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ON LOGARITHMICALLY SMALL ERRORS IN THE LATTICE POINT PROBLEM

M. M. Skriganov[a] **and A. N. Starkov**[b]

[a] Steklov Mathematical Institute
at St. Petersburg
Fontanka 27, St. Petersburg 191011
Russia
(E-mail: skrig@pdmi.ras.ru)

[b] Department of Mathematics
Moscow State University
Moscow 117234
Russia
(E-mail: alex@rdiees.msk.ru)

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ABSTRACT

In the present paper we give an improvement of a previous result of the paper [Skr, Theorem 2.2] on logarithmically small errors in the lattice point problem for polyhedra. This improvement is based on an analysis of hidden symmetries of the problem generated by the Weyl group for $SL(n,\mathbb{R})$. Let UP denote a rotation of a given compact polyhedron $P \subset \mathbb{R}^n$ by an orthogonal matrix $U \in SO(n)$, tUP a dilatation of UP by a parameter $t > 0$, and $N(tUP)$ the number of integer points $\gamma \in \mathbb{Z}^n$ which fall into the polyhedron tUP. We show that for almost all rotations U (in the sense of the Haar measure on $SO(n)$) the following asymptotic formula

$$N(tUP) = t^n \operatorname{vol} P + O((\log t)^{n-1+\varepsilon}), \quad t \to \infty,$$

holds with arbitrary small $\varepsilon > 0$.

1. Statement of the result. Let $\mathcal{O} \subset \mathbb{R}^n$ be a compact region of volume $\operatorname{vol} \mathcal{O}$ and $\Gamma \subset \mathbb{R}^n$ a lattice of determinant $\det \Gamma$. Consider the number $N(\mathcal{O}, \Gamma) = \#\{\mathcal{O} \cap \Gamma\}$ of lattice points $\gamma \in \Gamma$ which fall into $\mathcal{O}$, and put

$$N(\mathcal{O}, \Gamma) = \frac{\operatorname{vol} \mathcal{O}}{\det \Gamma} + \mathcal{R}(\mathcal{O}, \Gamma). \tag{1}$$

Let $t\mathcal{O}$ be the dilation of $\mathcal{O}$ by a parameter $t > 0$, and $\mathcal{O} + X$ the translation of $\mathcal{O}$ by a vector $X \in \mathbb{R}^n$, then (1) implies

$$N(t\mathcal{O} + X, \Gamma) = \frac{\operatorname{vol} \mathcal{O}}{\det \Gamma} t^n + \mathcal{R}(t\mathcal{O} + X, \Gamma) \tag{2}$$

We define *the error* $R(t\mathcal{O}, \Gamma)$ by setting

$$R(t\mathcal{O}, \Gamma) = \max_{X \in \mathbb{R}^n} |\mathcal{R}(t\mathcal{O} + X, \Gamma)|. \tag{3}$$

The classical lattice point problem deals with the behavior of the error $R(t\mathcal{O}, \Gamma)$ as $t \to \infty$ (see, e.g., a book [Kr] by Krätzel). Typically, say, for regions like a circle in $\mathbb{R}^2$ or a ball in $\mathbb{R}^n$, the error $R(t\mathcal{O}, \Gamma)$ is of order $\mathcal{O}(t^q)$ with an exponent $q = q(\mathcal{O}, \Gamma) > 0$. In a series of papers by Skriganov (see [Skr] and references therein) it was shown that in arbitrary dimensions there exist regions and lattices with logarithmically small errors in the asymptotics (2). It was also shown in [Skr] that such anomalies in the lattice point problem are intimately related with diophantine and ergodic properties of certain flows on homogeneous spaces of the group $SL(n, \mathbb{R})$. It is worth noting that number-theoretic applications are a major focus of interest for the modern ergodic theory on Lie groups (see, e.g., a recent survey [St] by Starkov; concerning general concepts and results of ergodic theory on Lie groups see, e.g., a book [Z] by Zimmer).

Now we wish to state one of the results from the paper [Skr]. Without loss of generality, in the relations (1)–(3) we may assume that the lattice Γ is unimodular, i.e., $\det \Gamma = 1$. The set of unimodular lattices in $\mathbb{R}^n$ can be naturally identified with the following homogeneous space

$$\mathcal{L}_n = SL(n, \mathbb{R})/SL(n, \mathbb{Z}). \tag{4}$$

The space (4) admits a unique $SL(n, \mathbb{R})$-invariant probability measure $\mu_{\mathcal{L}}$ (see, e.g. a book [Ra] by Raghunathan).

Theorem 1 [Skr, Theorem 2.1]. *Let $P \subset \mathbb{R}^n$ be an arbitrary compact polyhedron, then for almost all lattices $\Gamma \in \mathcal{L}_n$ (in the sense of the invariant measure $\mu_{\mathcal{L}}$) one has the bound*

$$R(tP, \Gamma) = O((\log t)^{n-1+\varepsilon}) \tag{5}$$

with arbitrary small $\varepsilon > 0$.

Thus, the appearance of logarithmically small errors for polyhedra is quite typical from metric point of view.

In the spirit of diophantine approximate on manifolds (see, e.g., a book [Sp] by Sprindžuk and recent paper [KM] by Kleinbock and Margulis) we may give the following

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Definition. *A smooth submanifold $\mathcal{M} \subset \mathcal{L}_n$ is called extremal (with respect to the lattice point problem) if the error $R(tP, \Gamma)$ is logarithmically small,*

$$R(tP, \Gamma) = O((\log t)^{\kappa}), \quad t \to \infty, \tag{6}$$

for each polyhedron $P \subset \mathbb{R}^n$ and almost all lattices $\Gamma \in \mathcal{M}$ (in the sense of a natural measure on $\mathcal{M}$).

¿From Theorem 1 with the help of Fubini's theorem we immediately conclude that for any submanifold $\mathcal{M} \subset \mathcal{L}_n$ and almost all $g \in SL(n, \mathbb{R})$ shifted manifolds $g\mathcal{M}$ are extremal with an exponent $\kappa = n - 1 + \varepsilon$ in (6). However, it is a difficult problem to verify whether or not a given manifold is extremal. In [Skr] the extremality of an orbit of the orthogonal group $\mathcal{M} = SO(n)\Gamma_0$ for each $\Gamma_0 \in \mathcal{L}_n$, was proved with an exponent $\kappa = 2(n-1) + \varepsilon$ in the bound (6). In the present paper we give an improvement of this result.

Theorem 2. *Let $P \subset \mathbb{R}^n$ be an arbitrary compact polyhedron and $\Gamma \in \mathcal{L}_n$ an arbitrary lattice, then for almost all rotations $U \in SO(n)$ (in the sense of the Haar measure on group $SO(n)$) one has the bound*

$$R(tP, U\Gamma) = R(tU^{-1}P, \Gamma) = O((\log t)^{n-1+\varepsilon}) \tag{7}$$

with arbitrary small $\varepsilon > 0$.

Considering orbits of the group $SO(n)$ on the homogeneous space (4), one can easily check (by Fubini's theorem) that the bound (7) implies the bound (5). Thus, Theorem 2 is a generalization of Theorem 1. There are some reasons to expect that the bounds (5) and (7) are best possible, and the exponent $n - 1$ can not be replaced by a smaller one.

In two dimensions the bound (7) was obtained by Khintchine [Kh] as early as in 1923 with the help of continnued fractions and on the base of previous results of Hardy and Littlewood [HL] (see [Skr] for other comments and references concerning the origins of the matter).

It is worth noting that in (5) and (7) the bound $O((\log t)^{n-1+\varepsilon})$ could be replaced by $O((\log t)^{n-1}\varphi(\log\log t))$ where $\varphi(\cdot)$ is an arbitrary positive monotone increasing function with $\int_1^\infty \frac{dt}{\varphi(t)} < \infty$. For simplicity, here we do not consider such minor improvements.

Finally, we would like to notice that our reduction of the $SL(n)$-asymptotics (given in Theorem 1) to the $SO(n)$-asymptotics (given in Theorem 2) actually goes back to a classical paper [Ho] by Hopf on geodesic flow on the unit tangent bundle T^1M of a manifold M of finite volume and negative curvature. One of main results in [Ho] was that this flow is ergodic, i.e., for almost all pairs $(z, v) \in T^1M$, $z \in M$, $v \in T^1_zM$, the corresponding geodesic trajectories are uniformly distributed. However, Hopf also found a stronger result: for *each* point $z \in M$ and almost all tangent vectors $v \in T^1_zM$ the corresponding trajectory is uniformly distributed.

2. Proof of Theorem 2. Recall main arguments from [Skr] which concern applications of ergodic theory on $SL(n, \mathbb{R})$ to the lattice point problem for polyhedra. Since such applications are not commonly known, we will try to give attention to details as much as possible. First, we introduce some notations. Let $\mathbb{R}^n$ be identified with the space of column vectors $X =^T (x_1, \ldots, x_n)$. From now on we denote by $^T(\cdot)$ the transpose of the matrix $(\cdot)$, and by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the standard metric and inner product on $\mathbb{R}^n$. Let $D \subset SL(n, \mathbb{R})$ be the subgroup of all diagonal matrices,

$$D = \{\delta_Y = \mathrm{diag}\,(2^{y_1}, \ldots 2^{y_n}) \in SL(n, \mathbb{R}) : Y =^T (y_1, \ldots, y_n) \in L\}, \tag{8}$$

where

$$L = \{Y =^T (y_1, \dots, y_n) \in \mathbb{R}^n : y_1 + \dots + y_n = 0\} \tag{9}$$

is the $n-1$ dimensional subspace in $\mathbb{R}^n$. Let $\Delta \subset D$ be a discrete subgroup of the form

$$\Delta = \{\delta_m = \mathrm{diag}\,(2^{m_1}, \dots, 2^{m_n}) \in SL(n, \mathbb{R}) : m =^T (m_1, \dots, m_n) \in \mathbb{Z}^n \cap L\}. \tag{10}$$

Thus, Δ is a free Abelian group of rank $n-1$. Let

$$\Delta_r = \{\delta_m \in \Delta : \|m\| < r\}, \quad r > 0 \tag{11}$$

be a system of finite subsets in Δ. Then $\Delta_r \to \Delta$, and $\#\{\Delta_r\} = c_n r^{n-1} + o(r^{n-1})$, $c_n > 0$, as $r \to \infty$.

The subgroup $\Delta \subset SL(n, \mathbb{R})$ acts on the homogeneous space (4) as follows

$$\delta : \mathcal{L} \ni \Lambda \to \delta\Lambda, \quad \delta \in \Delta. \tag{12}$$

By Moore's theorem the action (12) is ergodic (see, e.g., [Z]).

Denote by $\lambda(\Lambda)$ and $Nm\Lambda$ the homogeneous minima of a lattice $\Lambda \in \mathcal{L}_n$ relative to the Euclidean metric $\|X\|$, $X \in \mathbb{R}^n$, and the norm form $NmX = x_1, \dots, x_n$, $X =^T (x_1, \dots, x_n) \in \mathbb{R}^n$, respectively, i.e.,

$$\lambda(\Lambda) = \min\{\|\gamma\| : \gamma \in \Lambda \setminus \{0\}\} \tag{13}$$

and

$$Nm\Lambda = \inf\{|Nm\gamma| : \gamma \in \Lambda \setminus \{0\}\}. \tag{14}$$

The quantities (13) and (14) are related to topology of the homogeneous space (4). For example, by Mahler's compactness criterion, a subset $\mathcal{E} \subset \mathcal{L}_n$ is relatively compact if and only if $\inf\{\lambda(\Lambda) : \Lambda \in \mathcal{E}\} > 0$ (see, e.g., [Ra]).

Notice that from the definition (13) one can easily obtain the following inequality

$$\lambda(\alpha\Lambda) \le n|\alpha|\,\lambda(\Lambda), \tag{15}$$

where $\alpha = (\alpha_{ij}) \in GL(n, \mathbb{R})$ is an arbitrary matrix and $|\alpha| = \max\limits_{ij} |\alpha_{ij}|$ is a matrix norm.

Let $\Lambda^\perp$ denote the lattice dual to $\Lambda \in \mathcal{L}_n$. Recall that $\Lambda^\perp$ consists of all vectors $\gamma^\perp$ such that the inner product $\langle \gamma^\perp, \gamma \rangle \in \mathbb{Z}$ for each $\gamma \in \Lambda$.

We introduce the following diophantine characteristics of a lattice $L \in \mathcal{L}_n$

$$S(\Lambda, r) = \sum_{\delta_m \in \Delta_r} \{\lambda(\delta_m \Lambda)\}^{-n}, \quad r > 0, \tag{16}$$

and

$$\nu(\Lambda, \rho) = \min\{|Nm\gamma| : \gamma \in \Lambda,\ 0 < \|\gamma\| < \rho\}, \quad \rho \ge \lambda(\Lambda). \tag{17}$$

Actually, $S(\Lambda, r)$ and $\nu(\Lambda, \rho)$ (as r and $\rho \to \infty$) control the rate of approach of an orbit $\{\delta\Lambda : \delta \in \Delta\}$ to infinity (the *cusp* of the homogeneous space $\mathcal{L}_n$).

In terms of the characteristics (16) and (17) the error (3) for polyhedra can be estimated very precisely. Consider a compact convex polyhedron $P \subset \mathbb{R}^n$. Let

$$f = \{P = P^n \supset P^{n-1} \supset \dots P^0,\ \dim P^j = j,\ n \ge j \ge 0\} \tag{18}$$

be a flag of faces $P^j = \mathrm{Face}(P)$, $1 \le j \le n$, of P. Let $V_f \in SO(n)$ be an orthogonal matrix associated with the flag (18) by the relation

$${}^T V_f = [l^n, l^{n-1}, \dots, l^1]. \tag{19}$$

The columns of the matrix ${}^T V_f$ in (19) are orthogonal vectors l^j, $n \ge j \ge 1$, defined by the following rule: the unit vector l^j is the external normal to the face P^{j-1} parallel to the face P^j. Thus, coordinates of the vector l^j form the jth row of the matrix V_f.

Theorem 3 [Skr, Theorem 6.1]). *Let an arbitrary compact comvex polyhedron $P \subset \mathbb{R}^n$ and a lattice $\Gamma \in \mathcal{L}_n$ be given, then the error $R(tP, \Gamma)$ (see (3)) satisfies the bound*

$$R(tP, \Gamma) < c(P, \Gamma) \sum_f S(V_f \Gamma^{\perp}, r_f), \tag{20}$$

as $t \to \infty$, where

$$r_f = r_f(t) = \log \left(\frac{t^{n^2}}{\nu(V_f \Gamma^{\perp}, t^n)} \right), \tag{21}$$

the summation in (20) is taken over the finite set of all flags of faces f of the polyhedron P, and the constant $c(P, \Gamma)$ in (20) is independent of $t \to \infty$.

Theorem 3 relates the error $R(tP, \Gamma)$ to orbits $\{\delta \Lambda_f : \delta \in \Delta\}$ of lattices $\Lambda_f = V_f \Gamma^{\perp}$ associated with P and Γ under the action (12) of the discrete subgroup $\Delta \subset SL(n, \mathbb{R})$. Moreover, the rate of approach of those to infinity determines the behavior of the error $R(tP, \Gamma)$ as $t \to \infty$.

In order to use Theorem 3 we need certain bounds for characteristics (16) and (17).

Lemma 1 [Skr, Lemma 3.2]**.** *Let $\Lambda \in \mathcal{L}_n$ be an arbitrary lattice, then for almost all matrices $g \in SL(n, \mathbb{R})$ (in the sense of the Haar measure on $SL(n, \mathbb{R})$) one has the bound*

$$S(g\Lambda, r) = O(r^{n-1+\varepsilon}), \quad r \to \infty, \tag{22}$$

with arbitrary small $\varepsilon > 0$.

Notice that Lemma 1 is a corollary of ergodicity of the action (12) of the group Δ on the space $\mathcal{L}_n$. Indeed, the form (16) of the characteristic $S(\Lambda, r)$ suggests to apply here the individual ergodic theorem with discrete Abelian subgroup Δ. However, in (16) the function $\{\lambda(\Lambda)\}^{-n}$ is not integrable over $\mathcal{L}_n$. But for any $\varepsilon > 0$ the function $\{\lambda(\Lambda)\}^{\varepsilon - n}$ is already integrable over $\mathcal{L}_n$ (see [Skr, Lemma 3.1]), and the bound (22) easily follows.

In the present paper, using additional symmetries generated by the Weyl group for $SL(n, \mathbb{R})$, we give the following improvement of Lemma 1.

Lemma 2. *Let $\Lambda \in \mathcal{L}_n$ be an arbitrary lattice, then for almost all orthogonal matrices $U \in SO(n)$ (in the sense of the Haar measure on $SO(n)$) one has bound*

$$S(U\Lambda, r) = O(r^{n-1+\varepsilon}), \quad r \to \infty, \tag{23}$$

with arbitrarily small $\varepsilon > 0$.

Notice that Lemma 2 improves a previous result from [Skr, Lemma 4.4].

Concerning the characteristic (17) we need the following result

Lemma 3 [Skr, Lemma 4.3]**.** *Let $\Lambda \in \mathcal{L}_n$ be an arbitrary lattice, then for almost all orthogonal matrices $U \in SO(n)$ (in the sense of the Haar measure on $SO(n)$) one has the bound*

$$\nu(U\Lambda, \rho) > c_\varepsilon(U, \Lambda)(\log \rho)^{1-n-\varepsilon}, \quad \rho \to \infty, \tag{24}$$

with arbitrarily small $\varepsilon > 0$ and a constant $c_\varepsilon(U, \Lambda) > 0$.

Now Theorem 2 can be easily derived from Theorem 3 and Lemmas 2 and 3 by the same arguments as Theorem 1 was derived in [Skr] from Theorem 3 and Lemmas 1 and 3.

First, we observe that without loss of generality, the polyhedron P may be assumed to be convex. This follows from the fact that hyperplanes of the $n-1$ dimensional faces of a given polyhedron P divide P into a finite number of disjoint convex polyhedra, moreover, for almost all rotations $U \in SO(n)$ each of the hyperplanes contains at most one lattice point $\gamma \in U\Gamma$.

Thus, let a compact convex polyhedron $P \subset \mathbb{R}^n$ and a lattice $\Gamma \in \mathcal{L}_n$ be given. Replacing in Theorem 3 the lattice Γ by $U\Gamma$, $U \in SO(n)$, and using (21) and (24) we obtain the bound

$$r_f(t) = \log\left(\frac{t^{n^2}}{\nu(V_f U\Gamma^\perp, t^n)}\right) = O(\log t) \tag{25}$$

for each flag f of P and almost all rotations $U \in SO(n)$. Substituting (23) and (25) to (20), we obtain the bound in question (7).

The proof of Theorem 2 is complete.

3. Proof of Lemma 2. Recall necessary results on the group $SL(n,\mathbb{R})$ (see, e.g., a survey [VGO] by Vinberg, Gorbatsevich and Onischik). The Iwasawa decomposition for $SL(n,\mathbb{R})$ has the form

$$SL(n,\mathbb{R}) = B\,SO(n), \tag{26}$$

where $B \subset SL(n,\mathbb{R})$ is the Borel subgroup consisting of all lower triangular matrices $\beta = (\beta_{ij})$, $\beta_{ij} = 0$, $i < j$, with arbitrary diagonal elements $\beta_{ii} > 0$, $\beta_{11}\beta_{22}\ldots\beta_{nn} = 1$. More precisely, any matrix $g \in SL(n,\mathbb{R})$ can be uniquely represented in the form $g = \beta_g U_g$, $\beta_g \in B$, $U_g \in SO(n)$, and the projection $g \to U_g$ is a smooth mapping from $SL(n,\mathbb{R})$ onto $SO(n)$. Therefore, by Fubini's theorem, this projection maps any co-null subset from $SL(n,\mathbb{R})$ to a co-null subset in $SO(n)$ (in the sense of the corresponding Haar measures on $SL(n,\mathbb{R})$ and $SO(n)$).

The Weyl group for $SL(n,\mathbb{R})$ can be defined as the symmetric group S_n consisting of all $n!$ permutations of the coordinate axes in $\mathbb{R}^n$. As a matrix group W is contained in the group $O(n)$ of all orthogonal matrices in $\mathbb{R}^n$ of determinant ± 1.

Obviously, the vector ${}^T(1,\ldots,1)$ is an eigen-vector for each matrix $w \in W$, therefore, the action of the group W in $\mathbb{R}^n$ can be restricted on the subspace (9). By the closed Weyl chamber Φ one calls the following cone in the subspace (9)

$$\Phi = \{Y =^T (y_1,\ldots,y_n) \in L : y_1 \geq y_2 \geq \ldots \geq y_n\}. \tag{27}$$

One can easily verify that

$$L = \bigcup_{w\in W} \Phi_w, \quad \Phi_w = w\Phi. \tag{28}$$

Actually, Φ is a fundamental domain of the group W in L.

The adjoint action of the Weyl group

$$SL(n,\mathbb{R}) \ni g \to wgw^{-1}, \quad w \in W \tag{29}$$

generates symmetries for each of the groups $SL(n,\mathbb{R})$, $SO(n)$, and D. In the case of the diagonal group D these symmetries consist of all permutations of diagonal elements. From relations (8)–(9) we derive the equality

$$w\delta_Y w^{-1} = \delta_{wY}, \quad \delta_Y \in D, \quad Y \in L, \quad w \in W. \tag{30}$$

Applying the action (29) to (26), we obtain a series of Iwasawa decompositions

$$SL(n,\mathbb{R}) = B^{(w)} SO(n), \quad w \in W, \tag{31}$$

where $B^{(w)} = wBw^{-1}$ are conjugate Borel subgroups. Thus, for each $w \in W$ any matrix $g \in SL(n,\mathbb{R})$ can be uniquely represented in the form $g = \beta_g^{(w)} U_g^{(w)}$, $\beta_g^{(w)} \in B^{(w)}$, $U_g^{(w)} \in SO(n)$, and the projection $g \to U_g^{(w)}$ maps any co-null subset from $SL(n,\mathbb{R})$ to a co-null subset in $SO(n)$.

Now we wish to use an obvious fact that each Borel subgroup $B^{(w)}$, $w \in W$, is a non-expanding subgroup under the adjoint action of diagonal matrices d_Y, $Y \in \Phi_w$. Combining the definitions of the Borel subgroups, the Weyl chamber Φ, see (27), and the homogeneous minimum $\lambda(\cdot)$, see (13), together with the relations (8) and (30), by a direct calculation one can easily obtain the following.

Lemma 4. *Given a lattice $\Lambda \in \mathcal{L}_n$ and a matrix $\beta^{(w)} \in B^{(w)}$, one has the inequality*

$$\lambda(\delta_Y \beta^{(w)} \delta_Y^{-1} \Lambda) \le n|\beta^{(w)}| \lambda(\Lambda) \tag{32}$$

for all diagonal matrices δ_Y, $Y \in \Phi_w$.

¿From the definitions (11), (16) and the relation (28) we derive the inequality

$$S(\Lambda, r) \le \sum_{w \in W} S^{(w)}(\Lambda, r), \tag{33}$$

where

$$S^{(w)}(\Lambda, r) = \sum_{\delta_m \in \Delta_r, m \in \Phi_w} \{\lambda(\delta_m \Lambda)\}^{-n} \tag{34}$$

Using the Iwasawa decomposition (31) for a matrix $g \in SL(n,\mathbb{R})$ and the bound (32), we obtain that

$$\lambda(\delta_m g \Lambda) = \lambda(\delta_m \beta_g^{(w)} U_g^{(w)} \Lambda)$$

$$= \lambda(\delta_m \beta_g^{(w)} \delta_m^{-1} \delta_m U_g^{(w)} \Lambda) \le n|\beta_g^{(w)}| \lambda(\delta_m U_g^{(w)} \Lambda).$$

for all diagonal matrices δ_m, $m \in \Phi_w$. Therefore, in view of (34),

$$S^{(w)}(U_g \Lambda, r) \le c_w(g) S^{(w)}(g\Lambda, r) \le c_w(g) S(g\Lambda, r), \tag{35}$$

where the constant $c_w(g) = (n|\beta_g^{(w)}|)^{-n}$ is independent of $r > 0$.

¿From (35) we conclude that if g belongs to a co-null subset in $SL(n,\mathbb{R})$ where the bound (22) holds, then the corresponding orthogonal matrix U_g belongs to a co-null subset in $SO(n)$ where one has the bound

$$S^{(w)}(U_g \Lambda, r) = O(r^{n-1+\varepsilon}), \quad r \to \infty. \tag{36}$$

Since the Weyl group W is finite, the relations (33) and (36) imply the bound in question (23).

The proof of Lemma 2 is complete.

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