be a fixed natural number, $L_k^* = \{z : \arg z = 2\pi(k-1)/n\}$, $k = 1, \dots, n$ and let Φ be the group of isometries of $\overline{\mathbb{C}}$ consisting of superpositions of reflections with respect to lines $l_k = \{z : \operatorname{Im}(z \exp(\pi i(k-1)/n)) = 0\}$, $k = 1, \dots, n$. By a symmetrical structure $\{P_k\}_{k=1}^N$ we mean a collection of closed angles $P_k = \{z : \theta_{k_1} \le \arg z \le \theta_{k_2}\}$ satisfying the conditions:

a) $\cup_{k=1}^N = \overline{\mathbb{C}}$, $\quad b)$ $\operatorname{Int}(P_k \cap P_l) = \emptyset$ if $k \ne l$,

c) $\{\varphi(P_k)\}_{k=1}^N = \{P_k\}_{k=1}^N$ for any isometry $varphi \in \Phi$.

A collection of mappings $\{\lambda_k\}_{k=1}^N$ of the form $\lambda_k(z) = z \exp(i\theta_k)$, $\theta_k \in \mathbb{R}$, $k = 1, \dots, N$ will be called dissymetrization of the structure $\{P_k\}_{k=1}^N$ if the following conditions hold:

1) $\cup_{k=1}^N \lambda_k(P_k) = \overline{\mathbb{C}}$,

2) for any nonempty intersection $S_{k,l} = \lambda_k(P_k) \cap \lambda_l(P_l)$, $k, l = 1, \dots, N$, there exists the isometry $\varphi \in \Phi$ such that

$$\varphi(\lambda_k^{-1}(S_{k,l}) = \lambda^{-1}(S_{k,l}).$$

If the symmetrical structure $\{P_k\}_{k=1}^N$ and its dissymmetrization are defined, then by dissymmetrization of a set $A \subset \overline{\mathbb{C}}$ and a condenser $C = (E_0, E_1)$, which are symmetrical with respect to the group Φ, we mean the set

$$\operatorname{Dis} A = \cup_{k=1}^N \lambda_k(A \cap P_k)$$

and the condenser

$$\operatorname{Dis} C = (\operatorname{Dis} E_0, \operatorname{Dis} E_1),$$

respectively. By dissymmetrization of a function v which is automorphic with respect to Φ we mean the function

$$\operatorname{Dis} v(z) = v(\lambda_k^{-1}(z)), \quad z \in \lambda_k(P_k), \quad k = 1, \dots, N.$$

The success of the application of dissymmetrization for the solution of extremal problems depends mainly on a possiblity to construct objects of certain geometrical structure from other ones, which are invariant with respect to the transformations of the group Φ. In many cases the possibility of such construtions follows from Lemma 1 below, which was proved by V. N. Dubinin (cf.[2, §6]), it is presented here in the form convenient for our purposes.

Let $0 \le \theta \le \pi/n$, $L_{\pm k}(\theta) = L_k^* \exp(\pm i\theta)$,

$$A^*(\theta) = \cup_{k=1}^n \{z : |\arg z - \frac{2\pi k}{n}| \le \theta\}, \quad B^*(\theta) = \overline{\mathbb{C} \setminus A^*(\theta)}.$$

Let us consider the rays $L_{\pm k} = \{z : \arg z = \theta_{\pm k}\}$, $k = 1, \ldots, m;$, $m \le n$, such that

$$\theta_{-1} \le \theta_1 \le \theta_{-2} \le \ldots \le \theta_{-m} \le \theta_m \le \theta_{-1} + 2\pi$$

and

$$\sum_{k=1}^m (\theta_k - \theta_{-k}) = 2\theta n.$$

Let

$$A(\theta) = \cup_{k=1}^m \{z : \theta_{-k} \le \arg z \le \theta_k\}, \quad B(\theta) = \overline{\mathbb{C} \setminus A(\theta)}.$$

Lemma 1. *For any system of rays $\{L_{\pm k}\}_{k=1}^m$, $m \le n$, there exist a symmetrical structure $\{P_k\}_{k=1}^N$ and its dissymmetrization $\{\lambda_k\}_{k=1}^N$ satisfying the conditions:*

(1) any ray $L_{\pm k}(\theta)$ lies on the boundary of two angles from $\{P_k\}_{k=1}^N$,

(2) if $P_k \subset A^*(\theta)$, then $\lambda_k(P_k) \subset A(\theta)$ (therefore if $P_k \subset B^*(\theta)$, then $\lambda_k(P_k) \subset B(\theta)$),

(3) if a ray L_j, $j = \pm 1, \ldots, \pm m$, lies on the boundary of angles $\lambda_{k_1}(P_{k_1})$ and $\lambda_{k_2}(P_{k_2})$, then the rays $\lambda_{k_1}^{-1}(L_j)$ and $\lambda_{k_2}^{-1}(L_j)$ belong to the collection of rays $\{L_{\pm k}(\theta)\}_k^n$.

2.4. Hyperbolic realization of n-gons. We do not use the following lemma to prove our main results. But it may be useful in order to reduce the extremal problems on the n-gons to those on the hyperbolic plane.

Lemma 2. *For any n-gon D_n, $n \ge 2$, with vertices $a_1, \ldots, a_n$, and each point $z_0 \in D_n$ there exists a function $f(z)$, $f(z_0) = 0$, which maps D_n conformally and univalently onto a hyperbolic n-gon Δ_n with vertices $A_1 = 1$, $A_2, \ldots, A_n$, $|A_k| = 1$, such that $f(a_k) = A_k$, $k = 1, \ldots, n$.*

The assertion of the lemma may be known. It easily follows from the uniqueness of the mapping function and from the ordinary arguments of the continuity method (cf. [3, Chap. 5]).

Proof of the isoperimetric inequality

Let $n \ge 3$, $0 \le \beta \le 1$. Let us consider the family $\mathcal{D}_n(\beta)$ consisting of simply connected domains $D, 0 \in D$, such that for each of them there exist n rays issuing from the origin which divide the star D^* of the domain D at $z = 0$ (that means the maximal domain which is starshaped with respect to the origin) into n triangles having the form $e^{i\gamma}\Delta(\alpha, \beta, r)$, where $\gamma \in \mathbb{R}$ and $\Delta(\alpha, \beta, r)$ is defined in Sec. 2.1.

Let $\partial^* D$ denote the union of the circular arcs of ∂D^*. Let $D_n(\beta)$ be the regular n-gon from $\mathcal{D}_n(\beta)$ with vertices A_j^*, $j = 1, \ldots, n$.

We shall prove here the following generalization of Theorem 1.

Theorem 3. *Let $n \geq 3$, $0 \leq \beta \leq 1$ and let $D \in \mathcal{D}_n(\beta)$. Then*

$$\frac{R(D,0)}{\text{length}(\partial^* D)} \leq \frac{1}{2\pi} \frac{\Gamma^2(1-\frac{1}{n})}{\Gamma(\frac{1}{2}-\frac{1}{n}+\beta)\Gamma(\frac{3}{2}-\frac{1}{n}-\beta)}. \tag{3.1}$$

Equality is attianed in (3.1) only for the regular n-gons $a\mathcal{D}_n(\beta)$, where $a \in \mathbb{C}$.

Proof. Since $D \in \mathcal{D}_n(\beta)$ there exist n rays dividing the star of D into n triangles

$$\Delta_j = e^{i\gamma_j}\,\Delta(\alpha_j, \beta, r_j), \qquad j = 1, \ldots, n,$$

where $\gamma_j \in \mathbb{R}$, $r_j > 0$, $\alpha_j > 0$ and

$$\sum_{j=1}^{n} \alpha_j = 1. \tag{3.2}$$

An elementary geometrical computation shows that such partition is possible only under the conditions

$$0 < \alpha_j < \min\{\frac{1}{2}+\beta,\, \frac{3}{2}-\beta\}, \qquad j = 1, \ldots, n. \tag{3.3}$$

If l_j denotes the length of the circular arc included in $\partial\Delta_j$, then

$$r_j = \frac{l_j}{2\pi(\frac{1}{2}-(\alpha_j+\beta))} \frac{\cos \pi(\alpha_j+\beta)}{\sin \pi\alpha_j}. \tag{3.4}$$

By Theorem B,

$$m(D,0) \leq \sum_{j=1}^{n} \alpha_j^2 m(\Delta_j), \tag{3.5}$$

where $m(\Delta_j)$ denotes the correspondent reduced module of Δ_j.

The equality statement of Theorem B together with the Riemann–Schwarz symmetry principle shows that the equality is attained in (3.5) only for regular n-gons $a\,D_n(\beta)$, $a \in \mathbb{C}$.

Substituting the expressions of the reduced moduli (2.5) into (3.5) and replacing r_j in accordance with (3.4), we get

$$\begin{aligned} m(D,0) \leq \frac{1}{2\pi} \sum_{j=1}^{n} &\left[\alpha_j \log \frac{\Gamma(1-\alpha_j)\Gamma(\frac{1}{2}+\alpha_j+\beta)\cos \pi(\alpha_j+\beta)\; l_j}{\Gamma(1+\alpha_j)\Gamma(\frac{1}{2}-\alpha_j+\beta)\;(\frac{1}{2}-\alpha_j-\beta)\;\sin \pi\alpha_j}\right] \\ &- \frac{1}{2\pi}\log(2\pi). \end{aligned} \tag{3.6}$$

Applying the well-known relations

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \qquad \text{and} \qquad \Gamma(\frac{1}{2}+z)\Gamma(\frac{1}{2}-z) = \frac{\pi}{\cos \pi z},$$

we reduce (3.6) to the form

$$2\pi\, m(D,0) \leq F_n \;-\; \log(2\pi), \tag{3.7}$$

where

$$F_n = \sum_{j=1}^{n} \left[f_1(\alpha_j,\beta) + f_2(\alpha_j, l_j)\right], \tag{3.8}$$

$$f_1(\alpha,\beta) = \alpha \log \frac{\Gamma^2(1-\alpha)}{\Gamma(\frac{1}{2}-\alpha+\beta)\Gamma(\frac{3}{2}-\alpha-\beta)}, \tag{3.9}$$

$$f_2(\alpha,l) = \alpha \log(l/\alpha). \tag{3.9}$$

Now our aim is to find the maximum of F_n under the conditions (3.2), (3.3) and if the sum of lengths l_j is fixed. Without loss of generality we set

$$\sum_{j=1}^{n} l_j = 1 \tag{3.11}$$

and

$$0 < l_j < 1, \qquad j = 1, \ldots, n. \tag{3.12}$$

Under the conditions (3.3) and (3.12) the functions f_1, f_2 and therefore F_n are bounded above. Hence

$$\sup F_n \quad = \quad M_n \quad < \quad \infty.$$

Choose a sequence

$$\overline{x}_m = (\alpha_1^{(m)}, \ldots, \alpha_n^{(m)}, l_1^{(m)}, \ldots, l_n^{(m)}), \quad m = 1, 2, \ldots$$

converging to a point

$$\overline{x}_0 = (\alpha_1^{(0)}, \ldots, \alpha_n^{(0)}, l_1^{(0)}, \ldots, l_n^{(0)})$$

which maximizes F_n under the restrictions (3.2), (3.3), (3.11), (3.12). Without loss of generality we may assume that

$$f_1(\alpha_j^{(m)}, \beta) \to f_{1,j}; \qquad f_2(\alpha_j^{(m)}, l_j^{(m)}) \to f_{2,j}.$$

On the strength of choice of the sequence $\overline{x}_m$ and from upper boundness of f_1 and f_2 we have

$$-\infty < f_{1,j}, f_{2,j} < \infty, \qquad j = 1, \ldots, n, \tag{3.13}$$

and

$$M_n = \sum_{j=1}^{n} (f_{1,j} + f_{2,j}). \tag{3.14}$$

If $\alpha \to \min\{\frac{1}{2}+\beta,\ \frac{3}{2}-\beta\}$, then $f_1(\alpha,\beta) \to -\infty$. Therefore in accordance with (3.13),

$$\alpha_j^{(0)} < \min\{\frac{1}{2}+\beta,\ \frac{3}{2}-\beta\}, \quad j = 1, \ldots, n. \tag{3.15}$$

If $\alpha_j^{(m)} \to 0$ for some j, then

$$f_{1,j} = \lim_{m\to\infty} f_1(\alpha_j^{(m)}, \beta) = 0, \qquad f_{2,j} = \lim_{m\to\infty} f_2(\alpha_j^{(m)}, l_j^{(m)}) \le 0. \tag{3.16}$$

If $\alpha_j^{(m)} \to \alpha_j^{(0)} \ne 0$ but $l_j^{(m)} \to l_j^{(0)} = 0$, then

$$f_{2,j} = \lim_{m\to\infty} f_2(\alpha_j^{(m)}, l_j^{(m)}) = -\infty,$$

contrary to (3.13). Hence $l_j^{(0)} > 0$ if $\alpha_j^{(0)} > 0$.

Renumerating the triangles Δ_j if necessary, we may assume that

$$\alpha_j^{(0)} > 0 \ \text{ for } \ j = 1, \dots, n_1 \ \text{ and } \ \alpha_j^{(0)} = 0 \ \text{ for } \ j = n_1 + 1, \dots, n,$$

where $n_1 \le n$. Moreover, (3.2) and (3.15) imply $n_1 \ge 3$ if $\beta = 0$, or $\beta = 1$, $n_1 \ge 2$ if $\beta \ne \frac{1}{2}$ and $n_1 \ge 1$ if $\beta = \frac{1}{2}$.

If $n_1 = n$, then $\overline{x}_0$ is an inner point of the parallelepiped defined by the constrains (3.3) and (3.12).

Let $n_1 < n$. Using (3.14), (3.16) together with the fact that F_n is monotone with respect to l_j, we get

$$M_n \le \sum_{j=1}^{n_1} (f_{1,j} + f_{2,j}) = F_{n_1}(\overline{x}') \le F_{n_1}(\overline{x}'') \le M_{n_1}, \tag{3.17}$$

here

$$\begin{aligned}
\overline{x}' &= \left(\alpha_1^{(0)}, \dots, \alpha_{n_1}^{(0)}, l_1^{(0)}, \dots, l_{n_1}^{(0)}\right), \\
\overline{x}'' &= \left(\alpha_1^{(0)}, \dots, \alpha_{n_1}^{(0)}, \kappa l_1^{(0)}, \dots, \kappa l_{n_1}^{(0)}\right), \\
\kappa &= \left(\sum_{j=1}^{n_1} l_j^{(0)}\right)^{-1}.
\end{aligned}$$

Repeating the previous arguments, we shall come after finite number of steps to the function F_s, $s \le n$, of the form (3.8) which depends on the variables $\alpha_1, \dots, \alpha_s, l_1, \dots, l_s$, satisfying the conditions (3.2), (3.11) (with s instead of n) and the inequalities (3.3), (3.12). Moreover, this function takes its maximal value M_s in an inner point of the parallelepiped defined by these inequalities and according to (3.17)

$$M_n \ \le \ M_s. \tag{3.18}$$

We consider the Lagrange function corresponding to the maximum problem for the function F_s:

$$\Phi_s = F_s \ + \ \lambda_1 \sum_{j=1}^{s} \alpha_j \ + \ \lambda_2 \sum_{j=1}^{s} l_j,$$

and the system for the determination of the stationary points:

$$\begin{cases} f_0(\alpha_j, \beta) + \log(l_j/\alpha_j) = 1 - \lambda_1, \\ \alpha_j/l_j = \lambda_2, \qquad j = 1, \dots, s, \end{cases} \tag{3.19}$$

where

$$\begin{aligned}
f_0(\alpha, \beta) = \log \frac{\Gamma^2(1-\alpha)}{\Gamma(\frac{1}{2} - \alpha + \beta)\Gamma(\frac{3}{2} - \alpha - \beta)} \\
- \alpha(2\psi(1-\alpha) - \psi(\frac{1}{2} - \alpha + \beta) - \psi(\frac{3}{2} - \alpha - \beta)),
\end{aligned}$$

and $\psi(z)$ is the logarithmic derivative of the gamma function.

Since $f_0(\alpha, \frac{1}{2}) = 0$ hence if $\beta = \frac{1}{2}$ then arbitrary positive numbers α_j, l_j such that $\alpha_j/l_j =$ const, $j = 1, \dots, s$, are the solution of (3.19). In this case $r_j = r$ for all $j = 1, \dots, s$, and we have proved a particular case of the following well-known isoperimetric theorem: among all simply connected domains with given perimeter the disk has the maximal conformal radius.

Let us consider the case $\beta \neq \frac{1}{2}$. We have

$$\begin{aligned}\frac{\partial f_0(\alpha,\beta)}{\partial\alpha} &= -4\psi(1-\alpha) + 2\psi(\frac{1}{2}-\alpha+\beta) + 2\psi(\frac{3}{2}-\alpha-\beta)\\ &+ 2\alpha\psi'(1-\alpha) - \alpha\psi'(\frac{1}{2}-\alpha+\beta) - \alpha\psi'(\frac{3}{2}-\alpha-\beta),\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f_0(\alpha,\beta)}{\partial\alpha\partial\beta} &= 2[\psi'(\frac{1}{2}-\alpha+\beta) - \psi'(\frac{3}{2}-\alpha-\beta)]\\ &+ \alpha[\psi''(\frac{3}{2}-\alpha-\beta) - \psi''(\frac{1}{2}-\alpha+\beta)].\end{aligned}$$

Using the well-known monotonicity property of the functions $\psi'(t)$ and $\psi''(t)$, we obtain

$$\frac{\partial^2 f_0(\alpha,\beta)}{\partial\alpha\partial\beta} > 0 \ \text{ if } \ 0 \le \beta < \frac{1}{2},$$

and

$$\frac{\partial^2 f_0(\alpha,\beta)}{\partial\alpha\partial\beta} < 0 \ \text{ if } \ \frac{1}{2} < \beta \le 1.$$

These inequalities together with the relation $\partial f_0(\alpha, 1/2)/\partial\alpha = 0$ imply the inequality

$$\frac{\partial f_0(\alpha,\beta)}{\partial\alpha} < 0 \tag{3.20}$$

for $\beta \neq \frac{1}{2}$ and $0 < \alpha < \min\{\frac{1}{2}+\beta,\ \frac{3}{2}-\beta\}$.

It is easily seen that the coordinates α_j of the solution of system (3.19) are the solution of the equation

$$f_0(\alpha,\beta) = \lambda$$

for some fixed λ. By (3.20), this equation has at most one solution. Now using (3.2) and (3.11), we find the unique solution of the system (3.19):

$$\alpha_1 = \alpha_2 = \cdots = \alpha_s = \frac{1}{s}, \quad l_1 = l_2 = \cdots = l_s = \frac{1}{s},$$

that is the point of the maximum of the function F_s.

Substituting these values into (3.9), we get

$$M_s = g(\frac{1}{s}, \beta), \tag{3.21}$$

where

$$g(\alpha,\beta) = \log\frac{\Gamma^2(1-\alpha)}{\Gamma(\frac{1}{2}-\alpha+\beta)\Gamma(\frac{3}{2}-\alpha-\beta)}.$$

After some calculations we obtain

$$\frac{\partial g(\alpha,\beta)}{\partial\alpha} = -2\psi(1-\alpha) + \psi(\frac{1}{2}-\alpha+\beta) + \psi(\frac{3}{2}-\alpha-\beta),$$

$$\frac{\partial^2 g(\alpha,\beta)}{\partial\alpha\partial\beta} = \psi'(\frac{1}{2}-\alpha+\beta) - \psi'(\frac{3}{2}-\alpha-\beta).$$

The monotonicity property of the function $\psi'(t)$ yields the inequalities

$$\frac{\partial^2 g(\alpha,\beta)}{\partial\alpha\partial\beta} > 0 \quad \text{if } 0 \le \beta < \frac{1}{2}, \qquad \frac{\partial^2 g(\alpha,\beta)}{\partial\alpha\partial\beta} < 0 \quad \text{if } \frac{1}{2} < \beta \le 1,$$

which together with the relation $\partial g(\alpha,\frac{1}{2})/\partial\alpha = 0$ lead to the inequality

$$\frac{\partial g(\alpha,\beta)}{\partial\alpha} < 0 \qquad \text{if } \beta \ne \frac{1}{2}.$$

Hence $M_s < M_n$ if $s < n$, contrary to (3.18). Therefore all coordinates $\alpha_j^{(0)}$, $l_j^{(0)}$ of the limit $\overline{x}_0$ of the maximizer sequence $\overline{x}_n$ are positive.

Applying (3.7)–(3.10) and (3.21) with n instead of s, we get the inequality

$$m(D,0) \le \frac{1}{2\pi}\log\frac{\Gamma^2(1-\frac{1}{n})}{2\pi\Gamma(\frac{1}{2}-\frac{1}{n}+\beta)\Gamma(\frac{3}{2}-\frac{1}{n}-\beta)}, \tag{3.22}$$

that is equivalent to (3.1).

As follows from the remark on the case of the equality in (3.5) equality in (3.22) is attained only for the regular n-gon $kD_n(\beta)$, where $k = (\text{length}(\partial D_n(\beta)))^{-1}$.

Corollary 2. *If the perimeter of the regular n-gons $kD_n(\beta)$ is fixed, then its conformal radius (and therefore the conformal radius of the regular hyperbolic polygons) increases when the number of their sides increases.*

Remark. Applying the method of proving Theorem 3, the author has found isoperimetric inequalities for hyperbolic polygons with fixed Euclidean or hyperbolic area. We are going to publish these results in a further paper on this theme.

4. Dissymmetrization of some surfaces and the proof of Theorem 2

Let an n-gon D_n be realized as the disk U with vertices $A_j = e^{i\theta_j}$, $j = 1,\dots,n$, where

$$0 = \theta_1 < \theta_2 < \dots < \theta_n < 2\pi.$$

The full n-sides reflection $\widehat{D}_n$ is a surface $\mathcal{R}$ having one sheet S_0 over U and n sheets S_j, $j = 1,\dots,n$, over the domain $U' = \overline{\mathbb{C}}\setminus\overline{U}$ each of them is sticked down to S_0 along the circular arcs A_jA_{j+1}.

By $\mathcal{R}^*, S_j^*, \dots$ we denote objects corresponding to the regular n-gon D_n^* ($D_n(1/2)$ in Sec. 3) with vertices A_j^*, $j = 1,\dots,n$.

The symmetry group Φ introduced in Sec. 2.3 generates an analogous group of the surface $\mathcal{R}^*$ denoted here by Φ_n. Let $\mathcal{R}$ be an arbitrary surface described above. In what follows we construct a structure on $\mathcal{R}^*$ which is symmetrical with respect to Φ_n and the dissymmetrization of this structure which transforms the surface $\mathcal{R}^*$ into $\mathcal{R}$.

The rays $L_j(1) = \{z : \arg = \theta_j, |z| \geq 1\}$, $j = 1, \ldots, n$, issuing from the vertices A_j divide $\mathcal{R}$ into $n+1$ parts $\mathcal{R}_j$, $j = 0, \ldots, n$, where $\mathcal{R}_0$ is the plane $\mathbb{C}$ cut along the rays $L_j(1)$ and

$$\mathcal{R}_j = \{z : \theta_{j+1} < \arg z < \theta_j + 2\pi, |z| > 1\}$$

for $j = 1, \ldots, n$.

On the sheet $\mathcal{R}_0^*$ the symmetric structure $\{P'_k\}_{k=1}^{N_1}$ and its dissimmetrization $\{\lambda'_k\}_{k=1}^{N_1}$ are defined in accordance with Lemma 1 (with $\theta = 0$) for the rays $L_j = \{z : \arg z = \theta_j\}$, $j, \ldots, n$.

To define the symmetrical structure on the sheets $\mathcal{R}_j$, we use an auxiliary construction. Let Δ_j denote the image of $\mathcal{R}_j$ under the mapping

$$\psi_j : z \to e^{i\beta_j} z^{1/(n-1)},$$

where $\beta_j = (2\pi(j-1) - (\theta_j + \theta_{j+1}))/(n-1)$ and ψ_j is defined in $\mathbb{C} \setminus L_{j+1}$ such that $\psi_j(e^{i\theta_j}) = \exp((2\pi(j-1) - \theta_j)/(n-1))$.

In the regular case we have

$$\psi_j^* : z \to e^{i\beta_j^*} z^{1/(n-1)},$$

where

$$\beta_j^* = 2\pi(((j-1)/n) - (j/n(n-1))), \quad j = 1, \ldots, n.$$

Thus the mappings ψ_j, $j = 1, \ldots, n$, provide the correspondence between the collection of n parts $\mathcal{R}_j$ and the outer of the disk U in $\mathbb{C}_\zeta$ cut along the rays

$$L'_j = \{\zeta : \arg \zeta = (2\pi(j-1) - \theta_j)/(n-1)\}, j = 1, \ldots, n.$$

Similarly, the mappings ψ_j^* provide the correspondence between the collection $\{\mathcal{R}_j^*\}$ and $U' = \mathbb{C}_\zeta \setminus \overline{U}$ cut along the rays L_j^*.

Let $\{\widetilde{P}_k\}_{k=1}^{N_2}$ and $\{\widetilde{\lambda}_k\}_{k=1}^{N_2}$ be a symmetrical structure and its dissymmetrization which correspond to the rays L'_j in the sense of Lemma 1 with $\theta = 0$. For each number $k = 1, \ldots, N_2$ we define the indices j_k and l_k, $1 \leq j_k, l_k \leq n$, such that $\widetilde{P}_k \subset \Delta_{j_k}^*$ and $\widetilde{\lambda}_k(\widetilde{P}_k) \subset \Delta_{l_k}$. Let

$$P'' = \psi_{j_k}^{*\,-1}(\widetilde{P}_k), \quad \lambda''_k = \psi_{l_k}^{-1} \circ \widetilde{\lambda}_k \circ \psi_{j_k}^*, \quad k = 1, \ldots, N_2.$$

Clearly, each of the trasformations λ''_K reduces itself to the rotation about the origin and to the transition onto the suitable sheet of the surface $\mathcal{R}$.

To unify the notation, let us denote by P_j, $j = 1, \ldots, N$, $N = N_1 + N_2$, either an angle P'_k or a sector P''_m such that $P_j \neq P_l$ if $j \neq l$. Let λ_j denote the transformation corresponding to P_j. The collection of P_j forms the structure on $\mathcal{R}^*$ which is symmetric with respect to the group Φ_n, i.e., for $\{P_j\}_{j=1}^N$ the conditions (a)–(c) of Sec. 2.3 still hold with $\overline{\mathcal{R}^*}$ and Φ_n instead of $\overline{\mathbb{C}}$ and Φ.

In accordance with the terminology of Sec. 2.3, the collection of the transformations λ_j will be called the dissymmetrization of the symmetrical structure $\{P_j\}_{j=1}^N$ on $\mathcal{R}^*$. Again the conditions (1), (2) of Sec. 2.3 still hold with $\overline{\mathcal{R}}$ and Φ_n instead of $\overline{\mathbb{C}}$ and Φ.

Combaining previous results, we get.

Lemma 3. *For any surface $\mathcal{R}$ generated by an n-gon D_n, which is realized as the disk U, there exist a symmetrical structure $\{P_j\}_{j=1}^N$ on the regular surface $\mathcal{R}^*$ and the dissymmetrization $\{\lambda_j\}_{j=1}^N$ of this structure which transforms $\mathcal{R}^*$ into $\mathcal{R}$.*

For a given symmetrical structure on $\mathcal{R}^*$ and its dissymmetrization the dissymmetrization of sets, condensers or functions (which should be symmetric with respect to the group Φ_n) are defined in the same manner as for Dubinin's dissymmetrization in Sec. 2.3. For this kind of dissymmetrization we shall use the notation Dis_n.

Lemma 4. *Let v, C and D be respectively a real-valued function, condenser, and domain on $\mathcal{R}^*$, which are symmetric with respect to the group Φ_n. Then*

(1) $\operatorname{Dis}_n v$ *is a single-valued function on* $\mathcal{R}$. *Additionally, if v is continuous on $\mathcal{R}^*$ and Lipschitz' in D, then $\operatorname{Dis}_n v$ is continuous on $\mathcal{R}$ and Lipschitz' in $\operatorname{Dis}_n D$, and*

$$\int\int_D |\nabla v|^2 dA_z = \int\int_{\operatorname{Dis}_n D} |\nabla(\operatorname{Dis}_n v)|^2 \, dA_z,$$

(2) $\operatorname{cap} C \geq \operatorname{cap}(\operatorname{Dis}_n C)$,

(3) *if $0 \in D$ and $\operatorname{Dis}_n^{(0)} D$, $0 \in \operatorname{Dis}_n^{(0)} D$, is a connected component of $\operatorname{Dis}_n D$ then*

$$r(D, 0) \leq r(\operatorname{Dis}_n^{(0)} D, 0).$$

Equality is attained in 2) and 3) only if $\mathcal{R}$ coinsides with $\mathcal{R}^*$ up to rotation about the origin.

We omit the proof of the lemma which consists in the straightforward repeating corresponding simple arguments for Dubinin's dissymmetrization (see [2, §6]).

Proof of Theorem 2.. By the conformal invariance of the amplification coefficient, we may assume that the n-gon D_n is realized as the disk U and $z_0 = 0$.

Since $R(U, 0) = 1$ we need only prove that the regular surface $\mathcal{R}^*$ has the least conformal radius at the origin among all surfaces $\mathcal{R}$ under consideration, i.e.,

$$R(\mathcal{R}^*, 0) < R(\mathcal{R}, 0) \quad \text{if} \quad \mathcal{R} \neq e^{i\alpha}\mathcal{R}^*, \quad \alpha \in \mathbb{R}.$$

But this fact is just a particular case of the assertion (3) of Lemma 4.

To complete the proof, we should calculate $R(\mathcal{R}^*, 0)$ or equivalently $m(\mathcal{R}^*, 0)$. To do this, we use the equality statement of Theorem B. The segments $[0, A_k^*]$, $k = 1, \ldots, n$, divide $\mathcal{R}^*$ into n circular triangles congruent (by rotation about the origin) to the triangle $\Delta(\frac{1}{n}, \frac{3}{2}, 1)$ defined in Sec. 2.1. Using (2.5), we get

$$\begin{aligned} m(\mathcal{R}^*, 0) &= \frac{1}{2\pi n} m(\Delta(\frac{1}{n}, \frac{3}{2}, 1); 0|1, e^{2\pi/n}) \\ &= \frac{1}{2\pi} \log \frac{\Gamma(1 - \frac{1}{n})\Gamma(2 + \frac{1}{n})}{\Gamma(1 + \frac{1}{n})\Gamma(2 - \frac{1}{n})} = \frac{1}{2\pi} \log \frac{n+1}{n-1}, \end{aligned}$$

and the proof is complete.

Proof of Corollary 1. Inequality (1.8) for $n = 2, 3, \ldots$ follows directly from Theorem 2. Equalities (1.7) follow now from the fact that every bilateral or triangle is conformally equivalent to the regular one.

To prove the assertion about equality in (1.8), let us assume that D_n is realized as the disk U with vertices a_j, $j = 1, \ldots, n$ and suppose that

$$\kappa(D_n) = \frac{n+1}{n-1}. \tag{4.1}$$

Choose a sequence z_k, $|z_k| < 1$, $k = 1, 2, \ldots$ such that $z_k \to z_0$ and

$$\frac{R(\widehat{D}_n, z_k)}{R(D_n, z_k)} \to \kappa(D_n). \tag{4.2}$$

Let $D_n^{(k)}$ be the image of D_n under the Möbius mapping

$$\varphi_k(z) = \frac{z - z_k}{1 - \overline{z}_k z}, \quad k = 1, 2, \dots .$$

Without loss of generality we may assume that

$$\varphi_k(a_j) \to a_j^{(0)}, \quad j = 1, \dots, n.$$

Let $D^{(0)}$ denote the polygon U with vertices $a_j^{(0)}$ then $D^{(0)}$ has m sides, where $1 \le m \le n$. Let $\mathcal{R}^{(k)}$, $k = 1, 2, \dots$ and $\mathcal{R}^{(0)}$ denote the full n-sides reflections of $D_n^{(k)}$ and $D^{(0)}$, respectively. It is easily seen that the sequence of surfaces $\mathcal{R}^{(k)}$ converges to $\mathcal{R}^{(0)}$ as a kernel with the center at the origin. By the Caratheodory theorem,

$$R(\mathcal{R}^{(k)}, 0) \to R(\mathcal{R}^{(0)}, 0). \tag{4.3}$$

Using (4.2), (4.3) and applying Theorem 2, we have

$$\kappa(D_n) = R(\mathcal{R}^{(0)}, 0) \ge \frac{m+1}{m-1}. \tag{4.4}$$

Equality here can occur only if $D^{(0)}$ is a regular m-gon centered at the origin. (4.1) and (4.4) give at once

$$m = n > 3. \tag{4.5}$$

Let us show that $|z_0| < 1$. Suppose $z_0 = e^{i\theta_0}$, $\theta_0 \in \mathbb{R}$. Then $z_k = e^{i\theta_0} + \varepsilon_k$, where $\varepsilon_k \to 0$. Let $\theta \ne \theta_0$ be an arbitrary number from $[0, 2\pi)$. After simple calculations we find

$$\varphi_k(e^{i\theta}) = -e^{i\theta_0} + O(\varepsilon_k).$$

Hence in the case $|z_0| = 1$ the limit polygon $D^{(0)}$ has at most two sides, contrary to (4.5).

Therefore $|z_0| < 1$ and

$$\frac{R(\widehat{D}_n, z_k)}{R(D_n, z_k)} \to \frac{R(\widehat{D}_n, z_0)}{R(D, z_0)} = \kappa(D_n). \tag{4.6}$$

(4.1), (4.6) and Theorem 2 imply the desired conclusion.

Remark. The method used in this section can be applied to construct some other kinds of dissymmetrizations, for example, for regular surfaces consisting of n sheets over $\overline{\mathbb{C}}$ with winding points A_k^*, $k = 1, \dots, n$ and even for universal covering surfaces over $\overline{\mathbb{C}} \setminus (\cup_{k=1}^n A_n^*)$. We propose to return to these question in further papers.

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