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Correlation functions for a strongly correlated boson system

N.M. Bogoliubov
Russian Academy of Sciences
Steklov Mathematical Institute
St.Petersburg Department
E-mail: bogoliub@pdmi.ras.ru

A.G. Izergin
Russian Academy of Sciences
Steklov Mathematical Institute
St.Petersburg Department
E-mail: izergin@pdmi.ras.ru

N.A. Kitanine
Russian Academy of Sciences
Steklov Mathematical Institute
St.Petersburg Department
E-mail: kitanin@pdmi.ras.ru

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ABSTRACT

The correlation functions for a strongly correlated exactly solvable one-dimensional boson system on a finite chain as well as in the thermodynamic limit are calculated explicitly. This system which we call the phase model is the strong coupling limit of the integrable q-boson hopping model. The results are presented as determinants.

Introduction

In this paper we investigate a one-dimensional strongly correlated boson lattice system where the kinetic term depends on the occupation of the adjacent lattice sites. It was introduced and solved by the quantum inverse scattering method (QISM) in [1, 2, 3, 4]. The natural dynamical variables for this model are the so-called q-bosons [5] closely related to the quantum algebra formalism [6]. This model is called the q-boson hopping model. It can be treated as a quantization [7] of the classical Ablowitz-Ladik equation [8] which is one of the possible integrable lattice versions of the nonlinear Schrödinger equation [9]. In the scaling continuous limit the q-boson hopping model becomes the Bose gas model.

We calculate correlation functions in the special case of the lattice q-boson model corresponding to the infinite value of the coupling constant. It appears that this special case is related to the phase operators of the quantum nonlinear optics so we will call it the phase model [10]. Usually the infinite coupling limit corresponds to the free fermion point (the most well-known example is the Bose gas model). On the contrary the phase model cannot be regarded as a free fermion model.

The description of the correlation functions for the models solved by means of Bethe Ansatz is based on the representation for the correlators as the Fredholm determinants of linear integral operators. Such representations were obtained for the first time in [11, 12] for the simplest two point equal-time correlators of one-dimensional impenetrable bosons. Later they were generalized for the case of time-dependent correlators for the models which are the free-fermion points of models solved by means of the quantum inverse scattering method (the Bose gas in the infinite coupling limit [13] and the isotropic XX0 Heisenberg chain [14, 15]) and also for the case of finite interaction ([16, 17] for the one-dimensional Bose gas and [18] for XXZ chain). Such representations give an opportunity to get classical integrable equations for correlators which can be used, in particular, to calculate the long time and large distance asymptotics of correlation functions [19, 20, 21, 22].

The main result of this paper is the determinant representations for correlation functions. We consider the finite chain as well as the thermodynamic limit at zero and finite temperature. The form factors for the phase model were calculated in [23]. Some of the results in the thermodynamic limit were reported in [10].

This paper is organized as follows. In the first Section the q-boson model is defined and different limits are discussed. In Section 2 the Bethe Ansatz solution for the q-boson hopping model is presented. In Section 3 we discuss the solution of the phase model and its thermodynamics. In Section 4 the scalar products are calculated. In Section 5 we derive the determinant representation for the simplest correlation function which is the "darkness formation probability". In Section 6 we calculate the two-point time-dependent correlation functions for the finite lattice while the thermodynamic limit is obtained in the last Section. The derivation of the form factors is presented in Appendix.

1 Q-Boson hopping model

The q-boson hopping model is defined by the Hamiltonian

$$H_q = -\frac{1}{2}\sum_{n=1}^{M}(B_n^\dagger B_{n+1} + B_n B_{n+1}^\dagger - 2N_n), \tag{1.1}$$

with the periodic boundary conditions, $M+1 \equiv 1$. Operators B_n, $B_n^\dagger$, and $N_n = N_n^\dagger$ form the q-boson algebra:

$$[N_i, B_j^\dagger] = B_i^\dagger \delta_{ij}\,,\ [N_i, B_j] = -B_i\delta_{ij}, \tag{1.2}$$
$$[B_i, B_j^\dagger] = q^{-2N_i}\delta_{ij}.$$

The c-number q is taken to be $q = e^\gamma$. We consider real $\gamma > 0$. This algebra was investigated in [5] and the connection with the quantum $SU_q(2)$ algebra was established in [6].

The Hamiltonian (1.1)of the $q-$boson hopping model commutes with the operator of total number of particles,

$$\hat{N} = \sum_{n=1}^{M} N_n\,,\ [H_q, \hat{N}] = 0\,. \tag{1.3}$$

The Hamiltonian with the chemical potential $\bar{\mu}$

$$H_h = H_q - \bar{\mu}\hat{N}, \tag{1.4}$$

which has the same eigenstates will be considered further.

The $q-$boson algebra (1.2) possesses the representation in the local Fock spaces formed by the q-boson normalized states

$$B_j \mid 0\rangle_j = 0, \ \ N_j \mid 0\rangle_j = 0, \tag{1.5}$$

$$B_j^\dagger \mid n_j\rangle_j = [n_j+1]^{\frac{1}{2}} \mid n_j+1\rangle_j\,, \ \ B_j \mid n_j\rangle_j = [n_j]^{\frac{1}{2}} \mid n_j - 1\rangle_j,$$

and the normalized states on the whole lattice are

$$\mid 0\rangle = \prod_{j=1}^{M} \mid 0\rangle_j\,, \tag{1.6}$$

$$| n\rangle = \prod_{j=1}^{M} | n_j\rangle_j = \prod_{j=1}^{M}([n_j]!)^{-\frac{1}{2}}(B_j^\dagger)^{n_j} | 0\rangle.$$

The notation $[n]! = \prod_{k=1}^{n}[k]$ is used and the 'box' is

$$[n] = \frac{1-q^{-2n}}{1-q^{-2}}, \quad [n]! = \prod_{l=1}^{n}[l].$$

The $q-$boson operators (1.2) can be expressed in terms of the ordinary canonical bosons

$$b_j^\dagger,\ b_j, [b_i, b_j^\dagger] = \delta_{ij},\ N_j = b_j^\dagger b_j, \tag{1.7}$$

as

$$B_j = (B_j^\dagger)^\dagger = \sqrt{\frac{[N_j+1]}{N_j+1}}\, b_j\,. \tag{1.8}$$

If $q \to 1\,(\gamma \to 0)$, the $q-$bosons become ordinary bosons

$$B_j \to b_j\,,\ B_j^\dagger \to b_j^\dagger.$$

At $\gamma = 0$ (the free boson limit) the Hamiltonian (1.1) becomes the linear hopping model Hamiltonian

$$H_b = -\frac{1}{2}\sum_{n=1}^{M}(b_n^\dagger b_{n+1} + b_n b_{n+1}^\dagger - 2N_n), \tag{1.9}$$

while the $q-$boson algebra is the ordinary boson algebra.

In the continuum scaling limit the lattice constant $\delta \to 0$, $M\delta = L$, $\gamma = \frac{1}{2}c\delta$, and the q-boson Hamiltonian (1.1) is just the Hamiltonian of the Bose gas model with the repulsive δ-function interaction of strength c :

$$H = \int dx\{\partial_x b^\dagger(x)\partial_x b(x) + cb^\dagger(x)b^\dagger(x)b(x)b(x)\},$$

$$[b(x), b^\dagger(y)] = \delta(x-y).$$

Representation (1.8) shows that (1.1) involves non-linearities of all orders about the ordinary boson hopping model (1.9). The deformation parameter plays the role of the coupling constant. Really, for small $\gamma \to 0$ the hopping model (1.1) expands to

$$H_\gamma = H_b - \frac{\gamma}{2}\sum_{n=1}^{M}\{b_{n+1}b_n^\dagger N_n + b_{n+1}^\dagger(N_n + N_{n+1})b_n + N_{n+1}b_{n+1}b_n^\dagger\} + \mathcal{O}(\gamma^2).$$

We can consider the q-boson model as a strongly interacting boson model. On the other hand, the representation (1.8) suggests that the kinetic energy of (1.1) involves boson correlations, i.e., the hopping terms between adjacent sites depend on the occupation of those sites. These models are known as the strongly correlated ones.

From this point of view the limit $q \to \infty \, (\gamma \to \infty)$ is of interest. It follows from (1.5) and (1.8) that the operators $B, B^\dagger$ transform into operators $\phi, \phi^\dagger$:

$$B_j \to \phi_j = (N_j + 1)^{-\frac{1}{2}} b_j \,, \qquad (1.10)$$
$$B_j^\dagger \to \phi_j^\dagger = b_j^\dagger (N_j + 1)^{-\frac{1}{2}},$$

with the commutation relations

$$[N_i, \phi_j] = -\phi_i \delta_{ij} \,, \; [N_i, \phi_j^\dagger] = \phi_i^\dagger \delta_{ij} \,, \; [\phi_i, \phi_j^\dagger] = \pi_i \delta_{ij} \qquad (1.11)$$

where π_j is the local vacuum projector, $\pi_j = (\mid 0\rangle\langle 0 \mid)_j$. One may verify that ϕ, $\phi^\dagger$, and N can be expressed in terms of the Fock states, $\mid n\rangle$, as

$$\phi = \sum_{n=0}^{\infty} \mid n\rangle\langle n+1 \mid, \, \phi^\dagger = \sum_{n=0}^{\infty} \mid n+1\rangle\langle n \mid, N = \sum_{n=0}^{\infty} n \mid n\rangle\langle n \mid .$$

The introduced operator ϕ is "one-sided unitary" (or isometric), i.e.,

$$\phi\phi^\dagger = 1,$$

but

$$\phi^\dagger\phi = 1 - \mid 0\rangle\langle 0 \mid .$$

The operators $\phi, \phi^\dagger$ are studied intensively in quantum optics in connection with the phase-operator problem (see [25],[26] and references therein). The introduction of phase variables for bosons was discussed in [27]. The relative phase of boson fields is important in the theory of beam splitters [28] and Josephson junctions [29].

For $\gamma = \infty$, the Hamiltonian (1.4) becomes

$$H = -\frac{1}{2} \sum_{n=1}^{M} (\phi_n^\dagger \phi_{n+1} + \phi_n \phi_{n+1}^\dagger - 2N_n) - \bar{\mu}\hat{N}, \qquad (1.12)$$

and

$$[H, \hat{N}] = 0,$$

where the total number operator $\hat{N}$ is given by (1.3). This model is called the phase model. It should be mentioned that the case $M = 2$ with $\bar{\mu} = 0$ corresponds to the phase difference operator considered in [25] .

2 The Bethe Ansatz solution of the model

Define the $L-$operator for the q-bosons hopping model at lattice site n as

$$L_n(\lambda) = \begin{pmatrix} e^{\lambda} & \chi B_n^{\dagger} \\ \chi B_n & e^{-\lambda} \end{pmatrix}, \tag{2.1}$$

where $B_n, B_n^{\dagger}$ are the q-bosons (1.2), $\chi = \sqrt{1-q^{-2}}$, and $\lambda \in \mathcal{C}$ is the spectral parameter. This L-operator satisfies the bilinear relation:

$$R(\lambda,\mu)L_n(\lambda) \otimes L_n(\mu) = L_n(\mu) \otimes L_n(\lambda)R(\lambda,\mu), \tag{2.2}$$

with the (gauge transformed) trigonometric R-matrix

$$R(\lambda,\mu) = \begin{pmatrix} f(\mu,\lambda) & 0 & 0 & 0 \\ 0 & g(\mu,\lambda) & q & 0 \\ 0 & q^{-1} & g(\mu,\lambda) & 0 \\ 0 & 0 & 0 & f(\mu,\lambda) \end{pmatrix}, \tag{2.3}$$

with the matrix elements being defined by the functions

$$f(\lambda,\mu) = \frac{\sinh(\lambda-\mu+\gamma)}{\sinh(\lambda-\mu)}; \; g(\lambda,\mu) = \frac{\sinh\gamma}{\sinh(\lambda-\mu)}. \tag{2.4}$$

The monodromy matrix is introduced in the usual way as

$$T(\lambda) = L_M(\lambda)...L_1(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{2.5}$$

The 16 commutation relations of its matrix elements are then given by the R-matrix

$$R(\lambda,\mu)T(\lambda) \otimes T(\mu) = T(\mu) \otimes T(\lambda)R(\lambda,\mu), \tag{2.6}$$

Let us write explicitly the relations important for deriving the Bethe Ansatz

$$qA(\lambda)B(\mu) = f(\lambda,\mu)B(\mu)A(\lambda) + g(\mu,\lambda)B(\lambda)A(\mu), \tag{2.7}$$

$$qD(\lambda)B(\mu) = f(\mu,\lambda)B(\mu)D(\lambda) + g(\lambda,\mu)B(\lambda)D(\mu),$$

$$[B(\lambda), B(\mu)] = 0,$$

$$C(\lambda)B(\mu) - q^{-2}B(\mu)C(\lambda) = q^{-1}g(\lambda,\mu)\{A(\lambda)D(\mu) - A(\mu)D(\lambda)\}.$$

The matrix trace of the monodromy matrix is the transfer matrix

$$\tau(\lambda) = A(\lambda) + D(\lambda). \tag{2.8}$$

It follows from (2.6) that

$$[\tau(\lambda), \tau(\mu)] = 0. \tag{2.9}$$

The Hamiltonian (1.1) of the q-boson hopping model is expressed by means of trace identities in terms of the transfer matrix

$$-2\chi H_q = \frac{1}{2} e^{-2\lambda} \frac{\partial e^{M\lambda}\tau(\lambda)}{\partial \lambda} \mid_{\lambda=-\infty} - \frac{1}{2} e^{2\lambda} \frac{\partial e^{-M\lambda}\tau(\lambda)}{\partial \lambda} \mid_{\lambda=\infty} -2\hat{N}\chi^2, \tag{2.10}$$

and commutes with the transfer matrix

$$[H_q, \tau(\lambda)] = 0. \tag{2.11}$$

The eigenvectors of the transfer matrix and hence of the Hamiltonian are of the form

$$\mid \psi_N(\lambda_1, ..., \lambda_N)\rangle = \prod_{j=1}^{N} B(\lambda_j) \mid 0\rangle, \tag{2.12}$$

where the vacuum state $\mid 0\rangle$ is defined in (1.6), and parameters λ_j satisfy the Bethe equations:

$$e^{2M\lambda_j} = \prod_{k\neq j}^{N} \frac{f(\lambda_k, \lambda_j)}{f(\lambda_j, \lambda_k)}; \; j = 1, ..., N. \tag{2.13}$$

These states are called Bethe vectors. One can also construct dual Bethe vectors

$$\langle 0| \prod_{j=1}^{N} C(\lambda_j),$$

where $\{\lambda_j\}$ satisfy the same Bethe equation (2.13). The vacuum vector $|0\rangle$ and the dual vacuum $\langle 0|$ are eigenvectors of the operators $A(\lambda)$ and $D(\lambda)$, with the "vacuum" eigenvalues $a(\lambda)$ and $d(\lambda)$, respectively. For the q-boson hopping model one has

$$a(\lambda) = e^{M\lambda}, \quad d(\lambda) = e^{-M\lambda}. \tag{2.14}$$

The eigenvalues θ_N of the transfer matrix for the eigenvectors (2.12) are

$$\tau(\mu) \mid \psi_N\rangle = \theta_N(\mu, \{\lambda_j\}) \mid \psi_N\rangle,$$

$$q^N\theta_N(\mu,\{\lambda_j\}) = e^{M\mu}\prod_{j=1}^{N} f(\lambda_j,\mu) + e^{-M\mu}\prod_{j=1}^{N} f(\mu,\lambda_j). \tag{2.15}$$

In the explicit form the equations (2.13) are rewritten as

$$e^{iMp_j} = \prod_{k\neq j}^{N} \frac{\sin(\frac{p_j-p_k}{2}+i\gamma)}{\sin(\frac{p_j-p_k}{2}-i\gamma)}, \tag{2.16}$$

where we have introduced the momenta

$$p = -2i\lambda. \tag{2.17}$$

Later it will be convenient to use the parameters λ for arbitrary sets $\{\lambda\}$ and momenta p for the sets satisfying the Bethe equations. In the scaling limit these equations go into the Bethe equations for the Bose gas.

Eigenenergies of the Hamiltonian (1.1), $H_q \mid \psi_N\rangle = E_N \mid \psi_N\rangle$, can be found from the equations (2.15), (2.4) and (2.10)

$$E_N = \sum_{k=1}^{N} h(p_k);\;\; h(p) = 2\sin^2(p/2). \tag{2.18}$$

Eigenenergies of the Hamiltonian (1.4), $H_h \mid \psi_N\rangle = \tilde{E}_N \mid \psi_N\rangle$, are

$$\tilde{E}_N = \sum_{k=1}^{N}(h(p_k)-\bar{\mu}).$$

3 The phase model

The phase model is the model defined by the Hamiltonian (1.12). It belongs to the class of strongly coupled correlated boson models since the hopping terms between adjacent sites depend on the occupation of those sites which is evident from the representation (1.10) of the ϕ -operators in terms of the ordinary bosons.

The $L-$operator [10] of the phase model is obtained by putting $\gamma=\infty$ in (2.1)

$$L_n(\lambda) = \begin{pmatrix} e^{ip/2} & \phi_n^\dagger \\ \phi_n & e^{-ip/2} \end{pmatrix}, \tag{3.1}$$

where $\phi_n^\dagger, \phi_n$ are defined by (1.11). This operator satisfies the bilinear relation

$$R(p,s)L_n(p)\otimes L_n(s) = L_n(s)\otimes L_n(p)R(p,s),$$

with the 4×4 R-matrix $R(p,s)$. The non-zero elements of the R-matrix are

$$R_{11}(p,s) = R_{44}(p,s) = f(s,p),$$

$$R_{22}(p,s) = R_{33}(p,s) = g(s,p),$$

$$R_{23}(p,s) = 1$$

and

$$f(p,s) = i\frac{e^{i\frac{p-s}{2}}}{2\sin(\frac{s-p}{2})};\ \ g(p,s) = \frac{i}{2\sin(\frac{s-p}{2})}. \tag{3.2}$$

This R-matrix is obtained as the limit of the R-matrix (2.3).

The Bethe equations for the model are

$$\exp\{i(M+N)p_j\} = (-1)^{N-1}\exp\{i\sum_{k=1}^{N} p_k\}, \tag{3.3}$$

$(j=1,...,N)$ and are exactly solvable:

$$p_j = \frac{2\pi I_j + \sum_{k=1}^{N} p_k}{M+N}\,, \tag{3.4}$$

where I_j are integers or half-integers depending on N being odd or even. The Bethe vectors form a complete orthogonal basis.

The N-particle eigenenergies of the Hamiltonian $H-\bar{\mu}\hat{N}$ (1.12) are

$$E_N = \sum_{k=1}^{N}(h(p_k)-\bar{\mu});\ \ h(p) = 2\sin^2(p/2). \tag{3.5}$$

Here $\bar{\mu}$ is the chemical potential, $0\le\bar{\mu}\le 1$. For $M=2$ and $\bar{\mu}=0$ this result coincides with one obtained in [25].

The thermodynamics of the model is handled in the standard way. It will be considered for the case of the zero total momentum $P=\sum_{k=1}^{N} p_k = 0$. The state of the thermal equilibrium of the model at finite temperatures β^{-1} is determined through the solution of the nonlinear integral Yang-Yang equations which are drastically symplified in our case

$$\epsilon(p) = h(p) - \bar{\mu} - (2\pi\beta)^{-1}\int_{-\pi}^{\pi}\ln(1+e^{\beta\epsilon(p)})dp, \tag{3.6}$$

$$2\pi\rho(p)(1+e^{\beta\epsilon(p)}) = 1 + \int_{-\pi}^{\pi}\rho(p)dp.$$

The function $\rho(p)$ is a quasi-particle density while $\epsilon(p)$ is the excitation energy. The pressure is then

$$\mathcal{P} = (2\pi\beta)^{-1}\int_{-\pi}^{\pi}\ln(1+e^{\beta\epsilon(p)})dp, \tag{3.7}$$

and the density is

$$D = \frac{\partial\mathcal{P}}{\partial\bar{\mu}} = \int_{-\pi}^{\pi}\rho(p)dp. \tag{3.8}$$

So we have

$$\epsilon(p) = h(p) - \bar{\mu} - \mathcal{P} \tag{3.9}$$

and the quasi-particle density has the Fermi-like distribution

$$2\pi\rho(p) = (1+D)(1+e^{\beta\epsilon(p)})^{-1}. \tag{3.10}$$

At zero temperature ($\beta^{-1}=0$) the ground state is the Fermi zone, $-\Lambda \le p \le \Lambda$ ($\Lambda \le \pi$), filled by the particles with the negative energies $\epsilon_0(p)$. The pressure and density are now

$$\mathcal{P}_0 = -(2\pi)^{-1}\int_{-\Lambda}^{\Lambda}\epsilon_0(p)dp, \;\; D_0 = \int_{-\Lambda}^{\Lambda}\rho(p)dp, \tag{3.11}$$

where (see (3.9) and (3.10))

$$\epsilon_0(p) = h(p) - \bar{\mu} - \mathcal{P}_0, \;\; \epsilon_0(\pm\Lambda) = 0; \tag{3.12}$$

$$2\pi\rho_0(p) = (1+D_0).$$

The bare Fermi momentum Λ is expressed as a function of density

$$\Lambda = \frac{\pi D_0}{1+D_0}. \tag{3.13}$$

The Fermi velocity v is given as

$$v = \frac{\epsilon_0'(\Lambda)}{2\pi\rho_0(\Lambda)} = (1+D_0)^{-1}\sin\frac{\pi D_0}{1+D_0}.$$

If $\Lambda \to 0\,(\bar{\mu}\to 0)$, then $D_0 \to 0$ and $\mathcal{P}_0 \to 0$ as should be expected. If $\Lambda \to \pi\,(\bar{\mu}\to 1)$, all the vacancies are occupied by particles $D_0 \to \infty$, $\mathcal{P}_0 \to 1$, and the phase model becomes the classical XY chain in this limit [1, 4].

4 Scalar products and norms in the phase model

Here we calculate the scalar products of the states produced by the operators $B(\lambda)$ and the norms of Bethe vectors using the standard procedure [31, 9]. It is necessary to calculate the correlation functions. Consider first the scalar products in the q-boson hopping model (1.1)

$$\tilde{S}(\{\lambda_j^B\}, \{\lambda_k^C\}) = \langle 0|C(\lambda_1^C)\dots C(\lambda_N^C)B(\lambda_N^B)\dots B(\lambda_1^B)|0\rangle, \tag{4.1}$$

where $C(\lambda), B(\lambda)$ are the matrix elements of the monodromy matrix (2.5) and $\{\lambda_j^B\}$ and $\{\lambda_k^C\}$ are the sets of arbitrary spectral parameters (Bethe equations are not imposed). Using the commutation relations (2.7) one gets

$$\tilde{S}(\{\lambda_j^B\}, \{\lambda_k^C\}) = \sum_{AD} \prod_{\alpha=1}^{N} a(\lambda_\alpha^A) \prod_{\beta=1}^{N} d(\lambda_\beta^D) K_N \begin{pmatrix} \{\lambda^C\} & \{\lambda^B\} \\ \{\lambda^A\} & \{\lambda^D\} \end{pmatrix}. \tag{4.2}$$

Here the sum is taken over all the partitions of the set $\{\lambda_j^B\} \cup \{\lambda_k^C\}$ into two subsets

$$\{\lambda_j^B\} \cup \{\lambda_k^C\} = \{\lambda_\alpha^A\} \cup \{\lambda_\beta^D\}.$$

The number of elements in each set equals N: $\mathrm{card}\{\lambda_j^B\} = \mathrm{card}\{\lambda_k^C\} = \mathrm{card}\{\lambda_\alpha^A\} = \mathrm{card}\{\lambda_\beta^D\} = N$. Functions $a(\lambda)$ and $d(\lambda)$ are the vacuum eigenvalues of the operators $A(\lambda)$ and $D(\lambda)$ (2.14)

The coefficient $K_N \begin{pmatrix} \{\lambda^C\} & \{\lambda^B\} \\ \{\lambda^C\} & \{\lambda^B\} \end{pmatrix}$ (the "highest" coefficient) can be represented in terms of the partition function of the six-vertex model corresponding to the R-matrix (2.3). It was shown in [23] that this partition function is just the partition function of the six-vertex model corresponding to the XXZ R-matrix. The explicit expression for this partition function was given in [30]. This function depends on two sets of parameters $\{\lambda_a\}$ and $\{\nu_k\}$,

$$Z_N = (-1)^N \frac{\prod_{\alpha=1}^{N}\prod_{k=1}^{N} \sinh(\lambda_a - \nu_k - \frac{\gamma}{2}) \sinh(\lambda_a - \nu_k + \frac{\gamma}{2})}{\prod_{\alpha<\beta} \sinh(\lambda_a - \lambda_b) \prod_{k<l} \sinh(\nu_l - \nu_k)} \mathrm{det}_N \mathcal{M}, \tag{4.3}$$

where the $N \times N$ matrix $\mathcal{M}$ has the following form

$$\mathcal{M}_{\alpha,k} = \frac{\sinh\gamma}{\sinh(\lambda_a - \nu_k - \frac{\gamma}{2}) \sinh(\lambda_a - \nu_k + \frac{\gamma}{2})}. \tag{4.4}$$

The highest coefficient can be expressed as follows

$$K_N\begin{pmatrix} \{\lambda^C\} & \{\lambda^B\} \\ \{\lambda^C\} & \{\lambda^B\} \end{pmatrix} = e^{-2\gamma N^2}\left(\prod_{j,k=1}^{N}\sinh(\lambda_j^B-\lambda_k^C)\right)^{-1} Z_N(\{\lambda_j^B\},\{\lambda_k^C+\frac{\gamma}{2}\}). \tag{4.5}$$

To prove this proposition one should consider the inhomogeneous gauge transformed XXZ model on a lattice of N sites taking into account that the highest coefficient depends only on the R-matrix (see [9]). Other coefficients in (4.2) can be analogously represented as

$$K_N\begin{pmatrix} \{\lambda^C\} & \{\lambda^B\} \\ \{\lambda^A\} & \{\lambda^D\} \end{pmatrix} = e^{-2\gamma n_0(N-n_0)}\left(\prod_{j\in AC}\prod_{k\in DC} f(\lambda_j^{AC},\lambda_k^{DC})\right)\times$$

$$\left(\prod_{l\in AB}\prod_{m\in DB} f(\lambda_l^{AB},\lambda_m^{DB})\right) K_{n_0}\begin{pmatrix} \{\lambda^{AB}\} & \{\lambda^{DC}\} \\ \{\lambda^{AB}\} & \{\lambda^{DC}\} \end{pmatrix} K_{N-n_0}\begin{pmatrix} \{\lambda^{AC}\} & \{\lambda^{DB}\} \\ \{\lambda^{AC}\} & \{\lambda^{DB}\} \end{pmatrix}. \tag{4.6}$$

Now it is not difficult to represent the scalar products for the q-boson hopping model as mean values of determinants depending on dual fields using the approach of [16].

In this paper we will not, however, use this method concentrating our attention on the phase model only. It appears that the scalar products and correlation functions in the phase model can be represented without auxiliary fields. The matrix elements of the matrix $\mathcal{M}$ (4.4) the limit $\gamma\to\infty$ are given as

$$\mathcal{M}_{jk} = \frac{2e^{2\lambda_j^B}}{e^{2\lambda_k^C}-e^{2\lambda_j^B}} + O(e^{-\gamma}),$$

$$\det\mathcal{M} = 2^N\frac{\prod_{j<l}(e^{2\lambda_j^B}-e^{2\lambda_l^B})\prod_{k<m}(e^{2\lambda_m^C}-e^{2\lambda_k^C})}{\prod_{j,k}(e^{2\lambda_j^B}-e^{2\lambda_k^C})}e^{2\sum_{j=1}^N\lambda_j^B} + O(e^{-\gamma}),$$

for the highest coefficient (4.5) we obtain

$$K_N\begin{pmatrix} \{\lambda^C\} & \{\lambda^B\} \\ \{\lambda^C\} & \{\lambda^B\} \end{pmatrix} = \left(\prod_{j<l}(e^{2\lambda_j^B}-e^{2\lambda_l^B})\prod_{k<m}(e^{2\lambda_m^C}-e^{2\lambda_k^C})\right)^{-1}\times$$

$$\times\,\det{}_N\mathcal{K}(\{\lambda_j^B\},\{\lambda_k^C\}), \tag{4.7}$$

$$\mathcal{K}_{jk} = \frac{\exp((2N-1)\lambda_k^C+\lambda_j^B)}{e^{2\lambda_k^C}-e^{2\lambda_j^B}},$$

and other coefficients (4.6) are

$$K_N\left(\begin{array}{cc}\{\lambda^C\} & \{\lambda^B\}\\ \{\lambda^A\} & \{\lambda^D\}\end{array}\right)=\left(\prod_{j<l}(e^{2\lambda_j^B}-e^{2\lambda_l^B})\prod_{k<m}(e^{2\lambda_m^C}-e^{2\lambda_k^C})\right)^{-1}\times$$

$$\times(-1)^{([P]+[Q])}\mathrm{det}_{n_0}\mathcal{K}(\{\lambda^{DC}\},\{\lambda^{AB}\})\mathrm{det}_{N-n_0}\mathcal{K}(\{\lambda^{DB}\},\{\lambda^{AC}\}). \qquad (4.8)$$

Here P and Q are the transpositions $\{\lambda^B\}\to\{\lambda^{AB}\}\cup\{\lambda^{DB}\},\{\lambda^C\}\to\{\lambda^{AC}\}\cup\{\lambda^{DC}\}$ respectively.

It is convenient to consider the "normalized" scalar product in the phase model

$$S(\{\lambda_j^B\},\{\lambda_k^C\})=\langle 0|\prod_{k=1}^{N}\mathbf{C}(\lambda_k^C)\prod_{j=1}^{N}\mathbf{B}(\lambda_j^B)|0\rangle, \qquad (4.9)$$

where

$$\mathbf{B}(\lambda_j^B)=\frac{B(\lambda_j^B)}{d(\lambda_j^B)},\quad \mathbf{C}(\lambda_k^C)=\frac{C(\lambda_k^C)}{d(\lambda_k^C)}. \qquad (4.10)$$

From (4.2,4.8) one obtains

$$S(\{\lambda_j^B\},\{\lambda_k^C\})=\left(\prod_{j<l}(e^{2\lambda_j^B}-e^{2\lambda_l^B})\prod_{k<m}(e^{2\lambda_m^C}-e^{2\lambda_k^C})\right)^{-1}\times$$

$$\times\sum_{A,D}(-1)^{([P]+[Q])}\left(\prod_{l=1}^{N}r(\lambda_l^A)\right)\mathrm{det}_{n_0}\mathcal{K}(\{\lambda^{DC}\},\{\lambda^{AB}\})\mathrm{det}_{N-n_0}\mathcal{K}(\{\lambda^{DB}\},\{\lambda^{AC}\}),$$

where $r(\lambda)=a(\lambda)/d(\lambda)$. Using the formula for the determinant of the sum of two matrices we can express the scalar product as a determinant of an $N\times N$ matrix

$$S(\{\lambda_j^B\},\{\lambda_k^C\})=\left(\prod_{j<l}(e^{2\lambda_j^B}-e^{2\lambda_l^B})\prod_{k<m}(e^{2\lambda_m^C}-e^{2\lambda_k^C})\right)^{-1}\mathrm{det}_N F(\{\lambda_j^B\},\{\lambda_k^C\}), \qquad (4.11)$$

$$F_{jk}=\frac{r(\lambda_j^B)\exp((2N-1)\lambda_j^B+\lambda_k^C)-r(\lambda_k^C)\exp((2N-1)\lambda_k^C+\lambda_j^B)}{e^{2\lambda_j^B}-e^{2\lambda_k^C}}. \qquad (4.12)$$

For the phase model, $r(\lambda)=e^{2\lambda M}$ (2.14). Hence we obtain

$$F_{jk}=\frac{\exp((2N+2M-1)\lambda_j^B+\lambda_k^C)-\exp((2N+2M-1)\lambda_k^C+\lambda_j^B)}{e^{2\lambda_j^B}-e^{2\lambda_k^C}}. \qquad (4.13)$$

There is an important case when $\{\lambda^B\} = \{\lambda^C\}$. The diagonal elements of the matrix F are then

$$F_{jj} = (M+N-1)e^{2(M+N-1)\lambda_j},$$

Turn now to the norms of Bethe eigenvectors

$$|\psi(p_1,\ldots,p_N)\rangle_N = \prod_{j=1}^{N} \mathbf{B}(\lambda_j)|0\rangle, \quad \langle\psi(p_1,\ldots,p_N)| = \langle 0|\prod_{j=1}^{N} \mathbf{C}(\lambda_j), \quad p_j = -2i\lambda_j, \tag{4.14}$$

where $\{p\}$ is a solution of the Bethe equations (3.3).Now we have

$$F_{jj} = (M+N-1)e^{i(P-p_j)}, \qquad P = \sum_{j=1}^{N} p_j,$$

$$F_{jk} = -e^{i(P-\frac{p_j}{2}-\frac{p_k}{2})}, \quad j \neq k.$$

It is not difficult to calculate the determinant

$$\det F = e^{iP(N-1)}M(M+N)^{N-1},$$

and for the norm of any Bethe eigenstate we obtain the formula of the Gaudin type

$$\mathcal{N}^2(p_1,\ldots,p_N) = \langle\psi(p_1,\ldots,p_N)|\psi(p_1,\ldots,p_N)\rangle = \left(\prod_{j\neq k}\frac{1}{e^{ip_j}-e^{ip_k}}\right) e^{iP(N-1)}M(M+N)^{N-1}. \tag{4.15}$$

5 The darkness formation probability

Here we will calculate the darkness formation probability, i.e., the probability of the states with no particles on the first m sites of the lattice. Formally it can be defined as the normalized mean value

$$\tau(m,\{p\}) = \mathcal{N}^{-2}(p_1,\ldots,p_N)\langle\psi(p_1,\ldots,p_N)|\exp\{\alpha Q(m)\}|\psi(p_1,\ldots,p_N)\rangle\big|_{\alpha=-\infty}, \tag{5.1}$$

where $Q(m) = \sum_{j=1}^{m} N_j$ is the number of particles operator on the first m sites.

The monodromy matrix can be represented in the following form

$$T(\lambda) = T_2(\lambda)T_1(\lambda),$$

$$T_2(\lambda) = L_M(\lambda)\dots L_{m+1}(\lambda) = \begin{pmatrix} A_2(\lambda) & B_2(\lambda) \\ C_2(\lambda) & D_2(\lambda) \end{pmatrix}, \tag{5.2}$$

$$T_1(\lambda) = L_m(\lambda)\dots L_1(\lambda) = \begin{pmatrix} A_1(\lambda) & B_1(\lambda) \\ C_1(\lambda) & D_1(\lambda) \end{pmatrix}.$$

The bare vacuum (1.6) can be represented as

$$|0\rangle = |0\rangle_{\mathrm{II}} \otimes |0\rangle_{\mathrm{I}},$$

where $|0\rangle_{\mathrm{I}}$ and $|0\rangle_{\mathrm{II}}$ are the Fock vacua for the monodromy matrices $T_1(\lambda)$ and $T_2(\lambda)$, respectively. We denote $r_1(\lambda)$ and $r_2(\lambda)$ the ratios of the corresponding vacuum eigenvalues

$$r_1(\lambda) = a_1(\lambda)/d_1(\lambda) = \exp\{2m\lambda\} \quad, r_2(\lambda) = a_2(\lambda)/d_2(\lambda) = \exp\{2(M-m)\lambda\}.$$

The numerator of (5.1) is a particular value of the following matrix element

$$T(\{\lambda_j^B\},\{\lambda_k^C\}) = \langle 0| \prod_{k=1}^{N} \mathbf{C}(\lambda_k^C) \exp\{\alpha Q(m)\} \prod_{j=1}^{N} \mathbf{B}(\lambda_j^B)|0\rangle|_{\alpha=-\infty}, \tag{5.3}$$

where the operators $\mathbf{B}(\lambda)$ and $\mathbf{C}(\lambda)$ are defined by (4.10) and $\{\lambda_j^B\}, \{\lambda_k^C\}$ are arbitrary sets of N spectral parameters. Using the decomposition (5.2) one obtains

$$T(\{\lambda_j^B\},\{\lambda_k^C\}) = \prod_{j=1}^{N} r_1(\lambda_j^C)_{\mathrm{II}}\langle 0| \prod_{k=1}^{N} \mathbf{C_2}(\lambda_k^C) \prod_{j=1}^{N} \mathbf{B_2}(\lambda_j^B)|0\rangle_{\mathrm{II}}.$$

The scalar product here can be calculated by means of (4.11), with the result

$$T(\{\lambda_j^B\},\{\lambda_k^C\}) = \left(\prod_{j<l} (e^{2\lambda_j^B} - e^{2\lambda_l^B}) \prod_{k<m} (e^{2\lambda_m^C} - e^{2\lambda_k^C}) \right)^{-1} \det \mathcal{T}(\{\lambda_j^B\},\{\lambda_k^C\}), \tag{5.4}$$

the matrix elements of the $N \times N$ matrix $\mathcal{T}$ being equal to

$$\mathcal{T}_{jk} = \frac{1}{e^{2\lambda_k^C} - e^{2\lambda_j^B}} \times$$

$$\times \left(\exp((2N+2M-1)\lambda_k^C + \lambda_j^B) - \exp((2N+2M-2m-1)\lambda_j^B + (2m+1)\lambda_k^C) \right).$$

If the sets coincide, $\{\lambda^B\} = \{\lambda^C\} = \{\lambda\}$, then the diagonal elements should be understood in the sense of the l'Hôpital rule

$$\mathcal{T}_{jj} = (M+N-m-1)e^{2(M+N-1)\lambda_j}.$$

If the spectral parameters satisfy the Bethe equations the matrix elements $\mathcal{T}_{jk}$ can be represented in the form ($p_j = -2i\lambda_j$ and P is the sum of momenta)

$$\mathcal{T}_{jj} = (M + N - m - 1)e^{i(P-p_j)},$$

$$\mathcal{T}_{jk} = e^{i(P-p_k)} e^{i\frac{m+1}{2}(p_j-p_k)} \frac{\sin\frac{m+1}{2}(p_j - p_k)}{\sin\frac{1}{2}(p_j - p_k)}, \quad j \neq k.$$

Thus one obtains the following result for the darkness formation probability

$$\tau(m, \{p\}) = \frac{T(\{p\},\{p\})}{\mathcal{N}^2(\{p\})} = (1 + D)\det\tilde{T}(m, \{p\}), \tag{5.5}$$

$$\tilde{T}_{jk} = \delta_{j,k} - \frac{1}{M}\frac{1}{1+D}\frac{\sin\frac{m+1}{2}(p_j - p_k)}{\sin\frac{1}{2}(p_j - p_k)}, \tag{5.6}$$

where $D = N/M$.

6 Time-dependent correlation functions

In this section a determinant representation for the two-point time-dependent correlation function of fields on the finite lattice is obtained. The derivation is somewhat similar to the derivation in the case of the XX0 chain [15]. To this end one needs first a representation for the form factors

$$G_N(m, \{p\}, \{q\}) = \frac{\langle\psi(q_1, \ldots, q_{N-1})|\phi_m|\psi(p_1, \ldots, p_N)\rangle}{\mathcal{N}(q_1, \ldots, q_{N-1})\mathcal{N}(p_1, \ldots, p_N)}, \tag{6.1}$$

$$G_N^{(+)}(m, \{p\}, \{q\}) = \frac{\langle\psi(q_1, \ldots, q_{N+1})|\phi_m^\dagger|\psi(p_1, \ldots, p_N)\rangle}{\mathcal{N}(q_1, \ldots, q_{N+1})\mathcal{N}(p_1, \ldots, p_N)}, \tag{6.2}$$

where $|\psi(\{p\})\rangle$ are Bethe vectors and $\mathcal{N}(\{p\})$ are their norms (4.15)

$$\mathcal{N}(\{p\}) = \langle\psi(\{p\})|\psi(\{p\})\rangle^{1/2},$$

The determinant representations for the form factors of the phase model were obtained in [23] (a brief derivation of these results is given in Appendix A):

$$G_N(m, \{p\}, \{q\}) = M^{-1}(M + N)^{-N+\frac{3}{2}} \tilde{Z} e^{-im\Theta} \left(\frac{M+N}{M+N-1}\right)^{\frac{N}{2}-1} \times$$

$$\times (1+\frac{\partial}{\partial x})\mathrm{det}_{N-1}(D_1 - xD_2)|_{x=0}, \tag{6.3}$$

where is a complex number depending on P and Q, $|\tilde{Z}| = 1$The explicit form of $\tilde{Z}$ is not written here since since it appears to be of no importance for calculating the correltors. We use convenient notations:

$$\Theta = Q - P, \quad Q = \sum_j q_k, \quad P = \sum_j p_j.$$

Matrix elements of matrices D_1 and D_2 are

$$D_{1_{jk}} = e^{\frac{i}{2}p_j}(\frac{\cos\frac{1}{2}(\Theta - p_j)}{\tan\frac{1}{2}(p_j - q_k)} + \sin\frac{1}{2}(\Theta - p_j)), \tag{6.4}$$

$$D_{2_{jk}} = e^{\frac{i}{2}p_N}(\frac{\cos\frac{1}{2}(\Theta - p_N)}{\tan\frac{1}{2}(p_N - q_k)} + \sin\frac{1}{2}(\Theta - p_N)). \tag{6.5}$$

Analogously, for the form factor (6.2) we have (the star denotes the complex conjugation)

$$G_N^{(+)}(m, \{p\}, \{q\}) = M^{-1}(M+N)^{-N+\frac{1}{2}}\left(\frac{M+N+1}{M+N}\right)^{\frac{N}{2}}\tilde{Z}^* e^{im\Theta}\times$$

$$\times (1+\frac{\partial}{\partial x})\mathrm{det}_N(D_1^{(+)} - xD_2^{(+)})|_{x=0}, \tag{6.6}$$

where

$$D^{(+)}_{1_{jk}} = e^{-\frac{i}{2}q_k}(\frac{\cos\frac{1}{2}(\Theta - q_k)}{\tan\frac{1}{2}(q_k - p_j)} + \sin\frac{1}{2}(\Theta - q_k)), \tag{6.7}$$

$$D^{(+)}_{2_{jk}} = e^{-\frac{i}{2}q_{N+1}}(\frac{\cos\frac{1}{2}(\Theta - q_{N+1})}{\tan\frac{1}{2}(q_{N+1} - p_j)} + \sin\frac{1}{2}(\Theta - q_{N+1})). \tag{6.8}$$

Consider now the normalized mean value of the time-dependent product of two phase operators

$$f_N^+(\{p\}, m, t) = \frac{\langle 0|\prod_{k=1}^{N}\mathbf{C}(p_k)\phi_{m+1}(t)\phi_1^\dagger(0)\prod_{k=1}^{N}\mathbf{B}(p_k)|0\rangle}{\langle 0|\prod_{k=1}^{N}\mathbf{C}(p_k)\prod_{k=1}^{N}\mathbf{B}(p_k)|0\rangle}, \tag{6.9}$$

where

$$\phi_m(t) = \exp[iHt]\phi_m\exp[-iHt],$$

with the Hamiltonian H (1.12). Using the formulae (6.3), (6.6) we can represent this correlator as follows

$$
\begin{aligned}
f_N^+(\{p\},m,t) &= M^{-2}(M+N+1)^{-2N+1}\left(\frac{M+N+1}{M+N}\right)^{N-1}\times \\
&\times \sum_{\{q\}} \exp\left(im(\sum_{k=1}^{N+1} q_k - \sum_{j=1}^{N} p_j) + it(\sum_{j=1}^{N}\varepsilon(p_j) - \sum_{k=1}^{N+1}\varepsilon(q_k))\right)\times \\
&\times \left|(1+\frac{\partial}{\partial x})\det{}_N(D_1^{(+)}(\{p\},\{q\}) - xD_2^{(+)}(\{p\},\{q\}))|_{x=0}\right|^2,
\end{aligned}
\tag{6.10}
$$

where $\varepsilon(p)$ is the energy of quasiparticle $\varepsilon(p) = 2\sin^2\frac{p}{2} - \bar{\mu}$ (3.5).

The summation in (6.10) is taken over all the solutions $\{q\}$ of the Bethe equations (3.3) such that card$\{q\} = N+1$. It can be seen from (3.4) that $Q \equiv \sum q_j = \frac{2\pi K}{M}$ where K is an integer, $-\frac{M}{2} < K \leq \frac{M}{2}$. Then the sum over all the solutions of the Bethe equation can be rewritten as the sum over all such Q and the sum over all the sets of $N+1$ different q_k satisfying the following conditions

$$
q_k = \frac{2\pi I_k + Q}{M+N+1}, \qquad -\pi < q_k \leq \pi, \quad Q = \sum_{k=1}^{N+1} q_K.
$$

Taking into account that the form factor is an antisymmetric function of momenta $\{q\}$ one can make the following substitution

$$
\sum_{\{q\}} \cdots \longrightarrow \frac{1}{(M+N+1)(N+1)!}\sum_{Q}\sum_{l=0}^{M+N}\sum_{q_1}\cdots\sum_{q_{N+1}} \exp\left(-il(Q - \sum_{k=1}^{N+1} q_k)\right)\cdots
$$

The determinants in (6.10) can be then rewritten as follows

$$
\begin{aligned}
&\left|(1+\frac{\partial}{\partial x})\det{}_N(D_1^{(+)}(\{p\},\{q\}) - xD_2^{(+)}(\{p\},\{q\}))|_{x=0}\right|^2 = \\
&= \sum_{\mathcal{Q}}(-1)^{[\mathcal{Q}]}\prod_{a=1}^{N} e^{\frac{i}{2}q_{\mathcal{Q}_a}}\left(\frac{\cos\frac{1}{2}(\Theta - q_{\mathcal{Q}_a})}{\tan\frac{1}{2}(q_{\mathcal{Q}_a} - p_a)} + \sin\frac{1}{2}(\Theta - q_{\mathcal{Q}_a})\right)\times \\
&\times(1+\frac{\partial}{\partial x})\det{}_N(D_1^{(+)}(\{p\},\{q\}) - xD_2^{(+)}(\{p\},\{q\}))|_{x=0},
\end{aligned}
$$

where the sum is taken over all the permutations $\mathcal{Q}$ of $\{q_1, \ldots, q_{N+1}\}$. We can perform this summation using again the antisymmetry of the form factor

$$\frac{1}{(N+1)!}\sum_{\mathcal{Q}}(-1)^{[\mathcal{Q}]}\prod_{a=1}^{N} e^{\frac{i}{2}q_{\mathcal{Q}_a}}\Big(\frac{\cos\frac{1}{2}(\Theta - q_{\mathcal{Q}_a})}{\tan\frac{1}{2}(q_{\mathcal{Q}_a} - p_a)} + \sin\frac{1}{2}(\Theta - q_{\mathcal{Q}_a})\Big) \longrightarrow$$

$$\longrightarrow \prod_{a=1}^{N} e^{\frac{i}{2}q_a}\Big(\frac{\cos\frac{1}{2}(\Theta - q_a)}{\tan\frac{1}{2}(q_a - p_a)} + \sin\frac{1}{2}(\Theta - q_a)\Big).$$

Taking into account that $\det(D_1 - xD_2)$ is a linear function of x it is possible to rewrite (6.10) as

$$f_N^+(\{p\}, m, t) = \left(\frac{M+N+1}{M+N}\right)^{N-1}\sum_{l=0}^{M+N}\frac{M+N+1}{M^2}\sum_{\Theta} e^{i(m-l)\Theta} h_N^+(\{p\}, l, t, \Theta), \qquad (6.11)$$

$$h_N^+(\{p\}, l, t, \Theta) = \exp\left(-il\sum_{j=1}^{N} p_j + it\sum_{j=1}^{N}\varepsilon(p_j)\right) \times$$

$$\times \sum_{q_1}\cdots\sum_{q_{N+1}}\left(\frac{1}{M+N+1}\exp[ilq_{N+1} - it\varepsilon(q_{N+1})] + \frac{\partial}{\partial x}\right)\det W(x)|_{x=0}, \qquad (6.12)$$

where the $N \times N$ matrix $W(x)$ is defined as

$$W(z) = W^{(1)} - \frac{x}{M+N+1}W^{(2)}, \qquad (6.13)$$

$$W_{ab}^{(1)} = \frac{1}{(M+N+1)^2}\exp[ilq_a - it\varepsilon(q_a)]\left(\frac{\cos\frac{1}{2}(\Theta - q_a)}{\tan\frac{1}{2}(q_a - p_a)} + \sin\frac{1}{2}(\Theta - q_a)\right) \times$$

$$\times \left(\frac{\cos\frac{1}{2}(\Theta - q_a)}{\tan\frac{1}{2}(q_a - p_b)} + \sin\frac{1}{2}(\Theta - q_a)\right), \qquad (6.14)$$

$$W_{ab}^{(2)} = \frac{1}{(M+N+1)^2}\exp[i(l+\frac{1}{2})q_a - it\varepsilon(q_a)]\exp[i(l-\frac{1}{2})q_{N+1} - it\varepsilon(q_{N+1})] \times$$

$$\times \left(\frac{\cos\frac{1}{2}(\Theta - q_a)}{\tan\frac{1}{2}(q_a - p_a)} + \sin\frac{1}{2}(\Theta - q_a)\right)\left(\frac{\cos\frac{1}{2}(\Theta - q_{N+1})}{\tan\frac{1}{2}(q_{N+1} - p_b)} + \sin\frac{1}{2}(\Theta - q_{N+1})\right). \qquad (6.15)$$

The matrix $W^{(2)}$ is of rank one and hence $\det_N W(x)$ is a linear function of x.

Let us introduce the following functions

$$g(l,t) = \frac{1}{M+N+1}\sum_q \exp[ilq - it\varepsilon(q)], \tag{6.16}$$

$$e(l,t,p) = \frac{1}{M+N+1}\sum_q \frac{\exp[ilq - it\varepsilon(q)]}{\tan\frac{1}{2}(q-p)}, \tag{6.17}$$

$$d(l,t,p) = \frac{1}{(M+N+1)^2}\sum_q \frac{\exp[ilq - it\varepsilon(q)]}{\sin^2\frac{1}{2}(q-p)}. \tag{6.18}$$

The summation in these formulae is made over all the permitted values of q for Θ fixed, thus these three functions depend also on Θ. For $t=0$ they can be calculated explicitly.

$$g(l,0) = \delta_{l,0}, \tag{6.19}$$

$$e(l,0,p) = i(1-\delta_{l,0} - i\tan\frac{1}{2}(p-\Theta))e^{ilp}, \tag{6.20}$$

$$d(l,0,p) = \cos^{-2}\frac{1}{2}(\Theta-p)e^{ilp} + \frac{2}{M+N+1}\frac{\partial}{\partial p}e(l,0,p). \tag{6.21}$$

Using these functions we can perform the summations in (6.12), bearing in mind also the following relation

$$\cot\frac{1}{2}(q-p_a)\cot\frac{1}{2}(q-p_b) = \cot\frac{1}{2}(p_a-p_b)\left[\cot\frac{1}{2}(q-p_a) - \cot\frac{1}{2}(q-p_b)\right] - 1.$$

Performing the summation in each row and taking into account that $\det W(x)$ is a linear function of x we have

$$h_N^+(\{p\},l,t,\Theta) = \left(g(m,t) + \frac{\partial}{\partial x}\right)\det{}_N\left[S - xR^+\right]|_{x=0}. \tag{6.22}$$

The matrices S and R^+ are given as

$$S_{ab} = \frac{1}{M+N+1}\left\{\frac{1}{\tan\frac{1}{2}(p_a-p_b)}\left(\frac{1}{2}(e_+^+(l,t,p_a,\Theta) + e_+^-(l,t,p_a,\Theta))e_-(l,t,p_b) - \right.\right.$$

$$\left. -\frac{1}{2}(e_+^+(l,t,p_b,\Theta) + e_+^-(l,t,p_b,\Theta))e_-(l,t,p_a)\right) -$$

$$-g(l,t)e_-(l,t,p_a)e_-(l,t,p_b)+\frac{i}{2}(e_+^+(l,t,p_a,\Theta)-e_+^-(l,t,p_a,\Theta))e_-(l,t,p_b)+$$

$$\left.+\frac{i}{2}(e_+^+(l,t,p_b,\Theta)-e_+^-(l,t,p_b,\Theta))e_-(l,t,p_a)\right\}, \qquad \text{for} \quad a\neq b, \tag{6.23}$$

$$S_{aa}=\frac{1}{2}\left(d(l,t,p_a,\Theta)+\frac{1}{2}(e^{-i\Theta}d(l+1,t,p_a,\Theta)+e^{i\Theta}d(l-1,t,p_a,\Theta))\right)\times$$

$$\times e_-(l,t,p_a)e_-(l,t,p_a)+\frac{1}{M+N+1}\{-g(l,t)e_-(l,t,p_a)e_-(l,t,p_a)+$$

$$+i(e_+^+(l,t,p_a,\Theta)-e_+^-(l,t,p_a,\Theta))e_-(l,t,p_a)\}, \tag{6.24}$$

$$R_{ab}^+=\frac{1}{M+N+1}e_+^+(l,t,p_a,\Theta)e_+^-(l,t,p_b,\Theta). \tag{6.25}$$

The functions $e_-(l,t,p), e_+^+(l,t,p,\Theta), e_+^-(l,t,p,\Theta)$ are defined as

$$e_-(l,t,p)=\exp\left(-i\frac{l}{2}p+i\frac{t}{2}\varepsilon(p)\right), \tag{6.26}$$

$$e_+^+(l,t,p,\Theta)=\frac{1}{2}e_-(l,t,p)\times$$

$$\times\left((e(l,t,p)+e^{-i\Theta}e(l+1,t,p))-i(g(l,t)-e^{-i\Theta}g(l+1,t))\right), \tag{6.27}$$

$$e_+^-(l,t,p,\Theta)=\frac{1}{2}e_-(l,t,p)\times$$

$$\times\left((e(l,t,p)+e^{i\Theta}e(l-1,t,p))+i(g(l,t)-e^{i\Theta}g(l-1,t))\right). \tag{6.28}$$

In the equal-time case one has explicit expressions for these functions

$$e_-(l,0,p)=e^{-i\frac{l}{2}p}, \tag{6.29}$$

$$e_+^+(l,0,p,\Theta)=i(1-\delta_{l,0})e^{i\frac{l}{2}p}, \tag{6.30}$$

$$e_+^-(l,0,p,\Theta)=i(1-\delta_{l,1})e^{i(\Theta-p)}e^{i\frac{l}{2}p}. \tag{6.31}$$

Equations (6.11) and (6.22-6.25) give a determinant representation for the correlation function (6.9) on the finite lattice.

The calculation of the two-point time-dependent correlation function

$$f_N^-(\{p\},m,t)=\frac{\langle 0|\prod\limits_{k=1}^{N}\mathbf{C}(p_k)\phi_{m+1}^\dagger(t)\phi_1(0)\prod\limits_{k=1}^{N}\mathbf{B}(p_k)|0\rangle}{\langle 0|\prod\limits_{k=1}^{N}\mathbf{C}(p_k)\prod\limits_{k=1}^{N}\mathbf{B}(p_k)|0\rangle}, \tag{6.32}$$

is quite similar and we give only the final result:

$$f_N^-(\{p\}, m, t) = \left(\frac{M+N-1}{M+N}\right)^{N-1} \sum_{l=1}^{M+N-1} \frac{M+N-1}{M^2} \sum_{\Theta} e^{i(m-l)\Theta} h_N^-(\{p\}, l, t, \Theta), \quad (6.33)$$

$$h_N^-(\{p\}, l, t, \Theta) = \frac{\partial}{\partial x} \mathrm{det}_N \left[S + xR^-\right] |_{x=0}, \quad (6.34)$$

$$R_{ab}^- = \frac{1}{M+N-1} e_-(l, t, p_a) e_-(l, t, p_b). \quad (6.35)$$

One should note that in this case the factor $M+N+1$ entering the definitions of the matrix S (6.23,6.24) and the functions e, d and g (6.16-6.18) should be replaced by $M+N-1$.

7 The thermodynamic limit

Let us consider the thermodynamic limit (the total number of sites $M \to \infty$, number of particles $N \to \infty$, the density $D = N/M$ remains finite) of the correlation functions obtained in the previous sections. In this limit one should replace all the sums over momenta by the integrals and the determinants of $N \times N$ matrices by the Fredholm determinants of the corresponding integral operators acting on the functions on the interval $[-\pi, \pi]$ [9].

The result for the darkness formation probability (5.5) is

$$\tau(m) = (1+D)\det(\hat{I} - \hat{T}), \quad (7.1)$$

where $\hat{I}$ is the identity operator and $\hat{T}$ is an integral operator

$$(\hat{T}f)(p) = \int_{-\pi}^{\pi} T(p,q) f(q) dq,$$

with the kernel

$$T(p,q) = \frac{1}{1+D} \frac{\sin\frac{m+1}{2}(p-q)}{\sin\frac{1}{2}(p-q)} \rho(q). \quad (7.2)$$

Here $\rho(q)$ is the quasi-particle Fermi-like distribution function (3.10)

$$\rho(p) = \frac{1}{2\pi}(1+D)\left(1+\exp(\beta\epsilon(p))\right)^{-1}, \quad (7.3)$$

where β^{-1} is the temperature, the total density D is defined by (3.8) and the energy $\epsilon(p)$ is the solution of the non-linear integral equation (3.6). After symmetrizing the kernel the representation for the darkness formation probability can be rewritten as follows

$$\tau(m,\beta) = (1+D)\det(\hat{I}-\hat{M}), \tag{7.4}$$

where $\hat{M}$ is an integral operator with the kernel

$$M(p,q) = \frac{1}{2\pi}\sqrt{\nu(p,\beta)}\frac{\sin\frac{m+1}{2}(p-q)}{\sin\frac{1}{2}(p-q)}\sqrt{\nu(q,\beta)}, \tag{7.5}$$

and $\nu(p,\beta) = (1+\exp(\beta\epsilon(p)))^{-1}$ is the Fermi weight.

At zero temperature ($\beta^{-1}=0$) the Fermi weight becomes the step function equal to zero outside the Fermi zone and equal to one inside it. Thus one has

$$\tau_0(m) = (1+D_0)\det(\hat{I}-\hat{M}_0), \tag{7.6}$$

where $\hat{M}_0$ is an integral operator

$$(\hat{M}_0 f)(p) = \int\limits_{-\Lambda}^{\Lambda} M_0(p,q)f(q)dq, \tag{7.7}$$

with the kernel

$$M_0(p,q) = \frac{1}{2\pi}\frac{\sin\frac{m+1}{2}(p-q)}{\sin\frac{1}{2}(p-q)}, \tag{7.8}$$

and the Fermi momentum Λ is defined by the equations (3.12).

Consider now the two-point time-dependent correlation function (6.9) in the thermodynamic limit. The equation (6.11) takes the form

$$f^{(\pm)}(m,t,\beta) = \exp\left(\pm\frac{D}{1+D}\right)\frac{1+D}{2\pi}\sum_{l=0}^{\infty}\int\limits_{-\pi}^{\pi} e^{i(m-l)\Theta}h^{(\pm)}(l,t,\beta,\Theta)d\Theta. \tag{7.9}$$

The functions $h^{(\pm)}(l,t,\beta,\Theta)$ can be written as Fredholm determinants

$$h^{(+)}(l,t,\beta,\Theta) = \left(G(l,t)+\frac{\partial}{\partial x}\right)\det(\hat{I}+\hat{V}-x\hat{R}^{+})|_{x=0}, \tag{7.10}$$

$$h^{(-)}(l,t,\beta,\Theta) = \frac{\partial}{\partial x}\det(\hat{I}+\hat{V}+x\hat{R}^{-})|_{x=0}, \tag{7.11}$$

where $\hat{V}$ and $\hat{R}^{\pm}$ are integral operators

$$(\hat{V}f)(p) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} V(p,q)f(q)dq,$$

$$(\hat{R}^{\pm}f)(p) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} R^{\pm}(p,q)f(q)dq, \tag{7.12}$$

with kernels

$$V(p,q) = \frac{1}{\tan\frac{1}{2}(p-q)}\left(\frac{1}{2}(E_+^+(l,t,p,\beta,\Theta) + E_+^-(l,t,p,\beta,\Theta))E_-(l,t,q,\beta) - \right.$$

$$\left. -\frac{1}{2}(E_+^+(l,t,q,\beta,\Theta) + E_+^-(l,t,q,\beta,\Theta))E_-(l,t,p,\beta)\right) -$$

$$-G(l,t)E_-(l,t,p,\beta)E_-(l,t,q,\beta) + \frac{i}{2}(E_+^+(l,t,p,\beta,\Theta) - E_+^-(l,t,p,\beta,\Theta))E_-(l,t,q,\beta) +$$

$$+\frac{i}{2}(E_+^+(l,t,q,\beta,\Theta) - E_+^-(l,t,q,\beta,\Theta))E_-(l,t,p,\beta), \tag{7.13}$$

$$R^+(p,q) = E_+^+(l,t,p,\beta,\Theta)E_+^-(l,t,q,\beta,\Theta), \tag{7.14}$$

$$R^-(p,q) = E_-(l,t,p,\beta)E_-(l,t,q,\beta). \tag{7.15}$$

The functions $G(l,t)$, $E_-(l,t,p,\beta)$, $E_+^+(l,t,p,\beta,\Theta)$ and $E_+^-(l,t,p,\beta,\Theta)$ are defined as follows

$$G(l,t) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \exp(ilq - it\varepsilon(q))dq, \tag{7.16}$$

$$E(l,t,p,\Theta) = \frac{1}{2\pi}\text{v.p.}\int\limits_{-\pi}^{\pi} \frac{\exp(ilq - it\varepsilon(q))}{\tan\frac{1}{2}(q-p)}dq + \tan\frac{1}{2}(p-\Theta)\exp(ilp - it\varepsilon(p)), \tag{7.17}$$

$$E_-(l,t,p,\beta) = \sqrt{\nu(p,\beta)}\exp\left((-i\frac{l}{2}p + i\frac{t}{2}\varepsilon(p)\right)), \tag{7.18}$$

$$E_+^+(l,t,p,\beta,\Theta) = \frac{1}{2}E_-(l,t,p,\beta)\times$$

$$\times\left((E(l,t,p,\Theta) + e^{-i\Theta}E(l+1,t,p,\Theta)) - i(G(l,t) - e^{-i\Theta}G(l+1,t))\right), \tag{7.19}$$

$$E_+^-(l,t,p,\beta,\Theta) = \frac{1}{2}E_-(l,t,p,\beta)\times$$
$$\times\left((E(l,t,p,\Theta) + e^{i\Theta}E(l-1,t,p,\Theta)) + i(G(l,t) - e^{i\Theta}G(l-1,t))\right). \qquad (7.20)$$

One should note that the function $E(l,t,p,\Theta)$ is singular at the points $p = \Theta \pm \pi$ but the functions $E_+^+(l,t,p,\beta,\Theta)$ and $E_+^-(l,t,p,\beta,\Theta)$ entering the kernels have no singularities being well defined for all Θ and p.

In the case of equal-time correlators these functions can be calculated explicitly

$$G(l,0) = \delta_{l,0} \qquad (7.21)$$

$$E_-(l,0,p,\beta) = \sqrt{\nu(p,\beta)}e^{-i\frac{l}{2}p}, \qquad (7.22)$$

$$E_+^+(l,0,p,\beta,\Theta) = i(1-\delta_{l,0})\sqrt{\nu(p,\beta)}e^{i\frac{l}{2}p}, \qquad (7.23)$$

$$E_+^-(l,0,p,\beta,\Theta) = i(1-\delta_{l,1})e^{i(\Theta-p)}\sqrt{\nu(p,\beta)}e^{i\frac{l}{2}p}, \qquad (7.24)$$

and the kernels $V(p,q)$ and $R(p,q)$ are polynomials in $e^{i\Theta}$. Hence, $h^{(\pm)}(l,0,\beta,\Theta)$ can be represented as Taylor series in $e^{i\Theta}$ and

$$\int\limits_{-\pi}^{\pi} e^{i(m-l)\Theta}h^{(\pm)}(l,0,\beta,\Theta)d\Theta = 0 \qquad \text{for} \quad m > l, \qquad (7.25)$$

so that

$$f^{(\pm)}(m,0,\beta) = \exp\left(\pm\frac{D}{1+D}\right)\frac{1+D}{2\pi}\sum_{l=m}^{\infty}\int\limits_{-\pi}^{\pi} e^{i(m-l)\Theta}h^{(\pm)}(l,0,\beta,\Theta)d\Theta. \qquad (7.26)$$

The functions $h^{(\pm)}(l,0,\beta,\Theta)$ can be represented as determinants of simple operators:

$$h^{(\pm)}(l,0,\beta,\Theta) = \frac{\partial}{\partial x}\det(\hat{I} - \hat{v} + x\hat{r}^{\pm})|_{x=0}, \qquad (7.27)$$

where the kernels of the operators $\hat{v}$ and $\hat{r}^{\pm}$ are

$$v(p,q) = \sqrt{\nu(p)}\frac{\sin\frac{l+1}{2}(p-q) + \exp[i(\Theta - \frac{p+q}{2})]\sin\frac{l-2}{2}(p-q)}{\sin\frac{1}{2}(p-q)}\sqrt{\nu(q)}, \qquad (7.28)$$

$$r^+(p,q) = \sqrt{\nu(p)}e^{i(\Theta-q)}e^{i\frac{l}{2}(p+q)}\sqrt{\nu(q)}, \qquad (7.29)$$

$$r^-(p,q) = \sqrt{\nu(p)}e^{-i\frac{l}{2}(p+q)}\sqrt{\nu(q)}. \tag{7.30}$$

Again at zero temperature the Fermi weight $\nu(p,\beta)$ becomes the step function and the correlators have the following form

$$f_0^{(\pm)}(m,0) = \exp\left(\pm\frac{D_0}{1+D_0}\right)\frac{1+D_0}{2\pi}\sum_{l=0}^{\infty}\int_{-\pi}^{\pi}e^{i(m-l)\Theta}h_0^{(\pm)}(l,0,\Theta)d\Theta, \tag{7.31}$$

$$h_0^{(+)}(l,t,\Theta) = \left(G(l,t)+\frac{\partial}{\partial x}\right)\det(\hat{I}+\hat{V}_0-x\hat{R}_0^+)|_{x=0}, \tag{7.32}$$

$$h_0^{(-)}(l,t,\Theta) = \frac{\partial}{\partial x}\det(\hat{I}+\hat{V}_0+x\hat{R}_0^-)|_{x=0}, \tag{7.33}$$

where $\hat{V}_0$ and $\hat{R}_0^{\pm}$ are integral operators

$$(\hat{V}_0 f)(p) = \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda}V_0(p,q)f(q)dq,$$

$$(\hat{R}_0^{\pm} f)(p) = \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda}R_0^{\pm}(p,q)f(q)dq. \tag{7.34}$$

The kernels of these integral operators acting on the interval $[-\Lambda,\Lambda]$ are given by the equations (7.13-7.15) after putting formally $\nu(p,\beta)=1$.

Conclusion

In this paper we have represented the correlation functions for the phase model as determinants. In the thermodynamic limit these are the Fredholm determinants of "integrable integral operators" [33]. These representations will allow us to derive classical integrable equations for the correlators and to evaluate their large time and distance asymptotics.

The model considered is not a free fermion model but the correlation functions are represented in an explicit form not involving auxiliary dual fields. It is interesting to mention that the corresponding point exists also for the XXZ Heisenberg chain with an infinite anisotropy. It is natural that our technique can be applied also at this point, and the expressions for the correlators will be considerably simpler than in the general case. We are going to present this results somewhere.

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8 Appendix: Form factors

In this Appendix we calculate the matrix element

$$G(\{\lambda_j^B\},\{\lambda_k^C\}) = \langle 0| \prod_{k=1}^{N-1} \mathbf{C}(\lambda_k^C)\phi_M \prod_{j=1}^{N} \mathbf{B}(\lambda_j^B)|0\rangle. \tag{A.1}$$

Using (5.2) with $m = M-1$ and taking into account that in this case $B_2 = \phi_M^+, C_2 = \phi_M$ we obtain the representation

$$G(\{\lambda_j^B\},\{\lambda_k^C\}) = \sum_{\mathrm{I,II}} \prod_{\mathrm{I}} r_2(\lambda_I^B)\,{}_1\langle 0| \prod_{\mathrm{I}} \mathbf{C_1}(\lambda_I^C) \prod_{\mathrm{I}} \mathbf{B_1}(\lambda_I^B)|0\rangle_1 \times$$

$$\prod_{\mathrm{II}} r_1(\lambda_{\mathrm{II}}^C) \prod_{\mathrm{II}} \frac{1}{d_2(\lambda_{\mathrm{II}}^B)} \prod_{\mathrm{II}} \frac{1}{d_2(\lambda_{\mathrm{II}}^C)} \left(\prod_{\mathrm{I}} \prod_{\mathrm{II}} \frac{e^{2\lambda_I^B}}{e^{2\lambda_I^B} - e^{2\lambda_{\mathrm{II}}^B}} \right) \left(\prod_{\mathrm{I}} \prod_{\mathrm{II}} \frac{e^{2\lambda_{\mathrm{II}}^C}}{e^{2\lambda_{\mathrm{II}}^C} - e^{2\lambda_I^C}} \right), \tag{A.2}$$

where the sum is taken over all the partitions of the sets $\{\lambda^B\}$ and $\{\lambda^C\}$ into two subsets $\{\lambda_{\mathrm{I}}^B\}$, $\{\lambda_{\mathrm{II}}^B\}$ and $\{\lambda_{\mathrm{I}}^C\}$, $\{\lambda_{\mathrm{II}}^C\}$, respectively, satisfying the following conditions

$$\mathrm{card}\{\lambda_{\mathrm{I}}^B\} = \mathrm{card}\{\lambda_{\mathrm{I}}^C\} = n_1, \qquad n_1 = 0, 1, \ldots, N-1,$$

$$\mathrm{card}\{\lambda_{\mathrm{II}}^B\} = \mathrm{card}\{\lambda_{\mathrm{II}}^C\} + 1 = N - n_1 \equiv n_2.$$

Using (4.11) we can rewrite (A.2)

$$G(\{\lambda_j^B\},\{\lambda_k^C\}) = \left(\prod_{j<l} (e^{2\lambda_j^B} - e^{2\lambda_l^B}) \prod_{k<m} (e^{2\lambda_m^C} - e^{2\lambda_k^C}) \right)^{-1} \times$$

$$\exp(\sum_{j=1}^{N} \lambda_j^B + \sum_{k=1}^{N-1} \lambda_k^C) \sum_{\mathrm{I,II}} (-1)^{([P]+[Q])} \prod_{\mathrm{II}} \exp(2(M+n_1-1)\lambda_{\mathrm{II}}^C) \times$$

$$\prod_{j<l} (e^{2\lambda_{\mathrm{II}_j}^B} - e^{2\lambda_{\mathrm{II}_l}^B}) \prod_{m<k} (e^{2\lambda_{\mathrm{II}_k}^C} - e^{2\lambda_{\mathrm{II}_m}^C}) \det_{n_1} H_1(\{\lambda_I^B\},\{\lambda_I^C\}), \tag{A.3}$$

where

$$H_{1_{jk}} = \frac{\exp(2(N+M-1)\lambda_j^B) - \exp(2(N-n_1+1)\lambda_j^B + 2(M+n_1-2)\lambda_k^C)}{e^{2\lambda_j^B} - e^{2\lambda_k^C}}. \qquad (A.4)$$

Let us consider the function in the left hand side of (A.3)

$$G_2(\{\lambda_{\rm II}^B\},\{\lambda_{\rm II}^C\}) = \prod_{\rm II} \exp(2(M+n_1-1)\lambda_{\rm II}^C) \prod_{j<l}(e^{2\lambda_{{\rm II}_j}^B} - e^{2\lambda_{{\rm II}_l}^B}) \prod_{m<k}(e^{2\lambda_{{\rm II}_k}^C} - e^{2\lambda_{{\rm II}_m}^C}).$$

These products can be rewritten as determinants

$$G_2(\{\lambda_{\rm II}^B\},\{\lambda_{\rm II}^C\}) = \prod_{\rm II} \exp(2(M+n_1-1)\lambda_{\rm II}^C) {\rm det}_{N-n_1} h(\{\lambda_{\rm II}^B\}) {\rm det}_{N-n_1-1} g(\{\lambda_{\rm II}^C\}),$$

where the matrices g and h are given by the formulae

$$h_{jk} = e^{2(k-1)\lambda_j^B}, \qquad g_{jk} = e^{2(N-n_1-j-1)\lambda_k^C}.$$

It is convenient to introduce the $(N-n_1)\times(N-n_1)$ matrix $\tilde h$

$$\tilde h_{1k} = \tilde h_{k1} = \delta_{1k}, \qquad \tilde h_{jk} = g_{j-1,k-1} \quad \text{for} \quad j>1, k>1,$$

$${\rm det}_{N-n_1}\tilde h = {\rm det}_{N-n_1-1} g.$$

Then it is easy to see that

$$G_2(\{\lambda_{\rm II}^B\},\{\lambda_{\rm II}^C\}) = {\rm det}_{N-n_1} H_2(\{\lambda_{\rm II}^B\},\{\lambda_{\rm II}^C\}),$$

$$H_{2_{j1}} = 1,$$

$$H_{2_{jk}} = \frac{\exp(2(n_1+M-1)\lambda_{k-1}^C + 2(N-n_1)\lambda_j^B) - \exp(2\lambda_j^B + 2(M+N-2)\lambda_{k-1}^C)}{e^{2\lambda_j^B} - e^{2\lambda_{k-1}^C}}, \quad (A.5)$$

for $k>1$. The representation (A.3) can be rewritten now in the following form

$$G(\{\lambda_j^B\},\{\lambda_k^C\}) = \left(\prod_{j<l}(e^{2\lambda_j^B} - e^{2\lambda_l^B}) \prod_{k<m}(e^{2\lambda_m^C} - e^{2\lambda_k^C})\right)^{-1} \times$$

$$\exp(\sum_{j=1}^{N}\lambda_j^B + \sum_{k=1}^{N-1}\lambda_k^C) \sum_{\rm I,II} (-1)^{([P]+[Q])} {\rm det}_{n_1} H_1(\{\lambda_I^B\},\{\lambda_I^C\})\times$$

$$\times \det_{N-n_1} H_2(\{\lambda^B_{\rm II}\},\{\lambda^C_{\rm II}\}). \tag{A.6}$$

To use the Laplace formula for the determinant of the sum of two matrices it is necessary to introduce four dual fields [16]: $\psi^+_j, \psi_j, \quad j=1,2,3,4$

$$[\psi_j,\psi_k]=[\psi^+_j,\psi^+_k]=0, \quad [\psi_j,\psi^+_k]=\delta_{j,k},$$

acting in the dual Fock space with the vacuum $|0)$

$$\psi_j|0)=0, \quad (0|\psi^+_j=0.$$

Using the commutation relation for the dual fields it is easy to prove the following relations

$$\det_{n_1} H_1(\{\lambda^B_I\},\{\lambda^C_I\})=(0|\det_{n_1}\mathcal{G}_1(\{\lambda^B_I\},\{\lambda^C_I\})|0),$$

$$\det_{N-n_1} H_2(\{\lambda^B_{\rm II}\},\{\lambda^C_{\rm II}\})=(0|\det_{N-n_1}\mathcal{G}_2(\{\lambda^B_{\rm II}\},\{\lambda^C_{\rm II}\})|0), \tag{A.7}$$

$$\mathcal{G}_{1_{jk}}=\frac{\exp(\psi^+_2+\psi_1)}{e^{2\lambda^B_j}-e^{2\lambda^C_k}}(\exp(2(N+M-1)\lambda^B_j)-$$

$$-\exp(2(N+1)\lambda^B_j+2(M-2)\lambda^C_k+2(\lambda^C_k-\lambda^B_j)(\psi^+_1+\psi_2)), \tag{A.8}$$

$$\mathcal{G}_{2_{j1}}=1,$$

$$\mathcal{G}_{2_{jk}}=\frac{\exp(\psi^+_4+\psi_3)}{e^{2\lambda^B_j}-e^{2\lambda^C_{k-1}}}\exp(2(N+M-1)\lambda^C_{k-1}+2(\lambda^B_j-\lambda^C_{k-1})(\psi^+_3+\psi_4))-$$

$$-\exp(2\lambda^B_j+2(M+N-2)\lambda^C_{k-1}), \quad \text{for} \quad k>1. \tag{A.9}$$

Now (A.6) can be rewritten in the form

$$G(\{\lambda^B_j\},\{\lambda^C_k\})=\left(\prod_{j<l}(e^{2\lambda^B_j}-e^{2\lambda^B_l})\prod_{k<m}(e^{2\lambda^C_m}-e^{2\lambda^C_k})\right)^{-1}\times$$

$$\times\exp(\sum_{j=1}^{N}\lambda^B_j+\sum_{k=1}^{N-1}\lambda^C_k)(0|\det_N\mathcal{G}(\{\lambda^B\},\{\lambda^C\})|0), \tag{A.10}$$

$$\mathcal{G}_{j1}=1, \quad \mathcal{G}_{jk}=\mathcal{G}_{1_{j,k-1}}+\mathcal{G}_{2_{jk}} \quad \text{for} \quad k>1.$$

The determinant in (A.10) is represented as a sum of minors

$$(0|\det_N\mathcal{G}(\{\lambda^B\},\{\lambda^C\})|0)=\sum_{l=1}^{N}\exp(\sum_{j\neq l}\lambda^B_j-\sum_{k=1}^{N-1}\lambda^C_k)\times$$

$$\times (-1)^l (0|\mathrm{det}_{N-1} V(\{\lambda_1^B, \dots \lambda_{l-1}^B, \lambda_{l+1}^B, \dots, \lambda_N^B\}\{\lambda^C\})|0). \tag{A.11}$$

The mean value of the determinant of the matrix V can be expressed without dual fields,

$$(0|\mathrm{det}_N V(\{\lambda_j^B\}, \{\lambda_k^C\})|0) = \mathrm{det}_N F(\{\lambda_j^B\}, \{\lambda_k^C\}). \tag{A.12}$$

Now we can use this relation to rewrite $G(\{\lambda_j^B\}, \{\lambda_k^C\})$ also without dual fields:

$$(0|\mathrm{det}_N \mathcal{G}(\{\lambda^B\}, \{\lambda^C\})|0) = \mathrm{det}_N H(\{\lambda_j^B\}, \{\lambda_k^C\}),$$

$$H_{j1} = 1,$$

$$H_{jk} = \frac{\exp(2(N+M-1)\lambda_j^B) - \exp(2\lambda_j^B + 2(M+N-2)\lambda_{k-1}^C)}{e^{2\lambda_j^B} - e^{2\lambda_{k-1}^C}} \quad \text{for} \quad k > 1. \tag{A.13}$$

Finally we have

$$G(\{\lambda_j^B\}, \{\lambda_k^C\}) = \left(\prod_{j<l} (e^{2\lambda_j^B} - e^{2\lambda_l^B}) \prod_{k<m} (e^{2\lambda_m^C} - e^{2\lambda_k^C}) \right)^{-1} \times$$

$$\times \exp(\sum_{j=1}^{N} \lambda_j^B + \sum_{k=1}^{N-1} \lambda_k^C) \mathrm{det}_N H(\{\lambda_j^B\}, \{\lambda_k^C\}). \tag{A.14}$$

If $\{\lambda_j^B\}, \{\lambda_k^C\}$ are solutions of the Bethe equations (3.3) the matrix H can be rewritten as

$$H_{jk} = -\frac{e^{i(P-p_j)} + e^{i(Q-q_{k-1}+p_j)}}{e^{ip_j} - e^{iq_{k-1}}}, \tag{A.15}$$

where

$$p_j = -2i\lambda_j^B, \quad q_k = -2i\lambda_k^C, \quad P = \sum_{j=1}^{N} p_j, \quad Q = \sum_{k=1}^{N-1} q_k.$$

Using (4.15) we obtain the normalized form factor

$$G_N(M, \{p\}, \{q\}) = Z_M M^{-1} (M+N)^{-N+\frac{3}{2}} \left(\frac{M+N}{M+N-1} \right)^{\frac{N}{2}-1} \mathrm{det} H(\{p\}, \{q\}), \tag{A.16}$$

where Z_M is a complex number depending on P, Q and M, $|Z_M| = 1$.

If $\{\lambda_j^B\}, \{\lambda_k^C\}$ are solutions of the Bethe equation then, using the shift operator, it is also possible to calculate the normalized matrix elements of any operator ϕ_m.

$$G_N(m, \{p\}, \{q\}) = Z_m M^{-1} (M+N)^{-N+\frac{3}{2}} \left(\frac{M+N}{M+N-1}\right)^{\frac{N}{2}-1} \det H(\{p\}, \{q\}), \qquad (A.17)$$

$$Z_m = e^{i(M-m)(Q-P)} Z_M.$$

Notice that this representation is equivalent to the formulae (6.3),(6.4). It is evident that now one can also obtain a representation for the normalized matrix elements of the operators $\phi_m^\dagger$, using the following relation

$$G_N^{(+)}(m, \{p\}, \{q\}) = G_{N+1}^*(m, \{q\}, \{p\}),$$

where star means the complex conjugation. Thus one gets for this form factors the representation (6.6),(6.7).

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