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DIRAC OPERATOR ON κ-MINKOWSKI SPACE, BICOVARIANT DIFFERENTIAL CALCULUS AND DEFORMED $U(1)$ GAUGE THEORY

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ABSTRACT

Derivation of κ-Poincare bicovariant commutation relations between coordinates and 1-forms on κ-Minkowski space is given using Dirac operator and Allain Connes formula. The deformed $U(1)$ gauge theory and appearance of an additional spin 0 gauge field is discussed.

0. Introduction

Noncommutative geometry suggested by A. Connes [1] attracts these days a great interest of many researchers. Besides a lot of mathematical applications [1],[13] it also may be considered as a natural framework for quantisation of space and time [11]. One of the most promising results in this direction is the approach to gauge field theory developed in [2] where the Standard Model of gauge interactions was obtained from noncommutativity of space-time.

The basic notion of the approach studied in [1], [2] is the Connes triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is in general framework a noncommutative *-algebra which is considered as algebra of operators in the Hilbert space $\mathcal{H}$. D is a linear possibly unbounded operator in $\mathcal{H}$ with $D^* = -D$.

In classical case when $\mathcal{A} = \mathrm{Fun}(M)$ is commutative algebra of functions on the differential manifold M

$$D = \gamma^\mu \partial_\mu \,, \tag{0.1}$$

is the usual Dirac operator.

In (0.1) ∂_i are local derivatives and $\gamma^i = \gamma^i(x)$ are generators of local Clifford algebres

$$\gamma^\mu(x)\gamma^\nu(x) + \gamma^\nu(x)\gamma^\mu(x) = 2g^{\mu\nu}(x) \,, \tag{0.2}$$

where $g^{\mu\nu}(x)$ are local components of metric tensor.

Noncommutative differential calculus on the $\mathrm{Fun}(M)$ is defined by introduction exterior derivative operator which we shall denote by d_c. For each $f \in \mathrm{Fun}(M)$ it has the form

$$d_c f = [D, f] \,, \tag{0.3}$$

According to (0.1) formula (0.3) gives the following result

$$d_c f = \partial_\mu(f)\gamma^\mu \tag{0.4}$$

Correspondence of the definition (0.3) with the usual external derivative

$$df = \partial_i(f)dx^i \tag{0.5}$$

follows from the isomorhism between Γ and $Cl(\mathrm{Fun}(M))$ where Γ is the space of differential 1-forms over M or the space of sections of cotangent bundle $T^*(M)$ and $Cl(\mathrm{Fun}(M))$ is the space of sections of $Cl(M)$ the Clifford bundle over M [4]. This may be expressed by commutative diagram

$$\begin{array}{ccc} \mathrm{Fun}(M) & \xrightarrow{df=\partial_i(f)dx^i} & \Gamma \\ \big\downarrow{\scriptstyle id} & & \big\downarrow{\scriptstyle dx^i \to \gamma^i} \\ \mathrm{Fun}(M) & \xrightarrow{d_c f = [D,f]} & Cl(\mathrm{Fun}(M)) \end{array} \tag{0.6}$$

The fiber of $Cl(M)$ corresponding to $x \in M$ is a Clifford algebra defined by generators $\gamma^\mu(x)$ and relations (0.2). $Cl(\mathrm{Fun}(M))$ is a bimodule over $\mathrm{Fun}(M)$ generated by all sums of the form

$$\sum_i f_i[D, g_i] \,. \tag{0.7}$$

According to isomorphism (0.6) gauge connection 1-form $A_\mu dx^\mu$ which is used in construction of pure gauge action and the gauge interaction term $A_\mu \gamma^\mu$ in Dirac equation for spinor field have similar geometrical intrpretations. So studying deformations of field theory in quantum spaces it is natural to suppose that the diagram (0.6) has analog also in noncommutative case. We shall write corresponding diagram in the form,

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{d: f \to df} & \Gamma \\ {\scriptstyle id}\Big\downarrow & & \Big\downarrow{\scriptstyle df \to d_c f} \\ \mathcal{A} & \xrightarrow{d_c f = [D,f]} & Cl(\mathcal{A}) \end{array} \tag{0.8}$$

where $Cl(\mathcal{A})$ is bimodule over $\mathcal{A}$ generated by all sums of the form (0.7).

For the most interesting examples of noncommutative manifolds studied in the context of quantum group theory operator d and corresponding spaces of quantum 1-forms are defined axiomatically [3]. So in this case the condition of commutativity diagram (0.8) gives strong restrictions on the possible form of Dirac operator.

In this framework the Dirac equation for a massless spinor field coupled with gauge potential has the form

$$(D + gV)\psi = 0\,, \tag{0.9}$$

where $\psi \in \mathcal{H}$ and V is a noncommutative analog of $iA_\mu\gamma^\mu$ and g is a gauge charge. According to isomorphism between quntum Clifford and quantum cotangent bundles supposed by (0.8) it correspons to ω the gauge connection quantum 1-form which is the noncommutative analog of $iA_\mu dx^\mu$.

We suggest deformed $U(1)$ gauge transformation of spinorial fields in the form

$$\psi \to U\psi\,, \tag{0.10}$$

where U is a unitary element of $\mathcal{A}$

$$UU^* = U^*U = 1\,. \tag{0.11}$$

(Additional restriction on U will be discussed in the last section). The transformation (0.10) for ψ is compatible with the following transformation for V

$$V \to UVU^* + U[D, U^*] \tag{0.12}$$

that according to (0.8) is equivalent to the following law for ω

$$\omega \to \tilde{\omega} = U\omega U^* + UdU^*\,. \tag{0.13}$$

In the present paper we construct Dirac operators for quantum spaces admitting quantum group coaction. Bicovariant differential calculus on these spaces can be constructed according to [3]. Commutativity of the diagram (0.8) follows from general theory of quantum groups. As an example we consider the case of κ-Minkowski spase $\mathcal{M}_\kappa$ which corresponds to the κ-Poincare group and was intensively studied in the last few years. Description of different Minkowski space deformations is given in [11]. (Also Dirac operators on the quantum $SU(2)$ group and the quantum sphere were obtained in [8] and [12]). In paper [10] a Dirac operator on $\mathcal{M}_\kappa$ was defined. In this paper we present the one parametric family of Dirac operators on $\mathcal{M}_\kappa$ and among them lies the one suggested in [10].

The paper is organised as follows. In sect. 1 we study according to [5],[6] and cite7 differential geometry on $\mathcal{M}_\kappa$. Invariant Klein-Gordon operator on $\mathcal{M}_\kappa$ is constructed as a bilinear combination of generators of covariant algebra of vector fields expressed from standard generators of

κ-Poincare quantum algebra. Construction of exterior differential needs introduction of quantum derivatives which also are elements of quantum Poincare algebra. In sect. 2 we construct on $\mathcal{M}_\kappa$ Dirac operator and show commutativity of (0.8). In sect. 3 we suggest on $\mathcal{M}_\kappa$ equations of deformed electrodynamics. We also discuss gauge invariant action for deformed $U(1)$ gauge theory and illustrate how existance of extra dimension of space of quantum differential 1-forms can explain the appearance of spin 0 gauge fields discussed in [1],[2].

Everywhere in the paper we use Einstein rules of summation. In first two sections greece indices α, μ, ν numerate space-time components and take value 0,1,2,3 however latin indices m, n, numerate only the space components and take value 1,2,3. Everywhere $g^{\mu\nu}$ means the Minkowski space metric tensor $(1,-1,-1,-1)$.

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1. κ-Poincare group and κ-Minkowski space

The κ-Poincare quantum group was introduced in [6] (see also [5], [7]) and in one of equivalent forms it represents as a *-Hopf algebra generated by hermitian elements $\Lambda_\mu^{\ \nu}$, a^μ and relations

$$\begin{aligned}
[a^\mu, a^\nu] &= \frac{i}{\kappa}(\delta_0^\mu a^\nu - \delta_0^\nu a^\mu)\,, \\
[\Lambda_\mu^{\ \nu}, \Lambda_\alpha^{\ \beta}] &= 0\,, \\
[\Lambda_\mu^{\ \nu}, a^\alpha] &= \frac{i}{\kappa}[(\delta_0^\nu - \Lambda_0^{\ \nu})\Lambda_\mu^{\ \alpha} + g^{\nu\alpha}(\delta_\mu^0 - \Lambda_\mu^{\ 0})]\,, \\
\triangle(\Lambda_\mu^{\ \nu}) &= \Lambda_\mu^{\ \alpha} \otimes \Lambda_\alpha^{\ \nu}\,, \\
\triangle(a^\mu) &= a^\nu \otimes \Lambda_\nu^{\ \mu} + 1 \otimes a^\mu\,, \\
S(\Lambda_\mu^{\ \nu}) &= \Lambda^\nu_{\ \mu} = g_{\mu\alpha} g^{\nu\beta} \Lambda_\beta^{\ \alpha}\,, \\
S(a^\mu) &= -a^\nu \Lambda^\mu_{\ \nu}\,, \\
\varepsilon(\Lambda_\mu^{\ \nu}) &= \delta_\mu^\nu\,, \\
\varepsilon(a^\mu) &= 0\,,
\end{aligned} \tag{1.1}$$

The $\mathcal{P}_\kappa$ may be regarded as the quantum symmetry group of κ-Minkowski space $\mathcal{M}_\kappa$, which is defined by four hermitian generators x^μ and relations

$$[x^\mu, x^\nu] = \frac{i}{\kappa}(\delta_0^\mu x^\nu - \delta_0^\nu x^\mu)\,. \tag{1.2}$$

The corresponding right $\mathcal{P}_\kappa$ coaction is

$$\Phi_R(x^\mu) = x^\nu \otimes \Lambda_\nu^{\ \mu} + 1 \otimes a^\mu \tag{1.3}$$

(the left comodule structure also can be defined [5], [6]).

Coproduct, counit, and antipode [7]

$$\begin{aligned}
\triangle(x^\mu) &= 1 \otimes x^\mu + x^\mu \otimes 1\,, \\
\varepsilon(x^\mu) &= 0\,, \\
S(x^\mu) &= -x^\mu\,,
\end{aligned} \tag{1.4}$$

also define on $\mathcal{M}_\kappa$ structure of cocommutative Hopf algebra. As it was shown in [7] correspondence $a^\mu \to x^\mu$ and $\Lambda^\mu{}_\nu \to 1$ defines a Hopf algebra homomorphism from $\mathcal{P}_\kappa$ to $\mathcal{M}_\kappa$.

Its Hopf dual $\mathcal{M}_\kappa^*$ is defined by four hermitian generators P_μ and relations

$$
\begin{aligned}
[P_\mu, P_\nu] &= 0\,, \\
\triangle(P_0) &= P_0 \otimes 1 + 1 \otimes P_0\,, \\
\triangle(P_m) &= P_m \otimes 1 + e^{-\frac{P_0}{\kappa}} \otimes P_m\,, \\
S(P_0) &= -P_0\,, \\
S(P_m) &= -e^{P_0/\kappa} P_m\,, \\
\varepsilon(P_\mu) &= 0\,.
\end{aligned} \tag{1.5}
$$

and the pairing $(\cdot,\cdot) : \mathcal{M}_\kappa^* \otimes \mathcal{M}_\kappa \to \mathbb{C}$ is given by

$$
i(P_\mu, x^\nu) = \delta^\nu_\mu\,. \tag{1.6}
$$

As it was shown in [7] $\mathcal{M}_\kappa^*$ is a Hopf subalgebra of the quantum Poincare algebra which is dual to $\mathcal{P}_\kappa$. The corresponding pairing between $\mathcal{M}_\kappa^*$ and $\mathcal{P}_\kappa$ is given by formula

$$
i(P_\mu, a^\nu) = \delta^\nu_\mu\,. \tag{1.7}
$$

And according to this pairing $\mathcal{M}_\kappa^*$ acts on $\mathcal{M}_\kappa$ from the left.

$$
\begin{aligned}
\pi(x) &= ((id \otimes \pi), \Phi_R(x))\,, \\
\pi &\in \mathcal{M}_\kappa^*\,, \quad x \in \mathcal{M}_\kappa\,.
\end{aligned} \tag{1.8}
$$

Considering elements of $\mathcal{M}_\kappa$ as left multiplication operators we may obtain according to (1.5) the following relations

$$
\begin{aligned}
[P_0, x^\mu] &= \frac{1}{i}\delta^\mu_0\,, \\
[P_m, x^0] &= \frac{i}{\kappa} P_m\,, \\
[P_m, x^n] &= \frac{1}{i}\delta^n_m\,.
\end{aligned} \tag{1.9}
$$

Elements

$$
\begin{aligned}
e^4 &= i\kappa(\mathrm{ch}\frac{P_0}{\kappa} - \frac{1}{2\kappa^2} e^{P_0/\kappa}\vec{P}^2)\,, \\
e^0 &= i\kappa(\mathrm{sh}\frac{P_0}{\kappa} + \frac{1}{2\kappa^2} e^{P_0/\kappa}\vec{P}^2)\,, \\
e^m &= -i e^{P_0/\kappa} P_m\,,
\end{aligned} \tag{1.10}
$$

satisfy following commutation relations with elements of $\mathcal{M}_\kappa$

$$
\begin{aligned}
[e^\mu, x^\nu] &= \frac{i}{\kappa}(g^{0\mu}e^\nu - g^{\mu\nu}e^0 - g^{\mu\nu}e^4)\,, \\
[e^4, x^\mu] &= -\frac{i}{\kappa} e^\mu\,,
\end{aligned} \tag{1.11}
$$

and the additional relation

$$\square_\kappa \equiv e_\mu e^\nu = g^{\mu\nu} e_\mu e_\nu = \kappa^2 + (e^4)^2 \,. \tag{1.12}$$

Equations (1.11), (1.12) are invariant under the right $\mathcal{P}_\kappa$ coaction which on $\mathcal{M}_\kappa$ has the form (1.3) and on the elements (1.10) is defined by

$$\Phi_R(e_\mu) = e_\nu \otimes \Lambda^\nu{}_\mu \,, \qquad \Phi_R(e^4) = e^4 \otimes 1 \,. \tag{1.13}$$

So according to the general approach [9], cite14 we may consider the joint algebra generated by x^μ, e_μ, e^4 and relations (1.2), (1.11) and (1.12) as the algebra of vector fields on $\mathcal{M}_\kappa$.

Element $\square_\kappa$ from (1.12) is invariant under (1.13) and playes a role of massless Klein–Gordon operator on $\mathcal{M}_\kappa$ [10].

Quantum De Rham complex on $\mathcal{M}_\kappa$ with satisfied Leibnitz rule was constructed in [5]. The space of 1-forms Γ is generated by

$$\tau^\mu = dx^\mu \,, \qquad \tau^4 = \frac{i\kappa}{4}([\tau^\mu, x_\mu] + \frac{3i}{4}\tau^0) \,, \tag{1.14}$$

and relations

$$\begin{aligned} [\tau^\mu, x^\nu] &= \frac{i}{\kappa}(g^{0\mu}\tau^\nu - g^{\mu\nu}\tau^0 - g^{\mu\nu}\tau^4) \,, \\ [\tau^4, x^\mu] &= -\frac{i}{\kappa}\tau^\mu \,. \end{aligned} \tag{1.15}$$

External algebra relations and external derivative are given by $(i, j = 0, ...4)$

$$\tau^i \wedge \tau^j = -\tau^j \wedge \tau^i \,, \qquad d\tau^i = 0 \,. \tag{1.16}$$

Equations (1.14), (1.15) and (1.16) are invariant under the right $\mathcal{P}_\kappa$-coaction given on elements of $\mathcal{M}_\kappa$ by (1.3) and on elements Γ by

$$\Phi_R(\tau^\mu) = \tau^\nu \otimes \Lambda_\nu^{\ \mu} \,, \qquad \Phi_R(\tau^4) = \tau^4 \otimes 1 \,. \tag{1.17}$$

(In [5] the left variant of (1.17) was presented).

It is easy to see from (1.15) that for every $a \in \mathcal{M}_\kappa$

$$s^2 a = a s^2 \,, \tag{1.18}$$

where the metric form $s^2 \in \Gamma \otimes \Gamma$ defined by

$$s^2 = \tau_\mu \otimes \tau^\mu - \tau^4 \otimes \tau^4 \tag{1.19}$$

is invariant under the right $\mathcal{P}_\kappa$ coaction on $\Gamma \otimes \Gamma$.

According to (1.12) and (1.18) we may define corresponding to the e^4 and τ^4 component of metric tensor.

$$g^{44} = g_{44} = -1. \tag{1.20}$$

Commutation relations between 1-forms and elements of $\mathcal{M}_\kappa$ also may be represented in the standard form [3] $(i, j = 0, 1, 2, 3, 4)$

$$\tau^i a = f^i{}_j(a) \tau^j \,, \tag{1.21}$$

where τ^i_k are linear operators $\tau^i_k : \mathcal{A} \to \mathcal{A}$. From $\tau^i(ab) = (\tau^i a)b$ follows that

$$f^i{}_k(ab) = f^i{}_j(a) f^j{}_k(b) , \tag{1.22}$$

In the most interesting case when all $f^i{}_j \in \mathcal{M}^*_\kappa$ so that their action on elements of $\mathcal{A}$ is given by (1.8) that is equivalent to

$$\triangle(f^i{}_k) = f^i{}_j \otimes f^j{}_k . \tag{1.23}$$

Taking

$$\begin{aligned}
f^0{}_0 &= \mathrm{ch}\frac{P_0}{\kappa} + \frac{1}{2k^2} e^{P_0/\kappa} \vec{P}^2 , \\
f^0{}_m &= -\frac{1}{\kappa} P_m , \qquad f^m{}_0 = -\frac{1}{\kappa} e^{P_0/\kappa} P_m , \qquad f^n{}_m = \delta^n_m , \\
f^0{}_4 &= [\mathrm{sh}\frac{P_0}{\kappa} + \frac{1}{2\kappa^2} e^{P_0/\kappa} \vec{P}^2] , \\
f^4{}_0 &= [\mathrm{sh}\frac{P_0}{\kappa} - \frac{1}{2\kappa^2} e^{P_0/\kappa} \vec{P}^2] , \\
f^m{}_4 &= -\frac{1}{\kappa} e^{P_0/\kappa} P_m , \qquad f^4{}_m = \frac{1}{\kappa} P_m , \\
f^4{}_4 &= \mathrm{ch}\frac{P_0}{\kappa} - \frac{1}{2\kappa^2} e^{P_0/\kappa} \vec{P}^2 .
\end{aligned} \tag{1.24}$$

we can prove by direct calculations relations (1.15) for x^μ and then also relations (1.23).

According to (1.18) operators $f^i{}_j$ satisfy the following system of relations $(i, j, k, l = 0, 1, 2, 3, 4)$

$$f_k{}^j f^k_i = \delta^j_i , \tag{1.25}$$

where $f_i{}^j = g_{ik} g^{jl} f^k{}_l$.

The additional system

$$f^j{}_k f_i{}^k = \delta^j_i , \tag{1.26}$$

is also valid.

According to reltions $\tau^{i^*} = \tau^i$ $(i = 0, 1, 2, 3, 4)$ we may write for every $a \in \mathcal{M}_\kappa$

$$a\tau^i = (\tau^i a^*)^* = (f^i{}_j(a^*)\tau^j)^* = \tau^j (f^i{}_j(a^*)^* = f^j{}_k(f^i{}_j(a^*)^*)\tau^k , \tag{1.27}$$

and from eq.(1.25) follows that

$$f^i{}_j(a^*)^* = f_j{}^i(a) . \tag{1.28}$$

We shall use this relation in the last section.

2. Dirac operator and quantum Clifford bundle

Let us define now following elements of $\mathcal{M}^*_\kappa$:

$$\partial_0 = i\kappa f^4{}_0 , \qquad \partial_m = i\kappa f^4{}_m , \qquad \partial_4 = i\kappa (f^4{}_4 - 1). \tag{2.1}$$

Using these elements we may write the formula for the external derivation in the compact form $(i = 0, 1, 2, 3, 4)$

$$da = \partial_i(a)\tau^i \tag{2.2}$$

According to the Leibnitz rule

$$d(ab) = adb + da\,b\,, \tag{2.3}$$

the following system of relations must be satisfied ($i, j = 0, 1, 2, 3, 4$):

$$\partial_i(ab) = a\partial_i(b) + \partial_j(a) f^j{}_i(b)\,, \tag{2.4}$$

or

$$[\partial_i, a] = \partial_j(a) f^j{}_i\,. \tag{2.5}$$

Since all $\partial_i \in \mathcal{M}^*_\kappa$ this is equivalent to

$$\triangle(\partial_i) = 1 \otimes \partial_i + \partial_j \otimes f^j{}_i\,. \tag{2.6}$$

So to prove (2.2) we must check it for coordinates x^μ and then prove (2.6) that can be easely done by strict calculations.

According to the general approach [1], [2] we suppose now that the Hilbert space $\mathcal{H}$ is a subspace of $\mathbb{C}^4 \otimes \mathcal{M}_\kappa$. We take the Dirac operator in the form ($i = 0, ...4$)

$$D_\kappa = \gamma^i \partial_i\,, \tag{2.7}$$

where γ^i for $i = 0, ...3$ are the usual Dirac gamma matrices satisfying the standard relation:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\,, \tag{2.8}$$

($g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is a standard Minkowski space metric) and γ_4 is some undefinite matrix which however may be taken in the form $\gamma_4 = \lambda I_4$ where I_4 is a unit 4×4 matrix or $\gamma_4 = \lambda\gamma_5$ where $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. The choise $\gamma_4 = 0$ corresponds to the Dirac operator suggested in [10]. In this case connection between D_κ and $\Box_\kappa$ has the standard form

$$D^2_\kappa = \Box_\kappa. \tag{2.9}$$

The direct application of (0.3) gives according to (2.5) the following expressions for τ^i_c corresponding to τ^i elements of quantum Clifford bundle.

$$\tau^i = \gamma^j f_j{}^i\,, \tag{2.10}$$

Relations (1.21) are fullfilled also for τ^i_c and the diagram (0.8) is commutative.

3. The deformed $U(1)$ gauge theory

In this section we suppose that all up and down indices take values from 0 to 4.

By analogy with classical case we define gauge potentials as elements of $\mathcal{A}$ quantum algebra of functions. Let us introduce the $U(1)$ gauge field by the gauge connection 1-form

$$\omega = iA_k\tau^k \tag{3.1}$$

where A_μ for $\mu = 0, ..3$ are deformations of usual potential and A_4 may be interpreted as a spin 0 gauge field. Appearance of such scalar gauge fields in framework of noncommutayive geometry was intensively studied in [1] and [2].

According to (0.9), (2.7) and connection between ω and V we can write the gauge coupled Dirac equation for massless particle in the form

$$\gamma^k \nabla_k \psi = 0 \tag{3.2}$$

where

$$\nabla_k = \partial_k + igA_j f^j_{\ k}, \tag{3.3}$$

(g is a gauge charge).

The transformation low (0.13) gives

$$\tilde{A}_k = UA_j f^j_{\ k}(U^*) - i/gU\partial_k(U^*)\,. \tag{3.4}$$

Defining the curvature form

$$\Omega = d\omega + g\omega \wedge \omega\,, \tag{3.5}$$

we obtain according to (0.13) the following transformation law for it

$$\tilde{\Omega} = U\Omega U^*\,. \tag{3.6}$$

Defining field strenght tensor by

$$iF_{jk}dx^j \wedge dx^k = \Omega\,, \tag{3.7}$$

or according to (1.16)

$$F_{ij} = \partial_i(A_j) - \partial_j(A_i) + iA_k[f^k_{\ i}(A_j) - f^k_{\ j}(A_i)]\,, \tag{3.8}$$

we can using relations (3.6), (1.21) and (1.22) obtain for them the following transformation law

$$\tilde{F}_{ij} = UF_{kl}f^k_{\ i}f^l_{\ j}(U^*). \tag{3.9}$$

We also may obtain the tensor F_{ij} by commuting covariant derivatives

$$[\nabla_i, \nabla_j] = igF_{mn}f^m_{\ i}f^n_{\ j}\,. \tag{3.10}$$

It is easy to prove that the following Bianchi identiies

$$[\nabla_i, [\nabla_j, \nabla_k]] + [\nabla_k, [\nabla_i, \nabla_j]] + [\nabla_j, [\nabla_k, \nabla_i]] = 0\,. \tag{3.11}$$

are satisfied.

Defining deformed covariant derivatives of the strenght tensor as

$$\nabla_m F^{mk} = \partial_m F^{mk} + ig(A_j f^j_{\ m}(F^{mk}) - F^{mn} f_m^{\ j} f_n^{\ k}(A_j))\,, \tag{3.12}$$

it is easy to obtain for them the following transformation law

$$\tilde{\nabla}_m F^{mk} = U \nabla_m F^{mn} f_n^{\ k}(U^*). \tag{3.13}$$

In the limit $\kappa \to \infty$ as it follows from (1.24) (or generally from (1.21)) we have $f^m_n \to \delta^m_n$ so that transformation laws (3.9) and (3.13) may be considered as deformations of standard formulas. Eqs. (3.11) and

$$\nabla_m F^{mk} = 0, \tag{3.14}$$

may be interpreted as $\mathcal{P}_\kappa$-covariant equations of deformed electrodynamics in κ-Minkovski space.

It will be interesting to derive (3.14) according to some kind of variational principle. We have no recipe to do it. However defining elements

$$C = F^{ij} F^*_{ij}\,, \qquad C_+ = F_{ij} f^i{}_k f^j{}_l (F^{kl})\,, \qquad C_- = f^i{}_k f^j{}_l (F^*_{ij}) F^{kl^*} \tag{3.15}$$

we have

$$\tilde{C} = UCU^*\,, \qquad \tilde{C}_\pm = UC_\pm U^*\,. \tag{3.16}$$

Really for $\tilde{C}$

$$\tilde{C} = UF^{kl} f_k{}^i f_l{}^j (U^*) f_i{}^u f_j{}^v (U) F_{uv} U^*\,. \tag{3.17}$$

But since $f^i{}_j(1) = \delta^i_j$ then according to (0.11) and (1.22)

$$f_k{}^i f_l{}^j (U^*) f_i{}^u f_j{}^v (U) = f_k{}^u f_l{}^v (U^* U) = \delta^u_k \delta^v_l\,, \tag{3.18}$$

which proves first equation in (3.8). For C_+ we have,

$$\tilde{C}_+ = UF^*_{ij} f^i{}_k f^j{}_l (F^{uv}) f^k{}_c f^l{}_d f_u{}^c f_v{}^d (U^*)\,, \tag{3.19}$$

and relation (3.8) for C_+ follows now from (1.26) and commutativity of $\mathcal{M}^*_\kappa$. The proof for C_- follows now from the relation

$$C_- = C^*_+\,, \tag{3.20}$$

which can be easely proved according to (1.28).

Now in order to obtain a gauge invariant action from C and $C_\pm$ we have by analogy with the undeformed case to take integral over $\mathcal{M}_\kappa$. This means that there exist a subalgebra $L^1(\mathcal{M}_\kappa)$ of $\mathcal{M}_\kappa$ and a positive linear functional $h:\ L^1(\mathcal{M}_\kappa) \to \mathbb{C}$. It is natural to suppose that $L^1(\mathcal{M}_\kappa)$ is invariant under $\mathcal{P}_\kappa$-coaction (1.3),

$$\Phi_R(L^1(\mathcal{M}_\kappa)) = L^1(\mathcal{M}_\kappa) \otimes \mathcal{P}_\kappa\,. \tag{3.21}$$

Also it is natural to suppose that h is $\mathcal{P}_\kappa$ invariant so that for every $a \in L^1(\mathcal{M}_\kappa)$

$$(h \otimes id) \circ \triangle(a) = h(a) 1_{\mathcal{P}_\kappa}\,. \tag{3.22}$$

Let now $U(1)_{\mathcal{M}_k}$ be the group of all $U \in \mathcal{M}_\kappa$ satisfying (0.11) and additionaly preserving h so that for every $a \in L^1(\mathcal{M}_\kappa)$

$$UaU^* \in L^1(\mathcal{M}_k)\,, \tag{3.23}$$

and

$$h(UaU^*) = h(a)\,. \tag{3.24}$$

We see now that expressions $h(C)$, $h(C_\pm)$ and gauge coupled Dirac equation (3.2) are invariant under the $U(1)_{\mathcal{M}_\kappa}$ action.

Since now we have not any construction of h and $U(1)_{\mathcal{M}_\kappa}$ we cannot study completely the case of finite κ. However we may try to study the case $\kappa \to \infty$. Considering the following Lagrangian density $\mathcal{L} = -\frac{1}{4}C$ we get from (1.24), (2.1), (3.5), (3.7) in the limit $\kappa \to \infty$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu A_4 \partial^\mu A_4\,. \tag{3.25}$$

This express the general statement suggested in [1],[2] that noncommutativity of space-time produces the appearance of spin 0 gauge fields.

4. Conclusions

In this paper we have defined Dirac operator on κ-Minkovski space according to the A. Connes scheme. In the special case it coincides with the one suggested in [10]. We constructed equations for deformed electrodynamics on κ-Minkovski space. These equations may be concidered as covariant under the action of quantum Poincare group. We also mentoined the fact of natural appearance of scalar gauge field in the theory.

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