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VISUALIZATION OF WAVES IN THE MAXWELL DYNAMICAL SYSTEM (THE BC-METHOD)

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ABSTRACT

The present paper develops an approach to the inverse problems using their relations to boundary control theory (the BC-method). One of the main tools of the approach is the so-called amplitude formula (AF) based on geometric optics and properties of the controllability of dynamical systems. The AF makes the waves propagating inside a domain visible for an external observer performing measurements at the boundary. In the paper, a natural analog of the AF is obtained for a system governed by the Maxwell equations. As an auxiliary result, an approximate controllability of the electric component of the system is established under some assumptions of geometric nature.

0. Introduction

One of the main tools of the BC-method is the amplitude formula (AF) derived from relations of geometric optics and properties of the controllability of a dynamical system governed by the wave equation (see [2]). The AF expresses the images of waves (in semigeodesic coordinates) in terms of inverse data. Roughly speaking, it makes waves propagating *inside* a domain visible to an external observer performing measurements *at the boundary* of the domain. In our paper, an analog of the AF is obtained for the Maxwell system; possible applications to the inverse problem of electrodynamics are outlined.

1. Geometry

1.1. ρ-Metric. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth* boundary Γ; let ε and μ be smooth functions (permittivities) positive in $\overline{\Omega}$; denote $\rho := \varepsilon\mu$.

The function ρ determines a metric

$$ds^2 = \rho\,|dx|^2,$$

which converts Ω into a Riemannian manifold with distance dist_ρ.

The function

$$\tau(x) := \operatorname{dist}_\rho(x, \Gamma)$$

given in $\overline{\Omega}$ is called *an eikonal.* The subdomains

$$\Omega^\xi := \{x \in \Omega \mid \tau(x) < \xi\} \quad (\xi > 0)$$

enlarge as ξ increases; they exhaust Ω:

$$T_* := \inf\left\{\xi > 0 \mid \Omega^\xi = \Omega\right\} < \infty.$$

These subdomains are bounded by the surfaces

$$\Gamma^\xi := \{x \in \Omega \mid \tau(x) = \xi\} \quad (0 \le \xi \le T_*)$$

($\Gamma^0 = \Gamma$), which are equidistant to the boundary.

1.2. Patterns. Let $\omega \subset \Omega$ be *a separation set* of the domain Ω relative to the boundary Γ (in the sense of the ρ-metric, see [2]); as is known,

$$\omega = \overline{\omega}; \quad \operatorname{vol}\omega = 0; \quad \operatorname{dist}_\rho(\omega, \Gamma) =: T_\omega > 0. \tag{1.1}$$

For any point $x \in \Omega\setminus\omega$, there exists a unique point $\gamma(x) \in \Gamma$ on the boundary that is nearest to x : $\operatorname{dist}_\rho(x, \gamma(x)) = \operatorname{dist}_\rho(x, \Gamma) = \tau(x)$. The pair $(\gamma(x), \tau(x)) \in \Gamma \times [0, T_*]$ is said to be *the semigeodesic coordinates* (s.-g.c.) of a point x; let $x(\gamma, \tau)$ denote the point with s.-g.c. (γ, τ). Let γ^1, γ^2 be local coordinates in a neigborhood of the point $\gamma(x)$ (on Γ); then γ^1, γ^2, τ form a regular coordinate system in a neighborhood of the point x (in Ω). In these coordinates, the Euclidean length and volume elements have a known form:

$$|dx|^2 = g_{\alpha\beta}\, d\gamma^\alpha\, d\gamma^\beta + \rho^{-1}\, d\tau^2; \quad dx = \rho^{-1/2}\, J\, d\gamma^1\, d\gamma^2\, d\tau,$$

*throughout the paper "smooth" means "C^∞ – smooth."

where $J = J(\gamma,\tau) := [\det \|g_{\alpha\beta}(\gamma,\tau)\|]^{1/2}$ (see [1]).

The set

$$\Theta := \{(\gamma(x),\tau(x)) \mid x \in \Omega \setminus \omega\} \subset \Sigma := \Gamma \times [0,T_*]$$

is called *a pattern* of Ω. The boundary of Θ in Σ (*the coast* of a pattern) is a set of measure zero:

$$\operatorname{mes}_{d\Gamma d\tau} \overline{\Theta} \cap [\Sigma \setminus \Theta] = 0. \tag{1.2}$$

For fixed $T \in (0,T_*]$, the set

$$\Theta^T := \big\{(\gamma(x),\tau(x)) \mid x \in \Omega^T \setminus \omega\big\} \subset \Sigma^T := \Gamma \times [0,T]$$

is called a pattern of the subdomain Ω^T.

1.3. Images. Let y be an $\mathbb{R}^3$-vector field on Ω^T; we associate with it a field defined on $\Sigma^T = \Gamma \times [0,T]$ by the following rule:

(i) fix $x \in \Omega^T \setminus \omega$; let $[y(x)]^\wedge$ be the vector at the point $\gamma(x) \in \Gamma$ obtained by parallel translation (in ρ-metric!) of the vector $y(x)$ from x to $\gamma(x)$ along the shortest geodesic that joins x with Γ;

(ii) denote

$$\beta(\gamma,\tau) := \frac{J(\gamma,\tau)}{J(\gamma,0)} \cdot \Big(\frac{\varepsilon(x(\gamma,\tau))}{\varepsilon(x(\gamma,0))}\Big)^{1/2} \cdot \Big(\frac{\mu(x(\gamma,\tau))}{\mu(x(\gamma,0))}\Big)^{-1/2};$$

with every point $(\gamma,\tau) \in \Sigma^T$ we associate the vector

$$\widetilde{y}\,(\gamma,\tau) := \begin{cases} \beta^{1/2}(\gamma,\tau)[y(x(\gamma,\tau))]^\wedge, & (\gamma,\tau) \in \Theta^T; \\ \quad 0, & (\gamma,\tau) \in \Sigma^T \setminus \Theta^T. \end{cases}$$

The field $\widetilde{y}$ defined a.e. on Σ^T is called *an image* of the field y; the map $I^T : y \to \widetilde{y}$ is said to be *an image operator*.

1.4. Decomposition of fields. At the points of $\overline{\Omega}^T \setminus \omega$ (i.e., almost everywhere in $\overline{\Omega}^T$, see (1.1)), *the field of normals* $\nu = \nu(x) := |\nabla\tau(x)|^{-1}\nabla\tau(x)$ is well defined. Note that the $\nu\big|_\Gamma$ are inner normals of Γ.

A field y given in Ω^T may be decomposed as follows:

$$y = y_\nu + y_\theta \quad \text{a.e. in} \quad \Omega^T,$$

where

$$y_\nu := (y \cdot \nu)\,\nu, \quad y_\theta := y - y_\nu = \nu \times y \times \nu$$

are the *longitudinal* and *transversal* components of y.

If the field y is transversal (i.e., $y = y_\theta$), then, obviously, the image $\widetilde{y}$ of it is *tangential* to Γ:

$$\widetilde{y} \cdot \nu\big|_\Gamma = 0 \quad \text{a.e. on} \quad \Sigma^T.$$

1.5. The EP-condition. For $\xi < T_\omega$, the surface Γ^ξ is diffeomorphic to Γ; for $\xi \geq T_\omega$, the surface Γ, in general, is disconnected, but the smoothness of its components may be violated only at the points $\Gamma^\xi \cap \omega$.

One of the main results of the present paper (Theorem 2) will be established under additional assumptions on the ρ-metric. We say that the domain Ω^ξ satisfies an existence condition for a potential (the *EP-condition*) if

(α) the surface Γ^ξ is a union of finitely many connected components, and every component is piecewise smooth;

(β) every cycle (a simple closed curve) in $\overline{\Omega}^\xi$ may be continuously deformed into a cycle lying on Γ.

Proposition 1. *Let Ω^ξ satisfy the EP-condition, let y be a field such that*
(i) $\operatorname{rot} y = 0$ *in* Ω^ξ;
(ii) $\operatorname{div} \varepsilon y = 0$ *in* Ω^ξ;
(iii) $\nu \times y = 0$ *on* Γ;
(iv) $\nu \cdot y = 0$ *on* Γ^ξ.
Then $y = 0$ *in* Ω^ξ.

Indeed, together with (i), (iii), condition (β) ensures the existence of a potential $p : y = \nabla p$; moreover, the other conditions lead to the boundary-value problem

$$\operatorname{div} \varepsilon \nabla p = 0 \quad \text{in} \quad \Omega^\xi;$$

$$p\big|_\Gamma = 0; \quad \frac{\partial p}{\partial \nu}\bigg|_{\Gamma^\xi} = 0,$$

which implies that $p = 0$ and $y = 0$.

Note that the EP-condition is obviously satisfied for Ω^ξ with $\xi < T_\omega$.

2. Dynamics

2.1. The Maxwell system. Consider the dynamical system

$$\varepsilon u_t = \operatorname{rot} w, \quad \mu w_t = -\operatorname{rot} u \quad \text{in} \quad Q^T; \tag{2.1}$$

$$u\big|_{t=0} = w\big|_{t=0} = 0; \tag{2.2}$$

$$u_\theta\big|_{\Sigma^T} = f, \tag{2.3}$$

where $T > 0$; $Q^T := \Omega \times (0, T)$; $\Sigma^T = \Gamma \times [0, T]$; $f = f(\gamma, t)$ is an electric boundary control; let $\{u, w\} : u = u^f(x, t),\ w = w^f(x, t)$ be a solution of the system.

For $f \in \mathcal{F}^T := \{f \in L_2(\Sigma^T; \mathbb{R}^3) \mid f \cdot \nu\big|_\Gamma = 0\}$, the generalized solution is defined in [4]; for all $t \in [0, T]$, the solution is solenoidal:

$$\operatorname{div} \varepsilon u^f(\,\cdot\,, t) = \operatorname{div} \mu w^f(\,\cdot\,, t) = 0. \tag{2.4}$$

If

$$f \in \mathcal{M}^T := \left\{ f \in \mathcal{F}^T \cap C^\infty(\Sigma^T; \mathbb{R}^3) \Big| \left(\frac{\partial}{\partial t}\right)^k f\bigg|_{t=0} = 0, \quad k = 0, 1, \dots \right\},$$

then the solution is classical: $u^f,\ w^f \in C^\infty(\overline{Q}^{\,T}; \mathbb{R}^3)$.

2.2. The electric subsystem. The electric component u^f satisfies the system

$$u_{tt} - Lu = 0 \quad \text{in} \quad Q^T; \tag{2.5}$$

$$u\big|_{t=0} = u_t\big|_{t=0} = 0; \tag{2.6}$$

$$u_\theta\big|_{\Sigma^T} = f \tag{2.7}$$

with operator $L := -\varepsilon^{-1} \operatorname{rot} \mu^{-1} \operatorname{rot}$. For solenoidal fields we have

$$\begin{aligned} Lu &= -(\varepsilon\mu)^{-1} \operatorname{rot} \operatorname{rot} u + \dots \\ &= (\varepsilon\mu)^{-1} \left[\Delta u - \nabla \operatorname{div} u\right] + \dots \\ &= (\varepsilon\mu)^{-1} \left[\Delta u - \varepsilon^{-1} \nabla \operatorname{div} \varepsilon u + \dots\right] + \dots \\ &= (\text{see } (2.4)) = (\varepsilon\mu)^{-1} \Delta u + \dots \end{aligned} \tag{2.8}$$

(low order terms are omitted); therefore, Eq. (2.5) turns out to be *hyperbolic.*

2.3. Spaces and operators. The space of controls $\mathcal{F}^T := \{f \in L_2(\Sigma^T;\mathbb{R}^3)\big| f\cdot\nu\big|_\Gamma = 0\}$ is called *the outer space* of system (2.5)–(2.7). It contains a family of subspaces $\mathcal{F}^{T,\xi} := \{f \in \mathcal{F}^T \mid f(\,\cdot\,,t) = 0,\ 0 \le t \le T-\xi\}$ $(0 \le \xi \le T)$ formed by delayed controls.

The space $\mathcal{H} := \{y \in L_2(\Omega;\mathbb{R}^3) \mid \operatorname{div}\varepsilon y = 0\}$ with inner product

$$(u,y)_{\mathcal{H}} = \int\limits_{\Omega} dx \varepsilon(x)\, u(x)\, y(x)$$

is said to be *the inner space.* There is a family of subspaces $\mathcal{H}^\xi := \{y \in \mathcal{H} \mid \operatorname{supp} y \subset \overline{\Omega}^{\,\xi}\}$ $(0 \le \xi \le T)$ in it.

The "input→state" correspondence in the electric subsystem is realized by the *control operator* $W^T : \mathcal{F}^T \to \mathcal{H}$, $W^T f := u^f(\,\cdot\,,T)$.
By the hyperbolicity noted above, we have

$$W^T \mathcal{F}^{T,\xi} \subset \mathcal{H}^\xi \quad (0 \le \xi \le T). \tag{2.9}$$

The "input→output" correspondence is associated with *the response operator* $R^T : \mathcal{F}^T \to \mathcal{F}^T$, $\operatorname{Dom} R^T = \mathcal{M}^T$, $R^T f := (\nu \times \frac{1}{\mu}\text{rot } u^f)\big|_{\Sigma^T}$.

The operator $C^T : \mathcal{F}^T \to \mathcal{F}^T$, $C^T := (W^T)^* W^T$ is called *the connecting operator.* By definitions, we have

$$\big(u^f(\,\cdot\,,T),\, u^g(\,\cdot\,,T)\big)_{\mathcal{H}} = (C^T f, g)_{\mathcal{F}^T},$$

i.e., C^T connects the metrics of the inner and outer spaces.

The standard technique of the BC-method using A. Blagovestchenskii's relations enables us to express the connecting operator in terms of the response operator. Let R^{2T} be the response operator associated with system (2.5)–(2.7) with doubled final moment; we note that, by hyperbolicity, the operator R^{2T} is determined by the values of the permittivities ε, μ in Ω^T and is independent of the behavior of them in $\Omega \setminus \Omega^T$.

Theorem 1. *For $f \in \mathcal{F}^T \cap C_0^\infty(\Sigma^T;\mathbb{R}^3)$ the representation*

$$C^T f = \frac{1}{2}\,(S^T)^*\, R^{2T}\, J^{2T}\, S^T f \tag{2.10}$$

is valid, in which $S^T : \mathcal{F}^T \to \mathcal{F}^{2T}$ is the operator of odd continuation,

$$(S^T f)(\,\cdot\,,t) = \begin{cases} f(\,\cdot\,,t), & 0 \le t < T; \\ -f(\,\cdot\,,2T-t), & T \le t \le 2T, \end{cases}$$

and $J^{2T} : \mathcal{F}^{2T} \to \mathcal{F}^{2T}$ is the operator of integration,

$$(J^{2T} f)(\,\cdot\,,t) = \int\limits_0^t ds\, f(\,\cdot\,,s), \quad 0 \le t \le 2T.$$

The proof follows the line of arguments of [2; Theorem 3.1], using the Green's formula

$$(Lu,y)_{\mathcal{H}} - (u,Ly)_{\mathcal{H}} = \int\limits_{\Gamma} d\Gamma \left[(\nu \times \frac{1}{\mu}\text{rot } u)\cdot y_\theta - u_\theta \cdot (\nu \times \frac{1}{\mu}\text{rot } y)\right].$$

2.4. The dual system. The system

$$v_{tt} - Lv = 0 \quad \text{in} \quad Q^T; \tag{2.11}$$

$$v\big|_{t=T} = 0, \quad v_t\big|_{t=T} = y \in \mathcal{H}; \tag{2.12}$$

$$v_\theta\big|_{\Sigma^T} = 0 \tag{2.13}$$

is said to be *dual* to system (2.5)–(2.7). The solution $v = v^y(x,t)$ of this system is solenoidal:

$$\operatorname{div} \varepsilon v(\,\cdot\,, t) = 0, \quad 0 \le t \le T;$$

for this reason, the evolution of the system is governed by the hyperbolic equation

$$v_{tt} - (\varepsilon\mu)^{-1}\,\Delta v + \ldots = 0 \tag{2.14}$$

(see (2.8)).

The *operator of observation* $O^T : \mathcal{H} \to \mathcal{F}^T$, $O^T\, y := (\nu \times \frac{1}{\mu} \operatorname{rot}\, v^y)\big|_{\Sigma^T}$, may be associated with system (2.11)–(2.13).

Proposition 2. *For any $T > 0$, the relation*

$$O^T = (W^T)^* \tag{2.15}$$

holds.

This relation is established by integration by parts.

2.5. Controllability. The stationary state property of system (2.5)–(2.7) (the independence of ε, μ from time) implies

$$W^T\, \mathcal{F}^{T,\xi} = \left\{ u^f(\,\cdot\,, \xi) \mid f \in \mathcal{F}^T \right\} \quad (0 \le \xi \le T).$$

The set of states $\mathcal{U}^\xi := W^T\, \mathcal{F}^{T,\xi}$ lying in $\mathcal{H}^\xi$ (see (2.9)) is called *reachable* (at the moment $t = \xi$).

Theorem 2. *If a subdomain Ω^ξ satisfies the EP-condition, then the relation*

$$\operatorname{clos}_{\mathcal{H}} \mathcal{U}^\xi = \mathcal{H}^\xi \tag{2.16}$$

is valid.

This result may be established in accordance with the D. Russell's scheme (see [2, Theorem 4.1]). Omitting the proof, we only clarify the role of the EP-condition.

Assuming that $y \in \mathcal{H}^\xi \ominus \operatorname{clos} \mathcal{U}^\xi$ and using the vector generalization of the Holmgren–John–Tataru uniqueness theorem for Eq. (2.14) (see [5]), we can show that the field y satisfies all the conditions (i)–(iv) (see Proposition 1), which implies $y = 0$ and leads to Eq. (2.16).

In terms of control theory, property (2.16) means that the electric subsystem (2.5)–(2.7) is *approximately controllable* in the subdomain Ω^ξ captured by waves at the moment $t = \xi$.

2.6. An example due to V. Osmolovskii. The EP-condition is sufficient to establish property (2.16). If, for instance, Ω^ξ is homeomorphic to a ball from which a torus is removed, then condition (β) is violated. Do there exist nontrivial fields y satisfying (i)–(iv) of Proposition 1 in this case? As the following example due to V. Osmolovskii shows, such a possibility should not be excluded.

Let K, K_1, and K_2 be open circles on a plane $\{x^1, x^2\}$ such that $\overline{K}_{1,2} \subset K$, $\overline{K}_1 \cap \overline{K}_2 = \{\varnothing\}$; denote $D^\xi := K \setminus [\,\overline{K}_1 \cup \overline{K}_2\,]$. Let $\Phi = \Phi(x^1, x^2)$ be a solution of the problem

$$\Delta\Phi = 0 \quad \text{in} \quad D^\xi;$$

$$\Phi\big|_{\partial K_1} = c_1; \quad \Phi\big|_{\partial K_2} = c_2; \quad \frac{\partial \Phi}{\partial \nu}\bigg|_{\partial K} = 0,$$

with constants $c_{1,2}$ such that $c_1 c_2 \neq 0$, $c_1 \neq c_2$.

In the cylindrical domain $\Omega^\xi := D^\xi \times \mathbb{R}^1 \subset \mathbb{R}^3$, the function Φ determines a field

$$y = y(x^1, x^2, x^3) = \frac{\partial \Phi(x^1, x^2)}{\partial x^2} e_1 - \frac{\partial \Phi(x^1, x^2)}{\partial x^1} e_2 + 0 \cdot e_3$$

(e_i are the coordinate unit vectors). One can show that y satisfies (i)–(iv) (with $\varepsilon = 1$).

The authors do not know whether such a situation occurs in a *bounded* $\Omega \subset \mathbb{R}^3$. A positive answer would mean the existence of a physical effect: the lack of the controllability of system (2.5)–(2.7) (at the moment $t = \xi$) for "topological" reasons.

3. Visualization

3.1. The Weyl projectors. In the inner space of system (2.5)–(2.7), we consider the (orthogonal) *Weyl projectors* G^ξ projecting in $\mathcal{H}$ onto $\mathcal{H}^\xi$; denote $G^\xi_\perp := \mathbb{1}_\mathcal{H} - G^\xi$ ($\xi \geq 0$). The following result describes how they act.

Proposition 3. *For smooth $y \in \mathcal{H}$, the following representations are valid:*

$$(G^\xi\, y)(x) = \begin{cases} y(x) - \nabla p^\xi(x), & x \in \Omega^\xi; \\ 0, & x \in \Omega \setminus \Omega^\xi, \end{cases}$$

where $p = p^\xi$ is a solution of the problem

$$\operatorname{div} \varepsilon \nabla p = 0 \quad \textit{in} \quad \Omega^\xi; \tag{3.1}$$

$$p\big|_\Gamma = 0; \tag{3.2}$$

$$\nu \cdot \nabla p\big|_{\Gamma^\xi} = \nu \cdot y \tag{3.3}$$

and

$$(G^\xi_\perp\, y)(x) = \begin{cases} \nabla p^\xi(x), & x \in \Omega^\xi; \\ y(x), & x \in \Omega \setminus \Omega^\xi. \end{cases}$$

Note that projecting leads to the appearance of a discontinuity on Γ^ξ: at the points $x(\gamma, \xi) \in \Gamma^\xi \setminus \omega$ we have

$$(G^\xi_\perp\, y)(x(\gamma, \xi + 0)) - (G^\xi_\perp\, y)(x(\gamma, \xi - 0)) = (G^\xi\, y)(x(\gamma, \xi - 0))$$
$$= y(x(\gamma, \xi)) - \nabla p^\xi(x(\gamma, \xi - 0)) = (\text{see}\,(3.3)) = y_\theta(x(\gamma, \xi)) - (\nabla p^\xi)_\theta(x(\gamma, \xi - 0)), \tag{3.4}$$

i.e., a discontinuity proves out to be a *tangential* field on Γ^ξ.

3.2. The operator M^T. Let $\mathcal{L}_\theta := \{y \in L_2(\Omega;\mathbb{R}^3) \mid y = y_\theta\}$ be a space of *transversal* fields; denote $\mathcal{L}_\theta^T := \{y \in \mathcal{L}_\theta \mid \operatorname{supp} y \subset \overline{\Omega}^T\}$.

Define the operator $M^T : \mathcal{H}^T \to \mathcal{L}_\theta^T$ acting on smooth fields y by the following rule: if $x \in \Omega^T \setminus \omega$, $x = x(\gamma,\xi)$, then

$$(M^T y)(x) = (G^\xi y)(x(\gamma,\tau))\Big|_{\tau=\xi-0}^{\tau=\xi+0} = (\text{see (3.4)}) = y_\theta(x(\gamma,\xi)) - (\nabla p^\xi)_\theta(x(\gamma,\xi-0)).$$

One can establish the following properties of M^T.

Proposition 4. *The operator M^T is continuous; the relation*

$$M^T G^\tau = X^\tau M^T \quad (0 \le \tau \le T)$$

is valid, where X^τ is the cut-off projector in $\mathcal{L}_\theta^T$:

$$(X^\tau y)(x) = \chi(\tau - \tau(x))\, y(x), \quad x \in \Omega,$$

where $\chi(\dots)$ is the Heavyside function.

3.3. Wave projectors. In the inner space of system (2.5)–(2.7), consider a family of *wave projectors* P^ξ projecting in $\operatorname{clos}\mathcal{U}^T$ onto $\operatorname{clos}\mathcal{U}^\xi$; denote $P_\perp^\xi := P^T - P^\xi$ $(0 \le \xi \le T)$.

In the outer space, we introduce a family of operators $\Pi^{T,\xi} : \mathcal{F}^T \to \mathcal{F}^T$,

$$\Pi^{T,\xi} := (W^T)^* P_\perp^\xi W^T \quad (0 \le \xi \le T). \tag{3.5}$$

As the following results show, the operators $\Pi^{T,\xi}$ are determined by the connecting operator. Let $(C^T)^{1/2}$ be a positive square root of C^T; consider the projector $P_{\frac12}^\xi$ in $\mathcal{F}^T$ onto the subspace $\operatorname{clos}(C^T)^{1/2}\mathcal{F}^{T,\xi}$; denote $P_{\frac12\perp}^\xi := \mathbb{I}_{\mathcal{F}^T} - P_{\frac12}^\xi$.

Proposition 5. *For any $T > 0$ and $\xi \in [0,T]$, the following representation is valid:*

$$\Pi^{T,\xi} = (C^T)^{1/2} P_{\frac12\perp}^\xi (C^T)^{1/2}. \tag{3.6}$$

The proof is quite similar to that given in [2; Secs. 5.6, 5.7].

Assume that for all $\xi \in (0,T]$, the subdomains Ω^ξ satisfy the EP-condition. In this case, relation (2.16) is valid, which implies the relation

$$P^\xi = G^\xi \quad (0 \le \xi \le T). \tag{3.7}$$

Thus, the controllability of system (2.5)–(2.7) enables us to describe efficiently the action of the wave projectors (see Proposition 3).

The latter relation leads to the representation

$$O^T G_\perp^\xi W^T = (\text{see (2.15)}) = (W^T)^* G_\perp^\xi W^T = (\text{see (3.5), (3.7)}) = \Pi^{T,\xi} \quad (0 \le \xi \le T), \tag{3.8}$$

which is required in the sequel.

3.4. The visualizing operator. The mapping $V^T : \mathcal{F}^T \to \mathcal{F}^T$, $V^T := I^T\, M^T\, W^T$, associated with system (2.5)–(2.7), is called *the visualizing operator*. By virtue of the inclusion $I^T\, \mathcal{L}^T_\theta \subset \mathcal{F}^T$ (see Sec. 1.5), the operator V^T is well defined; $\operatorname{Ran} V^T \subset \mathcal{F}^T_\Theta := \{f \in \mathcal{F}^T \mid \operatorname{supp} f \subset \Theta^T\}$.

The remaining part of the paper is devoted to the proof of the following central result.

Theorem 3. *Let subdomains Ω^ξ satisfy the EP-condition for all $\xi \in (0, T]$, and let ε, μ be known on Γ; then the operator V^T is determined by the operator R^{2T}.*

3.5. Propagation of discontinuities. Turning to system (2.11)–(2.13), we fix $\xi \in (0, T)$ and put $v_t\big|_{t=T} = G^\xi_\perp\, y$ in (2.12) (for smooth $y \in \mathcal{H}$); let v^y_ξ be the corresponding solution. The new data are discontinuous on Γ^ξ (see (3.4)), which implies the discontinuity of the solution v^y_ξ propagating (in inverted time) along bicharacteristics and interacting with the boundary at the moment $t = T - \xi$. The evolution of the discontinuity is described by known relations of geometric optics (Rytov's law); the amplitude of its interaction with the boundary can be derived by means of the ray method (see [1; Chap. I, §5]):

$$\lim_{t\to T-\xi-0} \left(\nu \times \frac{1}{\mu}\operatorname{rot}\, v^y_\xi\right)\Big|_{\Sigma^T} (\gamma, t)$$

$$= \begin{cases} \eta(\gamma)\beta^{1/2}(\gamma,\xi)\left[(G^\xi_\perp\, y)(x(\gamma,\xi))\Big|_{\tau=\xi-0}^{\tau=\xi+0}\right]^\wedge, & (\gamma,\xi) \in \Theta^T, \\ 0, & (\gamma,\xi) \in \Sigma^T \setminus \overline{\Theta}^T, \end{cases} \tag{3.9}$$

where $\eta := (\varepsilon/\mu)^{1/4}\big|_\Gamma$.

3.6. The amplitude formula. Recalling the definitions of the operators O^T, I^T, and M^T, and taking into account (1.2), relation (3.9) can be written as follows:

$$(O^T\, G^\xi_\perp\, y)(\gamma, T - \xi - 0) = \eta(\gamma)(I^T\, M^T\, y)(\gamma,\xi) \quad \text{a.e.} \quad (\gamma,\xi) \in \Sigma^T. \tag{3.10}$$

Choose a control $f \in \mathcal{M}^T$ and put $y = u^f(\,\cdot\,, T) = W^T f$ in (3.10), so that $I^T\, M^T\, y = I^T\, M^T\, W^T\, f = V^T\, f$. By the EP-condition on Ω^ξ, we obtain

$$O^T\, G^\xi_\perp\, u^f(\,\cdot\,, T) = (W^T)^*\, G^\xi_\perp\, W^T\, f = (\text{see}\,(3.8)) = \Pi^{T,\xi}\, f. \tag{3.11}$$

Taking into account (3.10), (3.11), relation (3.9) of geometric optics can be transformed to the final form

$$(V^T\, f)(\gamma,\xi) = \eta^{-1}(\gamma)\left(\Pi^{T,\xi} f\right)(\gamma, T - \xi - 0) \quad \text{a.e.} \quad (\gamma,\xi) \in \Sigma^T.$$

This representation of V^T is called the *amplitude formula* (AF).

3.7. Proof of Theorem 3. Return to Theorem 3. Given R^{2T}, one can find the operator C^T (see (2.10)), which, in turn, determines the family $\Pi^{T,\xi}$ (see (3.6)). Possessing $\Pi^{T,\xi}$ and using the AF, one can restore the operator V^T. The theorem is proved. $\square$

If $\xi \le T < T_\omega$, the EP-condition is always satisfied; consequently, there are no problems with the reconstruction of V^T by R^{2T} for these moments of time.

4. Conclusion

4.1. Comparing the results of [2] with the results obtained in this paper, we observe an interesting feature that distinguishes a system governed by the wave equation from that governed by the Maxwell equations. Whereas in the first system one can restore the images of the waves themselves: $V^T f = I^T u^f(\,\cdot\,, T)$, in the Maxwell system one can visualize only the waves transformed by the operator $M^T : V^T f = I^T M^T u^f(\,\cdot\,, T)$.

4.2. The reconstruction of the visualizing operator by a response operator is only the first step toward the solution of the inverse problem that consists of restoring the permittivities ε, μ in Ω^T by a given R^{2T}. In accordance with the scheme of the BC-method, the next step is the construction of the operator $\widetilde{L}$ defined by the relation

$$V^T \frac{\partial^2}{\partial t^2} = \widetilde{L}\, V^T;$$

after that, the information about the ρ-metric should be extracted from $\widetilde{L}$ (see [2]).

4.3. Another possible way is to use the Gaussian beams (quasiphotons) as was done in [3]. There is reason to believe that the action of the operator M^T on a beam does not violate the character of localization of the beam. If this conjecture is true, then the visualization of beams on a pattern enables us to endow Θ^T with a metric that converts it into an isometrical copy of the subdomain Ω^T (with ρ-metric) (see [2]).

4.4. As is usual in the BC-method, the results obtained for dynamical problems can easily be adopted to the spectral ones: if the EP-condition is satisfied for any $T > 0$, then adequately-defined spectral data of the operator L (see (2.8)) determine the visualizing operator V^T.

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