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A Note on the Algebro-Geometric Integration of the Schlesinger Equations and on the Elliptic Solutions of the Painlevé VI Equation

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ABSTRACT

A new approach to the construction of isomonodromy deformations of 2×2 Fuchsian systems is presented. The method is based on a combination of the algebrogeometric scheme and Riemann-Hilbert approach of the theory of integrable systems. For a given number $2g+1$, $g \geq 1$, of finite (regular) singularities, the method produces a $2g$- parameter submanifold of the Fuchsian monodromy data for which the relevant Riemann-Hilbert problem can be solved in closed form via the Baker-Akhiezer function technique. This in turn leads to a $2g$-parameter family of solutions of the corresponding Schlesinger equations, explicitly described in terms of Riemann theta functions of genus g. In the case $g = 1$ the solution found coincides with the general elliptic solution of the particular case of the Painlevé VI equation first discovered by N. J. Hitchin [H1].

Introduction

Let

$$\Gamma = \left\{ P = (\lambda, w) : w^2 = \prod_{j=1}^{2g+1} (\lambda - a_j) \right\}, \quad a_j \neq a_k \quad \text{for} \quad j \neq k \,,$$

be a hyperelliptic curve of genus $g \geq 1$. We shall use the standard representation of Γ as a two-sheeted covering of $\mathbf{C}\mathrm{P}^1$ with the cuts along the intervals

$$[a_{2k+1}, a_{2k+2}], \quad k = 0, ..., g, \quad a_{2g+2} = \infty,$$

assuming that

$$\Re a_1 \leq \Re a_2 \leq \Re a_3 \leq ... \leq \Re a_{2g} \leq \Re a_{2g+1}, \tag{1}$$

and $\Re\lambda > \Re a_{2g+1}$ for $\lambda \in (a_{2g+1}, \infty)$. We define the upper sheet of Γ by the condition,

$$w \sim \lambda^{g+\frac{1}{2}}, \quad \lambda \to i\infty, \quad \arg \lambda = \frac{\pi}{2}. \tag{2}$$

We choose the basis $\{\mathcal{A}_k, \mathcal{B}_k\}_{k=1}^{g}$ of the group $H_1(\Gamma)$ so that $\mathcal{A}_k$ lies fully on the upper sheet and encircles the intervals $[a_{2j-1}, a_{2j}], j = 1, ..., k$ clockwise while $\mathcal{B}_k$ emerges on the upper sheet at the point a_{2k}, passes to the point a_{2k+1} and returns to the initial point a_{2k} through the lower sheet (see Figure 1). We normalize the corresponding basis of the holomorphic differentials, $\{\omega_j(P)\}$, by the relation

$$\int_{\mathcal{A}_k} \omega_j = 2\pi i \delta_{jk} \,, \quad 1 \leq j, k \leq g \,.$$

Consider $\mathbf{C}\mathrm{P}^1 \backslash \{a_1, ..., a_{2g+1}, a_{2g+2} = \infty\}$, the complex sphere with $2g+2$ punctures, and suppose that $2g+1$ matrices $A_j \in M(2, \mathbf{C})$ are given. Then a representation of the fundamental group $\pi_1(\mathbf{C}\mathrm{P}^1 \backslash \{a_1, ..., a_{2g+1}, \infty\})$ can be obtained via solutions of the 2×2 Fuchsian system,

$$\frac{dY(\lambda)}{d\lambda} = \sum_{j=1}^{2g+1} \frac{A_j}{\lambda - a_j} Y(\lambda), \qquad Y(\lambda) \in M(2, \mathbf{C}), \tag{3}$$

in the following way. Fix a base point $a_0 \in \mathbf{C}\mathrm{P}^1 \backslash \{a_1, ..., a_{2g+1}, \infty\}$ and realize π_1 as the collection of loops $\{\gamma = \gamma(t), \quad 0 \leq t \leq 1, \quad \gamma(0) = \gamma(1) = a_0\}$ in the usual way. Fix a matrix $Y_0 \in Gl(2, \mathbf{C})$. Now let $Y(\lambda)$ be the solution of (3) with $Y(a_0) = Y_0$. For each $\gamma \in \pi_1$, set

$$M(\gamma) = M(\gamma\,; a_0, Y_0) \equiv Y_0^{-1} Y(\gamma(1))$$

where $Y(\gamma(1))$ denotes the analytic continuation of the solution $Y(\lambda)$ along the loop $\gamma, \quad a_0 = \gamma(0) \mapsto \gamma(1) = a_0$. Then a simple calculation shows that the map

$$\pi_1 \ni \gamma \longmapsto M(\gamma) \in Gl(2, \mathbf{C})$$

is a linear representation of π_1, which is called *a monodromy representation* of system (3). Given $\{A_j, a_j\}$, the subgroup $\{M(\gamma), \gamma \in \pi_1\} \subset Gl(2, \mathbf{C})$ is called *the monodromy group* for the Fuchsian system (3). (This nomenclature of course involves a slight abuse of notation: the group $\{M(\gamma) = M(\gamma\,; a_0, Y_0)\}$ depends on the choice of a_0 and Y_0. However, all these groups are isomorphic via a conjugation, e.g. $M(\gamma\,; a_0, Y_0) = Y_0^{-1} M(\gamma\,; a_0, I) Y_0$, and when we say monodromy group we really mean the equivalence class of monodromy group modulo conjugation.) The monodromy group, which can of course be introduced in a similar way for a Fuchsian system of an arbitrary matrix dimension, is a central object of the global theory of Fuchsian systems. We refer the reader to the review [B] for a comprehensive account on the history as well as the most recent results concerning the general monodromy theory.

As we vary the a_j's, $1 \leq j \leq 2g+1$, and hence deform the punctured sphere

$$\mathbf{C}\mathrm{P}^1 \backslash \{a_1, ..., a_{2g+1}, \infty\},$$

it is clearly of basic interest to determine which coefficients $A_j = A_j(a_1, ..., a_{2g+1})$ give rise to the same monodromy group. Maps

$$\{a_j\}_{j=1}^{2g+1} \longmapsto \{A_j(a_1, ..., a_{2g+1})\}_{j=1}^{2g+1}$$

which give the same monodromy groups are called *isomonodromy deformations* of (3), and it turns out that $\{A_j(a_1, ..., a_{2g+1})\}$ is an isomonodromy deformation if and only if the A_j's satisfy the *Schlesinger equations*,

$$dA_j + \sum_{j \neq k} [A_j, A_k] \frac{da_j - da_k}{a_j - a_k} = 0, \qquad j = 1, ..., 2g+1, \tag{4}$$

(see [Sch]; see also [JMU], [Mal]). In the present paper we show how to use curves Γ of the above type to construct a $2g$-parameter family of solutions of (4).

In our approach, the complete set of the algebrogeometric data consists of the curve Γ and an abelian integral $\Omega(P)$ of the second kind,

$$\Gamma \to \{\Gamma, \Omega(P)\} \to \frac{dY(\lambda)}{d\lambda} = \sum_{j=1}^{2g+1} \frac{A_j}{\lambda - a_j} Y(\lambda). \tag{5}$$

The relevant monodromy group is parametrized by $2g$ independent parameters and is encoded in the abelian integral $\Omega(\lambda)$, more precisely, in its A, B-periods. All the generators M_j of the monodromy group are nontrivial, in fact (cf. (33)),

$$\text{the eigenvalues of } M_j = \pm i, \quad \forall j = 1, ..., 2g + 2. \tag{6}$$

The first arrow in diagram (5) is simply the selection of an abelian integral of the second kind on Γ. The second arrow constitutes an *explicit* solution of a certain matrix Riemann-Hilbert problem (see problem (**i** $-$ **iii**) below). This means that we have an example of a $2g$-parameter submanifold (see (33)- (34) below) of the Fuchsian monodromy data corresponding to $2g+1$ finite regular singularities for which the inverse monodromy problem can be solved in *closed form* (see (66), (88) below).

The method we propose is based on recent technical developments in the asymptotic analysis of integrable systems via the Riemann-Hilbert approach [DVZ, DIZ].

A general scheme for constructing algebrogeometric solutions of a generalized Fuchsian equation (irregular singularities may be included) of an arbitrary matrix dimension was suggested in [JM] (see also [Ko] and [KoM] where the approach of [JM] has been developed and used for studying the Einstein equations). Applied to the curve Γ, the method of [JM] leads to a *particular* solution of the Schlesinger equation (4) generated by the diagram,

$$\Gamma \to \frac{dY(\lambda)}{d\lambda} = \sum_{j=1}^{2g+1+m} \frac{A_j}{\lambda - a_j} Y(\lambda), \quad m > g. \tag{7}$$

As indicated, in this scheme one needs to *extend* the set of regular singularities by more than g extra points . Moreover, the monodromy group corresponding to the solution is very special and is determined a priori by the curve Γ. The monodromy group is in fact a finite abelian group isomorphic to the group $\mathbf{Z}_2$; the monodromy matrices associated with each of the singular ponts a_j are given by the equations,

$$M_j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv \sigma_1, \quad j = 1, ..., 2g + 1,$$

at the branch points, and

$$M_j = I, \quad j = 2g + 2, ..., 2g + 1 + m,$$

at the added points. We emphasize that all the monodromy matrices at the added singularities are *trivial.*

Said differently, in contrast to the method of [JM] the approach proposed in this paper allows us to construct a $2g$- parameter ($g \geq 1$) *family* of solutions of the Schlesinger equations (4) related to 2×2 Fuchsian system with a *fixed* number $2g + 1$ of finite (regular) singularities and with *nontrivial* monodromy in each of the singular points. The corresponding monodromy groups are infinite and non-abelian.

(The method of [JM] can, in principal, be modified, using some of the constructions presented below, to allow for the inclusion of the free parameters in the monodromy group.

Nevertheless, the extension of the set of regular singularities indicated above can not be avoided in the framework of [JM].)

In what follows, the new scheme is described in detail. In the particular case $g = 1$, this approach gives the general solution of the Painlevé VI equation with parameters $\left(\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8}\right)$ (see (109) below), whose elliptic function representation was first discovered in [H1] and then discussed extensively in the literature (see [H2], [Man], [LO] and references therein). We also believe that the constructions developed here have a connection to the results from topological field theory in [D], [Kr], [KrPh].

The main result of the paper is stated in Theorem 1 below.

Integral $\Omega(P)$.

Let $\Omega(P)$ be an abelian integral on Γ with possibly one pole, placed at $P = \infty$. In addition, suppose that

$$\Omega(a_{2g+1}) = 0,$$

and that

$$\Omega(P^*) = -\Omega(P),$$

where $P \to P^*$ denotes the involution of the sheets of Γ,

$$P = (\lambda, w), \qquad P^* = (\lambda, -w). \tag{8}$$

We note that $\Omega(P)$ can be represented as

$$\Omega(P) = \int_{a_{2g+1}}^{P} \frac{Q}{w} d\lambda, \tag{9}$$

where $Q = Q(\lambda)$ is a polynomial in λ.

With $\Omega(P)$ given, we define the sets of nonzero constants, $\{\ln c_k\}_{k=1}^{g}$ and $\{\ln d_k\}_{k=1}^{g}$ by the equations involving the A and B - periods of $\Omega(\lambda)$,

$$\int_{\mathcal{A}_k} d\Omega = \ln d_k, \qquad k = 1, ..., g, \tag{10}$$

$$\int_{\mathcal{B}_k} d\Omega = \ln \frac{c_k}{c_{k-1}}, \quad c_0 = 1, \qquad k = 1, ..., g, \tag{11}$$

These equations define the *period* map,

$$\mathcal{M} : \{\Omega(P)\} \to \{\ln d_k, \ln c_k\} \in \mathbf{C}^{2g}, \tag{12}$$

to the *monodromy* data $\{\ln d_k, \ln c_k\}$. Note that

$$\ker \mathcal{M} =$$

$$= \{\text{meromorphic functions on } \Gamma \text{ with possibly one pole at } \infty\}, \tag{13}$$

and that map (12) is onto. Indeed, as λ^0, λ^1,...,λ^g are analytic on the Riemann surface away from infinity, and behave like even powers of the local variable at infinity, it follows from the Weierstrass gap theorem (see e.g. [FK]) that there are no functions analytic on the finite part of the Riemann surface with odd order poles of degree $< 2g$ at infinity. But any nonzero vector in the null space of the map (12) gives rise to an analytic function on the finite part of the Riemann surface with an odd order pole at infinity. It follows that the null space of map $\mathcal{M}$ restricted to the integrals $\Omega(P)$ with poles of degree $< 2g$ at infinity is trivial and hence the map (12) is onto.

The surjectivity of the map $\mathcal{M}$ implies that the inverse period map,

$$\mathcal{M}^{-1} : \{\ln d_k, \ln c_k\} \to \{\Omega(P)\}/\ker \mathcal{M}, \tag{14}$$

is well defined, so that the set,

$$a_1, ..., a_{2g+1}; \ln c_1, ..., \ln c_g; \ln d_1, ..., \ln d_g,$$

becomes our *parameter set*:

$$\{a_1, ..., a_{2g+1}; \ln c_1, ..., \ln c_g; \ln d_1, ..., \ln d_g\} \to \{\Gamma, \Omega(P) + \ker \mathcal{M}\}.$$

The integral $\Omega(P)$ can be also represented as follows,

$$\Omega(P) = h(P) + \sum_{j=1}^{g} \alpha_j \int_{a_{2g+1}}^{P} \omega_j, \tag{15}$$

with the abelian integral $h(P)$ uniquely determined by this equation and the normalization conditions,

$$\int_{\mathcal{A}_j} dh = 0, \qquad j = 1, ..., g. \tag{16}$$

A simple calculation shows that

$$h(P) = \Omega_\infty(\lambda) + o(1), \quad P \to \infty, \tag{17}$$

where $\Omega_\infty(\lambda)$ denotes the principle part of the integral $\Omega(P)$ at $P = \infty$.

From (10) and (11) it follows that

$$\alpha_k = -\frac{i}{2\pi} \ln d_k, k = 1, ..., g, \tag{18}$$

and

$$\int_{\mathcal{B}_k} dh(P) = \frac{i}{2\pi} \sum_{j=1}^{g} B_{kj} \ln d_j + \ln \frac{c_k}{c_{k-1}}, \quad k = 1, ..., g, \tag{19}$$

where

$$B_{kj} = \int_{\mathcal{B}_k} \omega_j$$

is the canonical B-matrix of Γ.

Let

$$\Pi_0 = \left\{ (a_j; \ln c_j; \ln d_j) : \frac{i}{2\pi} \sum_{j=1}^{g} B_{kj} \ln d_j + \ln \frac{c_k}{c_{k-1}} = 0 \right\} \subset \mathbf{C}^{4g+1} \tag{20}$$

be a special submanifold of co-dimension g in the space of parameters $a_j, \ln c_j, \ln d_j$. From (19) it follows that

$$(a_1, ..., a_{2g+1}; \ln c_1, ..., \ln c_g; \ln d_1, ..., \ln d_g) \notin \Pi_0 \Longrightarrow \Omega_\infty(\lambda) \not\equiv 0. \tag{21}$$

Indeed, the assumption,

$$\Omega_\infty(\lambda) = 0,$$

would imply that the integral $h(P)$ is an abelian integral of the first kind with all the A-periods zero. This means $h(P)$ must be a constant (zero, in fact), and hence all the B-periods are zero as well. Because of (19), this would yield

$$(a_1, ..., a_{2g+1}; \ln c_1, ..., \ln c_g; \ln d_1, ..., \ln d_g) \in \Pi_0.$$

Implication (21) means that for generic a_j, $\ln c_j$, and $\ln d_j$, i.e. under the condition,

$$(a_1, ..., a_{2g+1}; \ln c_1, ..., \ln c_g; \ln d_1, ..., \ln d_g) \notin \Pi_0, \tag{22}$$

the integral $\Omega(P)$ is of the *second kind*.

The Riemann-Hilbert Problem.

Let $\mathcal{L}$ denote the oriented polygonal line (see Figure 2),

$$\mathcal{L} = [\infty, a_1] \cup [a_1, a_2] \cup [a_2, a_3] \cup ...$$

$$... \cup [a_{2k}, a_{2k+1}] \cup [a_{2k+1}, a_{2k+2}] \cup ... \cup [a_{2g}, a_{2g+1}] \cup [a_{2g+1}, \infty],$$

where we assume that

$$\Re\lambda < \Re a_1, \quad \text{for} \quad \lambda \in (\infty, a_1) \qquad \text{and} \quad \Re\lambda > \Re a_{2g+1}, \quad \text{for} \quad \lambda \in (a_{2g+1}, \infty).$$

The positive and the negative parts of the plane $\mathbf{C}$ with respect to the contour $\mathcal{L}$ we shall denote as $\mathbf{C}_+$ and $\mathbf{C}_-$ respectively. We also note that the domain $\mathbf{C}_+$ contains the infinite part of the positive imaginary axis.

Having choosen the monodromy set $\{\ln c_k, \ln d_k\}$, we define 2×2 matrix function $Y(\lambda)$ (cf. diagram (5)) as the solution of the following Riemann-Hilbert problem (RH) :

(i) $Y(\lambda)$ is analytic in $\mathbf{C}\mathrm{P}^1 \backslash \mathcal{L}$

(ii) The L_2-limits, $Y_\pm(\lambda)$, of $Y(\lambda)$ as $\lambda \to \mathcal{L}_\pm$ satisfy the jump conditions,

$$Y_-(\lambda) = Y_+(\lambda) \begin{pmatrix} 0 & \frac{c_g}{c_k} \\ -\frac{c_k}{c_g} & 0 \end{pmatrix}, \tag{23}$$

if $\lambda \in [a_{2k+1}, a_{2k+2}]$, $\quad k = 0, ..., g, \quad a_{2g+2} = \infty$ and

$$Y_-(\lambda) = Y_+(\lambda) \begin{pmatrix} \frac{d_g}{d_k} & 0 \\ 0 & \frac{d_k}{d_g} \end{pmatrix}, \tag{24}$$

if $\lambda \in [a_{2k}, a_{2k+1}]$, $\quad k = 0, ..., g, \quad a_0 = \infty, \quad d_0 = 1$.

(iii) The behavior of $Y(\lambda)$ as $\lambda \to \infty$ is normalized by the equation,

$$Y(\lambda) = \left\{ I + O\left(\frac{1}{\lambda}\right) \right\} \lambda^{-\frac{1}{4}\sigma_3} \begin{pmatrix} 1 & id_g \\ id_g^{-1} & 1 \end{pmatrix}, \quad \lambda \to \infty, \quad \lambda \in \mathbf{C}_+, \tag{25}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $\lambda^{\frac{1}{4}}$ denotes the branch defined in the neighborhood of ∞,

$$U_\infty = \{\lambda : |\lambda| > R > \max\{|a_j|, j = 1, ..., 2g+1\}\},$$

cut along the segment $[a_{2g+1}, \infty] \cap U_\infty$ and normalized by the condition, $\arg \lambda = \frac{\pi}{2}$ when λ lies on the positive imaginary axis.

The Fuchsian equation, the Monodromy data, and the Schlesinger equations.

Assume that the solution $Y(\lambda)$ of the RH problem $(\mathbf{i} - \mathbf{iii})$ exists. Then, one can see that in the neighborhoods of the points $\lambda = a_j$, $j = 1, ..., 2g+1$ the function $Y(\lambda)$ admits the following representations:

$$Y(\lambda) = \hat{Y}_j(\lambda)(\lambda - a_j)^{(-1)^{j+1}\frac{\sigma_3}{4}} \begin{pmatrix} 1 & i\rho_j \\ i\rho_j^{-1} & 1 \end{pmatrix} C_j(\lambda), \quad j = 1, ..., 2g+1, \tag{26}$$

where

$$\rho_j = \begin{cases} -\frac{c_g d_g}{c_{k-1} d_k} & \text{for } j = 2k \\ -\frac{c_g d_g}{c_{k-1} d_{k-1}} & \text{for } j = 2k-1 \end{cases} \tag{27}$$

and $C_j(\lambda)$ are the piecewise constant matrix functions given by the equations,

$$C_j(\lambda) = I, \, \forall \lambda \in \mathbf{C}_+, \tag{28}$$

$$C_j(\lambda) = \begin{cases} \left(\frac{d_g}{d_k}\right)^{\sigma_3} & \text{for } j = 2k \\ \left(\frac{d_g}{d_{k-1}}\right)^{\sigma_3} & \text{for } j = 2k-1 \end{cases}, \, \forall \lambda \in \mathbf{C}_- \, . \tag{29}$$

The matrix function $\hat{Y}_j(\lambda)$ is holomorphic and invertible in the neighborhood of the point $\lambda = a_j$,

$$U_j = \{\lambda : |\lambda - a_j| < r < \min\{|a_j - a_l|, 1 \le l \le 2g+1, \, l \ne j\}\},$$

and $(\lambda - a_j)^{\frac{1}{4}}$ denotes an arbitrary branch defined in the same neighborhood cut along the segment $[a_{j-1}, a_j] \cap U_j$ if $j = 2k$ and along the segment $[a_j, a_{j+1}] \cap U_j$ if $j = 2k-1$. A similar representation for $Y(\lambda)$ near $\lambda = \infty$ reads (cf. (25)) as

$$Y(\lambda) = \hat{Y}_\infty(\lambda)\lambda^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & id_g \\ id_g^{-1} & 1 \end{pmatrix} C_\infty(\lambda), \quad \hat{Y}_\infty(\infty) = I, \tag{30}$$

$$C_\infty(\lambda) = \begin{cases} I & \text{for } \lambda \in \mathbf{C}_+ \\ d_g^{\sigma_3} & \text{for } \lambda \in \mathbf{C}_- \end{cases}, \tag{31}$$

where $\hat{Y}_\infty(\lambda)$ is holomorphic and invertible at $\lambda = \infty$, and $\lambda^{\frac{1}{4}}$ is defined as above.

By standard Liouville theorem arguments (see e.g. [FT]) based on the λ-independence of the jump matrices of the RH problem $(\mathbf{i} - \mathbf{iii})$ and equations (26–29), we conclude that $Y(\lambda)$ satisfies the Fuchsian system (3) with the matrix residues A_j determined by the equations,

$$A_j = (-1)^{j+1}\hat{Y}_j(a_j)\frac{\sigma_3}{4}\hat{Y}_j^{-1}(a_j), \quad j = 1, ..., 2g+1. \tag{32}$$

Equations (26–30) yield the following set of the corresponding monodromy matrices,

$$\tilde{Y}(\lambda)|_{(\lambda - a_j) \mapsto (\lambda - a_j)e^{2\pi i}} = \tilde{Y}(\lambda)M_j,$$

$$M_j = (-1)^j \begin{pmatrix} 0 & \rho_j \\ -\rho_j^{-1} & 0 \end{pmatrix}, \quad j = 1, ..., 2g+1, \tag{33}$$

$$\tilde{Y}(\lambda)|_{\frac{1}{\lambda} \mapsto \frac{1}{\lambda}e^{2\pi i}} = \tilde{Y}(\lambda)M_\infty,$$

$$M_\infty = \begin{pmatrix} 0 & -d_g \\ d_g^{-1} & 0 \end{pmatrix}. \tag{34}$$

Here $\tilde{Y}(\lambda)$ denotes the analytic continuation of $Y(\lambda)$ from $\mathbf{C}_+$ to the universal covering of $\mathbf{C}\mathrm{P}^1\backslash\{a_1,...,a_{2g+1},\infty\}$.

We note that

$$\mathrm{trace}\, A_j = 0, \quad \text{for all} \quad j,$$

and that

$$\sum_{j=1}^{2g+1} A_j = \text{diagonal matrix},$$

due to the normalization condition, $\hat{Y}_\infty(\infty) = I$.

The monodromy matrices $M_1,...,M_{2g+1},M_\infty$ generate the monodromy group for the Fuchsian system (3), satisfy the cyclic relation,

$$M_\infty M_{2g+1}...M_1 = I,$$

and depend only on c_j, d_j. Hence, the 2×2 matrices,

$$A_j \equiv A_j(a_1,...,a_{2g+1}|c_1,...,c_g;d_1,...,d_g) =$$

$$= (-1)^{j+1}\hat{Y}_j(a_j)\frac{\sigma_3}{4}\hat{Y}_j^{-1}(a_j), \quad j=1,...,2g+1,, \tag{35}$$

which are evaluated explicitly through $Y(\lambda)$, give rise to a solution of the associated Schlesinger equations (4).

Solution for the Riemann-Hilbert Problem.

Here we follow the method, which for similar RH problems was first suggested in [DVZ], and then used in [DIZ] and subsequently in [IK].

Reduction to the canonical algebrogeometric RH problem.

Given the monodromy data $\{\ln c_k, \ln d_k\}$ let $\Omega(P)$ be the associated abelian integral. Define

$$Y(\lambda) = \Psi(\lambda)e^{\Omega(\lambda)\sigma_3}, \tag{36}$$

where $\Omega(\lambda)$ denotes the natural restriction of the integral $\Omega(P)$ on $\mathbf{C}_+\cup\mathbf{C}_-$: the integration path in (9) lies in $\mathbf{C}_+$ and in $\mathbf{C}_-$ for $\lambda\in\mathbf{C}_+$ and $\lambda\in\mathbf{C}_-$ respectively, and the integral itself is taking on the upper sheet of Γ.

Because of equations (10), (11), and (9) we have that

$$(\Omega_+ + \Omega_-)|_{[a_{2k+1},a_{2k+2}]} = -2\sum_{j=k+1}^{g}\int_{a_{2j}}^{a_{2j+1}} d\Omega =$$

$$= -\sum_{j=k+1}^{g}\int_{\mathcal{B}_j} d\Omega = \ln\frac{c_k}{c_g}, \quad k=0,...,g, \quad a_{2g+2}=\infty, \tag{37}$$

and

$$(\Omega_+ - \Omega_-)|_{[a_{2k},a_{2k+1}]} = -2\sum_{j=k}^{g-1}\int_{a_{2j+1}}^{a_{2j+2}} d\Omega =$$

$$= -\int_{\mathcal{A}_g} d\Omega + \int_{\mathcal{A}_k} d\Omega = \ln\frac{d_k}{d_g}, \quad k = 0, ..., g, \quad a_0 = \infty, \quad d_0 = 1. \tag{38}$$

Equation (37) implies that in terms of $\Psi(\lambda)$ the jump condition (23) takes the form

$$\Psi_-(\lambda) = \Psi_+(\lambda)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{39}$$

while the jump condition (24) is replaced by

$$\Psi_-(\lambda) = \Psi_+(\lambda)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \Psi_+(\lambda) \quad \text{i.e. no jump.} \tag{40}$$

Therefore, in terms of the function $\Psi(\lambda)$ the RH problem $(\mathbf{i}-\mathbf{iii})$ transforms to the following canonical RH problem (see Figure 3):

$(\mathbf{i^0})$ $\Psi(\lambda)$ is analytic in $\mathbf{C}\mathrm{P}^1\backslash\mathcal{L}_0$, $\quad \mathcal{L}_0 = \bigcup_{k=0}^{g}[a_{2k+1}, a_{2k+2}]$, $\quad a_{2g+2} = \infty$

$(\mathbf{ii^0})$ The L_2-limits, $\Psi_\pm(\lambda)$, of $\Psi(\lambda)$ as $\lambda \to \mathcal{L}_{0\pm}$ satisfy the jump condition,

$$\Psi_-(\lambda) = \Psi_+(\lambda)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{41}$$

$(\mathbf{iii^0})$ The behavior of $\Psi(\lambda)$ as $\lambda \to \infty$ in $\mathbf{C}_+$ is normalized by the relation,

$$\Psi(\lambda) = d_g^{\frac{\sigma_3}{2}}\begin{pmatrix} 1 & 0 \\ -i\Omega_0 & 1 \end{pmatrix}\left\{I + O\left(\frac{1}{\lambda}\right)\right\}\lambda^{-\frac{1}{4}\sigma_3}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}e^{-\Omega_\infty(\lambda)\sigma_3}, \tag{42}$$

where $\Omega_\infty(\lambda)$ as before denotes the principle part of $\Omega(P)$ at infinity, and the constant Ω_0 is given by the expansion,

$$\Omega(P) = \Omega_\infty(\lambda) - \frac{1}{2}\ln d_g + \frac{\Omega_0}{\lambda^{1/2}} + O\left(\frac{1}{\lambda^{3/2}}\right), \quad P \to \infty. \tag{43}$$

We intend to use solution of the RH problem $(\mathbf{i}-\mathbf{iii})$ to solve the Schlesinger equations (4). This means that the monodromy data $\{\ln c_j, \ln d_j\}$ should be taken to be *independent* of the a_j's. Thus, our main objective is to solve the RH problem $(\mathbf{i}-\mathbf{iii})$ for generic values of parameters a_j, $\ln c_j$, and $\ln d_j$, i.e. under condition (22).

The generic condition (22), by virtue of the implication (21), means that we should assume that,

$$\Omega_\infty(\lambda) \not\equiv 0, \tag{44}$$

in setting up the RH problem ($\mathbf{i^0} - \mathbf{iii^0}$). In fact, the solution of the RH problem ($\mathbf{i} - \mathbf{iii}$), which we will eventually obtain (see (66) below), will also be valid for the special submanifold Π_0.

Solution for the canonical RH problem.

Let

$$\beta(\lambda) = \left[(\lambda - a_{2g+1}) \prod_{l=1}^{g} \frac{\lambda - a_{2l-1}}{\lambda - a_{2l}}\right]^{\frac{1}{4}}. \tag{45}$$

The function $\beta(\lambda)$ is defined as a single-valued analytic function in $\mathbf{C}\mathrm{P}^1 \backslash \mathcal{L}_0$, normalized by the asymptotic condition,

$$\beta(\lambda) \sim \lambda^{1/4}, \quad \lambda \to i\infty, \quad \arg \lambda = \frac{\pi}{2}. \tag{46}$$

We also note that the boundary values of $\beta(\lambda)$ on the contour $\mathcal{L}_0$ satisfy the equation,

$$\beta_-(\lambda) = i\beta_+(\lambda), \quad \lambda \in \mathcal{L}_0. \tag{47}$$

The function $\beta^4(\lambda) - 1$ has precisely $g+1$ zeros $\lambda_0, \lambda_1, ..., \lambda_g$ in $\mathbf{C}$. At each λ_j either

$$\beta(\lambda) - \beta^{-1}(\lambda) = 0,$$

or

$$\beta(\lambda) + \beta^{-1}(\lambda) = 0.$$

Under the involution, $P \to P^*$, which switches the sheets of Γ (see (8)), function $\beta(\lambda)$ behaves so that

$$\beta(\lambda) + \beta^{-1}(\lambda) \to \pm i\left(\beta(\lambda) - \beta^{-1}(\lambda)\right).$$

Therefore the zero divisor ,

$$\mathcal{D}_0 = \sum_{j=0}^{g} P_j, \quad P_j = (\lambda_j, w_j) \in \Gamma \tag{48}$$

of the sum $\beta(\lambda)+\beta^{-1}(\lambda)$ considered as a function on Γ is well defined (though the function itself is not). Moreover,

$$\mathcal{D}_0^* = \sum_{j=0}^{g} P_j^*$$

is the zero divisor of the function $\beta(\lambda) - \beta^{-1}(\lambda)$.

We assume that the reduced divisor,

$$\mathcal{D} = \sum_{j=1}^{g} P_j, \tag{49}$$

is a nonspecial positive divisor of degree g. This property is generic for $(a_1, ..., a_{2g+1})$ in $\mathbf{C}^{2g+1}$, and is true, in particular, if the numbers $\lambda_1, ..., \lambda_g$ are distinct in $\mathbf{C}$ (see e.g. [Mum]).

Set

$$N(\lambda) = \begin{pmatrix} \beta + \beta^{-1} & -i\beta + i\beta^{-1} \\ i\beta - i\beta^{-1} & \beta + \beta^{-1} \end{pmatrix}. \tag{50}$$

From equation (47) it follows that matrix $N(\lambda)$ satisfies the jump condition (41). Nevertheless, because of the generic assumption (44), it does not solve the RH problem $(\mathbf{i^0} - \mathbf{iii^0})$: the asymptotic condition (42) at infinity is not satisfied. To make $N(\lambda)$ into a genuine solution of the problem $(\mathbf{i^0} - \mathbf{iii^0})$ we now use a certain *Baker-Akhiezer deformation* of $N(\lambda)$ (cf. [DIZ]).

Let us define on Γ the following pair of the Baker-Akhiezer functions:

$$\Psi_1(P) = \frac{\theta\left(\int_\infty^P \omega - \mathbf{V} - \mathbf{D}\right)}{\theta\left(\int_\infty^P \omega - \mathbf{D}\right)} e^{-h(P)}, \tag{51}$$

$$\Psi_2(P) = \frac{\theta\left(\int_\infty^P \omega - \mathbf{V} + \mathbf{D}\right)}{\theta\left(\int_\infty^P \omega + \mathbf{D}\right)} e^{-h(P)}, \tag{52}$$

where

1. $\theta(\mathbf{z}) = \sum_{\mathbf{m}\in\mathbf{Z}^g} \exp\left\{\frac{1}{2}(B\mathbf{m}, \mathbf{m}) + (\mathbf{z}, \mathbf{m})\right\}$, is the Riemann theta function (matrix B is given in (19)) which satisfies the canonical periodicity equation,

$$\theta(\mathbf{z} + 2\pi i\mathbf{n} + B\mathbf{m}) = \exp\left\{-\frac{1}{2}(B\mathbf{m}, \mathbf{m}) - (\mathbf{z}, \mathbf{m})\right\}\theta(\mathbf{z}). \tag{53}$$

2. The vector $\mathbf{D}$ belongs to the Jacobian of Γ and is defined by the equation

$$\mathbf{D} = \mathcal{A}(\mathcal{D}) + \mathbf{K},$$

where $\mathcal{A}(\mathcal{D})$ is the abelian mapping with base point at infinity,

$$\mathcal{A}(\mathcal{D}) = \sum_{j=1}^{g} \int_{\infty}^{P_j} \omega \,, \tag{54}$$

$\mathbf{K}$ is the associated vector of Riemann constants, and the divisor, $\mathcal{D} = \mathcal{D}_0 - P_0$, is the reduced zero divisor of the function $\beta(\lambda) + \beta^{-1}(\lambda)$ (see (49)). Also from (54) it follows that

$$\mathcal{A}(\mathcal{D}^*) = -\mathcal{A}(\mathcal{D}), \tag{55}$$

as points on the Jacobian.

3. The abelian integral $h(P)$ is defined in (17), (16).

4. The vector $\mathbf{V}$ is the vector of the B-periods of the integral h:

$$V_k = \int_{\mathcal{B}_k} dh = \quad (\text{ see } (19)) \quad =$$

$$= \frac{i}{2\pi} \sum_{j=1}^{g} B_{kj} \ln d_j + \ln \frac{c_k}{c_{k-1}}, \quad k = 1, ..., g. \tag{56}$$

Note that $V = 0$ if and only if the generic condition (22) is violated.

By standard arguments (see e.g. [BBEIM]) based on the periodicity property (53) of the theta function, the functions $\Psi_{1,2}(P)$ are single-valued on Γ. Moreover, by their definitions and in view of (55), the functions $\Psi_1(P)$ and $\Psi_2(P)$ only have poles at the divisors $\mathcal{D}$ and $\mathcal{D}^*$ respectively. They also have an essential singularity at $P = \infty$, which is described by the equations,

$$\Psi_1(P) = e^{-\Omega_\infty(\lambda)} \frac{\theta(\mathbf{V} + \mathbf{D})}{\theta(\mathbf{D})} \left\{ 1 + O\left(\frac{1}{\sqrt{\lambda}}\right) \right\}, \tag{57}$$

$$P \to \infty,$$

$$\Psi_2(P) = e^{-\Omega_\infty(\lambda)} \frac{\theta(\mathbf{V} - \mathbf{D})}{\theta(\mathbf{D})} \left\{ 1 + O\left(\frac{1}{\sqrt{\lambda}}\right) \right\}, \tag{58}$$

$$P \to \infty.$$

Therefore, deforming the matrix $N(\lambda)$ (see (50)) according to the rule,

$$N(\lambda) \to \Psi(\lambda) = \Lambda N_{deformed}(\lambda) \equiv$$

$$\equiv \Lambda \begin{pmatrix} (\beta + \beta^{-1})\Psi_1(\lambda) & -i(\beta - \beta^{-1})\Psi_1^*(\lambda) \\ i(\beta - \beta^{-1})\Psi_2(\lambda) & (\beta + \beta^{-1})\Psi_2^*(\lambda) \end{pmatrix}, \tag{59}$$

where

$$\Psi_{1,2}(\lambda) = \Psi_{1,2}(P), \quad P = (\lambda, w) \in \text{upper sheet of } \Gamma,$$
$$\Psi^*_{1,2}(\lambda) = \Psi_{1,2}(P^*), \quad P^* = (\lambda, -w) \in \text{lower sheet of } \Gamma,$$

and taking into account that

$$[\Psi_{1,2}]_\pm(\lambda) = [\Psi^*_{1,2}]_\mp(\lambda), \quad \lambda \in \mathcal{L}_0,$$

we come to a solution of $(\mathbf{i^0}, \mathbf{ii^0})$. In order to satisfy $(\mathbf{iii^0})$, and hence to produce a solution of the canonical RH problem $(\mathbf{i^0} - \mathbf{iii^0})$, the matrix Λ must be chosen to satisfy the normalization condition (42), i.e.

$$\Lambda = d_g^{\frac{\sigma_3}{2}} \begin{pmatrix} 1 & 0 \\ -i\Omega_0 & 1 \end{pmatrix} N_\infty^{-1}, \tag{60}$$

where N_∞ is determined by the asymptotic equation:

$$N_{deformed}(\lambda) = N_\infty \left\{ I + O\left(\frac{1}{\lambda}\right)\right\} \lambda^{-\frac{1}{4}\sigma_3} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{-\Omega_\infty(\lambda)\sigma_3}, \tag{61}$$
$$\mathbf{C}_+ \ni \lambda \to \infty.$$

From the equations (57) and (58) written up to the term $\lambda^{-1} = \left(\lambda^{-1/2}\right)^2$ it follows that

$$\Lambda = \frac{1}{\kappa_+ + \kappa_-} d_g^{\frac{\sigma_3}{2}} \begin{pmatrix} \theta_+ & i\theta_- \\ i\kappa_-\theta_+ & \kappa_+\theta_- \end{pmatrix}, \tag{62}$$

where

$$\theta_\pm = \frac{\theta(\mathbf{D})}{\theta\left(\mathbf{V} \pm \mathbf{D}\right)}, \tag{63}$$

$$\kappa_\pm = 1 + 2\sum_{l=1}^{g} \omega_{l1} \left(\partial_{z_l} \ln \frac{\theta\left(\mathbf{D} \pm \mathbf{V}\right)}{\theta\left(\mathbf{D}\right)} \pm \frac{i}{2\pi} \ln d_l\right), \tag{64}$$

$$\left(\partial_{z_l}\theta\left(\mathbf{F}\right) \equiv \partial_{z_l}\theta\left(\mathbf{z}\right)|_{\mathbf{z}=\mathbf{F}}\right)$$

and the constants $\omega_{l1} = \omega_{l1}(a_1, ..., a_{2g+1})$ are defined through the expansion,

$$\frac{\omega_l(P)}{d\lambda} = \sum_{k=1}^{\infty} \omega_{lk}\lambda^{-k-1/2}, \quad P \to \infty. \tag{65}$$

Equations (59) and (36) complete the solution of the original RH problem $(\mathbf{i} - \mathbf{iii})$.

The final expression for $Y(\lambda)$ is given by the formula,

$$Y(\lambda) = \Lambda \begin{pmatrix} (\beta + \beta^{-1})\frac{\theta\left(\int_\infty^\lambda \omega - \mathbf{V} - \mathbf{D}\right)}{\theta\left(\int_\infty^\lambda \omega - \mathbf{D}\right)} & -i(\beta - \beta^{-1})\frac{\theta\left(\int_\infty^\lambda \omega + \mathbf{V} + \mathbf{D}\right)}{\theta\left(\int_\infty^\lambda \omega + \mathbf{D}\right)} \\ i(\beta - \beta^{-1})\frac{\theta\left(\int_\infty^\lambda \omega - \mathbf{V} + \mathbf{D}\right)}{\theta\left(\int_\infty^\lambda \omega + \mathbf{D}\right)} & (\beta + \beta^{-1})\frac{\theta\left(\int_\infty^\lambda \omega + \mathbf{V} - \mathbf{D}\right)}{\theta\left(\int_\infty^\lambda \omega - \mathbf{D}\right)} \end{pmatrix} \times$$

$$\times\, e^{-\frac{i}{2\pi}\sum_{j=1}^{g}\ln d_j \int_{a_{2g+1}}^{\lambda}\omega_j\sigma_3}, \tag{66}$$

where all the abelian integrals are taken on the upper sheet of Γ, and the integration paths lie in $\mathbf{C}_+$ and in $\mathbf{C}_-$ for $\lambda \in \mathbf{C}_+$ and $\lambda \in \mathbf{C}_-$ respectively.

It is worth noticing that any ambiguity (see (13)) related to the choice of the abelian integrals $\Omega(P)$ and $h(P)$ has disappeared. The integral $h(P)$ is not involved in the final formula (66) at all. The rhs in (66) only depends on the basic parameters,

$$a_1, ..., a_{2g+1}; \ln c_1, ..., \ln c_g; \ln d_1, ..., \ln d_g.$$

In fact, as soon as formula (66) for the solution $Y(\lambda)$ of the RH problem $(\mathbf{i}-\mathbf{iii})$ is written down, it can be verified by direct calculation using the periodicity property (53) of the theta function.

The calculations that led to (66) were based on the asumption that certain quantities (e.g. $\theta(\mathbf{D})$) were nonzero. In order to guarantee that these calculations are valid, it is sufficient to make the following generic assumptions on a_j, $\ln c_j$, $\ln d_j$ in addition to (1):

$$\theta(\mathbf{D}_0 - \mathbf{V})\theta(\mathbf{D}_0)\theta(\mathbf{D}_0 + \mathbf{V}) \neq 0, \tag{67}$$

where

$$\mathbf{D}_0 = \mathcal{A}(\mathcal{D}_0) + \mathbf{K} \equiv \sum_{j=0}^{g}\int_{\infty}^{P_j}\omega + \mathbf{K}\,,$$

and $\mathcal{D}_0$, as in (48), denotes a complete zero divisor of the sum $\beta(\lambda)+\beta^{-1}(\lambda)$. This condition arises for the following reasons. First we notice that

$$\theta(\mathbf{D}_0) = \theta\left(\int_{\infty}^{P_0}\omega + \mathbf{D}\right) = \theta\left(\int_{\infty}^{P_0^*}\omega - \mathbf{D}\right),$$

and hence the inequality $\theta(\mathbf{D}_0) \neq 0$ means that the theta functions $\theta\left(\int_{\infty}^{P}\omega \pm \mathbf{D}\right)$ are not identical zero on Γ. This fact in turn (see e.g. [Mum]) is equivalent to the nonspeciality of the reduced divisors $\mathcal{D}$ and $\mathcal{D}^*$ (note that $\mathbf{D}^* = -\mathbf{D}$, in view of (55)) . Secondly, for our construction to be consistent, we need the a priori invertibility of the matrix N_∞ (see (60) and (61)). We have that

$$\det N_\infty = \lim_{\lambda\to\infty}\det N_{deformed}(\lambda).$$

On the other hand, since the determinants of all the jump matrices of the RH problem $(\mathbf{i^0}-\mathbf{iii^0})$ are equal to 1, the determinant of $N_{deformed}(\lambda)$ is holomorphic on $\mathbf{CP}^1$. Therefore, it is a constant. We can evaluate this constant taking $\lambda = \lambda_0$ in (59), which corresponds to the point P_0 (see (48)). The substitution makes the matrix $N_{deformed}(\lambda)$ either off diagonal (the point P_0 lies on the upper sheet of Γ) or diagonal (the point P_0 lies on the lower sheet of Γ). This in turn shows that up to a nonzero multiplier the determinant of N_∞ coincides with the product,

$$\theta(\mathbf{D}_0 - \mathbf{V})\theta(\mathbf{D}_0 + \mathbf{V}),$$

and, together with $\theta(\mathbf{D}_0) \neq 0$, condition (67) follows.

Note that the invertibility of N_∞ (i.e. the existence of matrix Λ) implies the inequality,

$$\kappa_+ + \kappa_- \neq 0\,. \tag{68}$$

In the special case,

$$(a_1, ..., a_{2g+1}; \ln c_1, ..., \ln c_g; \ln d_1, ..., \ln d_g) \in \Pi_0 \quad \Longleftrightarrow \quad \mathbf{V} = 0, \tag{69}$$

the theta functions disappear from all the equations above, and direct verification of formula (66) for the solution $Y(\lambda)$ of the RH problem $(\mathbf{i} - \mathbf{iii})$ becomes straightforward. In this case, formula (66) reduces to the formula,

$$Y(\lambda) = \Lambda^0 \begin{pmatrix} \beta + \beta^{-1} & -i\beta + i\beta^{-1} \\ i\beta - i\beta^{-1} & \beta + \beta^{-1} \end{pmatrix} e^{-\frac{i}{2\pi}\sum_{j=1}^{g} \ln d_j \int_{a_{2g+1}}^{\lambda} \omega_j \sigma_3}, \tag{70}$$

where

$$\Lambda^0 = \frac{1}{2} d^{\frac{\sigma_3}{2}} \begin{pmatrix} 1 & i \\ i\kappa_-^0 & \kappa_+^0 \end{pmatrix}, \tag{71}$$

and

$$\kappa_\pm^0 = 1 \pm \Omega_0 = 1 \pm \frac{i}{\pi} \sum_{l=1}^{g} \omega_{l1} \ln d_l. \tag{72}$$

If in addition we assume that

$$\ln c_j = \ln d_j = 0, \quad \forall j, \tag{73}$$

formula (70) in turn reduces to the trivial equation,

$$Y(\lambda) = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} \beta + \beta^{-1} & -i\beta + i\beta^{-1} \\ i\beta - i\beta^{-1} & \beta + \beta^{-1} \end{pmatrix} \equiv \beta^{-\sigma_3} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \tag{74}$$

which obviously provides a solution of the RH problem $(\mathbf{i} - \mathbf{iii})$ corresponding to the choice,

$$c_j = d_1 = 1, \quad \forall j.$$

Finally observe that as $Y(\lambda)$ is the (unique) solution of the RH problem $(\mathbf{i} - \mathbf{iii})$ which only depends on the a_j's, c_j's, and d_j's, the function $Y(\lambda)$ does not depend on the particular branches of $\ln c_k$, $\ln d_k$. Thus,

$$a_1, ..., a_{2g+1}; c_1, ..., c_g; d_1, ..., d_g,$$

are our effective parameters. It is, in fact, not difficult to see this directly from (66); one again has to take into account the periodicity property (53) of the theta function.

The explicit formulae obtained above for the solution $Y(\lambda)$ of the Riemann-Hilbert problem $(\mathbf{i} - \mathbf{iii})$ constitute the main result of the paper, which can be formulated as the following theorem.

Theorem 1. *Let g be a positive interger, $g \geq 1$, and let*

$$a_1, ..., a_{2g+1}; c_1, ..., c_g; d_1, ..., d_g,$$

be the complex numbers such that ,

$$a_j \neq a_k, \quad c_j \neq 0, \quad d_j \neq 0, \qquad \forall j, k, \quad j \neq k,$$

and

$$\Re a_1 \leq \Re a_2 \leq \Re a_3 \leq ... \leq \Re a_{2g} \leq \Re a_{2g+1}. \tag{75}$$

Let

$$\theta(\mathbf{z}) = \sum_{\mathbf{m} \in \mathbf{Z}^g} \exp\left\{\frac{1}{2}(B\mathbf{m}, \mathbf{m}) + (\mathbf{z}, \mathbf{m})\right\}, \quad B_{kj} = \int_{\mathcal{B}_k} \omega_j,$$

denote the Riemann theta function associated with the hyperelliptic curve,

$$\Gamma = \left\{P = (\lambda, w) : w^2 = \prod_{j=1}^{2g+1}(\lambda - a_j)\right\}.$$

(We shall assume that the basis $\{\mathcal{A}_k, \mathcal{B}_k\}_{k=1}^g$ of the group $H_1(\Gamma)$ and the corresponding basis of the holomorphic differentials $\omega_j(P)$ are choosen as indicated in the introduction.) Denote also

$$\mathcal{D}_0 = \sum_{j=0}^{g} P_j, \quad P_j \in \Gamma,$$

the zero divisor of the sum $\beta(\lambda)+\beta^{-1}(\lambda)$, where the function $\beta(\lambda)$ is defined by the equation (cf. (45) and (46)),

$$\beta(\lambda) = \left[(\lambda - a_{2g+1})\prod_{l=1}^{g}\frac{\lambda - a_{2l-1}}{\lambda - a_{2l}}\right]^{\frac{1}{4}}.$$

Suppose that in addition to (75) the following generic assumption on the parameters a_j, c_j, d_j is satisfied,

$$\theta(\mathbf{D}_0 - \mathbf{V})\theta(\mathbf{D}_0)\theta(\mathbf{D}_0 + \mathbf{V}) \neq 0, \tag{76}$$

where

$$\mathbf{D}_0 = \mathcal{A}(\mathcal{D}_0) + \mathbf{K} \equiv \sum_{j=0}^{g}\int_{\infty}^{P_j}\omega + \mathbf{K},$$

$$\mathbf{V} = (V_1, ..., V_g), \qquad V_k = \frac{i}{2\pi}\sum_{j=1}^{g} B_{kj}\ln d_j + \ln\frac{c_k}{c_{k-1}}, \quad k = 1, ..., g,$$

and $\mathbf{K}$ stands for the vector of Riemann constants associated with the abelian mapping $\mathcal{A}$. Then, the 2×2 matrix function

$$Y(\lambda) \equiv Y(\lambda | a_1, ..., a_{2g+1}; c_1, ..., c_g; d_1, ..., d_g),$$

defined by the formulae (62) - (66) solves the Riemann-Hilbert problem $(\mathbf{i}-\mathbf{iii})$. *In addition, the function* $Y(\lambda)$ *satisfies a* 2×2 *Fuchsian system,*

$$\frac{dY(\lambda)}{d\lambda} = \sum_{j=1}^{2g+1} \frac{A_j}{\lambda - a_j} Y(\lambda),$$

whose monodromy matrices at the singular points are given by the equations (cf.(33) and (34)),

$$M_j = (-1)^j \begin{pmatrix} 0 & \rho_j \\ -\rho_j^{-1} & 0 \end{pmatrix}, \quad \rho_j = \begin{cases} -\frac{c_g d_g}{c_{k-1} d_k} & \text{for } j = 2k \\ -\frac{c_g d_g}{c_{k-1} d_{k-1}} & \text{for } j = 2k-1 \end{cases} \tag{77}$$

at $\lambda = a_j$, $j = 1, ..., 2g+1$, *and*

$$M_\infty = \begin{pmatrix} 0 & -d_g \\ d_g^{-1} & 0 \end{pmatrix}, \tag{78}$$

at $\lambda = \infty$.

Solution for the Schlesinger equations.

Equation (66) for the function $Y(\lambda)$ allows us to evaluate explicitly the holomorphic factor $\hat{Y}_j(\lambda)$ in representation (26). In particular, we obtain that

$$\hat{Y}_j(a_j) = \beta_j^{\frac{(-1)^j}{2}} e^{\frac{1}{2}\sum_{l=[(j+1)/2]}^{g} V_l} \Lambda_j, \tag{79}$$

where

$$\beta_j = \prod_{l\neq j,\, l=1}^{2g+1} (a_j - a_l)^{\frac{(-1)^{l+1}}{2}} \equiv \left[(\lambda - a_j)^{\frac{(-1)^j}{2}} \beta^2 \right]_{\lambda = a_j}, \tag{80}$$

(the branch of $(\lambda - a_j)^{\frac{1}{2}}$ is the same as in (26))

and matrix Λ_j is given by the equations (cf. (62)- (65)),

$$\Lambda_j = \frac{(-1)^{j-1}}{\kappa_+ + \kappa_-} d_g^{\frac{\sigma_3}{2}} \begin{pmatrix} \theta_+^{(j)} & i\theta_-^{(j)} \\ i\kappa_- \theta_+^{(j)} & \kappa_+ \theta_-^{(j)} \end{pmatrix} \times$$

$$\times\, \sigma_3^{j-1} \begin{pmatrix} \kappa_+^{(j)} & i \\ i\kappa_-^{(j)} & 1 \end{pmatrix} \sigma_2^{j-1} (-\rho_j)^{-\frac{\sigma_3}{2}}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{81}$$

$$\theta_\pm^{(j)} = \frac{\theta(\mathbf{D})\theta(\mathbf{E}_j - \mathbf{V} \mp \mathbf{D})}{\theta(\mathbf{V} \pm \mathbf{D})\theta(\mathbf{E}_j \mp \mathbf{D})}, \tag{82}$$

$$\kappa_\pm^{(j)} = \beta_j^{(-1)^{j-1}} - 2\sum_{l=1}^{g} \omega_{l1}^{(j)} \left(\partial_{z_l} \ln \frac{\theta(\mathbf{D} \pm \mathbf{V} + \mathbf{E}_j)}{\theta(\mathbf{D} + \mathbf{E}_j)} \pm \frac{i}{2\pi} \ln d_l \right). \tag{83}$$

In (83), (82) the constants $\omega_{l1}^{(j)} = \omega_{l1}^{(j)}(a_1, ..., a_{2g+1})$ are defined through the expansion,

$$\frac{\omega_l(P)}{d\lambda} = \sum_{k=1}^{\infty} \omega_{lk}^{(j)} (\lambda - a_j)^{k-3/2}, \quad P \to a_j, \tag{84}$$

and the half-period $\mathbf{E}_j$ is defined by the equations,

$$(\mathbf{E}_j)_l = \pi i \delta_{l[j/2]} - \frac{1}{2} \sum_{r=[(j+1)/2]}^{g} B_{rl}. \tag{85}$$

It is worth mentioning that

$$\det \Lambda_j = \beta_j^{(-1)^{j+1}} e^{-\sum_{l=[(j+1)/2]}^{g} V_l}. \tag{86}$$

Indeed, a priori we have that $\det \hat{Y}_j(a_j) = 1$. Equation (86) then follows from (79).

From (35) we conclude that

$$A_j = (-1)^{j+1} \Lambda_j \frac{\sigma_3}{4} \Lambda_j^{-1}, \tag{87}$$

which together with equation (81) **yields the following explicit formula for the solution of the Schlesinger equations (4)**:

$$A_j \equiv \{A_{j;mn}(a_1, ..., a_{2g+1} | c_1, ..., c_g; d_1, ..., d_g)\}_{m,n=1,2} \ ,$$

$$A_{j;mn} = -d_g^{n-m} \frac{\beta_j^{(-1)^j}}{2(\kappa_+ + \kappa_-)^2} e^{\sum_{l=[(j+1)/2]}^{g} V_l} (K_{mn}^j \theta, \theta), \tag{88}$$

$$m, n = 1, 2, \quad j = 1, ..., 2g+1,$$

where $(K_{mn}^j \theta, \theta)$ denotes the quadratic forms,

$$(K_{mn}^j \theta, \theta) = K_{mn;11}^j \theta_+^{(j)^2} + 2 K_{mn;12}^j \theta_+^{(j)} \theta_-^{(j)} + K_{mn;22}^j \theta_-^{(j)^2},$$

and the 2×2 symmetric matrices K_{mn}^j, $m, n = 1, 2$, are given by the equations,

$$K_{11}^j = \begin{pmatrix} \kappa_- \kappa_+^{(j)} & \frac{(-1)^j}{4}(\kappa_- - \kappa_+)(\kappa_-^{(j)} - \kappa_+^{(j)}) \\ \frac{(-1)^j}{4}(\kappa_- - \kappa_+)(\kappa_-^{(j)} - \kappa_+^{(j)}) & \kappa_+ \kappa_-^{(j)} \end{pmatrix},$$

$$K_{12}^j = \begin{pmatrix} \kappa_+^j & \frac{(-1)^j}{2}(\kappa_-^{(j)} - \kappa_+^{(j)}) \\ \frac{(-1)^j}{2}(\kappa_-^{(j)} - \kappa_+^{(j)}) & -\kappa_-^j \end{pmatrix},$$

and

$$K_{22}^j = -K_{11}^j, \qquad K_{21}^j = \begin{pmatrix} \kappa_- & 0 \\ 0 & -\kappa_+ \end{pmatrix} K_{12}^j \begin{pmatrix} \kappa_- & 0 \\ 0 & -\kappa_+ \end{pmatrix}.$$

(We also recall that under the generic assumption (76) the inequality $\kappa_+ + \kappa_- \neq 0$ takes place.)

Equation (88) of course also describes an explicit solution of the inverse monodromy problem,

$$\{M_1, ..., M_{2g+1}; a_1, ..., a_{2g+1}, \infty\} \mapsto \{A_1, ..., A_{2g+1}; a_1, ..., a_{2g+1}, \infty\}, \tag{89}$$

where the monodromy matrices M_j are given in (77) .

We emphasize that solution (88) is nonsingular as long as condition (76) is assumed to hold. It is of interest to compare this result with the general theory of singularities of the solutions of the Schlesinger equations (see [Mal], [Miw]).

We also notice that the choice,

$$c_j = d_j = 1, \quad \forall j,$$

yields (cf. 74) the trivial solution of the Schlesinger equations,

$$A_j = (-1)^j \frac{\sigma_3}{4}.$$

Warning. One can not use the "intermediate" formula (70) to generate solutions of Schlesinger equations. Monodromy data corresponding to (70) *depend* on the a_j's.

Hitchin's solution of the sixth Painlevé equation.

In the case:

$$g = 1, \quad a_1 = 0, \quad a_2 = t, \quad a_3 = 1, \tag{90}$$

the Fuchsian system (3) reduces to the equation,

$$\frac{dY(\lambda)}{d\lambda} = \left(\frac{A_1}{\lambda} + \frac{A_2}{\lambda - t} + \frac{A_3}{\lambda - 1}\right) Y(\lambda), \tag{91}$$

$$\mathrm{trace} A_j = 0, \quad j = 1, 2, 3,$$

$$A_1 + A_2 + A_3 = \text{diagonal matrix},$$

with four regular singular points at 0, t, 1, and ∞. The local behavior of the solution $Y(\lambda)$ (considered as a function on the universal covering of $\mathbf{C}\mathrm{P}^1 \backslash \{0, t, 1, \infty\}$) at the regular points is given by the equations (cf. (26), (30)),

$$Y(\lambda) = \hat{Y}_j(\lambda)(\lambda - a_j)^{\frac{\nu_j}{2}\sigma_3} C_j, \quad \lambda \sim a_j, \quad j = 1, 2, 3, \tag{92}$$

$$Y(\lambda) = \hat{Y}_\infty(\lambda)\lambda^{-\frac{\nu_\infty}{2}\sigma_3} C_\infty, \quad \lambda \sim \infty, \quad \hat{Y}_\infty(\infty) = I, \tag{93}$$

where

$$\nu_j = \frac{(-1)^{j+1}}{2}, \quad j = 1, 2, 3, \qquad \nu_\infty = \frac{1}{2}, \tag{94}$$

the matrix function $\hat{Y}_j(\lambda)$ is holomorphic and invertible in some neighborhood of $\lambda = a_j$, and the invertible constant matrices C_j are defined by the equations,

$$C_1 = \begin{pmatrix} 1 & -ic_1d_1 \\ -\frac{i}{c_1d_1} & 1 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 1 & -ic_1 \\ -\frac{i}{c_1} & 1 \end{pmatrix}, \tag{95}$$

$$C_3 = \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \qquad C_\infty = \begin{pmatrix} 1 & id_1 \\ \frac{i}{d_1} & 1 \end{pmatrix}. \tag{96}$$

The generators of the corresponding monodromy group (see (33), (34)) are:

$$M_1 = \begin{pmatrix} 0 & c_1d_1 \\ -\frac{1}{c_1d_1} & 0 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0 & -c_1 \\ \frac{1}{c_1} & 0 \end{pmatrix}, \tag{97}$$

$$M_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad M_\infty = \begin{pmatrix} 0 & -d_1 \\ \frac{1}{d_1} & 0 \end{pmatrix}. \tag{98}$$

It is shown in [JM], that the isomonodromy equations (4) for Fuchsian systems of the type (91) lead to the sixth Painlevé equation,

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} +$$
$$+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right), \tag{99}$$

for the function

$$y(t) = \frac{t}{t + (t-1)u(t)}, \quad u = \frac{(A_3)_{12}}{(A_1)_{12}}. \tag{100}$$

The parameters α, β, γ, δ in (99) are related to the monodromy exponents ν_j in (92), (93) via the equations,

$$\alpha = \frac{1}{2}(\nu_\infty - 1)^2, \quad \beta = -\frac{1}{2}\nu_1^2, \quad \gamma = \frac{1}{2}\nu_3^2, \quad \delta = \frac{1}{2}(1 - \nu_2^2).$$

It follows, that in our case (see (94),

$$\alpha = \frac{1}{8}, \quad \beta = -\frac{1}{8}, \quad \gamma = \frac{1}{8}, \quad \delta = \frac{3}{8}, \tag{101}$$

as in [H1].

In view of equation (88), the formula for the ratio $u(t)$ (see (100)) can be rewritten as

$$u(t) = \frac{\beta_3}{\beta_1}e^{-V_1}\frac{(K_{12}^3\theta,\theta)}{(K_{12}^1\theta,\theta)} =$$
$$= \sqrt{\frac{1-t}{t}}\, e^{-V_1}\frac{\theta_+^{(3)}\kappa_+^{(3)} - \kappa_-^{(3)}\theta_-^{(3)}}{\theta_+^{(1)}\kappa_+^{(1)} - \kappa_-^{(1)}\theta_-^{(1)}}\;\frac{\theta_+^{(3)} + \theta_-^{(3)}}{\theta_+^{(1)} + \theta_-^{(1)}}. \tag{102}$$

Equations (82) - (85) yield then an explicit representation of the general (depending on the two constants c_1, d_1) solution of the PVI equation (99), (101) in terms of the elliptic theta functions and integrals. In fact, under the assumption (90), the canonical B-matrix reduces to the single number,

$$B = \frac{2}{\omega_a}\int_t^1 \frac{d\lambda}{\sqrt{\lambda(\lambda-t)(\lambda-1)}}, \quad \omega_a = \frac{i}{\pi}\int_1^\infty \frac{d\lambda}{\sqrt{\lambda(\lambda-t)(\lambda-1)}}, \tag{103}$$

and $\theta(\mathbf{z})$ reduces to the one dimensional theta function,

$$\theta(z) = \sum_{m=-\infty}^{+\infty} \exp\left\{\frac{1}{2}Bm^2 + zm\right\}. \tag{104}$$

Simultaniously, the vectors $\mathbf{V}$, $\mathbf{E}_j, j = 1,2,3$, and $\mathbf{D}$ reduce to the scalars,

$$V = \frac{i}{2\pi}B\ln d_1 + \ln c_1, \tag{105}$$

$$E_1 = -\frac{B}{2}, \qquad E_2 = \pi i - \frac{B}{2}, \qquad E_3 = \pi i,, \tag{106}$$

$$D = \frac{1}{\omega_a}\int_{\lambda_+}^\infty \frac{d\lambda}{\sqrt{\lambda(\lambda-t)(\lambda-1)}} + \frac{B}{2} + \pi i, \quad \lambda_+ = 1 + \sqrt{1-t^2}, \tag{107}$$

where the elliptic integrals are taken on the upper sheet of the elliptic curve Γ, and λ_+ (one of the two zeros of the sum $\beta + \beta^{-1}$) lies on the upper side of the cut from 1 to ∞.

It is worth noticing, that the differential

$$\frac{d\lambda}{\sqrt{\lambda(\lambda-t)(\lambda-1)}}$$

is invariant with respect to the birational transformation,

$$\lambda \mapsto t\frac{\lambda-1}{\lambda-t},$$

and it is anti-invariant (changes sign) with respect to the transformation,

$$\lambda \mapsto \frac{t}{\lambda}.$$

Taking this fact into account, one can easiely evaluate the integral in (107) and obtain the equation,

$$D = \frac{\pi i}{2} + \frac{B}{2}. \tag{108}$$

Let

$$\theta_1(v) \equiv \theta_1(v|\tau) = i\sum_{m=-\infty}^{\infty}(-1)^m \exp\{i\pi\tau(m-1/2)^2 + i\pi(2m-1)v\},$$

$$\theta_2(v) \equiv \theta_2(v|\tau) = \sum_{m=-\infty}^{\infty} \exp\{i\pi\tau(m-1/2)^2 + i\pi(2m-1)v\},$$

$$\theta_3(v) \equiv \theta_3(v|\tau) = \sum_{m=-\infty}^{\infty} \exp\{i\pi\tau m^2 + i\pi 2mv\},$$

$$\theta_4(v) \equiv \theta_4(v|\tau) = \sum_{m=-\infty}^{\infty} (-1)^m \exp\{i\pi\tau m^2 + i\pi 2mv\},$$

$$\Im\tau > 0,$$

be the standard Jacobian theta functions (see e.g. [BE]). The quarter period expression (108) of the quantity D yields the representation of the general solution (100) of the sixth Painlevé equation (99), (101) in the following closed form,

$$y(t) = \frac{t}{t+(t-1)u(t)}, \tag{109}$$

$$u(t) \equiv u(t|c_1, d_1) =$$

$$= \frac{\theta_3^2(1/4)}{\theta_1^2(1/4)} \frac{\epsilon_+^3 \theta_1^2(v) - \epsilon_-^3 \theta_2^2(v)}{\epsilon_+^1 \theta_1(v)\theta_3(v) - \epsilon_-^1 \theta_2(v)\theta_4(v)} \frac{\theta_1^2(v) + \theta_2^2(v)}{\theta_1(v)\theta_3(v) + \theta_2(v)\theta_4(v)},$$

where

$$\epsilon_+^j = 1 + (-1)^{\frac{j-1}{2}} \frac{2}{\omega} \left[\frac{\theta'_{4-j}(v)}{\theta_{4-j}(v)} - \frac{\theta'_{4-j}(1/4)}{\theta_{4-j}(1/4)} - \ln d_1 \right],$$

$$\epsilon_-^j = 1 + (-1)^{\frac{j-1}{2}} \frac{2}{\omega} \left[-\frac{\theta'_{5-j}(v)}{\theta_{5-j}(v)} - \frac{\theta'_{4-j}(1/4)}{\theta_{4-j}(1/4)} + \ln d_1 \right],$$

$$j = 1, 3,$$

$$v = \frac{1}{2\pi i}(\ln c_1 - \tau \ln d_1) + \frac{1}{4},$$

$$\omega = -2 \int_1^\infty \frac{d\lambda}{\sqrt{\lambda(\lambda-t)(\lambda-1)}},$$

$$\tau = \frac{2}{\omega} \int_t^\infty \frac{d\lambda}{\sqrt{\lambda(\lambda-t)(\lambda-1)}},$$

$$\theta_j(v) \equiv \theta_j(v|\tau).$$

These formulae for the solution $y(t)$ do not exactly coincide with the ones found in [H1]. Nevertheless, both the expressions describe the same Painlevé function since the monodromy data indicated in (97)-(98) are the same as in [H1]. Of course the two expressions can also be shown to be equal by the direct calculation.

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