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ON CONTINUOUS PARTITION LATTICE

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ABSTRACT

In this paper we deal with some class of continuous partition lattices, introduced by A. Bjorner in 1987 as a limit of finite partition lattices. We describe the direct way how to construct the continuous partition lattice in terms of measurable partitions of the measure space and present a simple realization of such lattices. Secondly, we give a representation of continuous partition lattice in a factor of type II_1 and in the continuous geometries in the sense of von Neumann. Thirdly, the continuous hyperfinite partition lattice is used to state and solve one natural asymptotical problem.

0. Introduction

In the paper [1] A. Björner, answering a question of G. C. Rota about an analog of von Neumann's continuous geometries for partition lattices, has constructed an example of continuous partition lattice as a completion of an inductive limit of finite partition lattices with special embeddings. There was no explicit model of the limit lattice. Later M. Haiman [3] has described this lattice as a lattice of a special kind of measurable partitions. The construction in [3] did not use appropriate language of measurable partition theory and consequently seems to be more tangled than it is necessary.

The first goal of this paper is to get a direct link between theory of measurable partitions and a theory of continuous lattices; particularly, we construct continuous analog of geometrical lattices for any orbit partition, not only for hyperfinite one. Secondly, we describe a representation of continuous partition lattice in a factor of type II_1. This representation can be considered as a linear realization of continuous matroids with continuous partition lattice being its flat lattice. Thirdly, continuous hyperfinite partition lattice is used for statement and solution of natural asymptotical problems. In the present paper we suggest the link between the theory of combinatorial partitions of finite sets and the theory of measurable partitions in measure space (Lebesgue space). This link is fruitful for both sides.

The paper consists of five sections. The first one presents some facts from the theory of measurable partitions; we emphasize the using of the notion from lattice theory which usually did not occur in that theory. The second section describes the Björner embedding scheme, along with some its generalizations, in the terms that will lead us to the representation of the continuous partition lattice in the lattice of measurable partitions of diadic compactum. This representation is also described in this section. In the third section authors introduce the representations of continuous partition lattice in the lattice of projectors of the factor of type II_1 and in the continuous geometry (in the sense of von Neumann). At last, in the fourth section some asymptotical problems concerning continuous partition lattice and ergodic theory are stated and solved.

The following important terminological remark must be done. Unfortunately the traditional ordering of partitions in combinatorics from the one side and functional analysis, measure and ergodic theory from the other one are opposite: in functional analysis and measure theory people use the order in which partition ξ is bigger than partition η if each (almost each) block of ξ is a subset of some block of the partition η; consequently a partition which is zero element of the lattice is trivial partition (=has one nonempty block of positive measure) and unit of the lattice is a partition such that almost all its blocks are one-point sets. This order coincides with the natural order in the lattice of mathematical expectations for the sigma-fields corresponding to partitions or simply to the order in the lattice of subspaces of the measurable functions which are constant on the blocks of the partitions. In the same time in combinatorics people use just the opposite (or dual) order, so the largest partition is a partition with one block and so on. Since we deal with both theories we used a "neutral" terminology: we use terms 'finer' and 'coarse' instead of 'greater' and 'less', as well as usual terms 'product' and 'intersection', and write $\prec$, $\succ$; $\vee$, $\wedge$ when we deal with measurable partitions; when we bear in mind partitions of finite set we use the standard notations 'greater', 'less', 'supremum', 'infimum' and write $>$, $<$, $\cup$, $\cap$.

1. Lattices of measurable partitions

1. Lattice of all classes of measurable partitions. We mention some notions of measurable partition theory making an emphasis on terms and notions of lattice theory. For underground in this field see [4], and also [6, 7].

Let $(X, \mathfrak{A}, \mu)$ be the Lebesgue space with continuous measure μ and σ-field $\mathfrak{A}$ of classes of all measurable sets, or, equivalently, $(X, \mathfrak{A}, \mu)$ is isomorphic in measure theoretical sense to the unit interval with Lebesgue measure. (An abstract definition of the Lebesgue space is the following: it is a complete separable measure space with complete σ-field of all classes (mod 0) of measurable sets (see [4])). The partition of X is said to be measurable if it is a partition on inverse images of points under some measurable mapping $f\colon X \to Y$, where Y is another Lebesgue space, for example $[0,1]$ with Lebesgue measure. *Element of partition* or *block* is an inverse image of a point. We write $B \in \xi$ to denote that the set B is a block of the partition ξ. The quotient space X/ξ (or *the space of blocks*) equipped with a quotient measure is again the Lebesgue space. A measurable partition is said to be *finite* if it has finite number of blocks. A partition with finite blocks is sometimes called *cofinite* one. A measurable set $A \in \mathfrak{A}$ is said to be measurable with respect to partition ξ (or ξ*-measurable*) if it is a union of blocks of ξ. Obviously, ξ-measurable sets generate σ-subfield $\mathfrak{A}_\xi$ of $\mathfrak{A}$. Two partitions are said to be equal mod 0 if they coincide on some subset of full measure. We say that two partitions belong to one class mod 0 if they coincide on some measurable subset of measure one. We will consider mainly the classes mod 0 of partitions instead of individual partitions; for example, the lattices of the classes mod 0 of partitions and so on. Sometimes we omit the word "class". It is easy to see that all previous definitions (finite partitions, cofinite ones and so on) are consistent with the notion of mod 0. The set of classes of partitions that are equal mod 0 will be denoted by $\mathrm{Part}(X, \mu) \equiv \mathrm{Part}(X)$. Now we define an order on $\mathrm{Part}(X)$. We said that partition ξ is finer than η if blocks of η are unions of blocks of ξ, that is $\mathfrak{A}_\xi \supset \mathfrak{A}_\eta$. One can see that definition of order remains correct after a pass to classes mod 0. Product and (measurable) intersection of any set of partitions are defined on $\mathrm{Part}(X)$ as exact upper and lower bounds, and they turn $\mathrm{Part}(X)$ into the *measurable partition lattice*. It has minimal and maximal elements, a trivial partition ν of one block X and a partition ε on single points, correspondingly.

As it was already mentioned, the subalgebra $\mathfrak{A}_\xi$ of ξ-measurable subsets correspond to each measurable partition ξ. Hence the lattice $\mathrm{Part}(X)$ is canonically isomorphic to the lattice of subalgebras of $\mathfrak{A}$. Also, an operator of taking the mean over blocks of ξ of functions in $L^1(X)$ can be defined (see [6]), so another one model of $\mathrm{Part}(X)$ is the lattice of conditional expectation operators in $L^1(X)$.

The fundamental result in the theory of measurable partitions is Rokhlin's theorem ([4]) stating that the canonical family of conditional measures on blocks of partition exists and is unique. Let ξ be a measurable partition of X. Then the Fubini–Rokhlin theorem states that for any $A \in \mathfrak{A}$

$$\mu(A) = \int_{X/\xi} \mu^c(A \cap c)\, d\mu^c,$$

where $\{\mu^c\}_{c\in\xi}$ is the canonical family. The measurable partition is said to be *homogeneous* if almost all its blocks are isomorphic and have a uniform conditional measure (continuous measure is considered to be uniform). An *independent complement* to partition ξ is partition η such that (i) $\mu(A \cap B) = \mu(A)\mu(B)$ for all $A \in \xi$ and $B \in \eta$ and (ii) $\xi \vee \eta = \epsilon$. Rochlin proved [4] that a homogeneous partition has an independent complement. The partition is said to be *semihomogeneous* [6] if almost all its blocks are isomorphic to some spaces with uniform measure (possibly to different ones). Any semihomogeneous partition is a product of homogeneous ones, and this decomposition is unique if each factor consists of all blocks with given number of points.

We shall mainly deal with semihomogeneous partitions with finite blocks.

2. Lattice of semihomogeneous measurable partitions. Remind the following definition. The partially ordered subset of a lattice is called an upper subsemilattice of it if an inherited order generates the supremum that coincides with the original one. The notion of lower subsemilattice is dual. The next Lemma holds:

Lemma 1.1. *The set of semihomogeneous partitions with finite blocks forms an upper subsemilattice of* $\mathrm{Part}(X)$ *without zero.* $\square$

Notice that semihomogeneous partitions with finite blocks do not form a sublattice of the lattice $\mathrm{Part}(X)$, since an intersection of two of them can have infinite blocks. There is no zero in this semilattice because the trivial partition ν has only one infinite block. So one have to join the trivial partition to homogeneous ones to form a lattice.

Definition. *The set* $\mathrm{Part}_0(X)$ *of partitions with not more than one continuous block, and semihomogeneous on its supplement with an intersection defined as* $\xi \mathring{\wedge} \eta = (\xi \wedge \eta)^\circ$, *where* $(\xi)^\circ$ *stands for pasting all continuous blocks of* ξ *together, is called the lattice of semihomogeneous partitions.*

The set $\mathrm{Part}_0(X)$ contains zero and consequently it is a lattice (but not a sublattice of $\mathrm{Part}(X)$, since intersections in these lattices are not the same). This lattice was defined in [3] using noninvariant terms and was called a lattice of continuous partitions.

Lattices $\mathrm{Part}(X)$ and $\mathrm{Part}_0(X)$ are not modular and have continuous chains. As far as we know, there is no common definition of rank function for such lattices. We shall use the following definition.

Definition. *Let* L *be a lattice with zero and one. We call a nonnegative function* $r\colon L \to \mathbb{R}$ *a normed rank function if it is monotone* ($r(\xi) < r(\eta)$ *if* $\xi \precneqq \eta$), *normed* ($r(0) = 0$ *and* $r(1) = 1$) *and invariant under lattice automorphisms (for any* $g \in \mathrm{Aut}\, L \;\; r(g\xi) = r(\xi)$).

Lemma 1.2. *There is no rank function on* $\mathrm{Part}(X, \mu)$. *There exists a rank function* r^* *on* $\mathrm{Part}_0(X, \mu)$ *defined by*

$$r^*(\xi) = \int_X \frac{1}{\#\xi(x)} dx, \tag{1.1}$$

where $\xi(x)$ *stands for the block of* ξ *containing* x, *and* $\#B$ *denotes the cardinality of* B.

Proof. Suppose that r is the rank function on $\mathrm{Part}(X)$. Consider partition $\xi \in \mathrm{Part}(X)$ such that blocks of ξ are continuous and X/ξ is continuous too. Let η be partition finer than ξ and have the same properties. According to the Rokhlin theorem (see [4]), these partitions are isomorphic, and hence an automorphism $g \in \mathrm{Aut}\,\mathrm{Part}(X)$ exists such that $g\xi = \eta$. Consequently, $r(\xi) = r(g\xi) = r(\eta)$, and this contradicts monotony of r.

The second part can be directly verified. $\square$

We shall use lower semimodularity of the rank r^* which can also be verified directly:

$$r^*(\xi) + r^*(\eta) \le r^*(\xi \vee \eta) + r^*(\xi \mathring{\wedge} \eta). \tag{1.2}$$

The semimodular inequality (1.2) makes it possible to define a rank metric ρ^*:

$$\rho^*(\xi, \eta) = r^*(\xi) + r^*(\eta) - 2r^*(\xi \mathring{\wedge} \eta), \tag{1.3}$$

since (1.2) allows to prove the triangle inequality.

Remarks. 1. Lattices $\mathrm{Part}(X)$ and $\mathrm{Part}_0(X)$ are not modular and have continuous chains. As far as we know, there is no common definition of semimodularity for such lattices. However the existence of semimodular rank function allows us to consider the lattice $\mathrm{Part}_0(X,\mu)$ as one of continuous analogues of semimodular lattices.

2. A number of other metrics on $\mathrm{Part}(X)$ have been used. See, for example, [6]. One of them is entropy metrics defined on partitions with finite blocks [4].

Theorem 1.3 (see [3]). $\mathrm{Part}_0(X)$ *is complete w.r.t. the metric* ρ^*.

The theorem follows from the fact that $\mathrm{Part}_0(X)$ is a complete lattice (in the sense that any family of its elements has product and measurable intersection) and the simple lemma stating that complete semimodular lattice is complete w.r.t. its semimodular metric (1.3). See [3] for details. $\square$

2. Orbit partitions lattice. Consider an measure preserving action of countable group G on the space (X,μ). An *orbit partition* is a partition τ_G of X such that $x,y \in X$ lie in one block if and only if there exists $g \in G$ such that $x = gy$. This partition is correctly defined and in general is not measurable (see [6]). If the group action is ergodic then measurable hull of τ_G is the trivial partition. The orbit partition is said to be *ergodic* in this case.

One can consider measurable refinements, say ξ, of orbit partition τ_G. Any such ξ has finite blocks with uniform conditional measure, i.e. it is semihomogeneous partition with finite blocks. Denote the set of such partitions by Part^τ. Just like the set of semihomogeneous partitions, the set Part^τ does not form a lattice.

Definition. *The lattice* Part_0^τ *of semihomogeneous partitions subordinated by orbit partition* τ *is the lattice of measurable partitions* ξ *such that* ξ *has at most one continuous block and each finite block of* ξ *lies in some block of* τ, *with ordinary product and intersection* $\overset{\circ}{\wedge}$.

The lattice Part_0^τ is a subsemilattice of $\mathrm{Part}(X)$ and is obtained from Part^τ by joining the trivial partition in the same way as $\mathrm{Part}_0(X)$ was obtained from the lattice of semihomogeneous partitions with finite blocks.

Tame or *hyperfinite* partition (with countable blocks) [6] is an intersection of decreasing sequence of semihomogeneous partitions.

Example 1.1. Let $X = \prod_1^\infty \mathbb{Z}/2$ and μ be the Haar (uniform) measure. Given a natural n, consider the partition ξ_n with blocks containing all sequences that coincide from n^{th} coordinate. Note that all ξ_n are homogeneous. Their intersection τ_0 is the (nonmeasurable) partition whose blocks are countable and contain all sequences that coincide from some (possibly different for various pairs of sequences) place. This partition is known as *tail partition*. Its measurable hull is trivial partition ν. The tail partition is an ergodic hyperfinite partition. The described realization of τ_0 is known as diadic model of hyperfinite partition.

Two fundamental facts take place:

Theorem 1.4. 1) *Each ergodic hyperfinite partition (with countable blocks) is isomorphic to the partition* τ_0, *in the other words, ergodic hyperfinite partition is unique modulo isomorphisms.*

2) *Orbit partition of an ergodic measure preserving action of an amenable group (in particular, of group* $\mathbb{Z}$*) is hyperfinite and (by* (1)*) is isomorphic to* τ_0.

The first part of this theorem follows from approximation type arguments (see f.e. [6]): each hyperfinite partition could be approximated by decreasing diadic sequence of measurable partitions. The second one is much deeper and has a long history. Uniqueness of the ergodic homogeneous hyperfinite partition was announced by von Neumann and Murray in their paper [4] and firstly proved by H. Dye [2]; more geometric proof could be obtained using the theorem about lacunary isomorphism - see [6]. It is not so hard to prove (see [4]) that if orbit partition for the measure preserving free action of the group is hyperfinite then this group is amenable. In the paper [16] the inverse theorem was proved: hyperfiniteness of orbit partition of amenable group. $\square$

The theorem above makes it possible to introduce the following

Definition. *The lattice* $\mathrm{Part}_0^{\tau_0}$ *is called the continuous hyperfinite partition lattice.*

In the next section we shall prove that the lattice Part_0^{τ} coincides with Björner's lattice.

2. The main construction

1. Björner's scheme and colored partitions. Now we describe Björner's construction in terms that will lead us to measurable partitions. We begin with construction of embedding of finite partition lattices. It will be more convenient for us to say about partitions of a $n+1$ element set as about *colored* partitions of a set of n elements. We call a partition of a finite set with not more than one marked block the colored partition. We denote the set of ordinary partitions of n-element set by $\Pi(n)$ and the set colored ones by $\Pi'(n)$. The bijection between $\Pi(n+1)$ and $\Pi'(n)$ is obvious: the extra element lies in the marked (or colored) block, if it exists, and if there is no marked block then the extra element lies in the single block. So the difference between colored partitions of Z and ordinary partitions of $(\#Z+1)$-element set is just terminological, but usage of colored partitions seems to simplify the exposition of Björner's construction.

The rank of combinatorial partition $\lambda \in \Pi(n)$ is $n - N(\lambda)$, where $N(\lambda)$ denotes the number of blocks in λ. This is the partial case of more general definition of rank of ξ as a length of maximal chain from lattice zero to ξ. This definition can be applied to a wide class of lattices with zero satisfying Jordan–Gölder condition that every maximal chain from one element of lattice to another one has the same number of edges. We consider not ordinary integer-valued rank but the *normed rank* r (defined by $r(\xi) = (n - N'(\xi))/n$, where $\xi \in \Pi'(X_n)$ and $N'(\xi)$ denotes the number of uncolored blocks of ξ).

We shall describe now briefly the Björner embedding scheme. Let $m = kn$. We shall show how to map colored partitions of n-element set $\mathbf{n}$ into those of m-element set $\mathbf{m}$. Since $m = kn$, one can consider $\mathbf{m}$ to be a direct product of $\mathbf{n}$ and some k-element set, say $\mathbf{k}$. (This structure of direct product can be chosen in various ways, so different embeddings can be obtained.) Whenever this structure is outlined the embedding is determined in the following way. The colored block B in partition of $\mathbf{n}$ gives the colored block $B \times \mathbf{k}$ and uncolored block B gives k uncolored blocks $B \times \{1\}, \ldots, B \times \{k\}$. Note that in terms of ordinary partitions this construction describes the embedding of $\Pi(n+1)$ into $\Pi(m+1)$ which was firstly presented by Björner [1].

Lemma 2.1 [1]**.** *The above embedding is the lattice homomorphism that preserves lattice zero and one and multiplies the rank of any partition by* k. $\square$

Now we show how this scheme leads us to measurable partitions. All statements concerning measurable partitions below in this section hold $\mod 0$. Let $Z = \{0,1\}$, $X_i = Z^i = \{0,1\}^i$ be the sequence of its Cartesian degrees, and projections $p_i \colon X_{i+1} \to X_i$ be defined in a natural way. Consider the projective limit

$$\varprojlim_i (X_i, p_i) = X = \{0,1\}^\infty.$$

One can interpret it as a space of sequences with terms in Z. This space is also known as the Cantor compactum.

Let $\pi_i\colon \Pi'(X_i) \to \Pi'(X_{i+1})$ be Björner's embeddings described above, with the given structure of Cartesian product. (In this case $m = 2^i$ and $k = 2$.) According to Lemma 2.1, one can consider an inductive limit defined in [1]:

$$\Pi = \varinjlim_i (\Pi'(X_i), \pi_i).$$

We shall show how to interpret Π *as a set of (measurable) partitions of* $X = \{0, 1\}$, or, in the other words, how to construct embedding $\pi^*\colon \Pi \to \mathrm{Part}(X)$. This construction repeats in general the construction of π_i. Consider the partition ξ in $\Pi'(X_i) \subset \Pi$. The partition $\pi^*(\xi) \in \mathrm{Part}(X)$ is determined by the following rule: two sequences $x, y \in X$ lie in the same block of $\pi^*(\xi) \in \mathrm{Part}(X)$ if either the first i coordinates of x and y (as elements of X_i) lie in the colored block of ξ or the first i coordinates of x and y lie in the same (uncolored) block and other coordinates coincide. It can be easily seen that this embedding agrees with embeddings π_i and that it sends combinatorial infimum to measure-theoretical product and combinatorial supremum to $\mathring{\wedge}$.

The normalization of rank allows us to lift the rank r on Π and to define the rank metric on Π by

$$\rho(\xi, \eta) = 2r(\xi \cup \eta) - r(\xi) - r(\eta). \tag{2.1}$$

This is metrics since r is semimodular. Note that in fact this is the same metrics that was used by von Neumann in his construction of continuous geometries [14], see Section 3.

Lemma 2.2. 1) *The image of* π^* *is the set of partitions* $\xi \in \mathrm{Part}(X)$ *such that*

(i) ξ *has not more than one continuous block which is a cylindrical set;*

(ii) *all other blocks of* ξ *are finite;*

(iii) *there exists* n *such that for any finite block* $B \in \xi$ *if* $x, y \in B$ *then* $(x_n, x_{n+1}, \dots) = (y_n, y_{n+1}, \dots)$.

2) $\pi^*(\Pi)$ *is a sublattice of* $\mathrm{Part}_0^{\tau_0}(X)$ *with inverse order.*

3) $r(\xi) = 1 - r^*(\pi^*(\xi))$, *and* $\rho(\xi, \eta) = \rho^*(\pi^*(\xi, \eta))$, *for any* $\xi, \eta \in \Pi$.

The first part of the Lemma follows from definition of π^*. The definition of $\mathrm{Part}_0^{\tau_0}$ (see Example 1.1) can be used to verify the second part. The third part can be verified directly. □

The above construction depends on the initial choice of embeddings. We stated at the beginning that $X = \{0, 1\}^\infty$. But one can take $X_i = \prod_{j=0}^{i} Y_j$, with $\#Y_i = k_i$, and the construction of π^* is the same. Lemma 2.2 also holds for this X. We denote the sequence $\{k_i\}$ by $\Bbbk$, the space $X = \varprojlim_i (X_i, p_i)$ by $X_\Bbbk$ and the lattice $\Pi = \varinjlim_i (\Pi'(X_i), \pi_i)$ by $\Pi_\Bbbk$.

Lemma 2.2 states that rank metrics is just the same in combinatorial and measure-theoretical cases. So we are ready to state

Theorem 2.3 (see also [3]). *The metric completion of* $\pi^*(\Pi_\Bbbk)$ *(w.r.t. the rank metrics) is hyperfinite partition lattice* $\mathrm{Part}_0^{\tau_0}(X_\Bbbk)$.

Proof. We need to show that (i) $\mathrm{Part}_0^{\tau_0}$ is closed subset of $\mathrm{Part}_0(X)$ and (ii) $\pi^*(\Pi)$ is dense in $\mathrm{Part}_0^{\tau_0}$. We begin with the proof of (i). Let $\xi \in \mathrm{Part}_0(X) \backslash \mathrm{Part}_0^{\tau_0}$. Then there exists a set $A_0 \in X$, $\mu(A_0) > 0$, such that the restriction $\xi|_{A_0}$ of ξ on A_0 is semihomogeneous and is not subordinated by τ_0. Since $\xi|_{A_0}$ is semihomogeneous, there exists its subset $A_1 \subset A_0$, $\mu(A_1) > 0$ such that $\xi|_{A_1}$ is homogeneous. Let all blocks of $\xi|_{A_1}$ have k elements. Since $\xi|_{A_1}$ is not subordinated by τ_0, there exists $A \subset A_1$, $\mu(A) > 0$, such that each block of $\xi|_A$ has elements from j blocks of τ_0, where $j \geq 2$ and $j \leq k$. Let η be any partition in $\mathrm{Part}_0^{\tau_0}$. Then $(\xi \mathring{\wedge} \eta)|_A$ have block with at least jk elements, and

$$\rho(\xi, \eta) \geq \int_A \left(\frac{1}{\#\xi(x)} + \frac{1}{\#\eta(x)} - 2\frac{1}{\#(\xi \mathring{\wedge} \eta)(x)} \right) dx \geq \mu(A) \frac{j-1}{jk} > 0,$$

and thus ξ can not be approximated by partitions in $\mathrm{Part}_0^{\tau_0}$.

Proceed now with statement (ii). Note that if two partitions ξ and η coincide on the set A then $\rho(\xi,\eta) \le 1-\mu(A)$. So one can approximate any $\xi \in \mathrm{Part}_0^{\tau_0}$ by the sequence $\xi \vee \xi_i$, where ξ_i are partitions from Example 1.1, with blocks containing all sequences that coincide from i^{th} coordinate. The observation that $\xi \vee \xi_i$ lie in $\pi^*(\Pi'(X_i)) \subset \pi^*(\Pi)$ finishes the proof. $\square$

Corollary. *The metric completion of $\Pi_{\Bbbk}$ are isometrically isomorphic for all sequences $\Bbbk$.*

Proof. Theorem 2.3 and Lemma 2.2 state that the metric completion of $\Pi_{\Bbbk}$ is $\mathrm{Part}_0^{\tau_0}(X_{\Bbbk})$, and part (1) Theorem 1.4 states that all such lattices are isomorphic. $\square$

2. Generalization of the Björner's scheme for Markov compactum. In the above subsection we dealt with X that is a direct product of some finite spaces. This construction can be essentially generalized, but the result (i.e. the completion of inductive limit) remains the same. We describe the general scheme without detailed proofs since they in the main repeat those presented above.

Consider two finite sets Z_1 and Z_2 and a subset X of their direct product such that the restrictions on X of coordinate projections π_1 and π_2 are surjective. (This means that Z_1 and Z_2 are minimal in a certain sense.) The embedding of $\Pi'(Z_1)$ into $\Pi'(X)$ is defined by the following rule: a colored block C gives a colored block $\pi_1^{-1}(C)$ and each uncolored block B gives a collection of uncolored blocks $\big\{(B\times\{j\})\cap X\big\}_{j\in Z_2}$. If there is no colored block in the original partition then the same is true for the image.

The above embedding can be constructed for any two partition lattices Π_n and Π_m, $n < m$. Exclude one element from $\mathbf{n}$ and one from $\mathbf{m}$ and denote the new sets by Z_1 and X, correspondingly. Choose an arbitrary Z_2 and construct the homomorphism $Z_1 \to X$ as it is described in the above paragraph. This gives the desired embedding since $\Pi_n \sim \Pi'(Z_1)$ and $\Pi_m \sim \Pi'(X)$.

Lemma 2.4. *Let $n < m$. Then the embedding $\Pi_n \to \Pi_m$ described above is a lower lattice homomorphism.*

The statement of the Lemma can be directly verified. $\square$

Remark. This embedding is not lattice homomorphism, in general, unlike the Björner case.

Let us consider stationary Markov compactum X which is defined by a finite state space Z and a matrix $A=(a_{x,y})$, $x,y\in Z$, $a_{x,y}\in\{0,1\}$. Remind that X is a set of sequences $(x_1,x_2,\dots)$ with terms in Z such that $a_{x_i,x_{i+1}}=1$ for all i. We assume that X is irreducible, that is any state is accessible from any one, or, in terms of matrix A, there exists n such that all elements of A^n are positive. Under these assumption there exists a measure μ with maximal entropy on X (see [6]).

The hyperfinite ergodic partition τ can be constructed on X, and it is isomorphic to τ_0 by Theorem 1.4. In this case τ has the same structure: any its block consists of all sequences that coincide from some place.

The lattice of homogeneous partitions subordinated to τ is defined as above and denoted by Part_0^{τ}. Again, it consists of partitions with not more than one continuous block and homogeneous on its complement, with all elements in finite blocks having the same tail. We define rank r^* and metrics ρ^* on Part_0^{τ} by (1.2) and (1.3), correspondingly.

Let p^i be the projections from X on the first i coordinates, $p^i(x_1,x_2,\dots)=(x_1,\dots,x_i)$. Let $X_i=p^i(X)$. All X_i are finite sets, and one can consider a colored partition lattices $\Pi'(X_i)$. Normed rank function on $\Pi'(X_i)$ will be denoted by r_i, and rank metrics (2.1) by ρ_i. According to Lemma 2.4, lower homomorphisms $\pi_i \colon \Pi'(X_i) \to \Pi'(X_{i+1})$ exist. We consider an inductive limit

$$\Pi = \varinjlim_i (\Pi'(X_i), \pi_i).$$

The embedding π^* is defined as above, and the following statement holds, which allows us to state the analogue of Theorem 2.3.

Lemma 2.5. 1) *The image of* π^* *is the set of partitions* ξ *in* $\mathrm{Part}(X)$ *such that*

(i) ξ *has not more than one continuous block which is a cylindrical set;*
(ii) *all other blocks of* ξ *are finite;*
(iii) *all sequences in one finite block coincide from some place* n *which is bounded for any given partition.*

2) $\pi^*(\Pi)$ *is an upper sublattice of* $\mathrm{Part}_0^{\tau}(X)$ *with inverse order.*
3) $\lim\limits_{i\to\infty} r_i(\xi) = 1 - r^*(\pi^*(\xi))$, *and* $\lim\limits_{i\to\infty} \rho_i(\xi,\eta) = \rho^*(\pi^*(\xi), \pi^*(\eta))$, *for any* $\xi, \eta \in \Pi$. □

Statement (3) of the above Lemma allows us to complete $\pi^*(\Pi)$ in topology generated by metrics ρ^*, since it is equivalent to metrics ρ_i for large i.

Theorem 2.6. *The metric completion of* $\pi^*(\Pi)$ *(with respect to metrics* ρ^* *is continuous hyperfinite partition lattice.*

The proof repeats in general the proof of Theorem 2.3. □

3. Continuous hyperfinite partition lattice and factors

1. Now we shall show that the continuous hyperfinite partition lattice can be naturally embedded in the projectors lattice of the factor of type II_1. We will use the well-known description of the factor of type II_1 which is associated with tensor cross-product of the space of functions on $X = \prod_1^\infty \mathbb{Z}_2$ and the group $G = \sum_1^\infty \mathbb{Z}_2$ which acts on X naturally as group of the shifts, that is $T_g x \overset{\mathrm{def}}{=} g + x$. Let μ_0 be uniform measure on X, then μ is G-invariant. The factor of type II_1 (see [9]) can be represented in $L^2(X \times G)$ by so called crossed product construction. The weak closure of the algebra spanned by operators M_φ and U_g, $\varphi \in L^\infty(X)$ and $g \in G$, defined by $M_\varphi f_h(x) = \varphi(x) f_h(x)$ and $U_g f_h(x) = f_{h+g}(gx)$, $f_g(\cdot) \in L^2(X \times G)$ is the factor of type II_1; it will be denoted by $\mathcal{A} = \mathcal{A}(X, G)$.

This construction can be symmetrized by an orbit construction, that is by considering a representation of $\mathcal{A}$ in $L^2_\mu(X \times X)$, where $\sup \mu = \{(x, y) \in X \times X : y = gx \text{ for some } g \in G\}$ and $\mu(Y, GY) = \mu(GY, Y) = \mu_0(Y)$ for any measurable set $Y \subset X$. In fact, there exists two representations, left and right one, defined by

$$M^L_\varphi f(x,y) = \varphi(x) f(x,y) \quad U^L_g f(x,y) = f(x, gy)$$
$$M^R_\varphi f(x,y) = \varphi(y) f(x,y) \quad U^R_g f(x,y) = f(gx, y).$$

Again, the weak closure of an algebra spanned by M^L_φ and U^L_g (or M^R_φ and U^R_g) is $\mathcal{A}(X, G)$. We use just this representation of the factor below. As for the representation of hyperfinite partition lattice, we assume it is the completion of the inductive limit (w.r.t. diadic scheme) of finite partition lattices. This means that the lattice $\Pi(X)$ consists of partitions of X with not more than one continuous block, semihomogeneous on its complement and subordinated by the orbit partition τ_G.

To represent the continuous partition lattice in the projector lattice of this factor we need to map each partition to some subspace of $L^2(X \times X, \mu)$ such that the projector on this subspace lies in $\mathcal{A}$. Let $\xi \in \Pi(X)$. We consider the subspace

$$L_\xi = \left\{ \sum f(x, gx) = 0, \ x \text{ lie in finite block of } \xi \right\}$$

where the sum is taken over $g \in G$ such that x and gx lie in the same block of ξ. The projector on the subspace L_ξ will be denoted by P_ξ. In fact, this is an operator of taking the conditional expectation over ξ. The projector P_ξ lie in the factor $\mathcal{A}$. The set of all projectors in $\mathcal{A}(X,G)$ will be denoted $\mathrm{Proj}(X,G)$.

Theorem 3.1. *The map $\xi \to P_\xi$ is injective embedding of the lattice $\xi \in \mathrm{Part}_0^{\tau_0}(X)$ to the lattice of the projectors of the factor $\mathcal{A}(X,G)$. The image is closed in the weak topology of the factor.*

We will give a proof of this theorem in Section 3.3 as a corollary of the fact that factor $\mathcal{A}$ is the weak closure of continuous geometry [13]. This allows us to consider only finite spaces, due to the hyperfiniteness of the partition lattice $\mathrm{Part}_0^{\tau_0}(X)$.

Remark. The Björner's continuous partition lattice can be naturally embedded in the factor of type II_1: $\mathrm{Part}_0^{\tau_0}(X) \subset \mathrm{Proj}(X,G) \subset \mathcal{A}(X,G)$.

2. Continuous geometry.

The continuous geometry $\mathcal{L}_\infty$ over complex numbers was defined by von Neumann in [10]. It can be naturally embedded into the projectors lattice $\mathrm{Proj}(X,G)$, namely $\mathrm{Proj}(X,G)$ is the completion of the continuous geometry in the weak topology; see [13]. We shall show that the hyperfinite partitions lattice can be embedded into the continuous geometry, and thus into projectors lattice. These embeddings both preserve order and rank. Nevertheless, they are not homeomorphisms in both cases, and even not lattice homomorphisms.

We remind the von Neumann's construction. Let $\mathcal{L}_{2^n}$ be the set of $2^n \times 2^n$-projectors. $\mathcal{L}_{2^n}$ can be naturally identified with the projective geometry of dimension $2^n - 1$. Following von Neumann, we introduce embedding ψ_n of $\mathcal{L}_{2^n}$ to $\mathcal{L}_{2^{n+1}}$ by the following rule:

$$\psi_n \colon (x_{ij}) \mapsto (\bar{x}_{kl}); \quad \bar{x}_{2(i-1)+s, 2(j-1)+t} = \begin{cases} x_{ij} & \text{when } s = t \\ 0 & \text{when } s \neq t; \end{cases} \qquad s,t = 1,2.$$

These embeddings preserve normed rank $\tilde{r}$. It makes possible to consider an inductive limit

$$\mathcal{L}_{(\infty)} = \varinjlim_{n \to \infty} \left(\mathcal{L}_{2^n}; \psi_n \right)$$

and to extend the definition of rank $\tilde{r}$ on L. The rank metrics $\tilde{\rho}$ is defined by $\tilde{\rho}(p,q) = \tilde{r}(p \vee q) - \tilde{r}(p \wedge q)$. The metric completion of $\mathcal{L}_{(\infty)}$ w.r.t. the rank metrics is the continuous geometry $\mathcal{L}_\infty$.

The common way to represent finite set partitions by linear subspaces is to map the partition ξ to the subspace $\left\{ x : \sum_{i \in B} x_i = 0 \ \ \forall B \in \xi \right\}$. However this mapping do not preserve (normed) rank, so we use a modified one. If $\xi \in \Pi'(n)$ then we introduce a subspace L_ξ of $\mathbb{C}^n$:

$$L_\xi = \left\{ x \in \mathbb{C}^n : \sum_{i \in B} x_i = 0 \text{ for all uncolored blocks } B \in \xi \right\}.$$

Thus we can define the mapping $\tilde{\pi}_n \colon \Pi'(X_n) \to \mathcal{L}_{2^n}$ by $\tilde{\pi}_n(\xi) = L_\xi$. The following Lemma precisely states how the mappings $\tilde{\pi}_n$ conform to Björner's embeddings.

Lemma 3.2. 1) *$\tilde{\pi}_n(\Pi'(X_n))$ is an upper subsemilattice of $\mathcal{L}_{2^n}$;*

2) *for any $\xi, \eta \in \Pi'(X_n)$*

(i) *$r(\xi) = \tilde{r}(\tilde{\pi}(\xi))$;*

(ii) *$\rho(\xi, \eta) = \tilde{\rho}(\tilde{\pi}(\xi), \tilde{\pi}(\eta))$;*

(iii) *$\psi_n(\tilde{\pi}_n(\xi)) = \tilde{\pi}_{n+1}(\pi_n(\xi))$, where $\pi_n : \Pi'(X_n) \to \Pi'(X_{n+1})$ are Björner's embeddings.*

The Lemma follows directly from definitions, so we just explain some details. Part (2ii) states that Björner's semimodular metrics and von Neumann's modular one are just the same, and part (2iii) states that Björner's and von Neumann's embeddings are the same too. However, one must bear in mind that intersections in these lattices do not coincide. Moreover, in fact $\psi_n(\Pi'(X_n))$ forms very small discrete set in $\mathcal{L}_{2^n}$.

Since Björner's embeddings agree with the von Neumann embedding scheme, the inductive limit Π of finite partition lattices can be embed to $\mathcal{L}_{(\infty)}$. Its metric completion $\mathrm{Part}_0^{\tau}(X)$ is embed to $\mathcal{L}_\infty$. This fact was noticed by Björner [1].

3. The lattice of projectors of the factor of type II_1.

The "trace metrics" ρ_{tr} on $\mathcal{L}_{2^n}$, defined by

$$\rho_{\mathrm{tr}}(P, Q) = \sqrt{\frac{1}{2^n}\mathrm{tr}(P-Q)^2},$$

was considered by von Neumann [14]. This metrics leads to the factor $\mathcal{A}$, as will be described below. The metrics ρ_{tr} is also preserved by von Neumann's embedding, and thus can be lifted on $\mathcal{L}_{(\infty)}$. The completion of $\mathcal{L}_{(\infty)}$ with respect to the trace metrics gives the lattice of projectors in factor $\mathcal{A}$, which is not isomorphic to $\mathcal{L}_\infty$, see [13]. However, we show below that the closure of Π in $\mathcal{A}$ is still $\mathrm{Part}_0^{\tau_0}(X)$.

To verify this we have to show that the convergence on Π is the same for both metrics. Given $\xi \in \Pi'(X_n)$, we consider the projector $P_\xi = (p_{ij}) \in \mathcal{L}_{2^n}$ on the subspace L_ξ. There exists a permutation σ such that $(p_{\sigma(i),\sigma(j)})$ is block diagonal matrix with not more that one unit block that correspond to colored block of ξ, and other blocks

$$A_k = \begin{pmatrix} 1-\frac{1}{k} & -\frac{1}{k} & \dots & -\frac{1}{k} \\ -\frac{1}{k} & 1-\frac{1}{k} & \dots & -\frac{1}{k} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{k} & -\frac{1}{k} & \dots & 1-\frac{1}{k} \end{pmatrix},$$

where k is the dimension of the block (that is the size of the corresponding block of ξ). Note that blocks of dimension 1 are zero blocks.

Lemma 3.3. *For any $\xi, \eta \in \Pi'(X_n)$,*

$$\rho(\xi,\eta) \le \big(\rho_{\mathrm{tr}}(P_\xi, P_\eta)\big)^2 \le 2\rho(\xi,\eta). \tag{3.1}$$

Proof. We use the notation $b_\xi(C)$ to denote the block of ξ that contains the set C (it is always supposed that C is a subset of some block of ξ when this notation is used). One can verify that

$$\rho(\xi,\eta) = \frac{1}{2^n} \sum_{\substack{B\in\xi\cap\eta \\ B \text{ uncolored}}} \#B \left(\frac{1}{\#\xi(B)} + \frac{1}{\#\eta(B)} - 2\frac{1}{\#(\xi\cup\eta)(B)}\right);$$

$$\big(\rho_{\mathrm{tr}}(P_\xi, P_\eta)\big)^2 = \frac{1}{2^n} \sum_{\substack{B\in\xi\cap\eta \\ B \text{ uncolored}}} \#B \left(\frac{1}{\#\xi(B)} + \frac{1}{\#\eta(B)} - 2\frac{\#B}{\#\xi(B)\,\#\eta(B)}\right). \tag{3.2}$$

Once this is verified it takes a little time to prove that inequalities (3.1) hold for each summand of (3.2) taken separately. $\square$

Proof of Theorem 3.1. Lemma 3.3 makes it clear that Theorem 3.1 holds. Indeed, according to Lemma 3.3 the convergences in metrics ρ and ρ_{tr} coincide, according to Lemma 3.3, and thus the completions in these metrics are the same. $\square$

4. An asymptotical problem: convergence of the random martingales

It seems to be very interesting to introduce a measure on continuous hyperfinite partition lattice $\mathrm{Part}_0^\tau(X)$. One of the possible approaches is to consider a weak limit of the uniform measures on finite partition lattices, as their size grows to infinity. The asymptotical behavior of these measures was investigated by one of the authors in [8] from the different point of view. Answers to some questions that arise in this connection are presented below, although the full description of the measure under consideration still is not known.

We use the notations of the first subsection of Section 2. Let $L^1(X,\mu)$ be the set of all measurable and integrable functions on X. An operator of conditional expectation E_ξ corresponds to each partition $\xi \in \mathrm{Part}_0^\tau(X)$ in a natural way. Particularly, it corresponds to each partition in $\Pi'(X_i)$. The question is whether there are some common properties of such conditional expectation operators, for sufficiently large i. It is not very surprising that these properties exist, and that these operators are close to operator of unconditional expectation E.

We remind that the number of partitions of n-element set is called n^{th} Bell number. We shall denote it by $B(n)$. The uniform probabilistic measure on $\Pi'(X_i)$ will be denoted by ν_i, so that $\nu_i(\xi) = 1/B(2^i+1)$. We also consider a product measure $\prod \nu_i$ on the set of all sequences $\{\xi_i\}$, where $\xi_i \in \Pi'(X_i)$.

Theorem 4.1. *For almost all sequences $\{\xi_i\}$, with respect to product measure $\prod \nu_i$ and for any function $f \in L^1(X,\mu)$ the limit*

$$\lim_{i\to\infty} E_{\xi_i} f(x) = Ef$$

exists for almost all points $x \in X$.

Proof. First of all we show that the statement of theorem holds for characteristic function f_A of some cylindrical set A. The easiest way to show it is to use the algorithm of construction of random n-element set partition due to A. Stam [5]. This algorithm is the following. First consider the random variable M with $P\{M = m\} = \frac{1}{eB(n)} \frac{m^n}{m!}$. This formula gives probabilistic distribution according to Dobinsky's formula

$$B(n) = \frac{1}{e} \sum_{m=1}^{\infty} \frac{m^n}{m!}.$$

Given the value of M, we consider M boxes and consequently put each element of the set into some box, without any preferences to any box. Stam proved [5] that the random partition of a set on blocks that lie in separate box is uniformly distributed.

We want to show that

$$\lim_{i\to\infty} E_{\xi_i} f_A(x) = Ef_A = \mu(A) \tag{4.1}$$

for almost all points $x \in X$. Now we apply Stam's algorithm independently for each i in order to construct the random sequence $\{\xi_i\}$. If the sequence $\#\xi_i(x)$ tends to infinity then (4.1) holds, since we put an element into some box independently of the fact that it lies in the set A and thus the values of conditional expectation of f_A over blocks $b_{\xi_i}(x)$ tend to the measure of A. The results of [8] show that, for large i, typical length of block containing $x \in X_i$ is $i \log 2$. Thus for almost all $x \in X$ (4.1) holds.

Now we pass to the case of arbitrary function in $L^1(X, \mu)$. We have to show that the sequence $E_{\xi_i} f(x)$ converges for almost all points x and for almost all sequences $\{\xi_i\}$. It suffices to show that $E_{\xi_i} f(x)$ is the Cauchy sequence under the same assumptions on x and $\{\xi_i\}$. All the statements below hold under these assumptions.

Suppose that f_n is the sequence of finite linear combinations of characteristic functions of cylindrical sets (and thus

$$\left|E_{\xi_i} f_n(x) - E_{\xi_j} f_n(x)\right| \to 0 \tag{4.2}$$

as $i, j \to \infty$) and $\left\|f_n - f\right\| \to 0$ as $n \to \infty$. We have

$$\left|E_{\xi_i} f(x) - E_{\xi_j} f(x)\right| \leq E_{\xi_i} |f - f_n|(x) + \left|E_{\xi_i} f_n(x) - E_{\xi_j} f_n(x)\right| + E_{\xi_j} |f - f_n|(x).$$

It suffices to show that $E_{\xi_i} g(x) \to 0$ as $\|g\| \to 0$ for almost all points x and sequences $\{\xi_i\}$. Here we make use of

Lemma 4.2. *For any real λ, integrable function $g\colon X \to \mathbb{R}$ and partition ξ the following inequality holds:*

$$\lambda \mu\{x : E_\xi g(x) > \lambda\} \leq \int\limits_{\{x : E_\xi g(x) > \lambda\}} g(x) dx \tag{4.3}$$

The Lemma shows that the measure of those x where $E_{\xi_i} |f - f_n| > \lambda$ is not greater than $\|f - f_n\|_1 / \lambda$. Choosing f_n sufficiently close to f (in the sense of L^1) one can see that the measure of those x where $E_{\xi_i} |f - f_n|(x)$ does not vanish is less than an arbitrary positive number and thus is zero. $\square$

Proof of Lemma 4.2. Note that function $E_\xi g$ is constant on the blocks of ξ. According to the Rokhlin theorem (see [4]), the integral in the right-hand part of (4.3) can be decomposed into integral

$$\int\limits_{\left\{c \in \xi : E_\xi g(x) > \lambda,\, x \in c\right\}} \int_c g(x)\, d\mu_c(x)\, dc.$$

The internal integral is just $E_\xi g(x)$ for $x \in c$ and so is $> \lambda$. So this integral is greater than $\lambda \mu\{x : E_\xi g(x) > \lambda\}$. $\square$

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