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Temperature correlators in the two-component one-dimensional gas

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ABSTRACT

The quantum nonrelativistic two-component Bose and Fermi gases with the infinitely strong point-like coupling between particles in one space dimension are considered. Time and temperature dependent correlation functions are represented in the thermodynamic limit as Fredholm determinants of integrable linear integral operators.

Introduction

The recent progress in calculating correlation functions of quantum solvable models is based on the fact that they are governed by classical integrable differential equations. That the language of classical differential equations is quite natural for the description of quantum correlation functions was realized a time ago [1, 2, 3, 4]. The idea of the approach suggested in [5, 6, 7] is to consider the Fredholm determinant in the representation for a correlation function of a quantum integrable model as a tau-function for a classical integrable system (see also the book [8] where the results for the simplest model of one-dimensional impenetrable bosons are reviewed). The necessary first step (which is also of interest by itself) in this approach is to represent the correlation function as the Fredholm determinant of a linear integral operator of a special kind (the "integrable" integral operator). It is to be mentioned that the first determinant representation of this kind was given in [9, 10] for the equal-time temperature correlators of the impenetrable bosons in one space dimension.

We consider the exactly solvable one-dimensional model of the nonrelativistic two-component Bose and Fermi gases with the infinitely large coupling constant, $c = \infty$. This model for any values of c was solved in the paper of Yang [11] where the eigenvectors of the Hamiltonian were constructed using relations known now as the famous Yang-Baxter relations. The general approach to the solution of the model with n internal degrees of freedom was given in paper [12], see also [13] (for the solution in the frame of the algebraic Bethe Ansatz see paper [14]). The Fredholm determinant representation for the equal-time temperature correlators of the Fermi gas with $c = \infty$ was derived in paper [15] where also the long distance asymptotics was constructed.

In our paper the Fredholm determinant representations for the time dependent temperature correlation functions are obtained, for both Bose and Fermi gases with $c = \infty$. To do this, we generalize the method used earlier for the one-component models [16, 17, 18, 19]. A presentation of our results was given in paper [20].

The contents of this paper is as follows. In Section 1, the explicit expressions for eigenstates (the nested Bethe eigenvectors) are given. At $c = \infty$ the auxiliary lattice problem for the nested Bethe Ansatz is reduced to the problem of finding eigenvectors for the cyclic shift operator and one can use the eigenvectors of the Heisenberg $XX0$ chain to solve it. Due to the simple form of the $XX0$ basis, all further calculations can be done explicitly. The form factors of the model in the finite box are calculated in Section 2. The normalized mean values of the products of two canonical fields with respect to Bethe eigenstates are obtained in Section 3. In Section 4, the expressions for the two point temperature correlation functions in the thermodynamic limit are derived which is the main result of our paper. Important particular cases are discussed in Section 5.

1 The eigenstates of the Hamiltonian

The model in the finite box of length L with the periodic boundary conditions is defined by the secondary quantized Hamiltonian

$$H = \int_0^L dx \left\{ (\partial_x \psi^+ \partial_x \psi) + c : (\psi^+ \psi)^2 : - h(\psi^+ \psi) + B(\psi^+ \sigma^z \psi) \right\}, \tag{1.1}$$

where c is a coupling constant, h is a chemical potential, B is a constant external transverse field and $: \, :$ means the normal ordering. The Pauli matrices are normalized as $[\sigma^x, \sigma^y] = 2i\sigma^z$. The one-dimensional fields $\psi_\alpha(x)$ and $\psi_\alpha^+(x)$ ($\alpha = 1, 2$) are quantum operators in the Fock space with the canonical commutation relations for the Bose gas ($\varepsilon = +1$ in the equations below) and with the canonical anticommutation relations for the Fermi gas ($\varepsilon = -1$):

$$\begin{aligned} &\psi_\alpha(x)\psi_\beta^+(y) - \varepsilon\, \psi_\beta^+(y)\psi_\alpha(x) = \delta_{\alpha\beta}\delta(x-y), \\ &\psi_\alpha^+(x)\psi_\beta^+(y) - \varepsilon\, \psi_\beta^+(y)\psi_\alpha^+(x) = 0, \quad \psi_\alpha(x)\psi_\beta(y) - \varepsilon\, \psi_\beta(y)\psi_\alpha(x) = 0. \end{aligned} \tag{1.2}$$

In what follows we consider both cases simultaneously. It appears that the dependence of the correlation functions on the statistics is rather simple.

The two-point temperature correlation functions are defined as the temperature normalized mean values,

$$\begin{aligned} G_\alpha^{(-)}(x,t;h,B) &= \frac{\mathrm{Tr}\left[e^{-H/T}\psi_\alpha^+(x,t)\psi_\alpha(0,0)\right]}{\mathrm{Tr}\left[e^{-H/T}\right]}, \\ G_\alpha^{(+)}(x,t;h,B) &= \frac{\mathrm{Tr}\left[e^{-H/T}\psi_\alpha(x,t)\psi_\alpha^+(0,0)\right]}{\mathrm{Tr}\left[e^{-H/T}\right]}, \end{aligned} \tag{1.3}$$

where

$$\psi_\alpha^+(x,t) = e^{itH}\psi_\alpha^+(x)e^{-itH}, \quad \psi_\alpha(x,t) = e^{itH}\psi_\alpha(x)e^{-itH} \tag{1.4}$$

and the traces are taken in the whole Fock space. Correlators G_1 and G_2 are related by changing the sign of the magnetic field,

$$G_1^{(\pm)}(x,t;h,B) = G_2^{(\pm)}(x,t;h,-B), \tag{1.5}$$

so that it is sufficient to calculate the correlators $G_1^{(\pm)}(x,t;h,B)$ only. Our aim is to calculate the correlators in the thermodynamic limit, i.e., at $L \to \infty$ keeping h and B fixed. To do this, we perform first the calculation for the finite box and then go to the limit.

The basis in the Fock space of the model is constructed by acting with operators $\psi_\alpha^+(x)$ onto the pseudovacuum $|0\rangle$ defined as

$$\psi_\alpha(x)|0\rangle = 0, \quad \langle 0|\psi_\alpha^+(x) = 0, \quad \langle 0|0\rangle = 1. \tag{1.6}$$

We say that a state belongs to the sector (N, M) of the Fock space if it contains $N-M$ particles of type 1 ($\alpha=1$) and M particles of type 2 ($\alpha=2$), so that the total number of particles is N. The number of particles of each type being conserved separately, an eigenstate of the Hamiltonian can be obtained as a linear superposition of the basis states from the same sector. In the sector (N, M) the eigenstates are enumerated by two sets, $\{k\}=k_1, \ldots, k_N$ and $\{\lambda\}=\lambda_1, \ldots, \lambda_M$, of unequal (in each set separately) real numbers. Thus, the eigenstates in the sector (N, M) can be written in the form

$$
\begin{aligned}
|\Psi_{N,M}(\{k\};\{\lambda\})\rangle = & \int_0^L dz_1 \ldots \int_0^L dz_N \sum_{\alpha_1,\ldots,\alpha_N=1,2} \chi_{N,M}^{\alpha_1\ldots\alpha_N}(z_1,\ldots,z_N|\{k\};\{\lambda\}) \\
& \times \psi_{\alpha_1}^{+}(z_1)\cdots\psi_{\alpha_N}^{+}(z_N)|0\rangle,
\end{aligned} \tag{1.7}
$$

where the wave function $\chi_{N,M}^{\alpha_1\ldots\alpha_N}(z_1,\ldots,z_N|\{k\};\{\lambda\})$ is not equal to zero only if M elements are equal to 2 and $N-M$ elements are equal to 1 in the set $\alpha_1,\ldots,\alpha_N$, i.e., $\sum_{j=1}^N \alpha_j = N+M$.

The wave functions for the finite coupling constant c were obtained in [11] (see also [13]). After imposing the periodic boundary conditions, the eigenfunctions of the Hamiltonian are given by the two-component (nested) Bethe Ansatz. The second component of the nested Ansatz (the "auxiliary lattice problem") is the Bethe Ansatz for the inhomogeneous XXX chain, which gives the higher weight eigenvectors. The complete basis is then obtained by acting with the Yangian generators onto the Bethe Ansatz vectors.

The model at $c=\infty$ is defined by taking the limit $c\to\infty$ in the expressions for the wave functions. This can be done differently. One way to do this is to take the limit in the final expressions for the Bethe eigenfunctions, keeping the XXX chain eigenfunctions for auxiliary lattice problem. We use another way. It is to take the limit in the expressions for the wave functions before imposing the periodic boundary conditions, and to satisfy the periodic boundary conditions after that. On this way the auxiliary lattice problem is reduced to the eigenstate problem for the cyclic shift operator on the lattice which can be solved explicitly, e.g., using eigenfunctions of the $XX0$ chain with the periodic boundary conditions. The advantage of this formulation is also the completeness of the $XX0$ eigenfunctions in the isospin space.

The Yang's wave functions for $c=\infty$ can be written in the form

$$
\begin{aligned}
& \chi_{N,M}^{\alpha_1\ldots\alpha_N}(z_1,\ldots,z_N|\{k\};\{\lambda\}) \\
& = \frac{1}{N!}\left[\sum_P (-\varepsilon)^{[P]} \xi_{N,M}^{\alpha_{P_1}\ldots\alpha_{P_N}}(\{\lambda\})\ \theta(z_{P1}<\ldots<z_{PN})\right] \det\nolimits_N\{e^{ik_a z_b}\},
\end{aligned} \tag{1.8}
$$

where $\xi_{N,M}^{\alpha_1\ldots\alpha_N}(\{\lambda\})$ are components of a 2^N-dimensional vector $\xi_{N,M}(\{\lambda\})$. The sum in (1.8) is taken over all the permutations of N numbers, $P:(1,2,\ldots,N)\to(P_1,P_2,\ldots,P_N)$; $[P]$ denotes the parity of the permutation.

The periodic boundary conditions for the wave function give the auxiliary lattice problem which is much simpler for $c = \infty$ than in the general case:

$$\xi_{N,M}^{\alpha_1\alpha_2\ldots\alpha_N}(\{\lambda\}) = (-\varepsilon)^{N+1} e^{ik_jL} \xi_{N,M}^{\alpha_2\ldots\alpha_N\alpha_1}(\{\lambda\}). \tag{1.9}$$

These equations should hold for all quasimomenta $k_1, \ldots, k_N$. They can be regarded as the eigenvalue problem for the cyclic shift operator C_N acting in a 2^N-dimensional vector space which can be identified with the space of states for the spin-$\frac{1}{2}$ chain with N sites.

The vectors $\xi_{N,M}$, which have to be eigenvectors of cyclic shift operator C_N, can be chosen as the eigenvectors of the $XX0$ Heisenberg spin-$\frac{1}{2}$ chain with the periodic boundary conditions. We write the expressions for $\xi_{N,M}$ in the form used in [19]:

$$\xi_{N,M}(\{\lambda\}) = \sum_{n_1=1}^{N} \cdots \sum_{n_M=1}^{N} \varphi_{N,M}(n_1, \ldots, n_M|\{\lambda\})\; \sigma^-_{(n_1)} \cdots \sigma^-_{(n_M)} |\Uparrow_N),$$

$$\varphi_{N,M}(n_1, \ldots, n_M|\{\lambda\}) = \frac{1}{M!}\left[\prod_{1\le j<l\le N} \mathrm{sgn}(n_l - n_j)\right] \det_M\{e^{i\lambda_a n_b}\}, \tag{1.10}$$

where $|\Uparrow_N) = \otimes_{n=1}^{N} |\uparrow_n)$ (all spins up) is the pseudovacuum for the chain with N sites, $(\Uparrow_N | \Uparrow_N) = 1$. The eigenvalues of C_N on the vectors (1.10) are $\exp\left(i\sum_{b=1}^{M}\lambda_b\right)$.

The periodic boundary conditions for the function $\varphi_{N,M}$ together with equation (1.9) result in the system of the Bethe equations,

$$\begin{array}{ll} e^{ik_aL} = (-\varepsilon)^{N-1} e^{i\sum_{b=1}^{M}\lambda_b}, & a = 1, \ldots, N, \\ e^{i\lambda_b N} = (-1)^{M+1}, & b = 1, \ldots, M. \end{array} \tag{1.11}$$

The permitted values $(k_a)_j \in (-\infty, +\infty)$ of the quasimomenta k_a $(a = 1, \ldots, N)$ are

$$(k_a)_j = \frac{2\pi}{L}\left(-\Big(\frac{1+\varepsilon}{2}\Big)\frac{N-1}{2} + j\right) + \frac{1}{L}\sum_{b=1}^{M}\lambda_b, \quad j \in Z, \tag{1.12}$$

while the permitted values $(\lambda_b)_l \in (-\pi, \pi]$ of the quasimomenta λ_b $(b = 1, \ldots, M)$ are

$$(\lambda_b)_l = \frac{2\pi}{N}\left(-\frac{N}{2} - \frac{1+(-1)^{N-M}}{4} + l\right), \quad l = 1, \ldots, N. \tag{1.13}$$

Each quasimomentum k_a (1.12) is a function of the total quasimomentum $\sum_{b=1}^{M}\lambda_b$ of the corresponding auxiliary lattice problem only. So one can decompose the quasimomentum k_a into two parts

$$k_a = \tilde{k}_a + \frac{1}{L}\sum_{b=1}^{M}\lambda_b \tag{1.14}$$

where $\tilde{k}_a$ does not depend on $\lambda_1, \ldots, \lambda_M$. We will often use this fact in what follows.

The eigenvalues $E_{N,M}(\{k\})$ of the Hamiltonian (1.1),

$$H \, |\Psi_{N,M}(\{k\};\{\lambda\})\rangle = E_{N,M}(\{k\}) \, |\Psi_{N,M}(\{k\};\{\lambda\})\rangle, \tag{1.15}$$

are given as

$$E_{N,M}(\{k\}) = \sum_{j=1}^{N}(k_j^2 - h + B) - 2MB. \tag{1.16}$$

In the case considered ($c = \infty$) they depend only on the total momentum of the auxiliary lattice problem which results in the additional degeneration of eigenstates in comparison with the case of finite c. In particular, the ground state at zero magnetic field B is degenerate. Indeed, the total momentum of the auxiliary lattice problem for the ground state should be equal to zero at $B = 0$, and the requirement of the minimal energy fixes only the total number of particles $N = N_1 + N_2$, the numbers $N_1 = N - M$ and $N_2 = M$ being otherwise arbitrary (of course, there exists also the additional degeneracy if even N and M are fixed). On the contrary, the ground state is not degenerate if $B \neq 0$. It is filled by the particles of the first kind ($N_1 = N$, $N_2 = 0$) if $B < 0$ and by the particles of the second kind ($N_1 = 0$, $N_2 = N$) if $B > 0$. Thus, there is the phase transition at point $B = 0$. It should be mentioned that the quantization of the total momentum is the same for the XXX and $XX0$ chains so that the same results can be obtained using the XXX chain formulation of the auxiliary lattice problem.

Consider now the properties of the eigenstates of the Hamiltonian. One can prove that two eigenstates belonging to different sectors as well as two eigenstates from the same sector but with different sets of quasimomenta are orthogonal, i.e.,

$$\langle\Psi_{N',M'}(\{k'\};\{\lambda'\})|\Psi_{N,M}(\{k\};\{\lambda\})\rangle = 0 \tag{1.17}$$

if $N' \neq N$ or $M' \neq M$ or $\{k'\} \neq \{k\}$ or $\{\lambda'\} \neq \{\lambda\}$ (two sets are considered to be different if one of them cannot be obtained from the other by a permutation of the elements). The normalization of eigenstates is easily computed to be

$$\begin{aligned}&\langle\Psi_{N,M}(k_1,\ldots,k_N;\lambda_1,\ldots,\lambda_M)|\Psi_{N,M}(k_1,\ldots,k_N;\lambda_1,\ldots,\lambda_M)\rangle\\ &\quad = N^M L^N.\end{aligned} \tag{1.18}$$

The eigenstates form the complete set,

$$\sum_{N=0}^{\infty}\sum_{M=0}^{N}\sum_{\substack{k_1<\ldots<k_N\\ \lambda_1<\ldots<\lambda_M}} \frac{|\Psi_{N,M}(\{k\};\{\lambda\})\rangle\langle\Psi_{N,M}(\{k\};\{\lambda\})|}{\langle\Psi_{N,M}(\{k\};\{\lambda\})|\Psi_{N,M}(\{k\};\{\lambda\})\rangle} = 1, \tag{1.19}$$

where the summations are performed over all different sets of solutions $k_1, \ldots, k_N$ (1.12) and $\lambda_1, \ldots, \lambda_M$ (1.13) of the Bethe equations (1.11).

Due to completeness of the eigenstates one can represent, e.g., the temperature correlation function $G_1^{(-)}(x,t;h,B)$ in the form

$$G_1^{(-)}(x,t;h,B) = \frac{\sum e^{-E_{N,M}(\{k\})/T}\langle\psi_1^+(x,t)\psi_1(0,0)\rangle_{N,M}}{\sum e^{-E_{N,M}(\{k\})/T}}, \tag{1.20}$$

where $\langle\psi_1^+(x,t)\psi_1(0,0)\rangle_{N,M}$ is the normalized mean value of the operator $\psi_1^+(x,t)\psi_1(0,0)$ with respect to the eigenstate $|\Psi_{N,M}(\{k\};\{\lambda\})\rangle$ and the summations in (1.20) are performed as in (1.19). The similar representation can be written for the correlator $G_1^{(+)}(x,t;h,B)$ as well.

2 Form factors of local field operators.

In this Section we calculate form factors of operators $\psi_\beta^+(x,t)$ and $\psi_\beta(x,t)$, i.e., their matrix elements between two eigenstates of the Hamiltonian. Since the form factors of operators $\psi_\beta^+(x,t)$ and $\psi_\beta(x,t)$ are related by means of complex conjugation, it is sufficient to calculate the form factors of operators $\psi_\beta(x,t)$. The nonvanishing form factors of $\psi_\beta(x,t)$ are

$$\mathcal{F}_{N,M}^{(\beta)}(x,t) \equiv \langle\Psi_{N,\tilde{M}}(\{q\};\{\mu\})|\psi_\beta(x,t)|\Psi_{N+1,M}(\{k\};\{\lambda\})\rangle, \tag{2.1}$$

where the notation

$$\tilde{M} = \begin{cases} M & \text{if} \quad \beta=1 \\ M-1 & \text{if} \quad \beta=2 \end{cases} \tag{2.2}$$

is introduced.

The form factor $\mathcal{F}_{N,M}^{(\beta)}(x,t)$ depends on the values of quasimomenta $\{k\},\{\lambda\}$ and $\{q\},\{\mu\}$ corresponding to the right and the left eigenstates in (2.1), respectively. Here and below the quasimomenta $\{k\},\{\lambda\}$ correspond to the eigenstates in the sector $(N+1,M)$, and the quasimomenta $\{q\},\{\mu\}$ are prescribed to the eigenstates in the sector $(N,\tilde{M})$. The Bethe equations for these momenta are

$$\begin{aligned}
&\{k\},\{\lambda\}: \begin{cases} e^{ik_aL} = \omega(-\varepsilon)^N & a=1,\ldots,N+1 \\ e^{i\lambda_b(N+1)} = (-1)^{M-1} & b=1,\ldots,M \end{cases} \\
&\{q\},\{\mu\}: \begin{cases} e^{iq_aL} = \nu(-\varepsilon)^{N-1} & a=1,\ldots,N \\ e^{i\mu_bN} = (-1)^{\tilde{M}-1} & b=1,\ldots,\tilde{M} \end{cases}
\end{aligned} \tag{2.3}$$

where ω and ν are eigenvalues of the cyclic shift operators of the corresponding auxiliary problems:

$$\omega = \exp\left(i\sum_{b=1}^{M}\lambda_b\right), \qquad \nu = \exp\left(i\sum_{b=1}^{\tilde{M}}\mu_b\right). \tag{2.4}$$

Using relation (2.5) and the eigenvalues (1.16) of the Hamiltonian, one extracts the time dependence of the form factor (2.1):

$$\mathcal{F}^{(\beta)}_{N,M}(x,t) = \exp\left\{ it\left(\sum_{j=1}^{N} q_j^2 - \sum_{j=1}^{N+1} k_j^2 + h_\beta \right) \right\} \; \mathcal{F}^{(\beta)}_{N,M}(x). \tag{2.5}$$

Here $h_1 = h - B$ and $h_2 = h + B$ are the chemical potentials of the particles of type 1 and of type 2, correspondingly. We denote

$$\mathcal{F}^{(\beta)}_{N,M}(x) \equiv \mathcal{F}^{(\beta)}_{N,M}(x,0). \tag{2.6}$$

The distance dependence of the form factor is described by function $\mathcal{F}^{(\beta)}_{N,M}(x)$. Using expressions (1.7) for eigenstates one represents it in terms of the wave functions as

$$\mathcal{F}^{(\beta)}_{N,M}(x) = (N+1)! \int_0^L dz_1 \ldots \int_0^L dz_N \sum_{\alpha_1,\ldots,\alpha_N} \bar{\chi}^{\alpha_1\ldots\alpha_N}_{N,\tilde{M}}(z_1,\ldots,z_N|\{q\};\{\mu\})$$
$$\times \chi^{\alpha_1\ldots\alpha_N\beta}_{N+1,M}(z_1,\ldots,z_N,x|\{k\};\{\lambda\}). \tag{2.7}$$

Here and below the bar denotes the complex conjugation. Substituting now the explicit expressions for the wave functions into (2.7) and producing the necessary calculation one comes to the expression

$$\mathcal{F}^{(\beta)}_{N,M}(x) = \frac{1}{N!} \int_0^L dz_1 \ldots \int_0^L dz_N \sum_{R\in S_N} \Big\{ \theta(z_{R_1} < \ldots < z_{R_N} < x) F_\beta(N)$$
$$+ \sum_{j=1}^{N-1} \theta(z_{R_1} < \ldots < z_{R_j} < x < z_{R_{j+1}} < \ldots < z_{R_N}) (-\varepsilon)^{N-j} F_\beta(j)$$
$$+ \theta(x < z_{R_1} < \ldots < z_{R_N}) (-\varepsilon)^N F_\beta(0) \Big\} \sum_{Q\in S_N} (-1)^{[Q]} e^{-i(q_{Q_1}z_1 + \cdots + q_{Q_N}z_N)}$$
$$\times \sum_{P\in S_{N+1}} (-1)^{[P]} e^{i(k_{P_1}z_1 + \cdots + k_{P_N}z_N + k_{P_{N+1}}x)}, \tag{2.8}$$

where

$$F_\beta(j) \equiv \sum_{\alpha_1,\ldots,\alpha_N} \bar{\xi}^{\alpha_1\ldots\alpha_N}_{N,\tilde{M}} \xi^{\alpha_1\ldots\alpha_j\beta\alpha_{j+1}\ldots\alpha_N}_{N+1,M}. \tag{2.9}$$

Further derivation requires more detailed explanation.

First, note that $\xi_{N+1,M}$ and $\xi_{N,\tilde{M}}$ are eigenvectors of the cyclic shift operators C_{N+1} and C_N with eigenvalues ω and ν, respectively (see (1.9) and (2.3), (2.4)), namely,

$$\xi^{\alpha_1\alpha_2\ldots\alpha_{N+1}}_{N+1,M} = \omega\, \xi^{\alpha_2\ldots\alpha_{N+1}\alpha_1}_{N+1,M}, \qquad \xi^{\alpha_1\alpha_2\ldots\alpha_N}_{N,\tilde{M}} = \nu\, \xi^{\alpha_2\ldots\alpha_N\alpha_1}_{N,\tilde{M}},$$
$$\omega^{N+1} = 1, \qquad \nu^N = 1. \tag{2.10}$$

Using relations (2.10) $(N-j)$ times for both vectors, one can move the superscript β in (2.9) to the right, preserving the same order of indexes $\alpha_1, \ldots, \alpha_N$ of $\bar{\xi}_{N,\tilde{M}}$ and $\xi_{N+1,M}$ with respect to which the summation in (2.9) is performed. This results in the expression

$$F_\beta(j) = (\bar{\omega}\nu)^{N-j} F_\beta, \qquad F_\beta \equiv F_\beta(N) \tag{2.11}$$

(one should take into account that $\bar{\omega} = \omega^{-1}, \bar{\nu} = \nu^{-1}$). Due to (2.11), one can move F_β in (2.8) out of the braces leaving the factors $(\bar{\omega}\nu)^{N-j}$ instead of $F_\beta(j)$ in the first sum (over the permutations $R \in S_N$).

Second, let us introduce the function

$$\rho(z) \equiv \theta(z) - \varepsilon\bar{\omega}\nu\theta(-z) \tag{2.12}$$

(which is an analogue of function $\mathrm{sgn}(z)$). It is not difficult to see that the following relation holds

$$\begin{aligned}
&\sum_{R\in S_N} \Big\{ \theta(z_{R_1} < \ldots < z_{R_N} < x) \\
&\quad + \sum_{j=1}^{N-1} \theta(z_{R_1} < \ldots < z_{R_j} < x < z_{R_{j+1}} < \ldots < z_{R_N})(-\varepsilon\bar{\omega}\nu)^{N-j} \\
&\quad + \theta(x < z_{R_1} < \ldots < z_{R_N})(-\varepsilon\bar{\omega}\nu)^{N} \Big\} \\
&\quad = \prod_{j=1}^{N} \rho(x - z_j)
\end{aligned} \tag{2.13}$$

for $z_1, \ldots, z_N, x \in [0, L]$ being all different. If two of z's do coincide, then the corresponding terms do not contribute into (2.8) due to the fact that the last sum with respect to the permutations $P \in S_{N+1}$ then vanishes. Hence the value $\rho(0)$ is unessential. Due to (2.11) and (2.13), the expression for $\mathcal{F}^{(\beta)}_{N,M}(x)$ can be rewritten in the form

$$\begin{aligned}
\mathcal{F}^{(\beta)}_{N,M}(x) &= \frac{1}{N!} F_\beta \int_0^L dz_1 \ldots \int_0^L dz_N \prod_{j=1}^{N} \rho(x - z_j) \\
&\times \sum_{\substack{Q\in S_N \\ P\in S_{N+1}}} (-1)^{[P]+[Q]} e^{i(k_{P_1} - q_{Q_1})z_1 + \cdots + i(k_{P_N} - q_{Q_N})z_N + ik_{P_{N+1}}x}.
\end{aligned} \tag{2.14}$$

Since the dependence on $z_1, \ldots, z_N$ in (2.14) is factorized, the integrals can be taken explicitly,

$$\int_0^L dz \rho(x - z) e^{i(k-q)z} = -i(1 + \varepsilon\bar{\omega}\nu) \frac{e^{i(k-q)x}}{(k - q)}. \tag{2.15}$$

The Bethe equations (2.3) for k and q were used in derivation of (2.15). Due to this relation we have

$$\mathcal{F}^{(\beta)}_{N,M}(x) = (-i)^N(1+\varepsilon\bar{\omega}\nu)^N F_\beta \left\{ \sum_{P\in S_{N+1}} (-1)^{[P]} \frac{1}{k_{P_1}-q_1} \times \cdots \times \frac{1}{k_{P_N}-q_N} \right\}$$
$$\times \exp\left\{ ix\left(\sum_{j=1}^{N+1} k_j - \sum_{l=1}^{N} q_l \right) \right\}. \tag{2.16}$$

The expression (2.16) is the desired representation for the form factor $\mathcal{F}^{(\beta)}_{N,M}(x)$. As usual, the dependence of the form factor on x is described by the "translational" exponential. It is multiplied by the determinant of a matrix depending on the momenta $\{k\}$ and $\{q\}$ only (the sum in the first braces in (2.16)). The factor F_β depends on the sets $\{\mu\}$ and $\{\lambda\}$ of the momenta of the auxiliary lattice problems only, being a special "scalar product" of two Bethe vectors defined on the lattices with N and $N+1$ sites:

$$F_\beta \equiv \sum_{\alpha_1,\ldots,\alpha_N} \bar{\xi}^{\alpha_1\ldots\alpha_N}_{N,\tilde{M}} \xi^{\alpha_1\ldots\alpha_N\beta}_{N+1,M}. \tag{2.17}$$

Now we are going to obtain the representation for F_β as a determinant of a matrix depending on $\{\mu\}$ and $\{\lambda\}$. To do this it is suitable to use the notations for the vectors $\bar{\xi}_{N,\tilde{M}}$ and $\xi_{N+1,M}$ as the bra and ket vectors, $(\xi_{N,\tilde{M}}|$ and $|\xi_{N+1,M})$, which are built upon the pseudovacua $(\Uparrow_N| := \otimes_{n=1}^{N}(\uparrow_n|$ and $|\Uparrow_{N+1}) := \otimes_{n=1}^{N+1}|\uparrow_n)$ of spin chains with N and $N+1$ sites, respectively. Let us introduce the new vector

$$(\widetilde{\xi}_{N,\tilde{M}}| \equiv (\xi_{N,\tilde{M}}| \otimes (\uparrow_{N+1}| \tag{2.18}$$

which is built upon the pseudovacuum $(\Uparrow_{N+1}| = \otimes_{n=1}^{N+1}(\uparrow_n|$. Then the scalar product (2.17) can be rewritten as

$$F_1 = \Big(\widetilde{\xi}_{N,M}(\{\mu\})\Big|\xi_{N+1,M}(\{\lambda\})\Big), \tag{2.19}$$

$$F_2 = \Big(\widetilde{\xi}_{N,M-1}(\{\mu\})\Big|\sigma^+_{(N+1)}\Big|\xi_{N+1,M}(\{\lambda\})\Big), \tag{2.20}$$

where the dependence on quasimomenta $\{\mu\}$ and $\{\lambda\}$ is written explicitly. Note that there are M quasimomenta in the set $\{\mu\}$ in (2.19) and $M-1$ quasimomenta in the set $\{\mu\}$ in (2.20); they satisfy different Bethe equations but nevertheless we denote these two sets by the same letter. At any particular case, it will be clear what the Bethe equations for the set $\{\mu\}$ are.

As mentioned above, the factors F_1 and F_2 given by (2.19) and (2.20) can be represented as determinants. This representation will be obtained now for F_1. The derivation of the representation for F_2 is quite similar.

Substituting the expressions for the Bethe vectors involved into (2.19) we get

$$F_1 = \sum_{m_1,\ldots,m_M=1}^{N} \sum_{n_1,\ldots,n_M=1}^{N+1} \overline{\varphi_{N,M}(m_1,\ldots,m_M|\{\mu\})}\, \varphi_{N+1,M}(n_1,\ldots,n_M|\{\lambda\})$$
$$\times \left(\Uparrow_{N+1} \left| \sigma^+_{(m_1)} \cdots \sigma^+_{(m_M)}\ \sigma^-_{(n_1)} \cdots \sigma^-_{(n_M)} \right| \Uparrow_{N+1} \right). \tag{2.21}$$

Since there are no $\sigma^+_{(N+1)}$ here, all the terms containing $\sigma^-_{(N+1)}$ do not contribute in (2.21). Hence the summations with respect to $n_1,\ldots,n_M$ can be performed also from 1 to N. Thus (2.21) is rewritten as

$$F_1 = M! \sum_{n_1,\ldots,n_M=1}^{N} \overline{\varphi_{N,M}(n_1,\ldots,n_M|\{\mu\})}\, \varphi_{N+1,M}(n_1,\ldots,n_M|\{\lambda\})$$
$$= \frac{1}{M!} \sum_{n_1,\ldots,n_M=1}^{N} \sum_{P\in S_M} (-1)^{[P]} e^{-i(\mu_{P_1} n_1 + \cdots + \mu_{P_M} n_M)}$$
$$\times \sum_{Q\in S_M} (-1)^{[Q]} e^{i(\lambda_{Q_1} n_1 + \cdots + \lambda_{Q_M} n_M)}, \tag{2.22}$$

which gives

$$F_1 = \sum_{P\in S_M} (-1)^{[P]} \left(\sum_{n_1=1}^{N} e^{i(\lambda_{P_1}-\mu_1)n_1} \right) \times \cdots \times \left(\sum_{n_M=1}^{N} e^{i(\lambda_{P_M}-\mu_M)n_M} \right). \tag{2.23}$$

This is, by definition, the determinant of $M \times M$-dimensional matrix. The determinant representation for F_2 can be derived in a similar way, only the terms with $\sigma^-_{(N+1)}$ contributing.

Let us now sum up the results of this Section.

For the form factors $\mathcal{F}^{(\beta)}_{N,M}(x,t)$ we obtain the following representation

$$\mathcal{F}^{(\beta)}_{N,M}(x,t) = (-i)^N (1+\varepsilon\bar{\omega}\nu)^N \mathrm{det}_M B_\beta\ \mathrm{det}_{N+1} D$$
$$\times \exp\left\{ \sum_{a=1}^{N+1} (-itk_a^2 + ixk_a) - \sum_{a=1}^{N} (-itq_a^2 + ixq_a) + ith_\beta \right\}, \tag{2.24}$$

where $h_1 = h - B$, $h_2 = h + B$. The matrix elements of $M \times M$-dimensional matrices $B_{1,2}$ are

$$(B_1)_{ab} = \sum_{n=1}^{N} e^{i(\lambda_a-\mu_b)n}, \qquad a,b = 1,\ldots,M,$$
$$(B_2)_{ab} = \sum_{n=1}^{N} e^{i(\lambda_a-\mu_b)n}, \quad (B_2)_{a,M} = 1,$$
$$a = 1,\ldots,M, \quad b = 1,\ldots,M-1, \tag{2.25}$$

and ω, ν are given by (2.4). The $(N+1)\times(N+1)$-dimensional matrix D has matrix elements

$$(D)_{ab}=\frac{1}{k_a-q_b},\quad (D)_{a,N+1}=1,\quad a=1,\dots,N+1,\quad b=1,\dots,N. \tag{2.26}$$

It should be noted, that the determinant of the $(N+1)\times(N+1)$-dimensional matrix D can be represented in terms of the determinant of the $N\times N$-dimensional matrix:

$$\det_{N+1}D=\left[1+\frac{\partial}{\partial z}\right]\det_N(A_1-zA_2)\bigg|_{z=0}. \tag{2.27}$$

The $N\times N$-dimensional matrices A_1 and A_2 have elements

$$(A_1)_{ab}=\frac{1}{k_a-q_b},\quad (A_2)_{ab}=\frac{1}{k_{N+1}-q_b},\qquad a,b=1,\dots,N. \tag{2.28}$$

Since the matrix A_2 is of rank equal to one, $\det_N(A_1-zA_2)$ is a linear function of z. Thus, the r.h.s. of (2.27) is equal to $\det_N(A_1-A_2)$. The relation (2.27) will be used in the next Section.

Let us note that the permitted values of q_a depend on the sum of the corresponding μ's. Hence the values of matrix elements of matrix D as well as the values of exponents in (2.24) are in fact different in cases $\beta=1$ and $\beta=2$ though being written down in the same way for the sake of simplicity.

Let us consider the particular cases $M=0$ and $M=N+1$. If $M=0$ then $\mathcal{F}^{(2)}_{N,M}(x,t)$ vanishes due to its definition while $\mathcal{F}^{(1)}_{N,M}(x,t)$ coincides with the form factor of the one-component gas. In this case we put $\det_{M=0}B_1\equiv 1$ (see the derivation of F_1 above), $\nu=\omega=1$, and the Bethe equations for $\{k\}$ and $\{q\}$ coincide with those in the one-component gas. Similarly, if $M=N+1$ then $\mathcal{F}^{(1)}_{N,M}(x,t)$ vanishes and $\mathcal{F}^{(2)}_{N,M}(x,t)$ coincides with the form factor of the one-component gas (in this case $\sum_{b=1}^{N+1}\lambda_b=0$, $\sum_{b=1}^{N}\mu_b=0$, hence $\nu=\omega=1$ and Bethe equations for $\{k\}$ and $\{q\}$ coincide with those in the one-component gas). The value of $\det_{M=N+1}B_2$ is a numerical factor related to the normalization of eigenstates (see (1.18)), being hence unessential.

3 Normalized mean values of bilocal operators.

In this section we derive the representations for the normalized mean values

$$\begin{aligned}&\langle\psi^+_\beta(x,t)\psi_\beta(0,0)\rangle_{N+1,M}\\ &\quad=\frac{\langle\Psi_{N+1,M}(\{k\};\{\lambda\})|\psi^+_\beta(x,t)\psi_\beta(0,0)|\Psi_{N+1,M}(\{k\};\{\lambda\})\rangle}{\langle\Psi_{N+1,M}(\{k\};\{\lambda\})|\Psi_{N+1,M}(\{k\};\{\lambda\})\rangle}\end{aligned} \tag{3.1}$$

and

$$
\begin{aligned}
&\langle \psi_\beta(x,t)\psi_\beta^+(0,0)\rangle_{N,M} \\
&\quad = \frac{\langle \Psi_{N,M}(\{q\};\{\mu\})|\psi_\beta(x,t)\psi_\beta^+(0,0)|\Psi_{N,M}(\{q\};\{\mu\})\rangle}{\langle \Psi_{N,M}(\{q\};\{\mu\})|\Psi_{N,M}(\{q\};\{\mu\})\rangle}
\end{aligned}
\tag{3.2}
$$

($\beta = 1, 2$) with respect to the eigenstates of the Hamiltonian. These quantities enter the representations for temperature correlation functions. Below the derivation of the representation in the case $\beta = 1$ is considered in detail, the derivation in the case $\beta = 2$ being quite similar.

Consider the normalized mean value (3.1) for $\beta = 1$. Inserting the complete set of eigenstates (see (1.19)) between operators $\psi_1^+(x,t)$ and $\psi_1(0,0)$ and taking into account the normalization condition (1.18) we get

$$
\begin{aligned}
\langle \psi_1^+(x,t)\psi_1(0,0)\rangle_{N+1,M} &= \frac{1}{L^{2N+1}N^M(N+1)^M} \sum_{\{q\},\{\mu\}} \overline{\mathcal{F}^{(1)}_{N,M}(x,t)}\;\mathcal{F}^{(1)}_{N,M}(0,0) \\
&= \frac{1}{L^{2N+1}N^M(N+1)^M} \sum_{\{q\},\{\mu\}} |1+\varepsilon\nu\bar{\omega}|^{2N} \left|\det{}_M B_1\right|^2 (\det{}_{N+1} D)^2 \\
&\times \exp\left\{ \sum_{a=1}^{N+1}(itk_a^2 - ixk_a) - \sum_{b=1}^{N}(itq_b^2 - ixq_b) - ith_1 \right\}.
\end{aligned}
\tag{3.3}
$$

Recall that $\omega = \exp i\Lambda$ and $\nu = \exp i\Theta$ where Λ and Θ are the total momenta of the corresponding auxiliary lattice problems:

$$
\Lambda \equiv \sum_{a=1}^{M}\lambda_a, \qquad \Theta \equiv \sum_{a=1}^{\tilde{M}}\mu_a. \tag{3.4}
$$

The summation over $\{q\}, \{\mu\}$ means the summation over all possible sets of quasimomenta $\{q\}$ and $\{\mu\}$. Since each q_a can be represented as $q_a = \tilde{q}_a + \Theta/L$ (see (1.14) and the comments there) where $\tilde{q}_a$ does not depend on μ's, one can perform the summation over $\{\tilde{q}\} \equiv \tilde{q}_1, \ldots, \tilde{q}_N$ independently of the summation over $\{\mu\}$. The expression under the sum in (3.3) is symmetric under the permutations of q's (and $\tilde{q}$'s as well) and μ's separately, being equal to zero whenever two q's or μ's coincide. Thus one can change the sum in (3.3) to the sum over all permitted values of each $\tilde{q}_a$ and each μ_b,

$$
\sum_{\{q\},\{\mu\}} \equiv \sum_{\substack{q_1<\cdots<q_N \\ \mu_1<\cdots<\mu_M}} \to \frac{1}{N!}\sum_{\tilde{q}_1}\cdots\sum_{\tilde{q}_N}\frac{1}{M!}\sum_{\mu_1}\cdots\sum_{\mu_M}, \tag{3.5}
$$

where the sums over each individual $\tilde{q}_a$ ($a = 1, \ldots, N$) and μ_b ($b = 1, \ldots, M$) are independent:

$$
\sum_{\tilde{q}_a} f(q_a) = \sum_{j\in Z} f((\tilde{q}_a)_j), \qquad \sum_{\mu_b} f(\mu_b) = \sum_{l=1}^{N} f((\mu_b)_l). \tag{3.6}
$$

The permitted values $(\tilde{q}_a)_j$ and $(\mu_b)_l$ are solutions of the Bethe equation (2.3). Explicitly,

$$
\begin{aligned}
(\tilde{q}_a)_j &= \frac{2\pi}{L}\left(-\Big(\frac{1+\varepsilon}{2}\Big)\frac{N-1}{2}+j\right), \qquad j\in Z, \\
(\mu_b)_l &= \frac{2\pi}{N}\left(-\frac{N}{2}-\frac{1+(-1)^{N-M}}{4}+l\right), \quad l=1,\ldots,N. \qquad (3.7)
\end{aligned}
$$

Taking into account (3.5), let us perform first the summations over $\tilde{q}_1,\ldots,\tilde{q}_N$ and then over $\mu_1,\ldots,\mu_M$ in (3.3). Note that the dependence on q's in (3.3) enters only the elements of matrix D and the exponential factor. The determinant of the matrix D can be represented as the sum over permutations,

$$
\det{}_{N+1}D = \sum_{P\in S_{N=1}} (-1)^{[P]} \prod_{a=1}^{N} \frac{1}{k_{P_a}-q_a}, \qquad (3.8)
$$

so that the dependence on each q_a in the r.h.s. of (3.3) is factorized. Therefore we can use the technique similar to that applied in the cases of the one-component impenetrable Bose-gas [16, 17] and of $XX0$ model [18, 19]. This procedure (which can be called "inserting the summation into determinant") results in our case in the following indentity:

$$
\begin{aligned}
&\frac{|1+\varepsilon\nu\bar{\omega}|^{2N}}{L^{2N+1}N!}\sum_{\tilde{q}_1}\cdots\sum_{\tilde{q}_N}(\det{}_{N+1}D)^2\exp\left\{\sum_{a=1}^{N+1}(itk_a^2-ixk_a)-\sum_{b=1}^{N}(itq_b^2-ixq_b)\right\} \\
&= \frac{\partial}{\partial z}\det{}_{N+1}(S^{(-)}+zR^{(-)})\Big|_{z=0}. \qquad (3.9)
\end{aligned}
$$

The matrix elements of the $(N+1)\times(N+1)$ matrices $S^{(-)}$ and $R^{(-)}$ are

$$
\begin{aligned}
(S^{(-)})_{ab} &= e_-(k_a)\; e_-(k_b)\;\frac{|1+\varepsilon\nu\bar{\omega}|^2}{L^2}\sum_{\tilde{q}}\frac{e^{-itq^2+ixq}}{(k_a-q)(k_b-q)}, \qquad q=\tilde{q}+\frac{\Theta}{L}, \\
(R^{(-)})_{ab} &= \frac{e_-(k_a)e_-(k_b)}{L}, \qquad (3.10)
\end{aligned}
$$

the function $e_-(k_a)$ is defined as

$$
e_-(k_a) = \exp\left(\frac{itk_a^2-ixk_a}{2}\right) \qquad (3.11)
$$

and z is a parameter. Since the rank of matrix $R^{(-)}$ is equal to one, $\det_{N+1}(S^{(-)}+zR^{(-)})$ is a linear function of z.

To go to the thermodynamic limit (which is done in the next Section) it is necessary to rewrite the expressions for the matrix elements of the matrix $S^{(-)}$. Note that matrix elements given by (3.10) for finite L are well-defined functions, since all possible zeros of the denominator of the expression under the sum over $\tilde{q}$ are canceled by zeros of the numerator coming from the factor $|1+\varepsilon\nu\bar{\omega}|^2$. It is due to the fact that the condition $k=q$ can be satisfied only if $\Lambda-\Theta=\pi\frac{1+\varepsilon}{2}(\mathrm{mod}\,2\pi)$, which follows from the Bethe equations. In the limit $L\to\infty$, however, the values of k and q become arbitrary, the summation over $\tilde{q}$ is changed for the integration $\frac{1}{L}\sum_{\tilde{q}}\to\frac{1}{2\pi}\int_{-\infty}^{\infty}d\tilde{q}$ and poles on the integration contour appear which should be taken into account before going to the limit $L\to\infty$. Therefore one has to rewrite the matrix elements of $S^{(-)}$ in terms of functions which are well defined in the thermodynamic limit.

The necessary calculations are given in Appendix. As the result, the matrix $S^{(-)}$ can be represented in the form

$$S^{(-)}=I+\frac{1+\varepsilon\,\cos(\Lambda-\Theta)}{2}\,V_1^{(-)}+\frac{\varepsilon\,\sin(\Lambda-\Theta)}{2}\,V_2^{(-)}, \tag{3.12}$$

where I is the unit $(N+1)\times(N+1)$ matrix, $(I)_{ab}=\delta_{ab}$, and matrices $V_{1,2}^{(-)}$ are

$$\begin{aligned}(V_1^{(-)})_{ab}&=\frac{2}{L}\,\frac{e_+^{(-)}(k_a)e_-(k_b)-e_-(k_a)e_+^{(-)}(k_b)}{k_a-k_b},\\ (V_2^{(-)})_{ab}&=\frac{2}{L}\frac{[e_-(k_a)]^{-1}e_-(k_b)-e_-(k_a)[e_-(k_b)]^{-1}}{k_a-k_b}.\end{aligned} \tag{3.13}$$

The functions involved are defined as

$$e_+^{(-)}(k)=e^{(-)}(k)e_-(k),\qquad e_-(k)=e^{\frac{itk^2-ixk}{2}}, \tag{3.14}$$

$$e^{(-)}(k)=\frac{2}{L}\sum_{\tilde{q}}\frac{e^{-itq^2+ixq}-e^{-itk^2+ixk}}{q-k},\qquad q=\tilde{q}+\frac{\Theta}{L}. \tag{3.15}$$

These functions are well defined in the thermodynamic limit.

The summation over $\tilde{q}$'s being fulfilled (3.9), one remains with the sum over the quasimomenta μ's in the expression (3.3) for the normalized mean value:

$$\begin{aligned}\langle\psi_1^+(x,t)\psi_1(0,0)\rangle_{N+1,M}&=\frac{1}{N^M(N+1)^M M!}\sum_{\mu_1}\cdots\sum_{\mu_M}|\mathrm{det}_M B_1|^2\\ &\times e^{-ith_1}\,\frac{\partial}{\partial z}\mathrm{det}_{N+1}(S^{(-)}+zR^{(-)})\Big|_{z=0}\end{aligned} \tag{3.16}$$

Let us perform the summation over $\mu_1,\ldots,\mu_M$. Note that each element of matrix $S^{(-)}$ depends on the sum $\Theta\equiv\mu_1+\cdots+\mu_M$ only. Moreover, since the summation over $\tilde{q}$ in (3.10) is governed by the rules (3.6), (3.7) and since $q=\tilde{q}+\Theta/L$, matrix elements $(S)_{ab}$ are periodic functions of Θ with the period 2π. Matrix $R^{(-)}$ does not depend on $\mu_1,\ldots,\mu_M$ at all, hence $\mathrm{det}_{N+1}(S^{(-)}+zR^{(-)})$ is also a periodic function of Θ. Further, $\exp i\Theta=\nu$ (see (2.4)) is an eigenvalue of cyclic shift operator C_N, hence Θ takes the values $2\pi n/N$ where $n = 0, 1, \ldots, N-1 \pmod N$ and we can write

$$1 = \sum_{n=0}^{N-1} \delta_{(N)} \left(N \frac{\mu_1 + \cdots + \mu_M}{2\pi} - n \right), \tag{3.17}$$

where $\delta_{(N)}$ is Kronecker symbol on Z_N, defined as

$$m \in Z, \quad \delta_{(N)}(m) = \begin{cases} 1 & \text{if } m = 0 \pmod N \\ 0 & \text{otherwise} \end{cases} \tag{3.18}$$

and can be represented as Fourier sum

$$\delta_{(N)}(m) = \frac{1}{N} \sum_{p=0}^{N-1} e^{\frac{2\pi i}{N} pm}. \tag{3.19}$$

Inserting (3.17) under the sum over $\mu_1, \ldots, \mu_M$ in (3.16) we can change $\Theta \to 2\pi n/N$ everywhere in the expressions for elements of the matrix $S^{(-)}$. Using the representation (3.19) we can rewrite (3.16) as

$$\begin{aligned} &\langle \psi_1^+(x,t)\psi_1(0,0)\rangle_{N+1,M} \\ &\quad = \frac{1}{N^{M+1}(N+1)^M M!} \sum_{\mu_1} \cdots \sum_{\mu_M} \sum_{n,p=0}^{N-1} e^{ip(\mu_1+\cdots+\mu_M) - \frac{2\pi i}{N} pn} \, |\mathrm{det}_M B_1|^2 \\ &\quad \times e^{-ith_1} \frac{\partial}{\partial z} \mathrm{det}_{N+1}(S_n^{(-)} + zR^{(-)}) \Big|_{z=0}, \end{aligned} \tag{3.20}$$

where matrices $S_n^{(-)}$ now do not depend on $\mu_1, \ldots, \mu_M$ at all and their elements are $(S_n^{(-)})_{ab} = (S^{(-)})_{ab}|_{\Theta=2\pi n/N}$. Matrices $S_n^{(-)}$ have the structure corresponding to (3.12),

$$S_n^{(-)} = I + \frac{1 + \varepsilon \, \cos(\Lambda - \frac{2\pi n}{N})}{2} V_{1,n}^{(-)} + \frac{\varepsilon \, \sin(\Lambda - \frac{2\pi n}{N})}{2} V_2^{(-)}, \tag{3.21}$$

where $V_{1,n}^{(-)} = V_1^{(-)}|_{\Theta=2\pi n/N}$, i.e., one should put $q = \tilde{q} + \frac{2\pi n}{NL}$ in the function $e^{(-)}$ (3.15) determining the elements of the matrix $V_1^{(-)}$.

Considering the determinant of matrix B_1 as the sum over permutations,

$$\mathrm{det}_M B_1 = \sum_{P \in S_M} (-1)^{[P]} \prod_{a=1}^{M} \left[\sum_{n_a=1}^{N} e^{i(\lambda_{P_a} - \mu_a) n_a} \right], \tag{3.22}$$

we see that the dependence on $\mu_1, \ldots, \mu_M$ in (3.20) can be factorized. Thus we can perform the procedure of "inserting summation into determinant" with respect to the summations over $\mu_1, \dots, \mu_M$. Picking up the relevant terms from (3.20) one comes to the indentity

$$\frac{1}{N^M (N+1)^M M!} \sum_{\mu_1} \cdots \sum_{\mu_M} e^{ip(\mu_1+\cdots+\mu_M)} \left|\det{}_M B_1\right|^2 = \det{}_M U_p^{(1,-)}, \tag{3.23}$$

where $M \times M$ matrices $U_p^{(1,-)}$ have elements

$$(U_p^{(1,-)})_{ab} = \frac{1}{N(N+1)} \sum_{n=1}^{N} \sum_{m=1}^{N} \sum_{\mu} e^{i(p+m-n)\mu + in\lambda_a - im\lambda_b}. \tag{3.24}$$

Finally, the normalized mean value of operator $\psi_1^+(x,t)\psi_1(0,0)$ takes the following form

$$\begin{aligned}\langle \psi_1^+(x,t)\psi_1(0,0)\rangle_{N+1,M} &= e^{-ith_1} \frac{1}{N} \sum_{n,p=0}^{N-1} e^{-\frac{2\pi i}{N}pn} \\ &\times \det{}_M U_p^{(1,-)} \left[\det{}_{N+1}(S_n^{(-)} + R^{(-)}) - \det{}_{N+1} S_n^{(-)}\right],\end{aligned} \tag{3.25}$$

where the property $\frac{\partial}{\partial z} f(z)|_{z=0} = f(1) - f(0)$ valid for a linear function $f(z)$ is used. Expression (3.25) is the final expression for the normalized mean value considered.

For the normalized mean value $\langle \psi_2^+(x,t)\psi_2(0,0)\rangle_{N+1,M}$ all the calculations presented above can be done similarly. The main difference is in the summation over the values of quasimomenta of the auxiliary lattice problem, i.e., over the set $\{\mu\}$, which in the latter case is $\{\mu\} = \mu_1, \dots, \mu_{M-1}$. The representation analogous to equation (3.25) for this normalized mean value is

$$\begin{aligned}\langle \psi_2^+(x,t)\psi_2(0,0)\rangle_{N+1,M} &= e^{-ith_2} \frac{1}{N} \sum_{n,p=0}^{N-1} e^{-\frac{2\pi i}{N}pn} \\ &\times \left[\det{}_M(U_p^{(2,-)} + P^{(-)}) - \det{}_M U_p^{(2,-)}\right] \\ &\times \left[\det{}_{N+1}(S_n^{(-)} + R^{(-)}) - \det{}_{N+1} S_n^{(-)}\right].\end{aligned} \tag{3.26}$$

Here the $M \times M$ matrix $P^{(-)}$ is of rank one

$$(P^{(-)})_{ab} = \frac{1}{N+1} \tag{3.27}$$

and matrices $U_p^{(2,-)}$ have the elements

$$(U_p^{(2,-)})_{ab} = \frac{1}{N(N+1)} \sum_{n=1}^{N} \sum_{m=1}^{N} \sum_{\mu} e^{i(p+m-n)\mu + in\lambda_a - im\lambda_b}. \tag{3.28}$$

It should be noted that though the matrices $U_p^{(2,-)}$ and $U_p^{(1,-)}$ look formally the same, their matrix elements are different (except, e.g., the case $p=0$, see below) since quasimomenta $\{\mu\}$ in these two cases satisfy different Bethe equations.

Consider the particular case $x=0$, $t=0$. Then $S_n = I$ and only the terms with $p=0$ contribute into the normalized mean values. It is easy to see that

$$(U_0^{(1,-)})_{ab} = (U_0^{(2,-)})_{ab} = \delta_{ab} - \frac{1}{N+1} \tag{3.29}$$

and it follows from (3.25), (3.26) that

$$\begin{aligned}
\langle \psi_1^+(0,0)\psi_1(0,0)\rangle_{N+1,M} &= \frac{(N+1)-M}{L}, \\
\langle \psi_2^+(0,0)\psi_2(0,0)\rangle_{N+1,M} &= \frac{M}{L},
\end{aligned} \tag{3.30}$$

which is quite obvious result since in this case the normalized mean values are just the expectation values of density operators for particles of type 1 and type 2, respectively.

Let us consider now another normalized mean value, given by (3.2). Inserting the complete set of eigenstates we get

$$\begin{aligned}
\langle \psi_1(x,t)\psi_1^+(0,0)\rangle_{N,M} &= \frac{1}{L^{2N+1}N^M(N+1)^M} \sum_{\{k\},\{\lambda\}} \mathcal{F}^{(1)}_{N,M}(x,t)\ \overline{\mathcal{F}^{(1)}_{N,M}(0,0)} \\
&= \frac{1}{L^{2N+1}N^M(N+1)^M} \sum_{\{k\},\{\lambda\}} |1+\varepsilon\nu\bar{\omega}|^{2N}\, |\mathrm{det}_M B_1|^2 \\
&\times \mathrm{det}_{N+1} D\ \left[1+\frac{\partial}{\partial z}\right] \mathrm{det}_N(A_1 - zA_2)\Big|_{z=0} \\
&\times \exp\left\{ \sum_{b=1}^{N} (itq_b^2 - ixq_b) - \sum_{a=1}^{N+1}(itk_a^2 - ixk_a) + ith_1 \right\},
\end{aligned} \tag{3.31}$$

where the relation (2.27) is used for one of the form factors. Again, as above, we can perform the summation as follows

$$\sum_{\{k\},\{\lambda\}} \equiv \sum_{\substack{k_1<\cdots<k_{N+1}\\ \lambda_1<\cdots<\lambda_M}} \to \frac{1}{(N+1)!} \sum_{\tilde{k}_1} \cdots \sum_{\tilde{k}_{N+1}} \frac{1}{M!} \sum_{\lambda_1} \cdots \sum_{\lambda_M}. \tag{3.32}$$

The procedure of "inserting the summation under determinant" with respect to the summation over $\tilde{k}_1, \ldots, \tilde{k}_{N+1}$ gives the following result:

$$\frac{|1+\varepsilon\nu\bar{\omega}|^{2N}}{L^{2N+1}(N+1)!} \sum_{\tilde{k}_1} \cdots \sum_{\tilde{k}_{N+1}} \mathrm{det}_{N+1} D\ \left[1+\frac{\partial}{\partial z}\right] \mathrm{det}_N(A_1 - zA_2)\Big|_{z=0}$$

$$\times \exp\left\{\sum_{b=1}^{N}(itq_b^2 - ixq_b) - \sum_{a=1}^{N+1}(itk_a^2 - ixk_a)\right\}$$
$$= \left[g(x,t) + \frac{\partial}{\partial z}\right] \det{}_N(S^{(+)} - zR^{(+)})\Big|_{z=0}. \tag{3.33}$$

Here we introduce the function

$$g(x,t) = \frac{1}{L}\sum_{\tilde{k}} e^{-itk^2+ixk}, \qquad k = \tilde{k} + \frac{\Lambda}{L}. \tag{3.34}$$

The $N \times N$ matrices $S^{(+)}$ and $R^{(+)}$ are

$$(S^{(+)})_{ab} = e_-(q_a)\ e_-(q_b)\ \frac{|1+\varepsilon\nu\bar{\omega}|^2}{L^2}\sum_{\tilde{k}}\frac{e^{-itk^2+ixk}}{(k-q_a)(k-q_b)}, \qquad k = \tilde{k} + \frac{\Lambda}{L},$$
$$(R^{(+)})_{ab} = \frac{|1+\varepsilon\nu\bar{\omega}|^2}{L^3}\left[e_-(q_a)\sum_{\tilde{k}}\frac{e^{-itk^2+ixk}}{k-q_a}\right]\left[e_-(q_b)\sum_{\tilde{k}'}\frac{e^{-itk'^2+ixk'}}{k'-q_b}\right] \tag{3.35}$$

The matrix $R^{(+)}$ is of rank equal to one, and $\det_N(S^{(+)} - zR^{(+)})$ is a linear function of the parameter z.

These matrices can be put into the form (see Appendix)

$$S^{(+)} = I + \frac{1+\varepsilon\ \cos(\Lambda-\Theta)}{2}\ V_1^{(+)} - \frac{\varepsilon\ \sin(\Lambda-\Theta)}{2}\ V_2^{(+)},$$
$$R^{(+)} = \frac{1+\varepsilon\ \cos(\Lambda-\Theta)}{2}\ R_1^{(+)}$$
$$-\varepsilon\ \sin(\Lambda-\Theta)\ R_2^{(+)} + \frac{1-\varepsilon\ \cos(\Lambda-\Theta)}{2}\ R_3^{(+)}, \tag{3.36}$$

where matrices involved are

$$(V_1^{(+)})_{ab} = \frac{2}{L}\,\frac{e_+^{(+)}(q_a)e_-(q_b) - e_-(q_a)e_+^{(+)}(q_b)}{q_a - q_b},$$
$$(V_2^{(+)})_{ab} = \frac{2}{L}\,\frac{[e_-(q_a)]^{-1}e_-(q_b) - e_-(q_a)[e_-(q_b)]^{-1}}{q_a - q_b},$$
$$(R_1^{(+)})_{ab} = \frac{e_+^{(+)}(q_a)e_+^{(+)}(q_b)}{L},$$
$$(R_2^{(+)})_{ab} = \frac{1}{2L}\left[\frac{e_+^{(+)}(q_a)}{e_-(q_b)} + \frac{e_+^{(+)}(q_b)}{e_-(q_a)}\right],$$
$$(R_3^{(+)})_{ab} = \frac{1}{Le_-(q_a)e_-(q_b)}. \tag{3.37}$$

Here

$$e_+^{(+)}(q) = e_-(q)e^+(q), \quad e_-(q) = e^{\frac{itq^2-ixq}{2}} \tag{3.38}$$

and

$$e^{(+)}(q) = \frac{2}{L}\sum_{\tilde{k}} \frac{e^{-itk^2+ixk} - e^{-itq^2+ixq}}{k-q}, \quad k = \tilde{k} + \frac{\Lambda}{L}. \tag{3.39}$$

Due to (3.33), the normalized mean value (3.31) can be rewritten as

$$\langle\psi_1(x,t)\psi_1^+(0,0)\rangle_{N,M} = \frac{1}{N^M(N+1)^M M!}\sum_{\lambda_1}\cdots\sum_{\lambda_M}|\det{}_M B_1|^2$$
$$\times e^{ith_1}\left[g(x,t) + \frac{\partial}{\partial z}\right]\det{}_N(S^{(+)} - zR^{(+)})\Big|_{z=0}. \tag{3.40}$$

The procedure of "inserting the summation into determinant" with respect to λ's is performed in a similar way as above. Using the fact that $\det_{N+1}(S^{(+)} - zR^{(+)})$ is a 2π periodic function of Λ and that $\Lambda = \frac{2\pi m}{N+1}$ where $m = 0, 1, \ldots, N \pmod{N+1}$ we insert

$$1 = \sum_{m=0}^{N} \delta_{(N+1)}\Big(m - (N+1)\frac{\lambda_1 + \cdots + \lambda_M}{2\pi}\Big) \tag{3.41}$$

under the sums over λ's in (3.40). Changing $\Lambda \to \frac{2\pi m}{N+1}$ in the elements of matrices $S^{(+)}, R^{(+)}$ and in function $g(x,t)$ (3.34) and using the representation (3.19) for the Kronecker symbol, we rewrite (3.40) as

$$\langle\psi_1(x,t)\psi_1^+(0,0)\rangle_{N,M}$$
$$= \frac{1}{N^M(N+1)^{M+1}M!}\sum_{\lambda_1}\cdots\sum_{\lambda_M}\sum_{r,m=0}^{N} e^{\frac{2\pi i}{N+1}rm - ir(\lambda_1+\cdots+\lambda_M)}\ |\det{}_M B_1|^2$$
$$\times e^{ith_1}\left[g_m(x,t) + \frac{\partial}{\partial z}\right]\det{}_N(S_m^{(+)} - zR_m^{(+)})\Big|_{z=0}, \tag{3.42}$$

where subscript m means that everywhere in the functions determining the elements of the matrices $S^{(+)}, R^{(+)}$ and in the function $g(x,t)$ one should put $2\pi m/(N+1)$ instead of Λ.

Similarly to (3.23), we have the relation

$$\frac{1}{N^M(N+1)^M M!}\sum_{\lambda_1}\cdots\sum_{\lambda_M} e^{-ir(\lambda_1+\cdots+\lambda_M)}\,|\det{}_M B_1|^2 = \det{}_M U_r^{(1,+)}, \tag{3.43}$$

where the $M \times M$ matrices $U_r^{(1,+)}$ have elements

$$(U_r^{(1,+)})_{ab} = \frac{1}{N(N+1)}\sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\lambda} e^{-i(r+m-n)\lambda + in\mu_a - im\mu_b}. \tag{3.44}$$

Finally, for the normalized mean value $\langle\psi_1(x,t)\psi_1^+(0,0)\rangle_{N,M}$ we obtain the representation

$$\langle\psi_1(x,t)\psi_1^+(0,0)\rangle_{N,M} = e^{ith_1}\frac{1}{N+1}\sum_{r,m=0}^{N} e^{\frac{2\pi i}{N+1}rm}$$
$$\times \det{}_M U_r^{(1,+)}\left[\det{}_N(S_m^{(+)}-R_m^{(+)}) + (g_m(x,t)-1)\det{}_N S_m^{(+)}\right]. \qquad (3.45)$$

For the sake of completeness, let us write down the representation for normalized mean value of operator $\psi_2(x,t)\psi_2^+(0,0)$:

$$\langle\psi_2(x,t)\psi_2^+(0,0)\rangle_{N,M-1} = e^{ith_1}\frac{1}{N+1}\sum_{r,m=0}^{N} e^{\frac{2\pi i}{N+1}rm}$$
$$\times\left[\det{}_{M-1}(U_r^{(2,+)}-P_r^{(+)}) + (\delta_{r,0}-1)\det{}_{M-1}U_r^{(2,+)}\right]$$
$$\times\left[\det{}_N(S_m^{(+)}-R_m^{(+)}) + (g_m(x,t)-1)\det{}_N S_m^{(+)}\right]. \qquad (3.46)$$

Here the $(M-1)\times(M-1)$ matrices $U_r^{(2,+)}$ have elements (recall that in this case there are $M-1$ μ's in the set $\{\mu\}$)

$$(U_r^{(2,+)})_{ab} = \frac{1}{N(N+1)}\sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{\lambda} e^{-i(r+m-n)\lambda+im\mu_a-in\mu_b} \qquad (3.47)$$

and the matrix $P_r^{(+)}$ is of rank equal to one

$$(P_r^{(+)})_{ab} = \frac{1}{N(N+1)^2}\sum_{\lambda}\sum_{m=1}^{N} e^{-ir\lambda-im(\lambda-\mu_a)}\sum_{\lambda'}\sum_{n=1}^{N} e^{-ir\lambda'+in(\lambda'-\mu_b)}. \qquad (3.48)$$

The results obtained in this Section allow us to calculate the correlation functions in the thermodynamic limit.

4 Temperature correlators in the thermodynamic limit

Here we present the derivation of the temperature correlation functions $G_{1,2}^{(\pm)}$ (1.3) in the thermodynamic limit $L\to\infty$. The correlation functions $G_1^{(\pm)}$ and $G_2^{(\pm)}$ are related by inversing the sign of the external field,

$$G_2^{(\pm)}(x,t;h,B) = G_1^{(\pm)}(x,t;h,-B), \qquad (4.1)$$

so that it is sufficient to calculate only one of them (an independent calculation of both correlators confirm, of course, the relation (4.1)). Our result is the determinant representation of the correlators which is the generalization of the results obtained earlier for one-component models [16, 17, 18, 19].

Consider first the partition function of the gas which is the denominator in (1.3). It can be written in the form

$$\mathrm{Tr}\left[e^{-H/T}\right] = 1 + \sum_{N=1}^{\infty}\sum_{M=0}^{N}\sum_{\tilde{q}_1<\ldots<\tilde{q}_N}\sum_{\mu_1<\ldots<\mu_M} e^{-E_{N,M}(\{q\})/T}, \tag{4.2}$$

where the energy in the sector (N, M) is given by

$$E_{N,M}(\{q\}) = \sum_{a=1}^{N}(q_a^2 - h + B) - 2MB. \tag{4.3}$$

As usual,

$$q_a = \tilde{q}_a + \frac{\Theta}{L}, \qquad \Theta = \mu_1 + \cdots + \mu_M. \tag{4.4}$$

For finite L, the partition function can be presented as an explicit linear combination of several determinants of "discrete" infinite dimensional matrices. This representation allows us to obtain strictly the thermodynamic limit ($L \to \infty$) of the partition function and also calculate the $1/L$ corrections to the free energy. The corresponding results will be given in a separate publication. Below we will consider the thermodynamic limit only.

Note that due to the summation over all possible values of $\tilde{q}_1 < \ldots < \tilde{q}_N$ in (4.2) it is possible, by making the same shift of all $\tilde{q}_a$, $\tilde{q}_a \to \tilde{q}'_a$, to present each q_a as

$$q_a = \tilde{q}'_a + \frac{\tilde{\Theta}}{L}, \qquad |\tilde{\Theta}| \le \frac{\pi}{L}. \tag{4.5}$$

So in the thermodynamic limit one can neglect Θ in (4.4). For the same reason one can neglect in the limit also the difference between the permitted values of $\tilde{q}_a$ in the sectors with odd or even number of particles N (see (1.12)). Under this conditions the summation over $\mu_1 < \ldots < \mu_M$ and over M can be done with the result

$$\sum_{M=0}^{N}\sum_{\mu_1<\ldots<\mu_M} e^{\frac{2B}{T}M} = (1 + e^{\frac{2B}{T}})^N. \tag{4.6}$$

The permitted values of quasimomenta $\tilde{q}_a$ ($a = 1, \ldots, N$),

$$(\tilde{q}_a)_{j+1} - (\tilde{q}_a)_j = \frac{2\pi}{L}, \tag{4.7}$$

fill the interval $(-\infty, \infty)$ densely in the thermodynamic limit and the sum with respect to $\tilde{q}_a$ should be changed for the integral

$$\frac{1}{L}\sum_{\tilde{q}} f(\tilde{q}) = \frac{1}{L}\sum_{j\in Z} f((\tilde{q})_j) \longrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tilde{q} f(\tilde{q}). \tag{4.8}$$

Thus, we have for the partition function

$$\begin{aligned}
\mathrm{Tr}\left[e^{-H/T}\right] &= 1+\sum_{N=1}^{\infty}\sum_{\tilde q_1<\ldots<\tilde q_N}(1+e^{\frac{2B}{T}})^N e^{-\frac{1}{T}\sum_{a=1}^{N}(q_a^2-h+B)}\\
&= \prod_{\tilde q}\left[1+2\cosh\tfrac{B}{T}\,e^{-\frac{q^2-h}{T}}\right]\\
&\cong \exp\left\{\frac{L}{2\pi}\int_{-\infty}^{\infty}dq\,\ln\left(1+2\cosh\tfrac{B}{T}\,e^{-\frac{q^2-h}{T}}\right)\right\}, \qquad L\to\infty.
\end{aligned} \tag{4.9}$$

Our purpose in what follows is to present the correlators as Fredholm determinants of linear integral operators. The typical expression for both the numerator and the denominator (the partition function) in (1.3) is of the form

$$F=\sum_{N=0}^{\infty}\sum_{q_1<\ldots<q_N}\det{}_N A=\sum_{N=0}^{\infty}\sum_{q_1<\ldots<q_N}\begin{vmatrix} A(q_1,q_1) & \cdots & A(q_1,q_N)\\ \vdots & \ddots & \vdots\\ A(q_N,q_1) & \cdots & A(q_N,q_N)\end{vmatrix} \tag{4.10}$$

(by definition, $\det_0 A=1$ for any matrix A). The sum in (4.10) is taken over all permitted values of $q_1,\ldots,q_N$. In the thermodynamic limit, as explained above, one can neglect the difference between the permitted values of q's in the sector (N,M) and put, e.g., $(q)_j=\frac{2\pi}{L}j$ (in other words, it is essential only that $(q)_{j+1}-(q)_j=\frac{2\pi}{L}$). Then, due to (4.8), in the limit $L\to\infty$ the expansion (4.10) is just the expansion for the Fredholm determinant,

$$F=\sum_{N=0}^{\infty}\int_{q_1<\ldots<q_N}dq_1\cdots dq_N\,\det{}_N\mathcal{A}=\det(\hat I+\hat{\mathcal{A}}). \tag{4.11}$$

Here, by definition, we denote $\det_N\mathcal{A}$ the determinants of the $N\times N$–dimensional matrices with matrix elements $\mathcal{A}(q_a,q_b)$:

$$\det{}_N\mathcal{A}:=\begin{vmatrix} \mathcal{A}(q_1,q_1) & \cdots & \mathcal{A}(q_1,q_N)\\ \vdots & \ddots & \vdots\\ \mathcal{A}(q_N,q_1) & \cdots & \mathcal{A}(q_N,q_N)\end{vmatrix}. \tag{4.12}$$

Functions $\mathcal{A}(q_a,q_b)$ are obtained from the elements $A(q_a,q_b)$ as

$$\mathcal{A}(q_a,q_b)=\lim_{L\to\infty}\frac{L}{2\pi}A(q_a,q_b). \tag{4.13}$$

The linear integral operator $\hat{\mathcal{A}}$ acts on functions $f(q)$ as

$$(\hat{\mathcal{A}}f)(q)=\int_{-\infty}^{\infty}dq'\mathcal{A}(q,q')f(q') \tag{4.14}$$

and the kernel $\mathcal{A}(q,q')$ is given just by the equation (4.13).

It should be kept in mind that in our notations $A(q_a,q_b)$ denote matrix elements of matrices A entering the expansions of the kind (4.10) which contain the sums. Quantities $\mathcal{A}(q_a,q_b)$ denote matrix elements of matrices entering the expressions similar to (4.11) containing the integrals. Finally, $\hat{\mathcal{A}}$ denote the integral operator with the kernel $\mathcal{A}(q,q')$.

The Fredholm determinant is well defined if the trace of operator $\hat{\mathcal{A}}$ is finite $\int dq\mathcal{A}(q,q)<\infty$. It is the case for all the operators considered below with the exception of the integral operator corresponding to the partition function which is divergent in the limit (see (4.9)). The partition function is the denominator in the expressions (1.3) for the correlation functions. The mean values in the numerators are also divergent. It will be seen that this divergency is described by the same operator as in the partition function, so that the final answer for the correlator is finite. To make sense to the intermediate formulae, one should keep this divergent operator regularized. We will make this regularization writing for the partition function at $L\to\infty$

$$\mathrm{Tr}\left[e^{-H/T}\right]=\det(\hat{I}+\hat{\mathcal{Z}}),\qquad \mathcal{Z}(q,q')=2\cosh\tfrac{B}{T}\,e^{-\frac{q^2-h}{T}}\delta_L(q-q'),\quad (4.15)$$

Here $\delta_L(q-q')$ is a regularization of the Dirac delta-function; e.g., one can put

$$\delta_L(q-q')\equiv\frac{\sin L(q-q')}{2\pi(q-q')}.\qquad (4.16)$$

Of course, this regularization reproduces only the leading term of the partition function; the $1/L$ corrections to the free energy should be calculated from the exact expression (4.2).

Turn now to the correlator $G_1^{(-)}$ (1.3). Let us calculate the numerator of $G_1^{(-)}$. There are three main steps in the calculation: i) take the thermodynamic limit; ii) sum over λ's (or μ's) and M; iii) sum over N by means of the Fredholm determinant formula (4.11), extracting explicitly the determinant $\det(\hat{I}+\hat{\mathcal{Z}})$ (which cancel exactly the denominator). The result after taking $L\to\infty$ is the well defined Fredholm determinant representation for the correlation function.

Consider the numerator of $G_1^{(-)}$. Using the representation (3.25) for the normalized mean value involved, we get

$$\begin{aligned}
&\mathrm{Tr}\left[e^{-H/T}\psi_1^+(x,t)\psi_1(0,0)\right]\\
&=\sum_{N=0}^{\infty}\sum_{M=0}^{N+1}\sum_{\tilde{k}_1<\ldots<\tilde{k}_{N+1}}\sum_{\lambda_1<\ldots<\lambda_M}e^{-\frac{E_{N+1,M}(\{k\})}{T}}\langle\psi_1^+(x,t)\psi_1(0,0)\rangle_{N+1,M}\\
&=e^{-it(h-B)}\sum_{N=0}^{\infty}\frac{1}{N}\sum_{p,n=0}^{N-1}e^{-\frac{2\pi i}{N}pn}\sum_{M=0}^{N+1}e^{\frac{2B}{T}M}\sum_{\lambda_1<\ldots<\lambda_M}\det{}_M U_p^{(1,-)}\\
&\times\sum_{\tilde{k}_1<\ldots<\tilde{k}_{N+1}}\left[\det{}_{N+1}(\widetilde{S}_n^{(-)}+\widetilde{R}^{(-)})-\det{}_{N+1}\widetilde{S}_n^{(-)}\right],
\end{aligned}\qquad (4.17)$$

where tildes over matrices mean that the temperature factors $\exp\{-\sum_{a=1}^{N+1}(k_a^2 - h + B)/T\}$ are included into the determinants, i.e., the matrix elements $(\;)_{ab}$ of the corresponding matrix is multiplied by the factor $\exp\{-\frac{k_a^2-h+B}{T}\}$. It means that the matrix $\widetilde{S}_n^{(-)}$ has the structure

$$
(\widetilde{S}_n^{(-)})_{ab} = e^{-\frac{k_a^2-h+B}{T}}\delta_{ab} + \cdots, \qquad a, b = 1, \ldots, N+1. \tag{4.18}
$$

Since in (4.17) the summation is over all values $\tilde{k}_1 < \ldots < \tilde{k}_{N+1}$, there is the periodicity in Λ (recall that $k_a = \tilde{k}_a + \Lambda/L$) and one can insert the Kronecker symbol on Z_{N+1} (3.41).

Then in the thermodynamic limit, $L \to \infty$, the numerator of $G_1^{(-)}$ acquire the form

$$
\begin{aligned}
&\mathrm{Tr}\left[e^{-H/T}\psi_1^+(x,t)\psi_1(0,0)\right] \\
&\quad = e^{-it(h-B)}\sum_{N=0}^{\infty}\frac{1}{N(N+1)}\sum_{r,m=0}^{N}\sum_{p,n=0}^{N-1} e^{\frac{2\pi i}{N+1}rm-\frac{2\pi i}{N}pn} \\
&\quad \times X_{N,p,r}\int\limits_{k_1<\ldots<k_{N+1}} dk_1\cdots dk_{N+1}\,\Xi_{N+1}^{(-)}(\eta_{n,m}),
\end{aligned} \tag{4.19}
$$

where we denote

$$
\begin{aligned}
X_{N,p,r} &:= \sum_{M=0}^{N+1} e^{\frac{2B}{T}M}\sum_{\lambda_1<\ldots<\lambda_M}\det{}_M U_{p,r}^{(1,-)}, \\
\Xi_{N+1}^{(-)}(\eta_{n,m}) &:= \det{}_{N+1}(\widetilde{\mathcal{S}}^{(-)}(\eta_{n,m}) + \widetilde{\mathcal{R}}^{(-)}) - \det{}_{N+1}\widetilde{\mathcal{S}}^{(-)}(\eta_{n,m}).
\end{aligned} \tag{4.20}
$$

The matrices $(U_{p,r}^{(1,-)})$ have matrix elements

$$
(U_{p,r}^{(1,-)})_{ab} = e^{-ir\lambda_a}(U_p^{(1,-)})_{ab} \tag{4.21}
$$

and elements of matrices $(U_p^{(1,-)})$ are given by (3.24). The matrix $\mathcal{S}^{(-)}(\eta_{n,m})$ has the structure

$$
\widetilde{\mathcal{S}}^{(-)}(\eta_{n,m}) = \mathcal{N} + \frac{1+\varepsilon\,\cos\eta_{n,m}}{2}\,\widetilde{\mathcal{V}}_1 + \frac{\varepsilon\,\sin\eta_{n,m}}{2}\,\widetilde{\mathcal{V}}_2, \tag{4.22}
$$

where

$$
\eta_{n,m} := \frac{2\pi m}{N+1} - \frac{2\pi n}{N}. \tag{4.23}
$$

The matrix elements of the matrix $\mathcal{N}$ are

$$
\mathcal{N}(k_a, k_b) = e^{-\frac{k_a^2-h+B}{T}}\,\delta_L(k_a - k_b) \tag{4.24}
$$

and the elements of the matrices $\widetilde{\mathcal{V}}_{1,2}$, $\widetilde{\mathcal{R}}^{(-)}$ are obtained in accordance with the rule (4.14):

$$\begin{aligned}
\widetilde{\mathcal{V}}_1(k_a,k_b) &= e^{-\frac{k_a^2-h+B}{T}}\ \frac{E_+(k_a)E_-(k_b)-E_-(k_a)E_+(k_b)}{\pi(k_a-k_b)},\\
\widetilde{\mathcal{V}}_2(k_a,k_b) &= e^{-\frac{k_a^2-h+B}{T}}\ \frac{[E_-(k_a)]^{-1}E_-(k_b)-E_-(k_a)[E_-(k_b)]^{-1}}{\pi(k_a-k_b)},\\
\widetilde{\mathcal{R}}^{(-)}(k_a,k_b) &= e^{-\frac{k_a^2-h+B}{T}}\ \frac{E_-(k_a)E_-(k_b)}{2\pi}.
\end{aligned} \tag{4.25}$$

(Since the elements of the matrices $\widetilde{\mathcal{V}}^{(-)}_{1,2}$ and $\widetilde{\mathcal{V}}^{(+)}_{1,2}$, which are obtained from $(\widetilde{V}^{(-)}_{1,2})_{ab}$ and $(\widetilde{V}^{(+)}_{1,2})_{ab}$ by means of (4.14), are described by the same functions, there is no need to distinguish between $\widetilde{\mathcal{V}}^{(-)}_{1,2}$ and $\widetilde{\mathcal{V}}^{(+)}_{1,2}$, hence we write simply $\widetilde{\mathcal{V}}_{1,2}$.) The functions $E_\pm$ are

$$E_+(k)=E(k)\ E_-(k), \qquad E_-(k)=e^{\frac{itk^2-ixk}{2}}, \tag{4.26}$$

where

$$E(k)=P.v.\int_{-\infty}^{\infty}dq\,\frac{e^{-itq^2+ixq}}{\pi(q-k)}. \tag{4.27}$$

These functions are the thermodynamic limits of the corresponding functions $e^{(\pm)}_+$, e_-, $e^{(\pm)}$ which is explained in detail in Appendix.

Consider the quantity $X_{N,p,r}$. Note that, by definition, $\det_M U^{(1,-)}_{p,r}\Big|_{M=0}=1$ and it is easy to see that $\det_M U^{(1,-)}_{p,r}\Big|_{M=N+1}=0$. Thus

$$X_{N,p,r}=1+\sum_{M=1}^{N}e^{\frac{2B}{T}M}\sum_{\lambda_1<\ldots<\lambda_M}\det{}_M U^{(1,-)}_{p,r}. \tag{4.28}$$

To obtain a useful expression for $X_{N,p,r}$ let us note that the following relation is valid

$$\sum_{\lambda_1<\ldots<\lambda_M}\det{}_M U^{(1,-)}_{p,r}=\sum_{1\le m_1<\ldots<m_M\le N}\det{}_M W^{(1)}_{p,r}, \tag{4.29}$$

where

$$\begin{aligned}
\det{}_M W^{(1)}_{p,r} &= \sum_{P\in S_M}(-1)^{[P]}\\
&\times\prod_{a=1}^{M}\left[\sum_{n_a=1}^{N}\left(\frac{1}{N+1}\sum_{\lambda}e^{-i\lambda(r+m_{Pa}-n_a)}\right)\left(\frac{1}{N}\sum_{\mu}e^{i\mu(p+m_a-n_a)}\right)\right].
\end{aligned} \tag{4.30}$$

To see that the relation (4.29) holds it is sufficient to note that: i) $\det_M U^{(1,-)}_{p,r}$ is symmetric with respect to permutations of λ's being equal to zero whenever two of them coincide; ii) $\det_M W^{(1)}_{p,r}$ is symmetric with respect to permutations of m's (being zero whenever two of them coincide). After changing $\sum_{\lambda_1<\ldots<\lambda_M}$ to $\frac{1}{M!}\sum_{\lambda_1}\cdots\sum_{\lambda_M}$ in the l.h.s. of (4.29) (and similarly for the summation over m's in the r.h.s.) the relation (4.29) becomes obvious.

Let us find the explicit expression for the elements of the matrix $W^{(1)}_{p,r}$. Using Bethe equations for λ's and μ's we have the following summation formulae (here $m := m_1, \ldots, m_M$; $n := n_1, \ldots, n_M$):

$$\begin{aligned}
&\frac{1}{N+1}\sum_{\lambda} e^{-i\lambda(r+m-n)} \\
&\quad = \delta_{r+m-n}\theta(r+m<N+1) + (-1)^{M+1}\delta_{r+m-n-N-1}\theta(r+m>N+1), \\
&\frac{1}{N}\sum_{\mu} e^{i\mu(p+m-n)} \\
&\quad = \delta_{p+m-n}\theta(p+m\le N) + (-1)^{M+1}\delta_{p+m-n-N}\theta(p+m>N). \qquad (4.31)
\end{aligned}$$

These expressions are valid only for the values of the integers involved: $p = 0, \ldots, N-1$; $r = 0, \ldots, N$; $m_1, \ldots, m_M, n_1, \ldots, n_M = 1, \ldots, N$. Here and below we use the short notation for the usual Kronecker symbol, $\delta_a = \delta_{a,0}$. Making use of (4.31) and summing over $n_1, \ldots, n_N$ in (4.30) we get

$$\begin{aligned}
(W^{(1)}_{p,r})_{ab} &= \left[\delta_{p-r+m_a-m_b} + (-1)^{M+1}\delta_{p-r+m_a-m_b+N+1}\right]\theta(p+m_a\le N) \\
&+ \left[\delta_{p-r+m_a-m_b+1} + (-1)^{M+1}\delta_{p-r+m_a-m_b-N}\right]\theta(p+m_a>N). \qquad (4.32)
\end{aligned}$$

Using the indentity for the determinant of an $N\times N$ matrix,

$$\det_N(I+A) = 1 + \sum_{j=1}^{N} A_{j,j} + \sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{N} \begin{vmatrix} A_{j_1,j_1} & A_{j_1,j_2} \\ A_{j_2,j_1} & A_{j_2,j_2}\end{vmatrix} + \cdots + \det_N A, \qquad (4.33)$$

we obtain, due to the relation (4.29), that

$$\begin{aligned}
X_{N,p,r} &= \frac{1}{2}\left[\det_N(I+e^{\frac{2B}{T}}W_-) + \det_N(I-e^{\frac{2B}{T}}W_-)\right] \\
&+ \frac{1}{2}\left[\det_N(I+e^{\frac{2B}{T}}W_+) - \det_N(I-e^{\frac{2B}{T}}W_+)\right] \qquad (4.34)
\end{aligned}$$

where $N\times N$ matrices $W_\pm$ are

$$\begin{aligned}
(W_\pm)_{ab} &= \left[\delta_{p-r+a-b} \pm \delta_{p-r+a-b+N+1}\right]\theta(p+a\le N) \\
&+ \left[\delta_{p-r+a-b+1} \pm \delta_{p-r+a-b-N}\right]\theta(p+a>N). \qquad (4.35)
\end{aligned}$$

To proceed further, let us look at the quantity $\Xi^{(-)}_{N+1}(\eta_{n,m})$ as a function of $\eta_{n,m}$. Obviously,

$$\Xi^{(-)}_{N+1}(\eta_{n,m}) = \sum_{s=-N}^{N} e^{is\eta_{n,m}}\Phi_s^{(-)}, \tag{4.36}$$

where $\Phi_s^{(-)}$ are some combinations of the elements of the matrices $\mathcal{N}$, $\widetilde{\mathcal{V}}_{1,2}$, $\widetilde{\mathcal{R}}^{(-)}$ which do not depend on n, m. After summation over n and m in (4.19) the terms with $s = 0, \ldots, N$ in (4.36) will produce δ_{p-r} while the terms with $s = -N, \ldots, -1$ will produce δ_{p-r+1}. Hence we should take into account only the terms in (4.34) for which either the condition $r = p$ or the condition $r = p+1$ is satisfied. It follows from (4.34) and (4.35) that

$$X_{N,p,r}|_{r=p} = (1 + e^{2B/T})^{N-p}, \qquad X_{N,p,r}|_{r=p+1} = (1 + e^{2B/T})^{p}. \tag{4.37}$$

Let us introduce the notation

$$\gamma = 1 + e^{2B/T}. \tag{4.38}$$

Then we can write the answer

$$X_{N,p,r} = 1 + (\gamma^{N-p} - 1)\delta_{p-r} + (\gamma^{p} - 1)\delta_{p-r+1} + \text{unessential terms} \tag{4.39}$$

where "unessential terms" are terms constructed from the Kronecker symbols containing in their arguments p, r and which are different from those written in (4.39) explicitly. These "unessential terms" will not contribute after the summation over n, m, p, r in (4.19).

Substituting (4.39) into (4.19) we obtain the following expression under the sum over N and $(N+1)$-fold integration over k's (except the overall factor $\exp\{-it(h-B)\}$)

$$\frac{1}{(N+1)N}\sum_{m=0}^{N}\sum_{n=0}^{N-1}\Bigg\{\gamma^N - 1 + \sum_{p=1}^{N-1}(\gamma^{N-p} - 1)\Big[e^{2\pi i\eta_{n,m}p} + e^{-2\pi i\eta_{n,m}p}\Big]\Bigg\}\Xi^{(-)}_{N+1}(\eta_{n,m}) + \Xi^{(-)}_{N+1}(0). \tag{4.40}$$

Due to the explicit dependence on N of the elements of the matrix under the determinant in (4.19), some efforts are needed in order to apply the Fredholm determinant formula (4.11) to the expression (4.19) as a whole. The first trick is to change the double sum (over n, m) in the first term in (4.40) to the ordinary integral, i.e., the following indentity holds

$$\frac{1}{(N+1)N}\sum_{m=0}^{N}\sum_{n=0}^{N-1}\Bigg\{\gamma^N - 1$$

$$+\sum_{p=1}^{N-1}(\gamma^{N-p}-1)\left[e^{2\pi i\eta_{n,m}p}+e^{-2\pi i\eta_{n,m}p}\right]\Bigg\}\Xi^{(-)}_{N+1}(\eta_{n,m})$$

$$=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}d\eta\left\{\gamma^N-1+\sum_{p=1}^{N-1}(\gamma^{N-p}-1)\left[e^{i\eta p}+e^{-i\eta p}\right]\right\}\Xi^{(-)}_{N+1}(\eta). \tag{4.41}$$

It is sufficient to sum over n,m in the l.h.s. of (4.41) and to integrate over η in the r.h.s. in order to see that the results coincide.

The second trick is to expand the summation with respect to p in the r.h.s. of (4.41) to the infinity. Adding the term at $p=N$ obviously do not produce any additional terms to the r.h.s. of (4.41) because $(\gamma^{N-p}-1)|_{p=N}=0$. The terms with $p>N$ will die after taking the integration with respect to η since then the quantity $e^{i\eta p}\Xi_N(\eta)$ (or $e^{-i\eta p}\Xi_N(\eta)$) contains $e^{i\eta}$ only in integer positive (or only in integer negative) powers, due to (4.36).

Therefore, the expression (4.40) can be rewritten as

$$\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}d\eta\left\{\gamma^N-1+\sum_{p=1}^{\infty}(\gamma^{N-p}-1)\left[e^{i\eta p}+e^{-i\eta p}\right]\right\}\Xi^{(-)}_{N+1}(\eta)+\Xi^{(-)}_{N+1}(0)$$

$$=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}d\eta\left\{\gamma^N+\sum_{p=1}^{\infty}\gamma^{N-p}\left[e^{i\eta p}+e^{-i\eta p}\right]\right\}\Xi^{(-)}_{N+1}(\eta)$$

$$=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}d\eta\left\{1+\sum_{p=1}^{\infty}\gamma^{-p}\left[e^{i\eta p}+e^{-i\eta p}\right]\right\}\Xi^{(-)}_{N+1}(\gamma;\eta). \tag{4.42}$$

Here we denote by $\Xi^{(-)}_{N+1}(\gamma;\eta)$ the expression for $\Xi^{(-)}_{N+1}(\eta)$ given by (4.20) where the replacement $\widetilde{\mathcal{S}}^{(-)}(\eta)\to\gamma\widetilde{\mathcal{S}}^{(-)}(\eta)$ has been made. The last expression in (4.42) is what we need in order to apply the Fredholm formula (4.11). Taking into account that $\gamma\mathcal{N}(k_a,k_b)=\mathcal{Z}(k_a,k_b)$, we get

$$\mathrm{Tr}\left[e^{-H/T}\psi_1^+(x,t)\psi_1(0,0)\right]=e^{-it(h-B)}\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}d\eta\,F(\gamma,\eta)$$

$$\times\left[\det\left(\hat{I}+\hat{\mathcal{Z}}+\gamma\hat{\hat{\mathcal{V}}}(\eta)+\hat{\hat{\mathcal{R}}}^{(-)}\right)-\det\left(\hat{I}+\hat{\mathcal{Z}}+\gamma\hat{\hat{\mathcal{V}}}(\eta)\right)\right]. \tag{4.43}$$

Here

$$\hat{\hat{\mathcal{V}}}(\eta):=\frac{1+\varepsilon\,\cos\eta}{2}\,\hat{\hat{\mathcal{V}}}_1+\frac{\sin\eta}{2}\,\hat{\hat{\mathcal{V}}}_2 \tag{4.44}$$

and $\hat{\mathcal{Z}}$, $\hat{\hat{\mathcal{V}}}_{1,2}, \hat{\hat{\mathcal{R}}}^{(-)}$ are integral operators with the kernels given by (4.15), (4.25). We have introduced the function

$$F(\gamma,\eta) = 1 + \sum_{p=1}^{\infty} \gamma^{-p}(e^{i\eta p} + e^{-i\eta p}). \tag{4.45}$$

It is worth mentioning that $1 \le \gamma < \infty$ for any real external field B. At the point $\gamma = 1$ ($B = -\infty$) the function $F(\gamma,\eta)$ is equal (up to the factor 2π) to the 2π-periodic delta-function of the variable η.

Finally, taking into account the representation (4.15) for $\mathrm{Tr}[e^{-H/T}]$ we obtain the following representation for the temperature correlation function considered

$$\begin{aligned} G_1^{(-)}(x,t;h,B) &= e^{-it(h-B)} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\eta\, F(\gamma,\eta) \\ &\times \left[\det\left(\hat{I} + \gamma\, \hat{\mathcal{V}}(\eta) + \hat{\mathcal{R}}^{(-)}\right) - \det\left(\hat{I} + \gamma\, \hat{\mathcal{V}}(\eta)\right) \right], \end{aligned} \tag{4.46}$$

where integral operator $\hat{\mathcal{V}}(\eta)$ is of the form

$$\hat{\mathcal{V}}(\eta) = \frac{1 + \varepsilon\,\cos\eta}{2}\, \hat{\mathcal{V}}_1 + \frac{\sin\eta}{2}\, \hat{\mathcal{V}}_2. \tag{4.47}$$

The integral operators $\hat{\mathcal{V}}_{1,2}$, $\hat{\mathcal{R}}^{(-)}$ do not depend on η and possess kernels

$$\begin{aligned} \mathcal{V}_1(k,k') &= \frac{E_+^T(k)E_-^T(k') - E_+^T(k')E_-^T(k)}{\pi(k-k')}, \\ \mathcal{V}_2(k,k') &= \frac{1}{\pi(k-k')} \left[\frac{\vartheta(k)}{E_-^T(k)} E_-^T(k') - E_-^T(k) \frac{\vartheta(k')}{E_-^T(k')} \right], \\ \mathcal{R}^{(-)}(k,k') &= \frac{E_-^T(k)E_-^T(k')}{2\pi}. \end{aligned} \tag{4.48}$$

The functions $E_+^T(k)$, $E_-^T(k)$ are

$$E_+^T(k) = E(k)\; E_-^T(k), \qquad E_-^T(k) = \sqrt{\vartheta(k)}\, e^{\frac{itk^2 - ixk}{2}}, \tag{4.49}$$

and the function $E(k)$ is given by (4.27). The Fermi weight $\vartheta(k)$ is

$$\vartheta(k) = \frac{e^{-B/T}}{2\cosh\frac{B}{T} + e^{\frac{k^2-h}{T}}}. \tag{4.50}$$

Let us now consider another temperature correlation function $G_1^{(+)}$ (1.3). Using the representation (3.45) for the normalized mean value, we represent the numerator of the temperature correlation function as follows

$$\begin{aligned}
&\mathrm{Tr}\left[e^{-H/T}\psi_1(x,t)\psi_1^+(0,0)\right]\\
&= \sum_{N=0}^{\infty}\sum_{M=0}^{N}\sum_{\tilde q_1<\ldots<\tilde q_N}\sum_{\mu_1<\ldots<\mu_M} e^{-\frac{E_{N,M}(\{q\})}{T}}\langle\psi_1(x,t)\psi_1^+(0,0)\rangle_{N,M}\\
&= e^{it(h-B)}\sum_{N=0}^{\infty}\frac{1}{N+1}\sum_{r,m=0}^{N} e^{\frac{2\pi i}{N+1}rm}\sum_{M=0}^{N} e^{\frac{2B}{T}M}\sum_{\mu_1<\ldots<\mu_M}\det{}_M U_r^{(1,+)}\\
&\times\sum_{\tilde q_1<\ldots<\tilde q_N}\left[\det{}_N(\widetilde S_m^{(+)}-\widetilde R_m^{(+)})+(g_m(x,t)-1)\det{}_N\widetilde S_m^{(+)}\right], \qquad (4.51)
\end{aligned}$$

where the tilde over the matrix means, as above, that the temperature factor $\exp\{-\sum_{a=1}^{N+1}(k_a^2-h+B)/T\}$ is included into the determinant.

In the thermodynamic limit the numerator of $G_1^{(+)}$ acquire the form

$$\begin{aligned}
&\mathrm{Tr}\left[e^{-H/T}\psi_1(x,t)\psi_1^+(0,0)\right]\\
&= e^{it(h-B)}\sum_{N=0}^{\infty}\frac{1}{N(N+1)}\sum_{r,m=0}^{N}\sum_{p,n=0}^{N-1} e^{\frac{2\pi i}{N+1}rm-\frac{2\pi i}{N}pn}\\
&\times X_{N,p,r}\int\limits_{q_1<\ldots<q_N} dq_1\cdots dq_N\ \Xi_N^{(+)}(\eta_{n,m}) \qquad (4.52)
\end{aligned}$$

where we denote

$$\begin{aligned}
&\Xi_N^{(+)}(\eta_{n,m})\\
&:= \det{}_N(\widetilde{\mathcal S}^{(+)}(\eta_{n,m})-\widetilde{\mathcal R}^{(+)}(\eta_{n,m}))+(G(x,t)-1)\det{}_N\widetilde{\mathcal S}^{(+)}(\eta_{n,m}). \qquad (4.53)
\end{aligned}$$

The function $G(x,t)$ is the thermodynamic limit of the function $g(x,t)$ (3.34):

$$G(x,t)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} dk e^{-itk^2+ixk}. \qquad (4.54)$$

The matrices $\widetilde{\mathcal S}^{(+)}(\eta_{n,m})$ and $\widetilde{\mathcal R}^{(+)}(\eta_{n,m})$ have the structure

$$\begin{aligned}
&\widetilde{\mathcal S}^{(+)}(\eta_{n,m})=\mathcal N+\frac{1+\varepsilon\,\cos\eta_{n,m}}{2}\,\widetilde{\mathcal V}_1-\frac{\varepsilon\,\sin\eta_{n,m}}{2}\,\widetilde{\mathcal V}_2,\\
&\widetilde{\mathcal R}^{(+)}(\eta_{n,m})=\frac{1+\varepsilon\,\cos\eta_{n,m}}{2}\,\widetilde{\mathcal R}_1^{(+)}\\
&-\varepsilon\,\sin\eta_{n,m}\,\widetilde{\mathcal R}_2^{(+)}+\frac{1-\varepsilon\,\cos\eta_{n,m}}{2}\,\widetilde{\mathcal R}_3^{(+)}. \qquad (4.55)
\end{aligned}$$

Matrix elements of the matrices $\mathcal N$ and $\widetilde{\mathcal V}_{1,2}$ (which are $N\times N$ dimensional matrices here) are given by the equations (4.24) and (4.25) (but with the momenta

q instead of k). The elements of the matrices $\widetilde{\mathcal{R}}^{(+)}_{1,2,3}$ (functions $\widetilde{\mathcal{R}}^{(+)}_{1,2,3}(q_a,q_b)$) are

$$\begin{aligned}
\widetilde{\mathcal{R}}^{(+)}_1(q_a,q_b) &= e^{-\frac{q_a^2-h+B}{T}}\,\frac{E_+(q_a)E_+(q_b)}{2\pi},\\
\widetilde{\mathcal{R}}^{(+)}_2(q_a,q_b) &= e^{-\frac{q_a^2-h+B}{T}}\,\frac{1}{4\pi}\left[\frac{E_+(q_a)}{E_-(q_b)}+\frac{E_+(q_b)}{E_-(q_a)}\right],\\
\widetilde{\mathcal{R}}^{(+)}_3(q_a,q_b) &= e^{-\frac{q_a^2-h+B}{T}}\,\frac{1}{2\pi E_-(q_a)E_-(q_b)}.
\end{aligned} \tag{4.56}$$

The functions $E_\pm$, entering (4.56) are defined in (4.26), (4.27).

In (4.52), the quantity $X_{N,p,r}$ is the same as in (4.20) above, since

$$\begin{aligned}
X_{N,p,r} &:= \sum_{M=0}^{N} e^{\frac{2B}{T}M} \sum_{\mu_1<\ldots<\mu_M} \det{}_M U^{(1,+)}_{p,r}(\{\mu\})\\
&= \sum_{M=0}^{N+1} e^{\frac{2B}{T}M} \sum_{\lambda_1<\ldots<\lambda_M} \det{}_M U^{(1,-)}_{p,r}(\{\lambda\}),
\end{aligned} \tag{4.57}$$

where

$$(U^{(1,+)}_{p,r}(\{\mu\}))_{ab} = e^{ip\mu_a}(U^{(1,+)}_{r}(\{\mu\}))_{ab} \tag{4.58}$$

and the elements of the matrices $U^{(1,+)}_r$ are given by (3.44).

Similarly to (4.36), we have

$$\Xi^{(+)}_N(\eta_{n,m}) = \sum_{s=-N}^{N} e^{is\eta_{n,m}}\Phi^{(+)}_s, \tag{4.59}$$

where $\Phi^{(+)}_s$ are some functions which do not depend on n,m. Therefore, the result (4.39) for $X_{N,p,r}$ is valid also in the case considered and we can repeat the calculation described by equations (4.40)–(4.42), except that we should write $\Xi^{(+)}_N$ instead of $\Xi^{(-)}_{N+1}$. The last expression in (4.42) is now written as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\eta \left\{1+\sum_{p=}^{\infty}\gamma^{-p}\left[e^{i\eta p}+e^{-i\eta p}\right]\right\}\Xi^{(+)}_N(\eta;\gamma). \tag{4.60}$$

Here $\Xi^{(+)}_N(\eta;\gamma)$ denotes the expression for $\Xi^{(+)}_N(\eta)$ given by (4.53) in which the replacement $\widetilde{\mathcal{S}}^{(+)}(\eta)\to\gamma\widetilde{\mathcal{S}}^{(+)}(\eta)$, $\widetilde{\mathcal{R}}^{(+)}(\eta)\to\gamma\widetilde{\mathcal{R}}^{(+)}(\eta)$ has been made.

Thus we obtain the Fredholm determinant representation for the numerator of the correlator $G^{(+)}_1$

$$\begin{aligned}
&\mathrm{Tr}\left[e^{-H/T}\psi_1(x,t)\psi_1^+(0,0)\right]\\
&= e^{it(h-B)}\frac{1}{2\pi}\int_{-\pi}^{\pi} d\eta\, F(\gamma,\eta)\Big[\det\left(\hat{I}+\hat{\mathcal{Z}}+\gamma\,\hat{\widetilde{\mathcal{V}}}(\eta)-\gamma\hat{\widetilde{\mathcal{R}}}^{(+)}(\eta)\right)\\
&+(G(x,t)-1)\det\left(\hat{I}+\hat{\mathcal{Z}}+\gamma\,\hat{\widetilde{\mathcal{V}}}(\eta)\right)\Big],
\end{aligned} \tag{4.61}$$

where $\hat{\hat{\mathcal{V}}}(\eta)$ is given by (4.44) and

$$\hat{\hat{\mathcal{R}}}^{(+)}(\eta) := \frac{1+\varepsilon\cos\eta}{2}\,\hat{\hat{\mathcal{R}}}_1^{(+)} + \sin\eta\,\hat{\hat{\mathcal{R}}}_2^{(+)} + \frac{1-\varepsilon\cos\eta}{2}\,\hat{\hat{\mathcal{R}}}_3^{(+)}. \tag{4.62}$$

The integral operators $\hat{\hat{\mathcal{Z}}}$, $\hat{\hat{\mathcal{R}}}_{1,2,3}^{(+)}$ possess kernels given by the functions (4.15), (4.25), (4.56) and the function $F(\gamma,\eta)$ is given by (4.45).

Finally, taking into account the representation (4.15) for $\mathrm{Tr}[e^{-H/T}]$ we obtain the following representation for the temperature correlation function

$$G_1^{(+)}(x,t;h,B) = e^{it(h-B)}\frac{1}{2\pi}\int_{-\pi}^{\pi} d\eta\, F(\gamma,\eta)$$
$$\times\left[\det\left(\hat{I}+\gamma\,\hat{\mathcal{V}}(\eta) - \gamma\,\hat{\mathcal{R}}^{(+)}(\eta)\right) + (G(x,t)-1)\det\left(\hat{I}+\gamma\,\hat{\mathcal{V}}(\eta)\right)\right] \tag{4.63}$$

where

$$\hat{\mathcal{V}}(\eta) = \frac{1+\varepsilon\cos\eta}{2}\,\hat{\mathcal{V}}_1 + \frac{\sin\eta}{2}\,\hat{\mathcal{V}}_2,$$
$$\hat{\mathcal{R}}^{(+)}(\eta) = \frac{1+\varepsilon\cos\eta}{2}\,\hat{\mathcal{R}}_1^{(+)} + \sin\eta\,\hat{\mathcal{R}}_2^{(+)} + \frac{1-\varepsilon\cos\eta}{2}\,\hat{\mathcal{R}}_3^{(+)}. \tag{4.64}$$

The integral operators $\hat{\mathcal{V}}_{1,2}$ possess kernels (4.48) and integral operators $\hat{\mathcal{R}}_{1,2,3}^{(+)}$ possess kernels

$$\mathcal{R}_1^{(+)}(q,q') = \frac{E_+^T(q)E_+^T(q')}{2\pi},$$
$$\mathcal{R}_2^{(+)}(q,q') = \frac{1}{4\pi}\left[E_+^T(q)\frac{\vartheta(q')}{E_-^T(q')} + \frac{\vartheta(q)}{E_-^T(q)}E_+^T(q')\right],$$
$$\mathcal{R}_3^{(+)}(q,q') = \frac{1}{2\pi}\frac{\vartheta(q)}{E_-^T(q)}\frac{\vartheta(q')}{E_-^T(q')}. \tag{4.65}$$

The functions E_+^T, E_-^T are given by equations (4.49) with the Fermi weight (4.50).

Let us now discuss our main results, the representations (4.46) and (4.63). The integral operators $\hat{\mathcal{V}}(\eta)$ and $\hat{\mathcal{R}}^{(+)}(\eta)$ involved into these representations can be put in the form usual to the integrable models, i.e., they are "integrable integral operators" in the sense of paper [7] (see also [8]). To show this we introduce a pair of functions

$$\ell_+^T(\eta|k) := \frac{1+\varepsilon\cos\eta}{2}\,E_+^T(k) + \frac{\sin\eta}{2}\,\frac{\vartheta(k)}{E_-^T(k)}, \qquad \ell_-^T(k) := E_-^T(k). \tag{4.66}$$

The kernels of the operators can be put into the form

$$\mathcal{V}(\eta|k,k') = \frac{\ell_+^T(\eta|k)\;\ell_-^T(k') - \ell_-^T(k)\;\ell_+^T(\eta|k')}{\pi(k-k')},$$
$$\mathcal{R}^{(-)}(k,k') = \frac{\ell_-^T(k)\;\ell_-^T(k')}{2\pi}, \qquad \mathcal{R}^{(+)}(\eta|k,k') = \frac{\ell_+^T(\eta|k)\;\ell_+^T(\eta|k')}{\pi(1+\varepsilon\cos\eta)}. \tag{4.67}$$

This form is important for deriving integrable partial differential equations for the correlators and for constructing the corresponding matrix Riemann-Hilbert problem. This in turn will make possible the evaluation of different asymptotics of the correlators considered.

5 Particular cases

Let us discuss now some particular cases of the representations obtained.

First, consider the "one-component" limit of the theory. Let $B \to -\infty$, $h \to -\infty$ in such a way that $h_1 = h - B$ is fixed. Note that $h_2 = h + B \to -\infty$ in this limit. It means that the energy of the particles of type 2 becomes (see (1.16)) infinite and therefore these particles are excluded from the spectrum of the theory. So one should has the one-component gas of particles of type 1. Indeed, the parameter $\gamma = 1$ in this limit. Remind that the function $F(\gamma = 1, \eta)$ is proportional to the 2π-periodic delta-function of the variable η, and the integrals over η in (4.46), (4.63) can be easily taken. For $\varepsilon = +1$ one gets in this way exactly the representations for the corresponding temperature correlation functions of the impenetrable one-component Bose gas obtained in papers [16, 17]:

$$
\begin{aligned}
&G_1^{(-)}(x,t;h,B)\Big|_{\substack{h,B\to-\infty\\ h-B=h_1}} = e^{-ith_1}\left[\det(\hat{I}+\hat{\mathcal{V}}_1+\hat{\mathcal{R}}^{(-)}) - \det(\hat{I}+\hat{\mathcal{V}}_1)\right],\\
&G_1^{(+)}(x,t;h,B)\Big|_{\substack{h,B\to-\infty\\ h-B=h_1}}\\
&\qquad = e^{ith_1}\left[\det(\hat{I}+\hat{\mathcal{V}}_1-\hat{\mathcal{R}}_1^{(+)}) + (G(x,t)-1)\det(\hat{I}+\hat{\mathcal{V}}_1)\right],
\end{aligned}
\tag{5.1}
$$

where the kernels of the integral operators $\hat{\mathcal{V}}_1$, $\hat{\mathcal{R}}^{(-)}$, $\hat{\mathcal{R}}_1^{(+)}$ are given by (4.48) and (4.65) with the Fermi weight

$$
\vartheta(k) = \frac{1}{1+e^{\frac{k^2-h_1}{T}}}. \tag{5.2}
$$

If $\varepsilon = -1$ (Fermi statistics) then the correlators simplify strongly, reproducing the well known simple results for the one-component Fermi gas. The correlator $G_2^{(-)}(x,t;h,B) = G_1^{(-)}(x,t;h,-B)$ vanish in the limit, what means that the particles of type 2 are really excluded from the theory.

Second, consider the particular case of the equal time ($t = 0$) temperature correlators of the two-component gas. It appears that in this case the integrals over η in (4.46) and (4.63) can be also taken with the result

$$
\begin{aligned}
&G_1^{(-)}(x,0;h,B) = \det\left(\hat{I}+\frac{\gamma+\varepsilon}{2}\,\hat{v}+\hat{r}^{(-)}\right) - \det\left(\hat{I}+\frac{\gamma+\varepsilon}{2}\,\hat{v}\right),\\
&G_1^{(+)}(x,0;h,B)\\
&\qquad = \det\left(\hat{I}+\frac{\gamma+\varepsilon}{2}\,\hat{v}+\varepsilon\,\hat{r}^{(+)}\right) + (\delta(x)-1)\det\left(\hat{I}+\frac{\gamma+\varepsilon}{2}\,\hat{v}\right),
\end{aligned}
\tag{5.3}
$$

where the integral operators $\hat{v}$, $\hat{r}^{(\pm)}$ possess kernels

$$\hat{v}(k,k') = -\sqrt{\vartheta(k)}\,\frac{2\sin(|x|\frac{k-k'}{2})}{\pi(k-k')}\,\sqrt{\vartheta(k')},$$
$$\hat{r}^{(\pm)}(k,k') = \sqrt{\vartheta(k)}\,\frac{\exp(\pm ix\frac{k+k'}{2})}{2\pi}\,\sqrt{\vartheta(k')}, \tag{5.4}$$

with the Fermi weight $\vartheta(k)$ given by (4.50). The equal-time correlators (5.3) satisfy the relation

$$G_1^{(+)}(x,0;h,B) = \delta(x) + \varepsilon\, G_1^{(-)}(-x,0;h,B), \tag{5.5}$$

which could be also easily derived from the canonical commutation relations (1.2). The representation (5.3) for the equal-time correlation function for the impenetrable Fermi gas ($\varepsilon = -1$) written in slightly different form has been obtained in [15].

Discuss now the case of the two-component gas at zero temperature, $T = 0$. In this case one has

$$\gamma\,\vartheta(k) = \theta(k_F^2 - k^2), \tag{5.6}$$

where θ is the step function and the Fermi momentum is $k_F = \sqrt{h + |B|}$.

Consider first zero magnetic field $B = 0$. In this case the ground state is degenerate (see Section 1), the parameter $\gamma = 2$, and one obtains

$$\begin{aligned}
G_1^{(-)}(x,t;h,0)\Big|_{T=0} &= G_2^{(-)}(x,t;h,0)\Big|_{T=0} \\
&= e^{-ith}\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} d\eta\,\frac{3}{5-4\cos\eta} \\
&\times\left[\det\left(\hat{I} + \hat{\mathcal{V}}_0(\eta) + \tfrac{1}{2}\,\hat{\mathcal{R}}_0^{(-)}\right) - \det\left(\hat{I} + \hat{\mathcal{V}}_0(\eta)\right)\right], \\
G_1^{(+)}(x,t;h,0)\Big|_{T=0} &= G_2^{(+)}(x,t;h,0)\Big|_{T=0} \\
&= e^{ith}\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} d\eta\,\frac{3}{5-4\cos\eta} \\
&\times\left[\det\left(\hat{I} + \hat{\mathcal{V}}_0(\eta) - \hat{\mathcal{R}}_0^{(+)}(\eta)\right) + (G(x,t)-1)\det\left(\hat{I} + \hat{\mathcal{V}}_0(\eta)\right)\right].
\end{aligned} \tag{5.7}$$

Here the integral operators act on the functions on the interval $(-k_F, k_F)$, e.g.,

$$\left(\hat{\mathcal{V}}_0(\eta)\cdot f\right)(k) = \int\limits_{-k_F}^{k_F} dk'\,\mathcal{V}_0(\eta|k,k')f(k'). \tag{5.8}$$

The kernels are given as

$$\mathcal{V}_0(\eta|k,k') = \frac{\ell_+(\eta|k)\ \ell_-(k') - \ell_-(k)\ \ell_+(\eta|k')}{\pi(k-k')},$$

$$\mathcal{R}_0^{(-)}(k,k') = \frac{\ell_-(k)\ \ell_-(k')}{2\pi}, \qquad \mathcal{R}_0^{(+)}(\eta|k,k') = \frac{\ell_+(\eta|k)\ \ell_+(\eta|k')}{\pi(1+\varepsilon\ \cos\eta)}, \tag{5.9}$$

where

$$\ell_+(\eta|k) := \frac{1+\varepsilon\ \cos\eta}{2}\ E_+(k) + \frac{\sin\eta}{2}\ \frac{1}{E_-(k)}, \qquad \ell_-(k) := E_-(k), \tag{5.10}$$

and functions $E_\pm(k)$ are given by (4.26).

Turn to the case $B<0$. In this case the ground state is the Fermi zone filled by the particles of type 1; one has $\gamma = 1$ and therefore $F(1,\eta) = 2\pi\Delta(\eta)$ where $\Delta(\eta)$ is a 2π-periodic delta-function. For the correlators one obtains

$$G_1^{(-)}(x,t;h,B)\Big|_{B<0,T=0}$$
$$= e^{-it(h-B)}\Big[\det\Big(\hat{I} + \hat{\mathcal{V}}_0(0) + \hat{\mathcal{R}}_0^{(-)}\Big) - \det\Big(\hat{I} + \hat{\mathcal{V}}_0(0)\Big)\Big],$$

$$G_1^{(+)}(x,t;h,B)\Big|_{B<0,T=0}$$
$$= e^{it(h-B)}\Big[\det\Big(\hat{I} + \hat{\mathcal{V}}_0(0) - \hat{\mathcal{R}}_0^{(+)}(0)\Big) + (G(x,t)-1)\det\Big(\hat{I} + \hat{\mathcal{V}}_0(0)\Big)\Big],$$

$$G_2^{(-)}(x,t;h,B)\Big|_{B<0,T=0} = 0,$$

$$G_2^{(+)}(x,t;h,B)\Big|_{B<0,T=0} = e^{it(h+B)}\frac{1}{2\pi}\int_{-\pi}^{\pi} d\eta$$
$$\times\Big[\det\Big(\hat{I} + \hat{\mathcal{V}}_0(\eta) - \hat{\mathcal{R}}_0^{(+)}(\eta)\Big) + (G(x,t)-1)\det\Big(\hat{I} + \hat{\mathcal{V}}_0(\eta)\Big)\Big], \tag{5.11}$$

where

$$\hat{\mathcal{V}}_0(0) = \frac{1+\varepsilon}{2}\hat{\mathcal{V}}_{1,0}, \qquad \hat{\mathcal{R}}_0^{(+)}(0) = \frac{1+\varepsilon}{2}\hat{\mathcal{R}}_{1,0}^{(+)} + \frac{1-\varepsilon}{2}\hat{\mathcal{R}}_{3,0}^{(+)}. \tag{5.12}$$

The integral operators act on the functions on the interval $(-k_F, k_F)$ by the rule (5.8). The kernels are given by

$$\mathcal{V}_{1,0}(k,k') = \frac{E_+(k)E_-(k') - E_-(k)E_+(k')}{\pi(k-k')},$$

$$\mathcal{R}_{1,0}^{(+)}(k,k') = \frac{E_+(k)E_+(k')}{2\pi}, \qquad \mathcal{R}_{3,0}^{(+)}(k,k') = \frac{1}{2\pi E_-(k)E_-(k')}, \tag{5.13}$$

and by (5.9).

If $B>0$, the ground state is formed by the particles of type 2, and the formulae for the correlators are obtained in an obvious way, due to the relation (4.1).

Conclusion

In this paper the determinant representation for the two-point time-dependent temperature correlation functions of the one-dimensional two-component bosons and fermions are obtained. The Fredholm determinant of the integrable integral operator enters our answers (4.46), (4.63). It makes possible deriving intergrable differential equations for the correlators and calculating their large time and distance asymptotics.

From the technical point of view, the essential thing in our approach is using the $XX0$ eigenfunctions for the two-component model with the infinitely strong coupling. This gives the opportunity of explicit calculation of the correlators. The same technique can be applied for calculating temperature correlation functions in the Hubbard model in the infinitely strong coupling limit. We are going to present the corresponding results in the next publication (with N. I. Abarenkova).

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Appendix

Here we consider matrix elements of the matrices $S^{(-)}$, $S^{(+)}$, $R^{(+)}$ entering the representations for normalized mean values for the model on the finite interval (Section 3). Our aim is to rewrite matrix elements of these matrices in terms of functions which are well defined in the thermodynamic limit. We prove also the representations (3.12), (3.36) for the matrices considered.

We begin from a pair of indentities

$$\cot \pi z = \frac{1}{z} + \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \left(\frac{1}{z-\pi j} - \frac{1}{\pi j} \right), \qquad \frac{\pi^2}{\sin^2 \pi z} = \sum_{j=-\infty}^{\infty} \frac{1}{(z-\pi j)^2}. \quad \text{(A.1)}$$

The value of the r.h.s. do not depend on organizing of summation. Let us fix the method of summation in such a way that the term $1/\pi j$ does not contribute to the sum in the first equation, e.g., combining terms with the same $|j|$ together and summing up in increasing order of $|j| = 0, 1, 2, \ldots$. The sums below should be understood with this prescription (where necessary).

In terms of quasimomenta k and q,

$$(k)_j = \frac{2\pi}{L} \left(-\left(\frac{1+\varepsilon}{2} \right) \frac{N}{2} + j \right) + \frac{\Lambda}{L},$$

$$(q)_j = \frac{2\pi}{L}\left(-\Big(\frac{1+\varepsilon}{2}\Big)\frac{N-1}{2}+j\right)+\frac{\Theta}{L}, \qquad j \in Z, \tag{A.2}$$

the both indentities (A.1) become ($q = \tilde{q} + \Lambda/L$):

$$\frac{2}{L}\sum_{\tilde{q}}\frac{1}{q-k} = \varepsilon\left[\tan\Big(\frac{\Lambda-\Theta}{2}\Big)\right]^{\varepsilon}, \qquad \frac{|1+\varepsilon\,\bar{\omega}\nu|^2}{L^2}\sum_{\tilde{q}}\frac{1}{(q-k)^2} = 1, \tag{A.3}$$

where $\omega := \exp i\Lambda$ and $\nu := \exp i\Theta$. The expression below, due to the first indentity in (A.3), can be rewritten as

$$\begin{aligned}
&\frac{|1+\varepsilon\bar{\omega}\nu|^2}{L}\sum_{\tilde{q}}\frac{e^{-itq^2+ixq}}{q-k}\\
&\quad = \frac{|1+\varepsilon\bar{\omega}\nu|^2}{L}\left\{\sum_{\tilde{q}}\frac{e^{-itq^2+ixq}-e^{-itk^2+ixk}}{q-k}+\sum_{\tilde{q}}\frac{e^{-itk^2+ixk}}{q-k}\right\}\\
&\quad = \frac{1+\varepsilon\,\cos(\Lambda-\Theta)}{2}e^{(-)}(k)+\frac{\varepsilon\,\sin(\Lambda-\Theta)}{2}[e_-(k)]^{-2}
\end{aligned} \tag{A.4}$$

where the functions $e_-(k)$ and $e^{(-)}(k)$ are introduced,

$$\begin{aligned}
e^{(-)}(k) &= \frac{2}{L}\sum_{\tilde{q}}\frac{e^{-itq^2+ixq}-e^{-itk^2+ixk}}{q-k}, \qquad q = \tilde{q}+\frac{\Theta}{L},\\
e_-(k) &= \exp\left(\frac{itk^2-ixk}{2}\right).
\end{aligned} \tag{A.5}$$

Let us consider the elements of the matrix $S^{(-)}$ given by

$$(S^{(-)})_{ab} = e_-(k_a)\; e_-(k_b)\;\frac{|1+\varepsilon\nu\bar{\omega}|^2}{L^2}\;\sum_{\tilde{q}}\frac{e^{-itq^2+ixq}}{(q-k_a)(q-k_b)}. \tag{A.6}$$

Consider first the off-diagonal elements, i.e., the case when $a \neq b$, hence $k_a \neq k_b$. For the off-diagonal elements of matrix $S^{(-)}$, due to (A.4), one has

$$\begin{aligned}
(S^{(-)})_{ab} &= e_-(k_a)\; e_-(k_b)\;\frac{|1+\varepsilon\nu\bar{\omega}|^2}{L^2(k_a-k_b)}\sum_{\tilde{q}}\left[\frac{e^{-itq^2+ixq}}{q-k_a}-\frac{e^{-itq^2+ixq}}{q-k_b}\right]\\
&= \frac{2}{L}\left\{\frac{1+\varepsilon\,\cos(\Lambda-\Theta)}{2}\,\frac{e_+^{(-)}(k_a)e_-(k_b)-e_-(k_a)e_+^{(-)}(k_b)}{k_a-k_b}\right.\\
&\left.+\frac{\varepsilon\,\sin(\Lambda-\Theta)}{2}\,\frac{[e_-(k_a)]^{-1}e_-(k_b)-e_-(k_a)[e_-(k_b)]^{-1}}{k_a-k_b}\right\}.
\end{aligned} \tag{A.7}$$

Let us consider now the diagonal elements of the matrix $S^{(-)}$. Using first the second indentity in (A.3) and then (A.4), one has for the diagonal elements

$$(S^{(-)})_{aa} = e^{itk_a^2-ixk_a}\left\{\frac{|1+\varepsilon\bar{\omega}\nu|^2}{L^2}\sum_{\tilde{q}}\frac{e^{-itq^2+ixq}-e^{-itk_a^2+ixk_a}}{(q-k_a)^2}+e^{-itk_a^2+ixk_a}\right\}$$

$$= \frac{2}{L}\left\{\frac{1+\varepsilon\,\cos(\Lambda-\Theta)}{2}\right.$$

$$\times e^{itk_a^2-ixk_a}\frac{2}{L}\sum_{\tilde{q}}\frac{e^{-itq^2+ixq}-e^{-itk_a^2+ixk_a}\Big(1+i(x-2tk_a)(q-k_a)\Big)}{(q-k_a)^2}$$

$$\left.+\frac{\varepsilon\,\sin(\Lambda-\Theta)}{2}i(x-2tk_a)\right\}+1. \tag{A.8}$$

It is easy to see that the expressions in (A.8) following $1+\varepsilon\cos(\Lambda-\Theta)$ and $\varepsilon\sin(\Lambda-\Theta)$ are the limiting cases of the corresponding expressions in (A.7) when $k_b \to k_a$. Thus, collecting (A.7) together with (A.8), one can conclude that the matrix $S^{(-)}$ has the structure (3.12) where matrices $V_{1,2}^{(-)}$ are given by (3.13) with the diagonal elements being understood in the sense of the l'Hôpitale rule.

Consider now matrix elements of the matrices $S^{(+)}$ and $R^{(+)}$ given by (3.35). In this case, instead of (A.3), the following indentities are useful

$$\frac{2}{L}\sum_{\tilde{k}}\frac{1}{k-q} = -\varepsilon\left[\tan\left(\frac{\Lambda-\Theta}{2}\right)\right]^{\varepsilon}, \qquad \frac{|1+\varepsilon\,\bar{\omega}\nu|^2}{L^2}\sum_{\tilde{k}}\frac{1}{(k-q)^2} = 1, \tag{A.9}$$

which are, of course, direct consequences of (A.1). Also, instead of (A.4), one has

$$\frac{|1+\varepsilon\bar{\omega}\nu|^2}{L}\sum_{\tilde{k}}\frac{e^{-itk^2+ixk}}{k-q}$$

$$= \frac{1+\varepsilon\,\cos(\Lambda-\Theta)}{2}e^{(+)}(q) - \frac{\varepsilon\,\sin(\Lambda-\Theta)}{2}[e_-(q)]^{-2} \tag{A.10}$$

where the function $e^{(+)}(q)$ is defined as

$$e^{(+)}(q) = \frac{2}{L}\sum_{\tilde{k}}\frac{e^{-itk^2+ixk}-e^{-itq^2+ixq}}{k-q}. \tag{A.11}$$

For the matrix $S^{(+)}$ all the calculation explained above on the example of the matrix $S^{(-)}$ can be done similarly. This leads to the structure (3.36) of the matrix $S^{(+)}$. The matrix $R^{(+)}$ defined by (3.35) can be put into the form (3.36) by means of the indentity (A.10) applied to each factor defining $R_{ab}^{(+)}$ in (3.35).

Let us now consider the thermodynamic properties of the functions containing sums over quasimomenta $\tilde{q}$ (or $\tilde{k}$) in their definitions. These functions are $e^{(\pm)}$

entering the matrix elements of the matrices $S^{(\pm)}$, $R^{(+)}$ and the function $g(x,t)$ entering the representations (3.45), (3.46).

Let us begin from the simplest example, the function g, which is defined as

$$g(x,t) = \frac{1}{L}\sum_{\tilde{k}} e^{-itk^2+ixk}, \qquad k = \tilde{k} + \frac{\Lambda}{L}. \tag{A.12}$$

In the limit $L \to \infty$ the values $(k)_j$ of the quasimomenta k fill the interval $(-\infty,+\infty)$ densely, $(k)_{j+1} - (k)_j = (\tilde{k})_{j+1} - (\tilde{k})_j = 2\pi/L$, and the sum over values of $\tilde{k}$ should be changed for the integral,

$$\frac{1}{L}\sum_{\tilde{k}} f(\tilde{k}) := \frac{1}{L}\sum_{j=-\infty}^{\infty} f((\tilde{k})_j) \to \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} dk\, f(k). \tag{A.13}$$

Thus, in the limit $L \to \infty$ for the function g one has

$$g(x,t) \to G(x,t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} dk\, e^{-itk^2+ixk}. \tag{A.14}$$

The dependence on Λ of the function g disappears in the limit $L \to \infty$. The same happens with respect to the dependence on m of functions $g_m := g|_{\Lambda=\frac{2\pi m}{N+1}}$. All functions g_m have the same thermodynamic limit which is equal to function G.

Consider the function $e^{(+)}(q)$ given by (A.11). Recall that the sum is understood in the sense of prescription described above. This prescription regularizes the sum on the "infinities" while the subtraction term, in the limit $L \to \infty$, remove the singularity coming from the pole on the integration counter. It means that such a sum turns into the integral in the sense of the principal value, i.e., for the function $e^{(+)}(q)$ one has

$$e^{(+)}(q) \to E(q) = P.v.\int\limits_{-\infty}^{\infty} dk\, \frac{e^{-itk^2+ixk}}{\pi(k-q)}. \tag{A.15}$$

Note that the dependence on Λ and Θ of the function $e^{(+)}(q)$ disappears in the limit $L \to \infty$ (function $E(q)$ does not depend on Λ and Θ) since k and q (function's argument) are assumed to run the interval $(-\infty,+\infty)$ continuously.

The function $e^{(-)}(k)$ given by (A.5) has the same thermodynamic limit as the function $e^{(+)}(k)$:

$$e^{(-)}(k) \to E(k) = P.v.\int\limits_{-\infty}^{\infty} dq\, \frac{e^{-itq^2+ixq}}{\pi(q-k)}. \tag{A.16}$$

Thus, all functions determining matrix elements of the matrices $S^{(\pm)}$ and $R^{(+)}$ are well defined in the thermodynamic limit.

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