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Discriminant set for the scaling limits in the third Painlevé transcendent

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ABSTRACT

The scaling limits in the third Painlevé transcendent are considered from the point of view of the associated with P3 linear equation. The discriminant set for the scaling limits with the reductions of the third Painlevé equation to the second Painlevé equation and one-parametric reductions to the first Painlevé equation are described.

Zero curvature representation.

The Lax pair for the third Painlevé equation P3 [1]

$$y'' = \frac{(y')^2}{y} - \frac{y'}{x} + \frac{1}{x}(ay^2 + b) + cy^3 + \frac{d}{y^3} \tag{1}$$

is found by Jimbo and Miwa [2]. Their Lax pair

$$\frac{\partial \Psi}{\partial \lambda} = A\,, \quad \frac{\partial \Psi}{\partial x} = U\,, \tag{2}$$

with the rational in λ matrices

$$A = \Big(\frac{x}{2} + \frac{\alpha}{\lambda} + \frac{z}{\lambda^2}\Big)\sigma_3 + \Big(\frac{u}{\lambda} - \frac{w}{\lambda^2}\big(z + \frac{x}{2}\big)\Big)\sigma_+ + \Big(\frac{v}{\lambda} + \frac{1}{w\lambda^2}\big(z - \frac{x}{2}\big)\Big)\sigma_-\,, \tag{3}$$

$$U = \Big(\frac{\lambda}{2} - \frac{z}{x\lambda}\Big)\sigma_3 + \frac{1}{x}\Big(u + \frac{w}{\lambda}\big(z + \frac{x}{2}\big)\Big)\sigma_+ + \frac{1}{x}\Big(v - \frac{1}{w\lambda}\big(z - \frac{x}{2}\big)\Big)\sigma_-\,, \tag{4}$$

where

$$u = \frac{w}{y}\big(z + \frac{x}{2}\big), \quad v = \frac{1}{wy}\big(z - \frac{x}{2}\big) + \frac{2}{w}\,\frac{\alpha z + \beta\frac{x}{2}}{z + \frac{x}{2}}\,, \tag{5}$$

is compatible if

$$\alpha = const\,, \quad \beta = -\frac{1}{x}\Big(2\alpha z + \frac{u}{w}\big(z - \frac{x}{2}\big) - vw\big(z + \frac{x}{2}\big)\Big) = const\,, \tag{6}$$

and the system holds

$$\begin{aligned} xz' &= 4\frac{z^2}{y} + (4\alpha + 1)z - \frac{x^2}{y} + 2\beta x\,, \\ xy' &= 2xy^2 + (4\alpha + 1)y + 4z\,, \\ x\frac{w'}{w} &= \frac{1}{y(z + \frac{x}{2})}\big(-2\alpha yz + (\alpha - 2\beta)xy + x^2 + 2xz\big)\,, \end{aligned} \tag{7}$$

Excluding z from the system (7), one obtains the third Painlevé equation (1)

$$y'' = \frac{(y')^2}{y} - \frac{y'}{x} + \frac{1}{x}\big(4(2\alpha + 1)y^2 + 8\beta\big) + 4y^3 - \frac{4}{y} \tag{8}$$

with the parameters

$$a = 8\big(\alpha + \frac{1}{2}\big)\,, \quad b = 8\beta\,, \quad c = 4\,, \quad d = -4\,.$$

Note that the original P3 equation (1) with the nontrivial parameters c and d is equivalent to the equation (8). Indeed, the scaling change of variables in (1)

$$y = pz\,, \quad x = qt\,, \tag{9}$$

with the parameters p, q satisfying the conditions

$$pq = \frac{2\varepsilon}{\sqrt{c}}\,, \quad \frac{q}{p} = \frac{2\sigma}{\sqrt{-d}}\,, \quad \varepsilon^2 = \sigma^2 = 1\,,$$

yield the equation (8)

$$z'' = \frac{(z')^2}{z} - \frac{z'}{t} + \frac{1}{t}\big(2\varepsilon\,\frac{a}{\sqrt{c}}\,z^2 + 2\sigma\,\frac{b}{\sqrt{-d}}\big) + 4z^3 - \frac{4}{z}\,.$$

Monodromy data parametrization.

The monodromy data of the linear system associated with the third Painlevé equation P3 (8) are described in [3, ?]. The system with rational coefficients (2), (3) with two irregular singular points at the infinity and at the origin allows one to introduce the following canonical solutions:

$$\Psi_k^\infty(\lambda) = \Big(I + \mathcal{O}(\lambda^{-1})\Big) e^{(\frac{x\lambda}{2} + \alpha \ln \lambda)\sigma_3}, \quad \lambda \to \infty, \quad \lambda \in \Omega_k^\infty, \tag{10}$$

where the canonical sectors near the infinity are

$$\Omega_k^\infty = \big\{\lambda \in \mathbb{C} : \arg\lambda \in \big(-\frac{\pi}{2} - \arg x + \pi(k-1); \frac{3\pi}{2} - \arg x + \pi(k-1)\big)\big\},$$

and

$$\Psi_k^0(\lambda) = B\big(I + \mathcal{O}(\lambda)\big) e^{-(\frac{x}{2\lambda} + \beta \ln \lambda)\sigma_3}, \quad \lambda \to 0, \quad \lambda \in \Omega_k^0, \tag{11}$$

$$B = \begin{pmatrix} \sqrt{\frac{w}{x}}\big(z + \frac{x}{2}\big) & \sqrt{\frac{w}{x}} \\ \frac{1}{\sqrt{wx}}\big(z - \frac{x}{2}\big) & \frac{1}{\sqrt{wx}} \end{pmatrix} e^{g(x)\sigma_3}, \quad g(x) = -\int^x \Big(\frac{\beta + \frac{1}{2}}{x} + \frac{2\alpha z}{x^2} + \frac{zw'}{xw}\Big)\, dx,$$

where the canonical sectors near the point zero are

$$\Omega_k^0 = \big\{\lambda \in \mathbb{C} : \arg\lambda \in \big(-\frac{3\pi}{2} + \arg x + \pi(k-1); \frac{\pi}{2} + \arg x + \pi(k-1)\big)\big\},$$

All the canonical solutions differ from each other in some right multipliers:

$$\Psi_{k+1}^\infty(\lambda) = \Psi_k^\infty(\lambda)\mathsf{S}_k, \quad \mathsf{S}_{2k-1} = \begin{pmatrix} 1 & s_{2k-1} \\ 0 & 1 \end{pmatrix}, \quad \mathsf{S}_{2k} = \begin{pmatrix} 1 & 0 \\ s_{2k} & 1 \end{pmatrix},$$

$$\mathsf{S}_{k+2} = e^{-2\pi i\alpha\sigma_3}\mathsf{S}_k e^{2\pi i\alpha\sigma_3}, \quad s_{k+2} = s_k e^{(-1)^k 4\pi i\alpha}, \tag{12}$$

$$\Psi_{k+1}^0(\lambda) = \Psi_k^0(\lambda)\mathsf{G}_k, \quad \mathsf{G}_{2k-1} = \begin{pmatrix} 1 & g_{2k-1} \\ 0 & 1 \end{pmatrix}, \quad \mathsf{G}_{2k} = \begin{pmatrix} 1 & 0 \\ g_{2k} & 1 \end{pmatrix},$$

$$\mathsf{G}_{k+2} = e^{2\pi i\beta\sigma_3}\mathsf{G}_k e^{-2\pi i\beta\sigma_3}, \quad g_{k+2} = g_k e^{(-1)^{k+1} 4\pi i\beta}, \tag{13}$$

$$\Psi_1^\infty(\lambda) = \Psi_1^0(\lambda)E, \quad E = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det E = 1. \tag{14}$$

The Stokes matrices S_k, G_k and the connection matrix E are not independent from each other. Comparing the monodromy matrices associated with the point zero and infinity

$$\Psi_3^\infty\big(e^{2\pi i}\lambda\big) = \Psi_1^\infty(\lambda)e^{2\pi i\alpha\sigma_3} = \Psi_1^0(\lambda)Ee^{2\pi i\alpha\sigma_3} = \Psi_{-1}^0(\lambda)\mathsf{G}_{-1}\mathsf{G}_0 Ee^{2\pi i\alpha\sigma_3},$$

$$\Psi_3^\infty\big(e^{2\pi i}\lambda\big) = \Psi_1^\infty\big(e^{2\pi i}\lambda\big)\mathsf{S}_1\mathsf{S}_2 = \Psi_1^0\big(e^{2\pi i}\lambda\big)E\mathsf{S}_1\mathsf{S}_2 = \Psi_{-1}^0(\lambda)e^{-2\pi i\beta\sigma_3}E\mathsf{S}_1\mathsf{S}_2,$$

and using (13), one can evaluate the so-called cyclic relation

$$E\mathsf{S}_1\mathsf{S}_2 = \mathsf{G}_1\mathsf{G}_2 e^{2\pi i\beta\sigma_3}Ee^{2\pi i\alpha\sigma_3}. \tag{15}$$

The matrix equation (15) is equivalent to the system of the scalar algebraic equations

$$a\big(1 + s_1 s_2 - e^{2\pi i(\alpha+\beta)}(1 + g_1 g_2)\big) + bs_2 - ce^{2\pi i(\alpha-\beta)}g_1 = 0,$$

$$as_1 + b\big(1 - e^{-2\pi i(\alpha-\beta)}(1 + g_1 g_2)\big) - de^{-2\pi i(\alpha+\beta)}g_1 = 0,$$

$$-ae^{2\pi i(\alpha+\beta)}g_2 + c\big(1 + s_1 s_2 - e^{2\pi i(\alpha-\beta)}\big) + ds_2 = 0,$$

$$-be^{-2\pi i(\alpha-\beta)}g_2 + cs_1 + d\big(1 - e^{-2\pi i(\alpha+\beta)}\big) = 0, \tag{16}$$

which should be supplemented by the condition $\det E = 1$. Treating (16) as the linear algebraic system for the entries a, b, c, d of the connection matrix E with the normalization condition $ad - bc = 1$, one can see that it is solvable when the corresponding determinant is trivial. The solvability condition is as follows:

$$\Big(e^{2\pi i\beta}(1+g_1g_2) - e^{-2\pi i\alpha}(1+s_1s_2)\Big)^2 + 2\big(1-e^{-2\pi i(\alpha+\beta)}\big)(1+s_1s_2) +$$
$$+2\big(1-e^{2\pi i(\alpha+\beta)}\big)(1+g_1g_2) + \big(e^{2\pi i\alpha} - e^{-2\pi i\beta}\big)^2 = 0\,. \tag{17}$$

The equation (17) describes the 3d-monodromy surface in the 4d-space of the Stokes multipliers. As easy to check the surface has the special points described by the equations

$$\alpha + \beta = n \in \mathbb{Z}\,, \quad s_1s_2 = g_1g_2\,. \tag{18}$$

If the point of the monodromy surface (17) is not special, the parameters a, b, c, d, $ad - bc = 1$, are uniquely determined by the values of the Stokes multipliers. If the point of the monodromy surface satisfies the conditions (18) the connection matrix E contains some essential free parameters.

Bäcklund and Schlesinger transformations.

Besides the isomonodromy deformation with respect to the continuous complex parameter x, one can introduce another (discrete) isomonodromy deformation with respect to parameters represented by the exponents of the formal monodromy α and β. Indeed, let us consider next Schlesinger transformations [3, ?] $\tilde\Psi = R\Psi$ with the matrices $R = \lambda^{1/2}R_1 + \lambda^{-1/2}R_0$ which change the parameters α, β in half-integers and obviously preserve the connection and Stokes matrices:

i) the first of the Schlesinger transformations is given by the matrix

$$R = \lambda^{\frac12}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \lambda^{-\frac12}\begin{pmatrix}-\frac{u}{xw} & \frac{u}{x}\\ -\frac{1}{w} & 1\end{pmatrix} \tag{19}$$

and transforms the matrix $\mathcal{A}$ in accord with the equation $\mathcal{A} = RAR^{-1} + R_\lambda R^{-1}$, so that the new parameter functions are as follows:

$$\tilde\alpha = \alpha + \frac12\,, \quad \tilde\beta = \beta + \frac12\,,$$
$$\tilde u = -(2\alpha+1)\frac{u}{x} - w\big(z + \frac{x}{2}\big) - \frac{u^2}{xw}\,, \quad \tilde v = -\frac{x}{w}\,,$$
$$\tilde z = -\frac{x}{2} - \frac{2\alpha u}{xw} - \frac{u^2}{xw^2} + \frac{uv}{x}\,,$$
$$\tilde w\big(\tilde z + \frac{x}{2}\big) = -u\Big(1 + \frac{2\alpha u}{x^2 w} + \frac{u^2}{x^2w^2} - \frac{uv}{x^2}\Big)\,,$$
$$\frac{1}{\tilde w}\big(\tilde z - \frac{x}{2}\big) = v - \frac{2\alpha}{w} - \frac{u}{w^2}$$

so that the Bäcklund transformation of the Painlevé transcendent is given by

$$\tilde y = \frac{1}{y} - \frac{\alpha+\beta+1}{\frac{x}{4}y' + \frac{x}{2}y^2 + (\alpha+\frac34)y + \frac{x}{2}}\,; \tag{20}$$

ii) the second of the Schlesinger transformations is described by the matrix

$$R = \lambda^{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \lambda^{-\frac{1}{2}} \begin{pmatrix} -\frac{u}{xw}\frac{z-\frac{x}{2}}{z+\frac{x}{2}} & \frac{u}{x} \\ -\frac{1}{w}\frac{z-\frac{x}{2}}{z+\frac{x}{2}} & 1 \end{pmatrix} \tag{21}$$

so that the new parameter functions are as follows:

$$\begin{aligned}
&\tilde{\alpha} = \alpha + \frac{1}{2}, \quad \tilde{\beta} = \beta - \frac{1}{2}, \\
&\tilde{u} = -(2\alpha+1)\frac{u}{x} - w\big(z + \frac{x}{2}\big) - \frac{u^2}{xw}\,\frac{z-\frac{x}{2}}{z+\frac{x}{2}}, \quad \tilde{v} = -\frac{x}{w}\,\frac{z-\frac{x}{2}}{z+\frac{x}{2}}, \\
&\tilde{z} = \frac{x}{2} - \frac{2\alpha u}{xw}\,\frac{z-\frac{x}{2}}{z+\frac{x}{2}} - \frac{u^2}{xw^2}\Big(\frac{z-\frac{x}{2}}{z+\frac{x}{2}}\Big)^2 + \frac{uv}{x}, \\
&\tilde{w}\big(\tilde{z} + \frac{x}{2}\big) = u\Big(1 - \frac{2\alpha u}{x^2 w}\,\frac{z-\frac{x}{2}}{z+\frac{x}{2}} - \frac{u^2}{x^2w^2}\Big(\frac{z-\frac{x}{2}}{z+\frac{x}{2}}\Big)^2 + \frac{uv}{x^2}\Big), \\
&\frac{1}{\tilde{w}}\big(\tilde{z} - \frac{x}{2}\big) = v - \frac{2\alpha}{w}\,\frac{z-\frac{x}{2}}{z+\frac{x}{2}} - \frac{u}{w^2}\Big(\frac{z-\frac{x}{2}}{z+\frac{x}{2}}\Big)^2,
\end{aligned}$$

so that the Bäcklund transformation of the Painlevé transcendent is given by

$$\tilde{y} = -\frac{1}{y} + \frac{\alpha - \beta + 1}{\frac{x}{4}\,y' + \frac{x}{2}\,y^2 + (\alpha + \frac{3}{4})y - \frac{x}{2}}; \tag{22}$$

iii) the third of the Schlesinger transformations is as

$$R = \lambda^{\frac{1}{2}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \lambda^{-\frac{1}{2}} \begin{pmatrix} 1 & -w\frac{z+\frac{x}{2}}{z-\frac{x}{2}} \\ -\frac{v}{x} & \frac{vw}{x}\frac{z+\frac{x}{2}}{z-\frac{x}{2}} \end{pmatrix}, \tag{23}$$

so that the new parameter functions are as follows:

$$\begin{aligned}
&\tilde{\alpha} = \alpha - \frac{1}{2}, \quad \tilde{\beta} = \beta - \frac{1}{2}, \\
&\tilde{u} = xw\,\frac{z+\frac{x}{2}}{z-\frac{x}{2}}, \quad \tilde{v} = -(2\alpha-1)\frac{v}{x} + \frac{1}{w}\big(z - \frac{x}{2}\big) + \frac{v^2 w}{x}\,\frac{z+\frac{x}{2}}{z-\frac{x}{2}}, \\
&\tilde{z} = -\frac{x}{2} + \frac{2\alpha vw}{x}\,\frac{z+\frac{x}{2}}{z-\frac{x}{2}} - \frac{v^2w^2}{x}\Big(\frac{z+\frac{x}{2}}{z-\frac{x}{2}}\Big)^2 + \frac{uv}{x}, \\
&\tilde{w}\big(\tilde{z} + \frac{x}{2}\big) = -u - 2\alpha w\,\frac{z+\frac{x}{2}}{z-\frac{x}{2}} + vw^2\Big(\frac{z+\frac{x}{2}}{z-\frac{x}{2}}\Big)^2, \\
&\frac{1}{\tilde{w}}\big(\tilde{z} - \frac{x}{2}\big) = v\Big(1 - \frac{2\alpha vw}{x^2}\,\frac{z+\frac{x}{2}}{z-\frac{x}{2}} + \frac{v^2w^2}{x^2}\Big(\frac{z+\frac{x}{2}}{z-\frac{x}{2}}\Big)^2 - \frac{uv}{x^2}\Big),
\end{aligned}$$

so that the Bäcklund transformation of the Painlevé transcendent is given by

$$\tilde{y} = \frac{1}{y} + \frac{\alpha + \beta}{\frac{x}{4}\,y' - \frac{x}{2}\,y^2 - (\alpha + \frac{1}{4})y - \frac{x}{2}}; \tag{24}$$

iv) the fourth of the Schlesinger transformations corresponds to the matrix

$$R = \lambda^{\frac{1}{2}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \lambda^{-\frac{1}{2}} \begin{pmatrix} 1 & -w \\ -\frac{v}{x} & \frac{vw}{x} \end{pmatrix} \tag{25}$$

so that the new parameter functions are as follows:

$$\begin{aligned}
&\tilde{\alpha} = \alpha - \frac{1}{2}\,, \quad \tilde{\beta} = \beta + \frac{1}{2}\,, \\
&\tilde{u} = xw\,, \quad \tilde{v} = -(2\alpha-1)\frac{v}{x} + \frac{1}{w}\big(z - \frac{x}{2}\big) + \frac{v^2 w}{x}\,, \\
&\tilde{z} = \frac{x}{2} + \frac{2\alpha v w}{x} - \frac{v^2 w^2}{x} + \frac{uv}{x}\,, \\
&\tilde{w}\big(\tilde{z} + \frac{x}{2}\big) = -u - 2\alpha w + v w^2\,, \\
&\frac{1}{\tilde{w}}\big(\tilde{z} - \frac{x}{2}\big) = v\Big(-1 - \frac{2\alpha v w}{x^2} + \frac{v^2 w^2}{x^2} - \frac{uv}{x^2}\Big)\,,
\end{aligned}$$

so that the Bäcklund transformation of the Painlevé transcendent is given by

$$\tilde{y} = -\frac{1}{y} - \frac{\alpha - \beta}{\frac{x}{4}\, y' - \frac{x}{2}\, y^2 - \big(\alpha + \frac{1}{4}\big) y + \frac{x}{2}}\,. \tag{26}$$

It is interesting that the Bäcklund transformations found earlier in [5] are superpositions of some pairs of the transformations presented above accompanied by the elementary changes $y \mapsto -y$ and $y \mapsto 1/y$.

So, introducing the chain $\Psi_n(\lambda, x)$ in accord with the definition of the Schlesinger transformation $\Psi_{n+1} \equiv \tilde{\Psi}_n$, one obtains the total system with the matrices A_n, U_n defined as in (3), (4) and R_n given by one or another of the Schlesinger transformation matrices (19)–(25)

$$\frac{\partial \Psi_n}{\partial \lambda} = A_n \Psi_n\,, \quad \frac{\partial \Psi_n}{\partial x} = U_n \Psi_n\,, \quad \Psi_{n+1} = R_n \Psi_n\,,$$

with the set of the compatibility conditions

$$\begin{aligned}
&\big(A_n\big)_x - \big(U_n\big)_\lambda + \big[A_n, U_n\big] = 0\,, \\
&\big(R_n\big)_\lambda + R_n A_n - A_{n+1} R_n = 0\,, \\
&\big(R_n\big)_x + R_n U_n = U_{n+1} R_n\,,
\end{aligned}$$

as being equivalent to the system (6), (7) with (5) and one of the systems (20)–(26).

Scaling limits in P3.

Let us consider the following scaling change of variables in the equation (1):

$$\begin{aligned}
&x = \delta^{-1} x_0 + t\,, \quad x_0 \neq 0\,, \\
&a = \delta^{-1} a_0 + a_1\,, \quad b = \delta^{-1} b_0 + b_1\,, \\
&c = c_0 = 4\,, \quad d = d_0 = -4\,,
\end{aligned} \tag{27}$$

parameterized by the constant parameters x_0, a_0, b_0, a_1, b_1, so that

$$\alpha = \delta^{-1}\alpha_0 + \alpha_1\,, \quad \beta = \delta^{-1}\beta_0 + \beta_1\,, \tag{28}$$

with

$$a_0 = 8\alpha_0\,, \quad a_1 = 8\alpha_1 + 4 \quad b_0 = 8\beta_0\,, \quad b_1 = 8\beta_1\,, \tag{29}$$

and t, $|t| < const$, as the new independent variable. The change (27) transforms the third Painlevé equation into such one:

$$y'' = \frac{(y')^2}{y} + \frac{a_0}{x_0}\, y^2 + \frac{b_0}{x_0} + cy^3 + \frac{d}{y} + \tag{30}$$
$$+\frac{\delta}{x_0}\Big(t\, \frac{(y')^2}{y} - ty'' - y' + a_1 y^2 + b_1 + cty^3 + t\, \frac{d}{y}\Big).$$

The reduced form of the equation arises when $\delta = 0$. The reduced form is integrable in terms of the elliptic functions since it has the first integral D satisfying the equation

$$(y')^2 = cy^4 + 2\, \frac{a_0}{x_0}\, y^3 + Dy^2 - 2\, \frac{b_0}{x_0}\, y - d\,, \quad D = const\,. \tag{31}$$

It is important that the limit $\delta \to +0$ can be interpreted in the isomonodromy sense. In other words, after proper quantization of the parameter $\delta \mapsto \delta_n$ in the spirit of works [6, ?], we can treat the scaling limit as the result of some chain of the iterated Bäcklund transformation action. In accord with the said, let us consider the scaling change (27) in the equation $\Psi_\lambda = A\Psi$. For what follows, the square of the eigenvalues of the matrix $A \equiv a_3\sigma_3 + a_+\sigma_+ + a_-\sigma_-$ are of the most importance:

$$\mu^2 = a_3^2 + a_+a_- = \frac{x^2}{4} + \frac{\alpha x}{\lambda} + \frac{B}{\lambda^2} - \frac{\beta x}{\lambda^3} + \frac{x^2}{4\lambda^4}\,, \tag{32}$$

where

$$B = \alpha^2 + xz + uv = \Big(\frac{xy'}{4y} - \frac{1}{4}\Big)^2 - \frac{x^2}{4}\Big(y^2 + \frac{1}{y^2}\Big) - x\Big(\alpha y - \frac{\beta}{y}\Big)\,. \tag{33}$$

Assuming that in the scaling limit (27)

$$B = \delta^{-2}\big(B_0 + \delta B_1\big) \tag{34}$$

with some functions B_0 and B_1, moreover

$$B_0 = \Big(\frac{x_0 y'}{4y}\Big)^2 - \frac{x_0^2}{4}\Big(y^2 + \frac{1}{y^2}\Big) - x_0\Big(\alpha_0 y - \frac{\beta_0}{y}\Big)\,, \tag{35}$$

or

$$(y')^2 = 4y^4 + 16\, \frac{\alpha_0}{x_0}\, y^3 + 16\, \frac{B_0}{x_0^2}\, y^2 - 16\, \frac{\beta_0}{x_0}\, y + 4\,, \tag{36}$$

so that $D = 16\,\frac{B_0}{x_0^2}$, one can introduce the reduced (or asymptotic) form of the square of the eigenvalues:

$$\mu_{as}^2 = \frac{x_0^2}{4} + \frac{\alpha_0 x_0}{\lambda} + \frac{B_0}{\lambda^2} - \frac{\beta_0 x_0}{\lambda^3} + \frac{x_0^2}{4\lambda^4} = \tag{37}$$
$$= \frac{x_0^2}{4\lambda^4}\big(\lambda - \lambda_1\big)\big(\lambda - \lambda_2\big)\big(\lambda - \lambda_3\big)\big(\lambda - \lambda_4\big)\,.$$

Comparing (37) and (36), one can rewrite (36) in terms of the asymptotic turning points λ_k

$$(y')^2 = 4\big(y - \lambda_1\big)\big(y - \lambda_2\big)\big(y - \lambda_3\big)\big(y - \lambda_4\big)\,. \tag{38}$$

Independence of the monodromy data from the scaling parameter $\delta \to +0$ implies the asymptotic stability of the turning point and the Stokes graph structure [7], i.e. independence of the function B_0 on the parameter δ. Furthermore, in the case of general position, the isomonodromy property of the scaling limit leads to the system of the Boutroux equations [7]

$$\mathrm{Re} \oint_{\ell_n} \mu_{as}(\lambda)\, d\lambda = 0 \,, \tag{39}$$

where ℓ_n form the complete set of the nonhomologous (basic) cycles on the Riemann surface of the algebraic curve $\mu_{as}^2(\lambda)$. The system (39) determines the limit value B_0 as the function of the parameters x_0, α_0 and β_0. As usual, we determine the discriminant set for the scaling limits in the third Painlevé transcendent as the set in the space of the parameters x_0, α_0 and β_0, points of which correspond to the degenerated algebraic curve $\mu_{as}^2(\lambda)$. Equivalently, one can consider the turning points λ_k as the basic parameters instead of x_0, α_0, β_0. They all are related with each other by the system of equations

$$\begin{aligned}
&\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = -\frac{4\alpha_0}{x_0}\,,\\
&\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = \frac{4B_0}{x_0^2}\,,\\
&\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4 = \frac{4\beta_0}{x_0}\,,\\
&\lambda_1\lambda_2\lambda_3\lambda_4 = 1\,.
\end{aligned} \tag{40}$$

Discriminant set.

In accord with (37) and (38), degeneration of the elliptical curve $\mu_{as}^2(\lambda)$ implies the degeneration of the asymptotic elliptical ansatz for the Painlevé transcendent of the very same kind. More interesting that, in the cases of more deep degeneration, the equation (30) admits some extra scaling changes which lead to the asymptotic ansatz for the third Painlevé transcendent expressible via the first or the second Painlevé transcendent. For these cases, the Boutroux system is trivial. For the less degenerated and non-degenerated cases, the transcendent Boutroux system determines the types of the trigonometric ansatz or the modules of the elliptic asymptotic ansatz.

Note first of all that, in the case of both nontrivial parameters $\alpha_0\beta_0 \neq 0$, one of them can be normalized in its absolute value by the redefining of the scaling parameter δ. Taking in account the reality of the parameters, we can assume that one of them is equal to ± 1.

1. Complete degeneration $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$ can take place if and only if $\alpha_0 = \pm\beta_0 \neq 0$. More exactly, the system (40) determines the multiple turning point

   $$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = e^{i\frac{\pi}{2}k}\,, \quad k = 0, 1, 2, 3, \tag{41}$$

   and the values of the parameters

   $$\frac{x_0}{\alpha_0} = -e^{-i\frac{\pi}{2}k}\,, \quad \frac{x_0}{\beta_0} = e^{i\frac{\pi}{2}k}\,, \quad \frac{\alpha_0}{\beta_0} = -e^{i\pi k}\,, \quad \frac{2B_0}{3x_0^2} = \lambda_1^2 = e^{i\pi k}\,.$$

   The system of the Boutroux equations (39) holds by the trivial reasons. The equation (38) is as

   $$(y')^2 = 4(y - \lambda_1)^4\,.$$

   It implies the additional scaling change

   $$y = \lambda_1 + \delta^{\frac{1}{3}}\rho\,, \quad t = \delta^{-\frac{1}{3}}\tau\,, \tag{42}$$

which turns the equation (30) into the following one:

$$\rho'' = 8\rho^3 + 16\lambda_1^2\frac{\tau}{x_0}\rho + \frac{a_1}{x_0}\lambda_1^2 + \frac{b_1}{x_0} + \tag{43}$$

$$+\frac{\delta^{\frac{1}{3}}}{\lambda_1}\Big(\big(\rho'\big)^2 - \rho\rho'' + 4\rho^4 + 24\lambda_1^2\frac{\tau}{x_0}\rho^2 + 3\lambda_1^2\frac{a_1}{x_0}\rho + \frac{b_1}{x_0}\rho\Big) +$$

$$+\frac{\delta^{\frac{2}{3}}}{x_0}\Big(\tau\rho'' - \rho' + 3a_1\rho^2 + 16\tau\rho^3\Big) + \frac{\delta}{x_0\lambda_1}\Big(\tau\big(\rho'\big)^2 + \tau\rho\rho'' - \rho\rho' + a_1\rho^3 + 4\tau\rho^4\Big),$$

In the limit $\delta = 0$, the equation (43) is equivalent to the second Painlevé equation. The reduction P3→P2 described above is found by Painlevé [1].

At the same time, the discriminant set has the vertices at the points

$$\alpha_0 = -\beta_0 = 1\,, \quad x_0 = \pm 1\,, \quad \text{and} \quad \alpha_0 = \beta_0 = 1\,, \quad x_0 = \pm i\,.$$

2. High degeneration occurs when $\lambda_1 = \lambda_2 = \lambda_3 \neq \lambda_4$. The system (40) determines six possible values of the parameters $\lambda_{1,4}$ in accord with the equation with real coefficients of the third degree about λ_1^2

$$\lambda_1^6 + 3\frac{\beta_0}{\alpha_0}\lambda_1^4 + 3\lambda_1^2 + \frac{\beta_0}{\alpha_0} = 0\,, \quad \lambda_4 = \frac{1}{\lambda_1^3}\,, \tag{44}$$

and six values of the parameter

$$x_0 = -\frac{4\lambda_1^3}{3\lambda_1^4 + 1}\alpha_0 = \frac{4\lambda_1}{\lambda_1^4 + 3}\beta_0\,. \tag{45}$$

Note that for any negative value $\frac{\beta_0}{\alpha_0} < 0$, the equation for λ_1^2 (44) has the only real (positive) and two complex conjugate roots. Therefore there are two real and four pairwise complex conjugate values of the ratio x_0/α_0. Similarly, for any positive value $\frac{\beta_0}{\alpha_0} > 0$, the equation for λ_1^2 (44) has the only real (negative) and two complex conjugate roots. Therefore there are two imaginary and four pairwise complex conjugate values of the ratio x_0/α_0.

The system of the Boutroux equations (39) simplifies up to the known conditions $\operatorname{Im}\alpha_0 = \operatorname{Im}\beta_0 = 0$. The equation (38) is as follows:

$$\big(y'\big)^2 = 4(y - \lambda_1)^3(y - \lambda_4)\,,$$

which implies the additional scaling change

$$y = \lambda_1 + \delta^{\frac{2}{5}}\rho\,, \quad t = \delta^{-\frac{1}{5}}\tau\,. \tag{46}$$

The change (46) transforms (30) into

$$\rho'' = 6(\lambda_1 - \lambda_4)\rho^2 + \frac{4}{\lambda_1}(\lambda_1^4 - 1)\frac{\tau}{x_0} + \tag{47}$$

$$+\frac{\delta^{\frac{1}{5}}}{x_0}(a_1\lambda_1^2 + b_1) + \frac{\delta^{\frac{2}{5}}}{\lambda_1}((\rho')^2 - \rho\rho'' + 8(2\lambda_1 + \frac{\alpha_0}{x_0})\rho^3 + 16\frac{\tau}{x_0}\lambda_1^3\rho) +$$

$$+\delta^{\frac{3}{5}}\frac{\rho}{\lambda_1 x_0}(3\lambda_1^2 a_1 + b_1) + \frac{\delta^{\frac{4}{5}}}{x_0}(4\frac{x_0}{\lambda_1}\rho^4 - \tau\rho'' - \rho' + 24\tau\lambda_1\rho^2) +$$

$$+\delta\frac{3a_1}{x_0}\rho^2 + \frac{\delta^{\frac{6}{5}}}{\lambda_1 x_0}(\tau((\rho')^2 - \rho\rho'') - \rho\rho' + 16\tau\lambda_1\rho^3) + \delta^{\frac{7}{5}}\frac{a_1}{\lambda_1 x_0}\rho^3 + \delta^{\frac{8}{5}}\frac{4\tau}{\lambda_1 x_0}\rho^4\,.$$

In the limit $\delta \to 0$, (47) goes into the first Painlevé equation.

In fact, we obtain six 1-parametric families of the reductions of P3 to P1 equation parameterized by the ratio $\frac{\beta_0}{\alpha_0}$, or, in other words, six edges of the discriminant set. In particular,

i) if $\frac{\beta_0}{\alpha_0} \to -1$, then the corresponding six values of the ratio $\frac{x_0}{\alpha_0}$ form two clusters near the points ± 1. More exactly,

$$\frac{x_0}{\alpha_0} = \pm\Big(1 - \frac{3}{8}(-4\varepsilon)^{\frac{2}{3}} + \mathcal{O}(\varepsilon)\Big), \quad \varepsilon = \frac{\beta_0}{\alpha_0} + 1 \to 0\,,$$

and we can choose any of three values of the cubic root.

ii) Similarly, in the limit $\frac{\beta_0}{\alpha_0} \to -0$, the ratio $\frac{x_0}{\alpha_0}$ can take the following asymptotic values:

$$\frac{x_0}{\alpha_0} = \begin{cases} \pm 4\big(-\frac{\beta_0}{\alpha_0}\big)^{\frac{3}{2}}\Big(1 + \mathcal{O}\big(\frac{\beta_0^2}{\alpha_0^2}\big)\Big) \to 0\,, \\ \frac{3^{3/4}}{2}\, e^{i\frac{\pi}{4} + i\frac{\pi}{2}k} + \mathcal{O}\big(\frac{\beta_0}{\alpha_0}\big), \quad k = 0, 1, 2, 3. \end{cases}$$

iii) in the reverse case $\frac{\beta_0}{\alpha_0} \to -\infty$, or $\frac{\alpha_0}{\beta_0} \to -0$, the ratio $\frac{x_0}{\beta_0}$ approaches the following values:

$$\frac{x_0}{\beta_0} = \begin{cases} \pm 4\big(-\frac{\alpha_0}{\beta_0}\big)^{\frac{3}{2}}\Big(1 + \mathcal{O}\big(\frac{\alpha_0^2}{\beta_0^2}\big)\Big) \to 0\,, \\ \frac{3^{3/4}}{2}\, e^{i\frac{\pi}{4} + i\frac{\pi}{2}k} + \mathcal{O}\big(\frac{\alpha_0}{\beta_0}\big), \quad k = 0, 1, 2, 3. \end{cases}$$

For the positive values of the ratio β_0/α_0, the corresponding values of the ratio x_0/α_0 or x_0/β_0 can be obtained from the formulae above by multiplying them in $e^{i\pi/2}$.

3. There are two types of trigonometric degeneration of the elliptical curve (37) and of the reduced equation (38).

i) Double degeneration is characterized by the equations

$$\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4.$$

In this case, when the reduced equation (38) is as

$$(y')^2 = 4(y - \lambda_1)^2 (y - \lambda_3)^2\,, \tag{48}$$

the system (40) implies

$$\lambda_1\lambda_3 = \sigma\,, \quad \frac{\alpha_0}{\beta_0} = -\sigma\,, \quad \sigma^2 = 1\,, \quad \text{so that} \quad x_0 = -\frac{2\lambda_1}{\lambda_1^2 + \sigma}\alpha_0\,. \tag{49}$$

The Boutroux system reduces to the only transcendent equation for the parameter λ_1:

$$\mathrm{Re}\,\Big(\frac{\lambda_1^2 - \sigma}{\lambda_1^2 + \sigma} - \ln\lambda_1\Big) = 0. \tag{50}$$

The equation is symmetric about the inversion $\lambda_1 \mapsto -\lambda_1$ and the complex conjugation $\lambda_1 \mapsto \bar\lambda_1$. Therefore, the corresponding graph on the x_0/α_0-plane is also symmetric with respect to reflection about the real and imaginary axes. Alteration of the value of the parameter $\sigma = \pm 1$ leads to rotation of the graph in square angle.

In particular, if

$$\frac{\alpha_0}{\beta_0} = -1\,,$$

then $\lambda_1 = e^{i\varphi}$, $\varphi \in \mathbb{R}$, is a solution of the equation (50) which describes two rays $(-\infty; -1] \cup [1; \infty)$ on the plane of the ratio x_0/α_0. Other solutions describe two conjugate smooth paths connecting the vertices of the discriminant set -1 and 1. The complete graph (50), (49) is connected and has the nodes at the vertices $\frac{x_0}{\alpha_0} = \pm 1$. Moreover, near the vertices, the graph is described asymptotically as follows:

$$\frac{x_0}{\alpha_0} = \begin{cases} -1 + \varepsilon e^{i\frac{\pi}{3} + i\frac{2\pi}{3}k} + \mathcal{O}\big(\varepsilon^{3/2}\big), \\ 1 + \varepsilon e^{i\frac{2\pi}{3}k} + \mathcal{O}\big(\varepsilon^{3/2}\big), \qquad k = 0, \pm 1, \quad \varepsilon \to +0 \,. \end{cases}$$

For the opposite value $\frac{\alpha_0}{\beta_0} = 1$, the "double" graph should be rotated in the right angle.

ii) The simple trigonometric degeneration is characterized by the only equality $\lambda_1 = \lambda_2 \neq \lambda_3 \neq \lambda_4$. The system (40) allows one to express the parameters λ_k via the values of α_0, β_0, x_0 in accord with the equations

$$\lambda_3\lambda_4 = \frac{1}{\lambda_1^2}\,, \qquad \lambda_3 + \lambda_4 = -2\lambda_1 - \frac{4\alpha_0}{x_0}\,,$$

accompanied by the equation of the fourth order for λ_1

$$\lambda_1^4 + \frac{2\alpha_0}{x_0}\lambda_1^3 + \frac{2\beta_0}{x_0}\lambda_1 - 1 = 0\,,$$

while the system of Boutroux equations reduces to the only relation

$$\begin{aligned} &\mathrm{Re}\Big\{ x_0\sqrt{(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)} + \\ &+\beta_0 \ln\Big[\frac{2}{\lambda_3 - \lambda_4}\Big(\frac{\lambda_3 + \lambda_4}{2} + \frac{1}{\lambda_1^2}\sqrt{(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)} - \frac{1}{\lambda_1^3}\Big)\Big] + \\ &+\alpha_0 \ln\Big[\frac{2}{\lambda_3 - \lambda_4}\Big(\lambda_1 - \frac{\lambda_3 + \lambda_4}{2} + \sqrt{(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)}\Big)\Big]\Big\} = 0\,, \end{aligned} \tag{51}$$

which describes some graph on the complex plane of the variable x_0/α_0. This graph is symmetric with respect to reflections about the coordinate axes and contains eight infinite branches asymptotically described by the equations

$$\mathrm{Re}\Big\{\pm i x_0 - \frac{\alpha_0 + \beta_0}{4}\ln x_0 + \mathcal{O}(1)\Big\}, \quad x_0 \to \infty\,, \tag{52}$$

and

$$\mathrm{Re}\Big\{\pm x_0 - \frac{\alpha_0 - \beta_0}{4}\ln x_0 + \mathcal{O}(1)\Big\}, \quad x_0 \to \infty\,. \tag{53}$$

If $\alpha_0 = -\beta_0$, then (52) degenerates to describe two real rays, mentioned above. If $\alpha_0 = \beta_0$, then (53) degenerates to describe two imaginary rays.

Besides the infinite branches, the equation (51) describes several finite branches connecting some of the points x_0/α_0 (44), (45). Besides them, there is the segment of the real axis (if $\frac{\beta_0}{\alpha_0} < 0$) or of the imaginary axis (if $\frac{\beta_0}{\alpha_0} > 0$). If $\alpha = \pm\beta$, some of the finite branches collapse or coincide with each other yielding the contours of the "double" graph described above.

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