POMI PREPRINT — 14/1997

METRICS OF NONPOSITIVE CURVATURE ON GRAPH-MANIFOLDS AND ELECTROMAGNETIC FIELDS ON GRAPHS

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August 1997

*The author was supported by RFFI Grant 96-01-00674 and CRDF Grant RM1-169.

§0. Introduction

The aim of this work is to establish a precise analogy between metrics of nonpositive curvature on 3-dimensional graph-manifolds on the one hand and the interaction of an electromagnetic field ∇ with a scalar charged field ξ on the other hand. This analogy is quite unexpected, however, various geometric effects related to those metrics coincide with effects arising in a discrete model of the interaction of fields ∇ and ξ.

Metrics of nonpositive sectional curvature on a graph-manifold have rather special structure, and most essential geometric information encoded in a metric can be described by a finite number of parameters. These parameters satisfy a compatibility equation, which is similar to the Laplace equation on graphs. However, there are essential distinctions from the Laplace equation, and seeking for a continual analog of the compatibility equation we came to the Lagrange-Euler equations for the action

$$S(\nabla,\xi)=\mathrm{YM}\,(\nabla)+E_\nabla(\xi)-m^2\|\xi\|^2,$$

which describes the interaction of a complex-valued scalar field ξ and an electromagnetic field ∇, where $\mathrm{YM}\,(\nabla)=\|\nabla\circ\nabla\|^2$ is the Yang-Mills action of ∇, $E_\nabla(\xi)=\|\nabla\xi\|^2$ the energy, m the mass of ξ. The field ξ is a section of the trivial line bundle $\mathcal{E}$ over a graph, and ∇ is a connection on $\mathcal{E}$. For a discrete space such as a graph (i.e. a collection of vertices and a collection of edges between them) the definitions of the connection ∇, the covariant differential $\nabla\xi$ and the curvature $\nabla\circ\nabla$ are possible in the framework of the spectral calculus of A. Connes developed in [Con1]. This calculus serves as the basis of the analogy between geometric and physical phenomena to be described.

The suggested approach to the discretization problem also unveil new striking features of the Lagrange-Euler equations such as the mechanism of exclusions and degenerations, see § 4, and necessity to dispense with the electric charge conservation low for typical metrics (this low is recovered when taking the classical limit $\Delta s\to 0$), see 2.9, 2.10.

Acknowledgement. The author is grateful to L. Khalfin for the attention to this work and valuable remarks, V. Kobel'skii for numerous discussions and V. Schroeder for the invitation in the University of Zürich, where this work was finished, and the hospitality.

§1. The compatibility equation

We consider a closed orientable (and oriented) graph-manifold $M=\cup_{v\in V}M_v$, which consists of a finite set V of building blocks $M_v=F_v\times S^1$, where F_v is a compact surface with boundary, different from the disk and the annulus. The blocks M_v are glued along boundary tori $T_w=(\partial F_v)_w\times S^1$, $w\in\partial v$, where ∂v is the set of the boundary components of M_v, and the index w points on such a component. For more details about graph-manifolds see Appendix A.

Any metric of nonpositive sectional curvature of the manifold M locally splits along each block M_v into the metric product $ds_F^2+dl^2$, where l is the coordinate along the factor S^1, and ds_F^2 is a metric of nonpositive curvature on a surface (the global splitting along M_v may be absent and, as the rule, it does not exist). Furthermore, all fibers $f\times S^1$, $f\in F_v$, are closed geodesics of the same length l_v, and the boundary tori can always be chosen to be flat and totally geodesic. The most essential information about the metric is encoded in collections of lengths $\{l_v\}_{v\in V}$ of fibers of blocks M_v and angles $\{\omega_w\}_{w\in W}$ between the fibers of adjacent blocks M_v, $M_{v'}$, $w=(v,v')$ on the gluing torus T_w, and, therefore, it is described by a finite set of parameters. The sets V of blocks and W of gluing tori are correspondingly the vertex set and the set of oriented edges of a graph Γ, which is called *the graph* of the manifold M. Here a vertex $v\in V$ is initial for an edge $w\in W$ iff $w\in\partial v$ (notation: $v=\partial_-w$). We denote by ∂_+w the terminal vertex of the edge

w. It is not excluded that $\partial_- w = \partial_+ w$, i.e. different boundary components of M_v may be glued with each other; in that case the edge w is a loop in Γ. However, we require that for any edge $w = (v, v') \in W$ the fibers S^1_v, $S^1_{v'}$ of corresponding adjacent blocks M_v, $M_{v'}$ are not homotopic on the gluing torus T_w.

The collections of lengths $\{l_v\}$ and angles $\{\omega_w\}$ correspond to a metric of nonpositive sectional curvature on M iff they are a solution to *the compatibility equation*

$$k_v l_v - \sum_{w\in\partial v} \frac{\cos\omega_w}{b_w} l_{\partial_+ w} = 0, \quad v \in V, \tag{0.1}$$

satisfying the conditions $l_v > 0$ for all $v \in V$ and $0 < \omega_w = \omega_{-w} < \pi$ for all $w \in W$. Here the coefficients $k_v \in \mathbb{Q}$, $b_w = b_{-w} \in \mathbb{N}$ are topological invariants of the manifold M (see Appendix A). In this case we say that the collections $\{l_v\}$, $\{\omega_w\}$ define *an isometric state* of the system (0.1).

0.2. Examples. (a) Dipole. The graph Γ consists of two vertices v_0, v_1 connected by an edge. The compatibility equation has the form

$$\begin{aligned} k_0 l_0 - \frac{\cos\omega}{b} l_1 &= 0 \\ -\frac{\cos\omega}{b} l_0 + k_1 l_1 &= 0. \end{aligned} \tag{0.2.1}$$

Isometric states exist in the next cases.

(i) $k_0 = k_1 = 0$. Then $\omega = \pi/2$, and the ratio l_0/l_1 may be arbitrary;

(ii) $0 < k_0 k_1 b^2 < 1$. Then $\cos^2\omega = k_0 k_1 b^2$, $\mathrm{sgn}\,(\cos\omega) = \mathrm{sgn}\,(k_j)$ and

$$l_0^2/l_1^2 = k_1/k_0.$$

In what follows this example plays a key role.

(b) Monopole. The graph Γ is a loop with vertex v. The compatibility equation (0.1) has the form

$$(k - \frac{2}{b}\cos\omega) l = 0 \tag{0.2.2}$$

for the corresponding coefficients k, b. Isometric states exist iff $|k|b < 2$. In that case $\cos\omega = kb/2$ and $l > 0$ is arbitrary.

Comparison with the Laplace equation on graphs

The Laplacian $\Delta : L^2(V) \to L^2(V)$ on a graph $\Gamma = \Gamma(V, W)$ is defined as

$$\Delta f(v) = f(v) - \frac{1}{|\partial v|} \sum_{w\in\partial v} f(\partial_+ w),$$

where $L^2(V)$ is the (real) Hilbert space with the scalar product

$$\langle f, f' \rangle = \sum_{v\in V} |\partial v| f(v) f'(v).$$

Then $\Delta = d^* d$, the differential $d : L^2(V) \to L^2_{\mathrm{odd}}(W)$ is $df(w) = f(\partial_+ w) - f(\partial_- w)$, and $L^2_{\mathrm{odd}}(W)$ is the Hilbert space of odd functions on W with the scalar product

$$\langle \phi, \psi \rangle = \frac{1}{2} \sum_{w \in W} \phi(w)\psi(w).$$

Solutions to the Laplace equation $\Delta f = 0$ minimize the action $E(f) = \|df\|^2$.

The compatibility equation (0.1) looks like the Laplace equation with "variable coefficients". However, the factors $\cos\omega_w$ cardinally change properties of the solutions and indicate the hidden presence of a connection ∇, an electromagnetic field: it will be clear, that the usual differential d has to be replaced by a covariant differential ∇.

Recall that an electromagnetic field is described as a connection ∇ on the principal (trivial) bundle $M^4 \times \mathbb{U}(1) \to M^4$; a scalar charged field is a section $\xi : M^4 \to M^4 \times \mathbb{C}$ of the bundle $M^4 \times \mathbb{C} \to M^4$, where M^4 is the Minkowski space. Their interaction minimizes the action

$$S(\nabla, \xi) = \|F\|^2 + \|\nabla\xi\|^2 - m^2\|\xi\|^2,$$

where $F = \nabla \circ \nabla$ is the curvature of ∇. In other words, the connection ∇ and the section ξ satisfy to the Lagrange-Euler equation for the action S

$$d^* F + \xi^* \nabla\xi - \xi(\nabla\xi)^* = 0 \tag{+}$$

$$(\nabla^*\nabla - m^2)\xi = 0. \tag{++}$$

One should add to these the Bianchi's identity $dF = 0$, which gives (after a choice of coordinates in M^4) the first pair of the Maxwell equations

$$\mathrm{div}\ \mathbf{H} = 0, \qquad \mathrm{rot}\ \mathbf{E} = -\partial_t \mathbf{H}.$$

The equation (+) gives the second pair

$$\mathrm{div}\ \mathbf{E} = J_0, \qquad \mathrm{rot}\ \mathbf{H} = \partial_t \mathbf{E} + \mathbf{J},$$

and (++) is the wave equation.

Roughly speaking, a metric of nonpositive curvature on a graph-manifold M may be interpreted as a scalar charged field $\xi : V \to V \times \mathbb{C}$ on the vertex set of the graph Γ of M interacting with a connection ∇ on the bundle $V \times \mathbb{C} \to V$. The connection ∇ may be interpreted as a collection of conformal structures on gluing tori; the field ξ contains the information about the lengths of fibers; all together they minimize the action

$$S(\nabla, \xi) = \|F\|^2 + \|\nabla\xi\|^2 - m^2\|\xi\|^2.$$

Therefore, the isometric states of the equation (0.1) are extremals of the action S, and the equation (0.1) itself turns out to be an analog of the wave equation (++).

§2. Geometry of a two-point space

The program described at the end of the previous section can be realized in the framework of the spectral calculus of A. Connes. Here we collect necessary facts of that formalism restricting to the case of finite dimensional algebras. Furtheremore, all notions are illustrated by an example of the dipole, the simplest nontrivial example, to which the formalism is applicable (this justifies the title of the section). At the same time, this example turns out to be fundamental for us, because the case of general graphs can be reduced to the case of dipole using the decomposition principle (see § 5). Detailed discussion of the spectral calculus can be found in [Con1-4], [V-Gr], see also [Ch-C], [KPPW], [Sch-Z], [C-M], [Ka], [K-W], [Sit].

Space

The role of a space is played by an involutive algebra $\mathcal{A}$. For the dipole $V = \{v_0, v_1\}$ the algebra $\mathcal{A}$ is the function algebra $\{V \to \mathbb{C}\} = \mathbb{C}^2$ with involution $a = (a_0, a_1) \mapsto a^* = (\overline{a}_0, \overline{a}_1)$.

Riemannian metric and the spectral differential

A geometry on V is specified by a representation of $\mathcal{A}$ is a Hilbert space $\mathcal{H}$, "the tangent bundle", and a selfadjoint compact operator of unit length $ds \in L(\mathcal{H})$. *The spectral triple* $\{\mathcal{A}, \mathcal{H}, ds\}$ plays the role of a Riemannian manifold. In the case of the dipole the algebra $\mathcal{A}$ is represented in the Hilbert space $\mathcal{H} = \mathbb{C}^2$ with the scalar product $\langle a, b\rangle = a_0\overline{b}_0 + a_1\overline{b}_1$ by multiplication operators

$$a = (a_0, a_1) \mapsto \pi(a) = \operatorname{diag}(a_0, a_1) : \mathbb{C}^2 \to \mathbb{C}^2.$$

The unit length operator is defined as

$$ds = \begin{pmatrix} 0 & -i\Delta s \\ i\Delta s & 0 \end{pmatrix}, \qquad \Delta s \in \mathbb{R}.$$

The inverse operator $D = ds^{-1}$ (the Dirac operator) makes possible to differentiate functions $a \in \mathcal{A}$: the spectral differential is defined as the operator

$$da := i[D, \pi(a)] \in L(\mathcal{H}).$$

The operator da is an element of the space $\Omega^1_D(\mathcal{A}) \subset L(\mathcal{H})$ of 1-forms, which consists of the operators

$$\omega = \sum_j a_0^j da_1^j = i\sum_j a_0^j [D, a_1^j], \quad a_i^j \in \mathcal{A}$$

(we identify a and $\pi(a)$ to save notations). Obviously, $(da)^* = da^*$.

For $a = (a_0, a_1)$ we have

$$da = \begin{pmatrix} 0 & \frac{1}{\Delta s}(a_1 - a_0) \\ \frac{1}{\Delta s}(a_1 - a_0) & 0 \end{pmatrix}.$$

The metric on V is defined by the Connes' formula

$$\operatorname{dist}(v_0, v_1) = \sup\{|a(v_0) - a(v_1)| \mid a \in \mathcal{A}, \|da\| \le 1\}, \tag{2.1}$$

which gives $\operatorname{dist}(v_0, v_1) = |\Delta s|$.

The line bundle and gauge transformations

The trivial line bundle $V \times \mathbb{C} \to V$ is completely characterized by its sections which form the trivial right $\mathcal{A}$-module $\mathcal{E}$. The algebra $\mathcal{A}$ acts on $\mathcal{E}$ from the right, $(\xi, a) \mapsto \xi a \in \mathcal{E}$ for $\xi \in \mathcal{E}$, $a \in \mathcal{A}$. Though the module $\mathcal{E}$ is isomorphic to the algebra $\mathcal{A}$ (considered as $\mathcal{A}$-module) there is no canonical isomorphism. An isomorphism $\mathcal{E} \to \mathcal{A}$ is implied by the choice of a basis $e \in \mathcal{E}$. The group of gauge transformations $\mathcal{U} = \{u \in \mathcal{A} \,|\, u^*u = uu^* = 1\}$ acts on $\mathcal{E}$ as $(\xi, u) \mapsto \xi u^*$, $\xi \in \mathcal{E}$. We consider $\mathcal{U}$ as the set of distinguished bases e of the bundle $\mathcal{E}$: the coordinate of a section $\xi \in \mathcal{E}$ is the function $a = a(\xi, e) \in \mathcal{A}$, defined by the relation $\xi = ea$. When changing the basis $e \mapsto e' = eu^*$ the coordinate transforms as follows $a(\xi, e') = ua(\xi, e)$.

In the case of the dipole the group $\mathcal{U}$ consists of the elements of form $u = (e^{i\alpha_0}, e^{i\alpha_1})$, α_0, $\alpha_1 \in \mathbb{R}$. For $\xi \in \mathcal{E}$ and a basis $e \in \mathcal{U}$ we write $a(\xi, e) = (\xi_0, \xi_1)$.

Connections on $\mathcal{E}$

A connection on $\mathcal{E}$ is a linear mapping $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A})$ satisfying the Leibniz rule

$$\nabla(\xi a) = \nabla\xi \cdot a + \xi \otimes da,$$

$\xi \in \mathcal{E}$, $a \in \mathcal{A}$. For a basis $e \in \mathcal{U}$ a connection ∇ is given by the vector potential $\Phi \in \Omega^1_D(\mathcal{A})$ defined by the relation

$$\nabla e = e \otimes \Phi.$$

Therefore, for $\xi = ea \in \mathcal{E}$ we have

$$\nabla\xi = \nabla e \cdot a + e \otimes da = e \otimes (da + \Phi a).$$

The gauge transformation group $\mathcal{U}$ acts on vector potentials of the connection ∇ as follows. For $e' = eu^*$, $\nabla e' = e' \otimes \Phi'$ we have

$$\Phi' = udu^* + u\Phi u^*.$$

Indeed, on the one hand, we have $\nabla e' = e' \otimes \Phi' = eu^* \otimes \Phi' = e \otimes u^*\Phi'$, on the other hand, $\nabla e' = \nabla(eu^*) = \nabla e \cdot u^* + e \otimes du^* = e \otimes (du^* + \Phi u^*)$.

In the case of a dipole representing Φ as an operator

$$\Phi = i \begin{pmatrix} 0 & \phi_{01}\Delta s \\ \phi_{10}\Delta s & 0 \end{pmatrix}, \quad \phi_{01}, \phi_{10} \in \mathbb{C},$$

for $\xi = (\xi_0, \xi_1) \in \mathcal{E}$ we have

$$\nabla\xi = e \otimes \begin{pmatrix} 0 & \frac{1}{\Delta s}(\xi_1 - \xi_0) + i\phi_{01}\Delta s\xi_1 \\ \frac{1}{\Delta s}(\xi_1 - \xi_0) + i\phi_{10}\Delta s\xi_0 & 0 \end{pmatrix}.$$

The space of k-forms $\Omega^k_D(\mathcal{A})$.

The space of k-forms $\Omega^k_D(\mathcal{A}) \subset L(\mathcal{H})$ consists of the operators of form

$$\omega = \sum_j a^j_0 da^j_1 \dots da^j_k = i^k \sum_j a^j_0 [D, a^j_1] \dots [D, a^j_k], \tag{2.2}$$

$a^j_i \in \mathcal{A}$. The exterior differential $d : \Omega^k_D(\mathcal{A}) \to \Omega^{k+1}_D(\mathcal{A})$ can be computed by the formula

$$d\omega = \sum_j da^j_0 \dots da^j_k \tag{2.3}$$

(it is well defined for the dipole case because if $\sum_j a_0^j[D,a_1^j]\dots[D,a_k^j] = 0$, then $\sum_j [D,a_0^j][D,a_1^j]\dots[D,a_k^j] = 0$, and the usual problem related to the ambiguity of the representation (2.2) here does not arise, see [Con1–2]).

Using $da \cdot b = d(ab) - adb$ for $a,\ b \in \mathcal{A}$ we have $\sigma\omega \in \Omega_D^{l+k}(\mathcal{A})$ for $\sigma \in \Omega_D^l(\mathcal{A})$, $\omega \in \Omega_D^k(\mathcal{A})$. Furtheremore, the Leibniz rule holds

$$d(\sigma\omega) = d\sigma \cdot \omega + (-1)^l \sigma d\omega.$$

Curvature of a connection ∇

A connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1(\mathcal{A})$ extends to a differentiation $\nabla : \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^k(\mathcal{A}) \to \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^{k+1}(\mathcal{A})$ by the formula

$$\nabla(e \otimes \omega) := \nabla e \cdot \omega + e \otimes d\omega$$

for $\omega \in \Omega_D^k(\mathcal{A})$, where $e \in \mathcal{U}$ is a basis of $\mathcal{E}$. For $\sigma \in \Omega_D^l(\mathcal{A})$ we have

$$\begin{aligned}\nabla(e \otimes \omega \cdot \sigma) &= \nabla e \cdot \omega \cdot \sigma + e \otimes d\omega \cdot \sigma + (-1)^k e \otimes \omega \cdot d\sigma \\ &= \big(\nabla(e \otimes \omega)\big)\sigma + (-1)^k e \otimes \omega \cdot d\sigma.\end{aligned}$$

The curvature of a connetion ∇ is defined as the operator

$$\nabla^2 : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^2(\mathcal{A}).$$

With respect to a basis $e \in \mathcal{U}$, we have $\nabla^2 e = e \otimes \theta$, where $\theta = d\Phi + \Phi^2 \in \Omega_D^2(\mathcal{A})$ is *the curvature operator.* Indeed,

$$\nabla^2 e = \nabla(e \otimes \Phi) = \nabla e \cdot \Phi + e \otimes d\Phi = e \otimes (d\Phi + \Phi^2).$$

2.4. Lemma. *The curvature ∇^2 is an $\mathcal{A}$-linear operator, i.e.*

$$\nabla^2(\xi \cdot a) = \nabla^2 \xi \cdot a$$

for each $\xi \in \mathcal{E}$ and $a \in \mathcal{A}$.

Proof. Let $e \in \mathcal{U}$ be a basis of the module $\mathcal{E}$. It suffices to show that $\nabla^2(ea) = \nabla^2 e \cdot a$ for $a \in \mathcal{A}$. We have

$$\begin{aligned}\nabla^2(ea) &= \nabla\big(e \otimes (da + \Phi a)\big) \\ &= e \otimes \Phi(da + \Phi a) + e \otimes d(da + \Phi a) \\ &= e \otimes (\Phi da + \Phi^2 a + d^2 a + d\Phi a - \Phi da) \\ &= e \otimes (d\Phi + \Phi^2)a. \quad \square\end{aligned}$$

This implies the gauge invariance of the curvature. Let θ' be the curvature operator of a connection ∇ with respect to a basis $e' = eu^*$, $\nabla^2 e' = e' \otimes \theta'$. Then

$$\theta' = u\theta u^*.$$

Indeed, we have $e' \otimes \theta' = e \otimes u^*\theta'$. On the other hand, $\nabla^2 e' = \nabla^2 e \cdot u^*$ by Lemma 2.4. Hence $u^*\theta' = \theta u^*$.

To compute the curvature operator θ in the case of the dipole, one introduces the projectors $p = (1, 0)$, $1 - p = (0, 1) \in \mathcal{A}$. Then

$$\Phi = -i\phi_{01}\Delta s^2 pdp - i\phi_{10}\Delta s^2(1 - p)dp.$$

Using that $(dp)^2 = \Delta s^{-2} \cdot 1$, one obtains $d\Phi = -i(\phi_{01} - \phi_{10}) \cdot 1$. Further, $\Phi^2 = -\phi_{01}\phi_{10}\Delta s^2 \cdot 1$. Thus

$$\theta = d\Phi + \Phi^2 = i(\phi_{10} - \phi_{01} + i\phi_{01}\phi_{10}\Delta s^2) \cdot 1.$$

In particular, we see that for an unitary connection ∇, i.e. $\Phi^* + \Phi = 0$, or $\overline{\phi}_{01} = \phi_{10}$ the norm of the curvature operator

$$\|\theta\|^2 = \left(\left| \frac{1}{\Delta s} + i\phi_{01}\Delta s \right|^2 - \frac{1}{\Delta s^2} \right)^2$$

achieves its minimum along the circle $U_{\Delta s} \subset \mathbb{C}$ of radius $1/\Delta s^2$ centered at $z_{\Delta s} = i/\Delta s^2$ which goes to the real line $\mathbb{R} = \{\operatorname{Im} \phi_{01} = 0\}$ as $\Delta s \to 0$. The gauge transformation group $\mathcal{U}$ transitively acts on $U_{\Delta s}$.

The Bianchi identity

Let θ be the curvature operator of a connection ∇, $\theta = d\Phi + \Phi^2$. Then *the Bianchi identity* takes place

$$d\theta = [\theta, \Phi].$$

Indeed, $d\theta = d^2\Phi + d\Phi \cdot \Phi - \Phi \cdot d\Phi$. Using $d^2 = 0$ and $d\Phi = \theta - \Phi^2$ we obtain

$$d\theta = (\theta - \Phi^2)\Phi - \Phi(\theta - \Phi^2) = \theta\Phi - \Phi\theta.$$

In the case of the dipole, the operator θ is a scalar operator. Thus the both parts of the Bianchi identity are equal to zero,

$$d\theta = 0 = [\theta, \Phi].$$

Hermitian structures on $\mathcal{E}$

Let $\Lambda : \mathcal{E} \to \mathcal{E}$ the multiplication operator by a positive selfadjoint element of the algebra $\mathcal{A}$. With respect to a basis $e \in \mathcal{U}$ the operator Λ is defined by

$$a(\Lambda\xi, e) = \lambda a(\xi, e),$$

where $\xi \in \mathcal{E}$, $\lambda = \lambda(\Lambda, e) \in \mathcal{A}$ is the matrix of the map Λ with respect to e, $\lambda^* = \lambda$. For $e' = eu^*$ we have $\lambda' = u\lambda u^*$, where $\lambda' = \lambda(\Lambda, e')$.

A hermitian structure on $\mathcal{E}$ plays the role of a fiberwise hermitian metric and is defined by

$$(\xi, \eta)_\Lambda = (\Lambda\xi, \eta) \in \mathcal{A},$$

where $(\xi, \eta) = \eta^*\xi$ is the standard hermitian structure on $\mathcal{E}$. We have

$$(\Lambda\xi, \eta) = (\xi, \Lambda^*\eta) = \eta^*\Lambda\xi.$$

With respect to a basis e we have $\xi = ea$, $\eta = eb$ and $(\xi,\eta)_\Lambda = b^*\lambda a$. In the case of the dipole a hermitian structure is given by positive numbers λ_0, λ_1 and $(\xi,\eta)_\Lambda(v_0) = \overline{b}_0\lambda_0 a_0$, $(\xi,\eta)_\Lambda(v_1) = \overline{b}_1\lambda_1 a_1$ for $a = (a_0, a_1)$, $b = (b_0, b_1)$.

Connections compatible with a hermitian structure

On says that a connection ∇ on $\mathcal{E}$ is *compatible* with a hermitian structure $(\ ,\ )_\Lambda$ or is Λ-hermitian, if

$$d(\xi,\eta)_\Lambda = (\nabla\xi,\eta)_\Lambda + (\xi,\nabla\eta)_\Lambda \tag{2.5}$$

for all ξ, $\eta \in \mathcal{E}$. Here both sides of the equality are 1-forms, i.e. elements of $\Omega^1_D(\mathcal{A})$. For $\xi = ea$, $\eta = eb$ we have

$$(\nabla\xi,\eta)_\Lambda = \big(e\otimes(da+\Phi a),\eta\big)_\Lambda := (e,\eta)_\Lambda(da+\Phi a)$$

and

$$(\xi,\nabla\eta)_\Lambda = \big(\xi, e\otimes(db+\Phi b)\big)_\Lambda := (db+\Phi b)^*(\xi,e)_\Lambda.$$

2.6. Lemma. *A connection ∇ is compatible with a hermitian structure Λ iff with respect to some (and hence any) basis one has*

$$d\lambda = \lambda\Phi + \Phi^*\lambda, \tag{2.7}$$

where Φ is the vector potential of the connection ∇ with respect to e, $\lambda = \lambda(\Lambda, e)$.

Proof. For $\xi = ea$, $\eta = eb \in \mathcal{E}$ we have

$$\begin{aligned}
d(\xi,\eta)_\Lambda &= d(b^*\lambda a) = db^*\lambda a + b^* d\lambda a + b^*\lambda da\\
(\nabla\xi,\eta)_\Lambda &= (e,\eta)_\Lambda(da+\Phi a) = b^*\lambda da + b^*\lambda\Phi a\\
(\xi,\nabla\eta)_\Lambda &= (db+\Phi b)^*(\xi,e)_\Lambda = db^*\lambda a + b^*\Phi^*\lambda a,
\end{aligned}$$

where we used $(db)^* = db^*$. Thus (2.5) is equivalent to (2.7). $\square$

2.8. Remark. The relation (2.7) is the unique one from the relations above which the author did not find in the literature on the spectral calculus accessible to him. However, the using of general hermitial structures on $\mathcal{E}$ and the formula (2.7) plays an important role in what follows.

For the case of the dipole a simple calculation shows that (2.7) is equivalent to

$$\lambda_0(1 + i\phi_{01}\Delta s^2) = \lambda_1(1 + i\overline{\phi}_{10}\Delta s^2), \tag{2.9}$$

where $\lambda = (\lambda_0, \lambda_1)$, $\Phi = -i\phi_{01}\Delta s^2 p dp - i\phi_{10}\Delta s^2(1-p)dp$.

2.10. Remark. The vector potential A of an electromagnetic field ∇ is a 1-form on the Minkowski space M^4 with values in the Lie algebra $u(1) = i\mathbb{R}$ of $\mathbb{U}(1)$, thus $A^* + A = 0$. It follows that for any hermitian structure λ on $\mathcal{E} = \{M^4 \to M^4\times\mathbb{C}\}$ we have $d\lambda = \lambda A + A^*\lambda = 0$, i.e. the function λ is constant. One can interpret this as the conservation low of the electric charge.

Hermitian metrics

For a hermitian structure Λ on the bundle $\mathcal{E}$ the product $(\xi,\eta)_\Lambda \in \mathcal{A}$ is an operator in $\mathcal{H}$. The corresponding *hermitian metric* on $\mathcal{E}$ is obtained by integration of the fiberwise hermitian product

$$\langle\xi,\eta\rangle = \operatorname{Tr}(\xi,\eta)_\Lambda \in \mathbb{C}$$

for $\xi,\,\eta\in\mathcal{E}$. In the case of the dipole taking a coordinates $\xi=ea$, $\eta=eb$, $\lambda=\lambda(\Lambda,e)=(\lambda_0,\lambda_1)$, we obtain

$$\langle\xi,\eta\rangle=\mathrm{Tr}\,(b^*\lambda a)=\overline{\eta}_0\lambda_0\xi_0+\overline{\eta}_1\lambda_1\xi_1.$$

Any hermatian metric on $\mathcal{E}$ is canonically extended to hermitian metrics on $\mathcal{A}$-modules $\mathcal{E}\otimes_{\mathcal{A}}\Omega^k_D(\mathcal{A})$, $\hat{\mathcal{E}}=\{\mathcal{E}\to\mathcal{A}\}$, $\mathcal{E}\otimes\Omega^k_D(\mathcal{A})\otimes\hat{\mathcal{E}}$ (the tensor products are taken over $\mathcal{A}$). This makes possible to define the norms of operators $\nabla\xi\in\mathcal{E}\otimes_{\mathcal{A}}\Omega^1_D(\mathcal{A})$, $\nabla^2\in\mathcal{E}\otimes\Omega^2_D(\mathcal{A})\otimes\hat{\mathcal{E}}$ (for more details see Appendix B) and to introduce the action $S=S(\nabla,\xi)$.

The interaction Lagrangian of a charged scalar field ξ and an electromagnetic field ∇

A connection ∇ on $\mathcal{E}$ compatible with a hermitian structure $(\ ,\ )_\Lambda$ can be considered as analog of an electromagnetic field, and the sections $\xi\in\mathcal{E}$ describes scalar charged particles. The energy of such a field ξ in the presence of ∇ is defined as $E_\nabla(\xi)=\langle\nabla\xi,\nabla\xi\rangle$.

To compute $E_\nabla(\xi)$ in the case of the dipole, we introduce a basis $e\in\mathcal{U}$ and obtain $\xi=ea=(\xi_0,\xi_1)$, $\nabla\xi=e\otimes\sigma$, $\sigma=da+\Phi a\in\Omega^1_D(\mathcal{A})$, where

$$\sigma=\begin{pmatrix}0 & \frac{1}{\Delta s}(\xi_1-\xi_0)+i\phi_{01}\Delta s\xi_1\\ \frac{1}{\Delta s}(\xi_1-\xi_0)+i\phi_{10}\Delta s\xi_0 & 0\end{pmatrix}.$$

Thus for $\lambda=(\lambda_0,\lambda_1)$ we have

$$\begin{aligned}E_\nabla(\xi)&=\langle e\otimes\sigma,e\otimes\sigma\rangle=\langle\sigma,\lambda\sigma\rangle\\&=\lambda_0\Big|\frac{1}{\Delta s}(\xi_1-\xi_0)+i\phi_{01}\Delta s\xi_1\Big|^2+\lambda_1\Big|\frac{1}{\Delta s}(\xi_1-\xi_0)+i\phi_{10}\Delta s\xi_0\Big|^2.\end{aligned}$$

The Yang-Mills action of ∇ is $\mathrm{YM}\,(\nabla)=\langle\nabla^2,\nabla^2\rangle$, where the curvature ∇^2 is considered as an element of the space $\mathcal{E}\otimes\Omega^2_D(\mathcal{A})\otimes\hat{\mathcal{E}}$ equipped with the Λ-hermitian metric. For the dipole case we have

$$\mathrm{YM}\,(\nabla)=\langle\theta\lambda,\lambda\theta\rangle=(\lambda_0^2+\lambda_1^2)\left|\phi_{10}-\phi_{01}+i\phi_{01}\phi_{10}\Delta s^2\right|^2.$$

The interaction of ξ and ∇ minimizes the action

$$S(\nabla,\xi)=\mathrm{YM}\,(\nabla)+E_\nabla(\xi)-m^2\langle\xi,\xi\rangle.$$

For the dipole case we finally obtain

$$\begin{aligned}S(\nabla,\xi)=&(\lambda_0^2+\lambda_1^2)\left|\phi_{10}-\phi_{01}+i\phi_{01}\phi_{10}\Delta s^2\right|^2\\&+\lambda_0\left|\frac{1}{\Delta s}(\xi_1-\xi_0)+i\phi_{01}\Delta s\xi_1\right|^2+\lambda_1\left|\frac{1}{\Delta s}(\xi_1-\xi_0)+i\phi_{10}\Delta s\xi_0\right|^2\\&-m^2\left(\lambda_0|\xi_0|^2+\lambda_1|\xi_1|^2\right),\end{aligned}\tag{2.11}$$

and the condition (2.9) for ∇ to be Λ-hermitian is fulfilled. Here (ξ_0,ξ_1) are the coordinates of ξ with respect to a basis e, (λ_0,λ_1) coefficients of a hermitian structure Λ, $\Phi=-i\phi_{01}\Delta s^2pdp-i\phi_{10}\Delta s^2(1-p)dp$ the vector potential of ∇. The action S is defined in an invariant manner independently on the choice of $e\in\mathcal{U}$, thus the expression (2.11) is invariant for the action of $\mathcal{U}$.

§3. The Lagrange-Euler equation for the action S

The Lagrange-Euler equations for the action S have the form

$$d^*_\nabla \nabla^2 = -J \tag{3.1.1}$$

$$\nabla^* \nabla \xi = m^2 \xi, \tag{3.1.2}$$

where $J = \nabla\xi \otimes \hat{\xi} - \xi \otimes \hat{\nabla}\hat{\xi}$ is the current, and they correspond to the usual equations of classical electrodynamics: (3.1.1) is the analog of the second pair of the Maxwell equations (+), (3.1.2) the analog of the wave equation (++). In the case of the dipole, the action S is given by (2.11), and the Lagrange-Euler equations have the form

$$\frac{1}{\Delta s^2}(\psi_{01}\psi_{10} - 1)(\lambda_0\psi_{10} + \lambda_1\overline{\psi}_{01}) = \overline{\xi}_0\xi_1 - \frac{1}{2}(\psi_{10}|\xi_0|^2 + \overline{\psi}_{01}|\xi_1|^2) \tag{3.2.1}$$

$$\begin{gathered} \frac{1}{2}(\lambda_0(\overline{\psi}_{01}\overline{\psi}_{10} + 1) - \lambda_0\rho^2)\xi_0 - \lambda_0\psi_{01}\xi_1 = 0 \\ -\lambda_1\psi_{10}\xi_0 + \frac{1}{2}(\lambda_1(\psi_{01}\psi_{10} + 1) - \lambda_1\rho^2)\xi_1 = 0, \end{gathered} \tag{3.2.2}$$

where we used the notations $\psi_{01} = 1 + i\phi_{01}\Delta s^2$, $\psi_{10} = 1 - i\phi_{10}\Delta s^2$, $\rho^2 = m^2\Delta s^2$. We are keeping the coefficients λ_0, λ_1 in the equations (3.2.2) to stress the analogy with the compatibility equation (0.1) in view of the Λ-hermitian condition (2.9), which takes the form

$$\lambda_0\psi_{01} = \lambda_1\overline{\psi}_{10}. \tag{3.2.3}$$

We show how to obtain the equations (3.1.1)–(3.2.2) in Appendix B. Here we analize the equations (3.2.1), (3.2.2) comparing them with the compatibility equation for the case of the elementary graphs: dipole (0.2.1) and monopole (0.2.2). The general case is considered in § 5.

Representing $\xi_0 = |\xi_0|e^{i\alpha_0}$, $\xi_1 = |\xi_1|e^{i\alpha_1}$ we interpret the numbers $|\xi_0|$, $|\xi_1|$ as to be proportional to the fiber lengths l_0, l_1 of blocks M_{v_0}, M_{v_1} (see Introduction). Since these lengths are positive, the only nondegenerated solutions to the Lagrange-Euler equations are of the interest, i.e. for which $\xi_0, \xi_1 \neq 0$. The necessary condition for that is vanishing of the determinant of the system (3.2.2),

$$\big((\overline{\psi}_{01}\overline{\psi}_{10} + 1) - \rho^2\big)\big((\psi_{01}\psi_{10} + 1) - \rho^2\big) - 4\psi_{01}\psi_{10} = 0. \tag{3.3}$$

In view of (3.2.3) we can assume that $\psi_{01} = |\psi_{01}|e^{i\gamma}$, $\psi_{10} = |\psi_{10}|e^{-i\gamma}$, and we put

$$\psi := \lambda_0|\psi_{01}| = \lambda_1|\psi_{10}|.$$

Then the nondegeneration condition (3.3) has the form

$$\big(\psi^2 - (\rho^2 - 1)\lambda_0\lambda_1\big)^2 - 4\lambda_0\lambda_1\psi^2 = 0$$

or

$$\big(\psi^2 - (\rho^2 + 1)\lambda_0\lambda_1\big)^2 = 4\lambda_0^2\lambda_1^2\rho^2,$$

hence

$$\psi^2 = \lambda_0\lambda_1(\rho \pm 1)^2. \tag{3.4}$$

It follows from the imaginary part of the equations (3.2.1), (3.2.2) that

$$\alpha_1 - \alpha_0 + \gamma = n\pi, \quad n \in \mathbb{Z}, \tag{3.5}$$

and we obtain from (3.2.1) and (3.2.2) the next (real) system

$$\frac{1}{\Delta s^2}\left(\frac{\psi^2}{\lambda_0\lambda_1} - 1\right)\left(\frac{\lambda_0}{\lambda_1} + \frac{\lambda_1}{\lambda_0}\right)\psi = (-1)^n|\xi_0||\xi_1| - \frac{1}{2}\left(\frac{|\xi_0|^2}{\lambda_1} + \frac{|\xi_1|^2}{\lambda_0}\right)\psi \tag{3.6.1}$$

$$\begin{aligned} &\frac{1}{2}\left(\frac{\psi^2}{\lambda_1} + \lambda_0(1-\rho^2)\right)|\xi_0| + (-1)^{n+1}\psi|\xi_1| = 0 \\ &(-1)^{n+1}\psi|\xi_0| + \frac{1}{2}\left(\frac{\psi^2}{\lambda_0} + \lambda_1(1-\rho^2)\right)|\xi_1| = 0 \end{aligned} \tag{3.6.2}$$

The case of a massless field ξ

We consider, first, the case $m = 0$ for the mass m and, consequently, $\rho = 0$. Then $\psi^2 = \lambda_0\lambda_1$ by (3.4), and for a nonzero solution $(|\xi_0|, |\xi_1|)$ of (3.6.2) we have

$$\lambda_0^{1/2}|\xi_0| = (-1)^n\lambda_1^{1/2}|\xi_1|.$$

Thus n is even and

$$|\xi_0|^2/|\xi_1|^2 = \lambda_1/\lambda_0, \tag{3.7}$$

cp. (0.2.1),(ii). Equation (3.6.1) is fulfilled automatically, its left and right hand side both vanish.

Therefore, the solutions to the Lagrange-Euler equation (3.1.1), (3.1.2) for the action (2.11) in the case of a massless field ξ are as follows

$$\alpha_1 - \alpha_0 + \gamma = 2n\pi; \quad \psi^2 = \lambda_0\lambda_1; \quad |\xi_0|^2/|\xi_1|^2 = \lambda_1/\lambda_0.$$

These data define a flat connection ∇ and a parallel field ξ, $\nabla\xi = 0$, i.e. $S(\nabla, \xi) = 0$.

Comparing of the spectral and the geometric models

To interpret a solution (l_0, l_1, ω) to the compatibility equation (0.2.1), defining a nonpositively curved metric on a graph-manifold as a field configuration (∇, ξ) extremal for the action S, we put

$$l_0 = |\xi_0|,\ l_1 = |\xi_1|,\ \lambda_0 = |k_0|,\ \lambda_1 = |k_1|,\ k_0|\psi_{01}| = \cos\omega/b,\ k_1|\psi_{01}| = \cos\omega/b$$

assuming that $k_0k_1 > 0$. These data define a Λ-hermitian connection ∇ on $\mathcal{E}$ and a field ξ, related by the gauge invariance condition $\alpha_1 - \alpha_0 + \gamma = 2n\pi$. It is this choice that lead to a necessity to consider nonconstant hermitian structures λ on $\mathcal{E}$ and to use the relation (2.7). Now the nondegeneration condition $\psi^2 = \lambda_0\lambda_1$ is equvalent to the condition $\cos^2\omega = k_0k_1b^2$ and (3.7) is exactly the condition $l_0^2/l_1^2 = k_1/k_0$ for the solutions (ii) of (0.2.1). Under this choice, the equation (3.6.2) becomes the equation (0.2.1), if the condition $\psi^2 = \lambda_0\lambda_1$ is fulfilled, which in turn ensures that the connection ∇ is flat, $\mathrm{YM}(\nabla) = 0$.

The decomposition principle (see §5) makes possible to extend this equivalence of the spectral and geometric models to the case of arbitrary (finite) graphs, and in that sense a nonpositively curved metric on a graph-manifold can be interpreted as a field configuration (∇, ξ) extremal for the action S, where ∇ is a flat Λ-hermitian connection on $\mathcal{E}$ and $\xi \in \mathcal{E}$ is a covariantly constant field, $\nabla\xi = 0$, or, in physical terminology, as the interaction of an electromagnetic field with a massless scalar charged field on the graph of the manifold.

However, this interpretation, even being attractive, is not complete. Its drawback is first of all the necessity of the condition $k_0k_1 > 0$ since otherwise the spectral model disappears, $\lambda_0 = \lambda_1 = 0$. On the other hand, on a graph-manifold often exist nonpositively curved metrics for which it is necessary to consider the case $k_0 = k_1 = 0$, $\omega = \pi/2$ by the decomposition principle. Such metrics cannot be interpreted in terms of massless fields. Another disadvantage of such interpretation is that the equations (3.6.1), (3.6.2) for $\rho = 0$ do not reflect the obvious geometric prohibition $\omega \neq 0$. The condition $\omega = 0$ implies a collaps of the metric, because it means that the fibers of the adjacent blocks being homotopically distinct closed geodesics on the flat gluing torus T_w coincide. These problems will be solved in the next section, where we consider the fields ξ with nonzero masses, due to a remarkable mechanism of exclusions and degenerations hidden in equations (3.6.1), (3.6.2).

§4. The mechanism of exclusions and degenerations

Here we describe a mechanism which allows to interpret in terms of the Lagrange-Euler equations (3.6.1), (3.6.2) for the action $S = S(\nabla, \xi)$, (2.11), degenerations $\omega = \pi/2$ and $\omega = 0$ for metrics on nonpositive curvature on a graph-manifold. It requires to consider fields with nonzero masses. Then the equations (3.6.1), (3.6.2) amazingly unveil a geometric information about the interpreted metric on the graph-manifold, which is completely absent in the compatibility equation (0.1). Namely, the equation (0.1) being linear and homogeneous in lengths does not fix the lengths of fibers only defining its ratios. However, if $m \neq 0$, then the equations (3.6.1), (3.6.2) define the values $|\xi_0|$, $|\xi_1|$, and for that there is a natural geometric interpretation.

The exclusions and degenerations mechanism is described during the proof of the following theorem. We put $\rho = m\Delta s$ do not assuming that ρ is positive. Recall that m is the mass of a field ξ and $|\Delta s|$ is interpreted as the distance between the vertices v_0, v_1.

4.1. Theorem. *The Lagrange-Euler equations (3.6.1), (3.6.2) for the action S given by (2.11) possess nondegenerated solutions (i.e. $|\xi_0||\xi_1| \neq 0$) if and only if $\rho = 0$, $1 < |\rho| < 2$. For $|\rho| = 1, 2$ any solution is degenerated, and for the remaining $\rho \in \mathbb{R}$ there is no solution.*

Proof. Since the case $\rho = 0$ is already considered we assume that $\rho \neq 0$. Furthemore, we assume that $\rho > 0$, the case $\rho < 0$ is treated similarly. For a solution $(|\xi_0|, |\xi_1|)$ to (3.6.2) by nondegeneracy condition (3.4) we have

$$\lambda_0(1 \pm \rho)|\xi_0| = (-1)^n\psi|\xi_1|. \tag{4.1.1}$$

We first show that in the case $\psi^2 = \lambda_0\lambda_1(\rho+1)^2$ for the condition (3.4) our system possesses no any solution. Indeed, $\psi = \lambda_0^{1/2}\lambda_1^{1/2}(\rho+1)$, and it follows from (4.1.1) that n is even and $|\xi_0|^2/|\xi_1|^2 = \lambda_1/\lambda_0$. Then the left hand side of (3.6.1) is positive,

$$\frac{1}{\Delta s^2}\rho(\rho+1)(\rho+2)\frac{\lambda_0^2+\lambda_1^2}{\lambda_0^{1/2}\lambda_1^{1/2}} > 0,$$

whereas the right hand side is

$$(-1)^n|\xi_0||\xi_1| - \frac{1}{2}\left(\frac{|\xi_0|^2}{\lambda_1} + \frac{|\xi_1|^2}{\lambda_0}\right)\psi = -\lambda_0^{1/2}\lambda_1^{-1/2}\rho|\xi_0|^2 \leq 0.$$

Thus we suppose in the sequel that $\psi^2 = \lambda_0\lambda_1(\rho-1)^2$.

(a) The case $0 < \rho < 1$**.** Then $\psi = \lambda_0^{1/2}\lambda_1^{1/2}(1-\rho)$ and it follows from (4.1.1) that $\lambda_0^{1/2}|\xi_0| = (-1)^n\lambda_1^{1/2}|\xi_1|$. Thus n is even and $|\xi_0|^2/|\xi_1|^2 = \lambda_1/\lambda_0$. This time the left hand side of (3.6.1) is negative,

$$\frac{1}{\Delta s^2}\rho(\rho-2)(1-\rho)\frac{\lambda_0^2+\lambda_1^2}{\lambda_0^{1/2}\lambda_1^{1/2}} < 0,$$

whereas the right hand side is

$$(-1)^n|\xi_0||\xi_1| - \frac{1}{2}\left(\frac{|\xi_0|^2}{\lambda_1} + \frac{|\xi_1|^2}{\lambda_0}\right)\psi = \lambda_0^{1/2}\lambda_1^{-1/2}\rho|\xi_0|^2 \geq 0.$$

(b) The case $\rho = 1$**.** Then $\psi = 0$ and the system (3.6.2) puts no any restriction on $|\xi_0|$, $|\xi_1|$. The equation (3.6.1) is then reduced to the condition $|\xi_0||\xi_1| = 0$, which gives a degenerated critical configuration (∇, ξ) for the action S.

We assume now that $\rho > 1$. Then $\psi = \lambda_0^{1/2}\lambda_1^{1/2}(\rho-1)$, and it follows from (4.1.1) that n is odd and $|\xi_0|^2/|\xi_1|^2 = \lambda_1/\lambda_0$. The left hand side of (3.6.1) is

$$\frac{1}{\Delta s^2}\left(\frac{\psi^2}{\lambda_0\lambda_1} - 1\right)\left(\frac{\lambda_0}{\lambda_1} + \frac{\lambda_1}{\lambda_0}\right)\psi = \frac{1}{\Delta s^2}\rho(\rho-1)(\rho-2)\frac{\lambda_0^2+\lambda_1^2}{\lambda_0^{1/2}\lambda_1^{1/2}}, \tag{4.1.2}$$

and the right hand side is

$$(-1)^n|\xi_0||\xi_1| - \frac{1}{2}\left(\frac{|\xi_0|^2}{\lambda_1} + \frac{|\xi_1|^2}{\lambda_0}\right)\psi = -\lambda_0^{1/2}\lambda_1^{-1/2}\rho|\xi_0|^2. \tag{4.1.3}$$

Thus for $\rho > 2$ there is no critical configuration (∇, ξ) for S, and for $\rho = 2$ the unique critical configuration is a degenerated one, $|\xi_0| = |\xi_1| = 0$, $\psi = \lambda_0\lambda_1$.

It remains to consider

(c) The case $1 < \rho < 2$**.** Then equations (3.6.1), (3.6.2) give the critical configuration (∇, ξ) for which $\alpha_1 - \alpha_0 + \gamma = (2n+1)\pi$, $\psi = \lambda_0^{1/2}\lambda_1^{1/2}(\rho-1)$ and

$$\lambda_0|\xi_0|^2 = \lambda_1|\xi_1|^2 = (\lambda_0^2+\lambda_1^2)(\rho-1)(2-\rho)/\Delta s^2.$$

Obviously, we have $|\xi_0|^2/|\xi_1|^2 = \lambda_1/\lambda_0$ and the connection ∇ is not flat. This finishes the proof of the theorem. □

Interpretation of metrics as fields with nonzero masses

Interpreting a solution (l_0, l_1, ω) to the compatibility equation (0.2.1) with k_0, $k_1 > 0$ (the case k_0, $k_1 < 0$ is treated similarly) as a field configuration (∇, ξ) extremal for the action S with a field ξ of nonzero mass m, we assume by Theorem 4.1 that $1 < \rho < 2$ (in the case k_0, $k_1 < 0$ one should take $-2 < \rho < -1$) and put

$$\lambda_0|\xi_0|^2 = k_0 l_0^2,\ \lambda_1|\xi_1|^2 = k_1 l_1^2,\ \psi = \cos\omega/b,\ \lambda_0(\rho-1) = k_0,\ \lambda_1(\rho-1) = k_1.$$

These data define a Λ-hermitian connection ∇ on $\mathcal{E}$ and a field ξ of nonzero mass m, $\rho = m\Delta s$ related by the gauge invariance condition $\alpha_1 - \alpha_0 + \gamma = (2n+1)\pi$. Now the nondegeneration condition $\psi^2 = \lambda_0\lambda_1(\rho - 1)^2$ is equivalent to the condition $\cos^2\omega = k_0k_1b^2$ and the relation $|\xi_0|^2/|\xi_1|^2 = \lambda_1/\lambda_0$ is exactly the condition $l_0^2/l_1^2 = k_1/k_0$. But now, in contrast to the case $m = 0$, these data define the lengths l_0, l_1,

$$\lambda_0 l_0^2 = \lambda_1 l_1^2 = (\lambda_0^2 + \lambda_1^2)(2-\rho)/\Delta s^2 \tag{4.2}$$

making the configuration (∇, ξ) extremal for the action S. This raises the question how to understand these equalities in geometric terms? The equation (0.1) is homogeneous in lengths of fibers, and this reflects the possibility of scaling the metric by homotheties.

To understand (4.2) correctly we will consider special nonpositively curved metrics on a graph-manifold M, whose restrictions on each block M_v are geometric structures modelled on $H^2 \times \mathbb{R}$, i.e. metrics ds_F^2 in the local decompositions $ds_{M_v}^2 = ds_F^2 + dl^2$ are metrics of the constant curvature -1. We call such metrics *geometrizations* of M (see [BK1–3]).

The knowledge of the lengths l_0, l_1 of fibers of adjacent blocks and of the angle ω between them uniquely defines a flat metric on the gluing torus T_w and hence it fixes the length of a closed geodesic z_w on T_w representing the boundary component $(\partial F_v)_w$ of the surface F_w from the decomposition $M_v = F_v \times S^1$ (the choice of z_w is in general not unique and it is equivalent to the choice of a Waldhausen basis, see Appendix A; however, it does not depend on the metric and one can assume a Waldhausen basis to be fixed).

In turn, there is a metric of constant curvature -1 on the surface F_v with geodesic boundary for which the length of each component z_w, $w \in \partial v$ is a given positive number. But then the lengths l_0, l_1 are uniquely defined by the length of the geodesic z_w, if a conformal structure on the torus T_w given by the ratio l_0/l_1 and the angle ω is fixed.

Namely this property is reflected by (4.2). The length of z_w is proportional to $1/|\Delta s|$, and this is compatible with interpretation $|\Delta s|$ as a "distance" between blocks M_{v_0}, M_{v_1}: metrics collapse as $|\Delta s| \to \infty$, and become infinitely "thick" as $|\Delta s| \to 0$. In other words, the parameter m governs the shape of geometrizations of the manifold M when $\rho = m\Delta s$ is fixed.

Degeneration $\rho = 1$

Another fact confirming that the spectral model with nonzero mass is adequate to the geometric model is its possibility to interpret the degeneration $k_0 = k_1 = 0$, when $\omega = \pi/2$ and there is no restriction on the fiber lengths l_0, l_1 (recall that such interpretation is impossible by massless fields). This case naturally corresponds to the case $\rho = 1$, when $\psi = 0$ and $|\xi_0||\xi_1| = 0$. Then one can take λ_0, λ_1 as arbitrary positive numbers, and we obtain a critical configuration (∇, ξ) defined by the metric in question. Furthermore, any limit transition k_0, $k_1 \to 0+$ can be interpreted by a corresponding limit transition $\rho \to 1+$.

An adequate description of a much more strong degeneration $\omega \to 0$ of metrics is possible for more general graph-manifolds, whose graph is different from the dipole. The point is that if a graph-manifold with the dipole graph admits a nonpositively curved metric, then $k_0 = k_1 = 0$ and $\omega = \pi/2$ (see Appendix A), which corresponds to the case $\rho = 1$.

Monopole

We briefly describe the spectral model for the graph Γ which is a loop with vertex v, see (0.2.b). Here $\mathcal{A} = \mathbb{C}$, $\mathcal{H} = \mathbb{C}^2$, the representation of $\mathcal{A}$ in $\mathcal{H}$ is given by matrices $\pi(a) = \mathrm{diag}\,(a, a)$. The rest data are obiously obtained from the already considered dipole case: $\phi_{01} = \phi_{10}$, $\lambda_0 = \lambda_1 (= \lambda)$, $\xi_0 = \xi_1 (= \xi)$ etc. The Lagrange-Euler equation for the action S possess the same mechanism of exclusions and degenerations as in the dipole case. We leave the details to the reader. Notice only that $k_0 = k_1 = k/2$ by the decomposition principle and the correspondence between the geometric and the spectral models is given by

$$\lambda|\xi| = kl, \ \ \psi = \cos\omega/b, \ \ \lambda(\rho - 1) = k/2$$

(we assume that $k > 0$). Then for a critical configuration we have

$$\lambda^2(\rho - 1)^2 = \psi^2 = \cos^2\omega/b^2.$$

§5. The decomposition principle

The decomposition principle was found by V. Kobel'skii in 1994 and widely used in works [BK1–3]. It is based on the following simple topological property of the graph-manifolds (see Appendix A). Let M_v be a block of a graph-manifold M, which is a toral sum $M' + M''$ of graph-manifolds M', M'' along blocks $M'_{v'} \subset M'$, $M''_{v''} \subset M''$, where the vertex v is the conjunction of the vertices v', v''. Then we have $k_v = k_{v'} + k_{v''}$ for the corresponding invariants (charges) of the manifolds M, M', M''. The decomposition principle is related to the graphs of graph-manifolds, more precisely, to *the labeled graphs* being equipped with charges k_v, $v \in V$ and intersection indices b_w, $w \in W$, which are the coefficients of the compatibility equation (0.1). To simplify the statement, we restrict to the case when the graph Γ of a manifold M is *the triplet*, i.e. consists of two edges $w_0 \cup w_1$ with the common vertex $\partial_+ w_0 = v_1 = \partial_- w_1$. We use notations $v_0 = \partial_- w_0$, $v_2 = \partial_+ w_1$ and k_0, k_1, k_2 for the corresponding charges, b_0, b_1 for the corresponding intersection indices. Any decomposition $k_1 = k_1^+ + k_1^-$ represents the labeled graph Γ as the conjunction of the labeled graphs $\Gamma_0 = w_0$, $\Gamma_1 = w_1$ along the vertex v_1, where Γ_0 is equipped with the charges k_0, k_1^+ and the intersection index b_0, and Γ_1 with k_1^-, k_2, b_1 correspondingly.

The decomposition principle asserts that

(a) if the graphs Γ_0, Γ_1 possess isometric states $(l_0, l_1^+; \omega_0)$ and $(l_1^-, l_2; \omega_1)$ correspondingly and $l_1^+ = l_1^- =: l_1$, then $(l_0, l_1, l_2; \omega_0, \omega_1)$ is an isometric state of the graph Γ;

(b) conversely, any isometric state $(l_0, l_1, l_2; \omega_0, \omega_1)$ of Γ canonically defines the decomposition $\Gamma = \Gamma_0 + \Gamma_1$ with $k_1 = k_1^+ + k_1^-$, where the numbers k_1^+, k_1^- are defined by the relations $l_0^2/l_1^2 = k_1^+/k_0$, $l_1^2/l_2^2 = k_2/k_1^-$ (if k_0 or k_1 vanishes, then $k_1^+ = 0$ or correspondingly $k_1^- = 0$).

In other words, this principle corresponds to a representation of system (0.1) for the graph Γ

$$
\begin{array}{lllll}
 & k_0 l_0 & -\frac{\cos\omega_0}{b_0} l_1 & & = 0 \\
(5.1) & -\frac{\cos\omega_0}{b_0} l_0 & +k_1 l_1 & -\frac{\cos\omega_1}{b_1} l_2 & = 0 \\
 & & -\frac{\cos\omega_1}{b_1} l_1 & +k_2 l_2 & = 0
\end{array}
$$

as the system for two dipoles Γ_0, Γ_1

$$
\begin{array}{lllll}
 & k_0 l_0 & -\frac{\cos\omega_0}{b_0} l_1^+ & & = 0 \\
 & -\frac{\cos\omega_0}{b_0} l_0 & +k_1^+ l_1^+ & & = 0 \\
(5.2) & & & & \\
 & & k_1^- l_1^- & -\frac{\cos\omega_1}{b_1} l_2 & = 0 \\
 & & -\frac{\cos\omega_1}{b_1} l_1^- & +k_2 l_2 & = 0
\end{array}
$$

with additional conditions $k_1 = k_1^+ + k_1^-$ and $l_1^+ = l_1 = l_1^-$.

We stress that a decomposition of labeled graphs $\Gamma = \Gamma_0 + \Gamma_1$ in general does not exist on the level of graph-manifolds, i.e. the corresponding representation $M = M_0 + M_1$ as a toral sum may not exist and as the rule it does not exist.

This principle is obviously extended to any graph-manifold M permitting to decompose its labeled graph into elementary labeled graphs – monopoles and dipoles. Of course, it is meaningful only for M admitting metrics of nonpositive curvature. Roughly speaking, this principle describes a parametrization of the set of nonpositively curved metrics (geometrizations) on M.

Spectral model

Here we describe the spectral model of a geometrization of an arbitrary graph-manifold M. As the motivation we use the above decomposition principle.

Let $\Gamma = \Gamma(V, W)$ be the graph of M with the vertex set V and the set of oriented edges W. The algebra $\mathcal{A}$ ("the space") is defined as the algebra of functions $v \mapsto \{\partial v \to \mathbb{C}\}$ on the set V, where $\partial v \subset W$ is the set of edges w with the common initial vertex $\partial_- w = v$. As "the tangent bundle" we take the Hilbert space of sections $\eta : W \to W \times \mathbb{C}$ of the trivial line bundle $W \times \mathbb{C} \to W$ with the scalar product

$$\langle \eta, \eta' \rangle = \sum_{w \in W} \overline{\eta}'(w) \eta(w).$$

The representation π of $\mathcal{A}$ in $\mathcal{H}$ is given by

$$(\pi(a)\eta)_v (w) = a_v(w) \eta(w)$$

for $w \in \partial v$. The unit length operator ds is defined as

$$ds(\eta)(w) = i \Delta s(w) \cdot \eta(-w),$$

where $\Delta s : W \to \mathbb{R} \setminus 0$ is an odd function, $\Delta s(-w) = -\Delta s(w)$.

These data define a spectral triple $\{\mathcal{A}, \mathcal{H}, ds\}$ associated with the graph Γ. In the dipole case this definition coincides with the one given in § 2. The Connes' formula (2.1) applied to edges defines an intrinsic metric on Γ, which is called *spectral*. Different edges w, w' between the same vertices may have different lengths. It is not difficult to see that for any intrinsic metric on Γ there is an odd function $\Delta s : W \to \mathbb{R} \setminus 0$, for which the lengths of any edge with respect to the given metric and the corresponding spectral one coincide. This fact may serve as a starting point for approximation by (discrete) spectral models of classical ones.

The space of scalar charged fields $\mathcal{E}$, the gauge transformation group $\mathcal{U}$, a hermitian structure Λ, a Λ-hermitian connection ∇ on $\mathcal{E}$, its curvature $\nabla \circ \nabla$ and the hermitian metrics on $\mathcal{A}$-modules $\mathcal{E}$, $\mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A})$, $\mathcal{E} \otimes \Omega^2_D(\mathcal{A}) \otimes \hat{\mathcal{E}}$ are described in § 2. Hence we can define the action

$$S(\nabla, \xi) = \mathrm{YM}\,(\nabla) + E_\nabla(\xi) - m^2 \|\sigma\|^2$$

and derive its Lagrange-Euler equation (3.1.1), (3.1.2).

Reduction of the gauge group

The model described above is, obviously, the direct sum of (almost independent) spectral models for the dipoles Γ_w, $w \in W$. Only relations between the last are established via the function Δs defining the spectral metric on the graph Γ. The function Δs is not a dynamical variable, hence the Lagrange-Euler equation for S breaks down into a system of $|W|/2$ independent pairs of equations (3.2.1), (3.2.2), parametrized by (nonoriented) edges $(w, -w)$, $w \in W$. The gauge group $\mathcal{U}$ is the group of maps $\{\partial V \to \mathbb{U}(1)\}$, where $\partial V = \cup_{v \in V} \partial v$. To make the situation less trivial and adopted for approximations, we restrict $\mathcal{U}$ to its subgroup $\mathcal{U}_0 = \{V \to \mathbb{U}(1)\}$. It means that for a fixed basis $e \in \mathcal{U}_0$ and for every $\xi = (\xi_w) \in \mathcal{E}$, $v \in V$ we have $\arg(\xi_w) = \arg(\xi_{w'})$ for any w, $w' \in \partial v$.

Much more essential relations between the dipoles Γ_w arise when we fix a correspondence between the spectral and geometrical models.

Correspondence between the spectral and geometric models

To simplify statements we assume that the graph Γ of M does not have edges, which are loops. Fix a metric of nonpositive curvature (geometrization) on M (the necessary and sufficient conditions for the existence of such metrics are established in [BK2]). By the decomposition principle it defines the decomposition of the labeled graph $\{\Gamma, B, K\}$ into the dipoles $\{\Gamma_w, b_w, K_w\}$, $w \in W$, where $K = \{k_v \mid v \in V\}$ is the collection of charges of the manifold M, $B = \{b_w \mid w \in W\}$ the intersection indices, $b_w = b_{-w} \in \mathbb{N}$, $K_w = \{k_w, k_{-w}\}$ charges of the dipole Γ_w, $k_w := k_{\partial_- w}$. Furthemore, the condition

$$k_v = \sum_{w \in \partial v} k_w \tag{5.3}$$

holds for each vertex $v \in V$.

The geometrization defines for each dipole Γ_w its isometric state $(l_w, l_{-w}; \omega_w)$, where $l_w := l_{\partial_- w}$, see Example 0.2(a). We assume for simplicity that all these isometric states are nondegenerated, i.e.

$$0 < \cos^2 \omega_w = k_w k_{-w} b_w^2 < 1. \tag{5.4}$$

The case of negative charges k_w, k_{-w} in general cannot be excluded. To describe such cases we pick an orientation $w = w_s$ of every nonoriented edge $(w, -w)$ such that $\mathrm{sgn}\,(\Delta s(w)) =$

$\operatorname{sgn}(k_w) = \operatorname{sgn}(k_{-w})$. We let $W_s \subset W$ be the set of the choosen oriented edges. Furthemore, if the graph Γ is not simply connected, then for every its nontrivial circuit $Z \subset \Gamma$ the following balance condition is fulfilled. Assume that Z is oriented and the orientations $w = w_Z$ of its edges are compatible with the orientation of Z (the orientation w_Z nothing to do with the orientation w_s above). Then

$$\prod_{w \in Z} \frac{k_w}{k_{-w}} = 1. \tag{5.5}$$

This relation, obviously, follows from the equality

$$l_{-w}^2 / l_w^2 = k_w / k_{-w} \tag{5.6}$$

for each edge $w \in W$.

Assuming for simplicity that for any edge $w \in W_s$ the number $\rho_w = m\Delta s(w)$ satisfies the condition $1 < \rho_w < 2$, we put as in § 4

$$\lambda_w |\xi_w|^2 = k_w l_w^2; \tag{5.7}$$

$$\lambda_w(\rho_w - 1) = k_w; \tag{5.8}$$

$$\psi_w = \cos\omega_w / b_w; \tag{5.9}$$

$$\arg(\xi_w) - \arg(\xi_{-w}) + \gamma_w = (2n+1)\pi, \tag{5.10}$$

where $\lambda_w := \lambda_{\partial_- w}$. For a fixed basis $e \in \mathcal{U}_0$ these conditions define a field $\xi \in \mathcal{E}$ with coordinates $\xi_w := \xi_{\partial_- w} \in \mathbb{C}$, a hermitian structure $\Lambda = (\lambda_w)$ on $\mathcal{E}$ and a Λ-hermitian connection ∇ on $\mathcal{E}$, whose coefficients are defined via parameters ψ_w,γ_w as in § 3. Condition (5.10) of the gauge invariance ensures that the configuration (∇, ξ) is well defined, i.e. does not depend on the choice of $e \in \mathcal{U}_0$. It follows from (5.4), (5.8) and (5.9) that the nondegeneration condition $\psi_w^2 = \lambda_w \lambda_{-w}(\rho_w - 1)^2$ holds, see (3.4), and from (5.6) and (5.7) that $|\xi_{-w}|^2/|\xi_w|^2 = \lambda_w/\lambda_{-w}$, i.e. the field ξ satisfies the wave equation (3.1.2).

Solution of the Maxwell equations

The field configuration (∇, ξ) defined above is critical for the action S, if it satisfies the equation (3.1.1). It means that for any edge $w \in W_s$ the condition

$$\lambda_w |\xi_w|^2 = \lambda_{-w}|\xi_{-w}|^2 = (\lambda_w^2 + \lambda_{-w}^2)(\rho_w - 1)(2 - \rho_w)/\Delta s^2(w) \tag{5.11}$$

has to be satisfied, see proof of Theorem 4.1, sect.(c). Such conditions for different edges w, $w' \in \partial v \cap W_s$ are not independent, because for the fiber lengths we have the equality $l_{\partial_- w} = l_v = l_{\partial_- w'}$. Thus condition (5.11) means that there is some relation between parameters ρ_w, $\rho_{w'}$. In other words, the field configuration (∇, ξ) defined by the geometrization (l_w, ω_w), $w \in W$ of the manifold M by (5.7)–(5.10) is critical for the action S, if it is possible to satisfy the mentioned relation between ρ_w, $\rho_{w'}$ for all pairs w, $w' \in \partial v \cap W_s$, $v \in V$ without a contradiction.

5.12. Theorem. *For any geometrization (l_w, ω_w), $w \in W$ of the manifold M there exists an (odd) function $\Delta s : W \to \mathbb{R} \setminus 0$ such that the field configuration (∇, ξ) defined by (5.7)–(5.10) is critical for the action S.*

Proof. If there is an edge w with $k_w = 0$, then $k_{-w} = 0$ also, and we put $\rho_w = \pm 1 = m\Delta s(w)$. Since in this case there is no any restriction on the lengths l_w, l_{-w}, we take as λ_w, λ_{-w} in (5.8) arbitrary positive numbers, and then (5.7)–(5.9) are satified with ξ_w, $\xi_{-w} = 0$, $\psi_w = 0$. Such edges $(w, -w)$ separate the graph into several pieces, and for simplicity we consider the case when it is only one piece, moreover, we assume that condition (5.4) with k_w, $k_{-w} > 0$ for all $w \in W_s$ is satisfied for the geometrization of M. This defines the sign of the function Δs, $\Delta s(w) > 0$ for all $w \in W_s$. Take w, $w' \in \partial v \cap W_s$ for some vertex $v \in V$. It follows from the equality $l_w = l_{w'}$ that the condition (5.11) for (∇, ξ) to be critical has the form

$$\frac{k_w^2 + k_{-w}^2}{k_w} \cdot \frac{2 - \rho_w}{\rho_w - 1} \cdot \frac{1}{\Delta s^2(w)} = \frac{k_{w'}^2 + k_{-w'}^2}{k_{w'}} \cdot \frac{2 - \rho_{w'}}{\rho_{w'} - 1} \cdot \frac{1}{\Delta s^2(w')},$$

where we used relations (5.7), (5.8). Recalling that $\rho_w = m\Delta s(w)$, we obtain

$$\frac{2 - \rho_{w'}}{(\rho_{w'} - 1)\rho_{w'}^2} = \frac{2 - \rho_w}{(\rho_w - 1)\rho_w^2} \cdot K_{w,w'}, \tag{5.12.1}$$

where

$$K_{w,w'} = \frac{k_w^2 + k_{-w}^2}{k_{w'}^2 + k_{-w'}^2} \cdot \frac{k_{w'}}{k_w}. \tag{5.12.2}$$

Let $\rho = \rho(\delta)$ be the function on $(0, \infty)$ inverse to the function

$$\delta(\rho) = \frac{2 - \rho}{(\rho - 1)\rho^2}.$$

(It is easy to see that the function $\delta = \delta(\rho)$ is monotone on $1 < \rho < 2$ and $\rho(\delta) \to 2$ as $\delta \to 0+$, $\rho(\delta) \to 1$ as $\delta \to \infty$).

Putting $\rho_w = \rho(\delta)$ for some $\delta > 0$, we obtain that (5.12.1) is satisfied for $\rho_{w'} = \rho(K_{w,w'} \cdot \delta)$. Therefore, values of the function Δs are also fixed by $\Delta s(w) = \rho(\delta)/m$, $\Delta s(w') = \rho(K_{w,w'} \cdot \delta)/m$.

If the graph Γ of the manifold M is simply connected, then the choice $\Delta s(w) = \rho(\delta)/m$ for an edge $w \in W_s$ for some $\delta > 0$ well and uniquely defines the value $\Delta s(w')$ for any edge $w' \in W_s$ such that the condition (5.12.1) is satisfied for every pair of adjacent edges w, w', providing that the configuration (∇, ξ) is critical.

Assuming that the graph Γ is nonsimply connected, let us consider some its nontrivial (oriented) circuit Z. Defining Δs as above, we obtain that the function Δs is well defined, if the condition

$$\prod_{w \in Z} K_{w,w'} = 1$$

is satisfied, where w' is the next edge of Z after w. But, obviously, we have

$$\prod_{w \in Z} K_{w,w'} = \prod_{w \in Z} \frac{k_w}{k_{-w}},$$

and the function Δs is well defined due to condition (5.5). $\square$

Appendix A

We collect here necessary definitions, results and facts on geometrizations of graph-manifolds from [BK1–3].

In the present work saying about a graph-manifold M we mean a 3-dimensional closed orientable manifold, for which there exists a nonempty minimal collection E of disjoint embedded incompressible tori such that the closure M_v of each connected component of the complement to E is homeomorphic to $F_v \times S^1$, where F_v is a compact surface with boundary different from the disc and the annulus. The manifolds M_v are called *blocks* of M. The set V of the blocks is the vertex set of the graph Γ of M, whose set of nonoriented edges is E. We denote by W the set of oriented edges of Γ.

Waldhausen bases

We fix an orientation of M. This defines the orientation of every block M_v, for which we also fix an orientation of the factor S^1 in the decomposition $M_v = F_v \times S^1$.

For $w \in \partial v \subset W$ let $L_w \simeq \mathbb{Z}^2$ be the first homology group of the corresponding boundary component $(\partial M_v)_w = T_w$, which is the torus T^2. Choose a basis $\{(z_w, f_w) \,|\, w \in \partial v\}$ of the group $H_1(\partial M; \mathbb{Z}) = \oplus_{w\in\partial v} L_w$ in such a way that the basis (z_w, f_w) be compatible with the orientation induced on ∂M_v, the elements f_w represent oriented fibers, and the sum $\oplus_{w\in\partial v} z_w$ belongs to the kernel of the inclusion homomorphism

$$H_1(\partial M_v; \mathbb{Z}) \to H_1(M_v; \mathbb{Z}).$$

The collection $(z, f) = \{(z_w, f_w) \,|\, w \in \partial v,\ v \in V\}$ is called *the Waldhausen basis.*

To an oriented edge w of the graph Γ there corresponds a gluing map of the boundary components of adjacent blocks, which induces an isomorphism $g_w : L_{-w} \to L_w$; in the chosen bases, the g_w has the matrix

$$\begin{pmatrix} a_w & b_w \\ c_w & d_w \end{pmatrix} \in GL(2, \mathbb{Z}),$$

i.e.

$$g_w(z_{-w}) = a_w z_w + c_w f_w$$
$$g_w(f_{-w}) = b_w z_w + d_w f_w.$$

We have $\det g_w = a_w d_w - b_w c_w = -1$ because M is orientable. Furthermore, $g_{-w} = g_w^{-1}$ and $b_{-w} = b_w$.

Invariants of a graph-manifold

The (nonoriented) *intersection index* $b_e = |b_w| = |b_{-w}|$ of an edge $e = (w, -w)$ is a topological invariant of the manifold M. We have $b_e \neq 0$, since the collection E is minimal.

The charge

$$k_v = \sum_{w\in\partial v} d_w / b_w$$

of a block M_v (of a vertex v of Γ) is also an invariant of the oriented manifold M. The charges change sign if the orientation of M is changed.

The charges k_v and the intersection indices b_w are the coefficients of the compatibility equation (0.1).

A.1. Example. For a graph-manifold M whose graph is the dipole there is only one gluing map

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}.$$

Thus the intersection index of M is equal to $|b|$, and the charges are $k_0 = d/b$, $k_1 = -a/b$.

A criterion of geometrizability

A geometric structure of type $H^2 \times \mathbb{R}$ on a block M_v is a metric for which the universal covering $\widetilde{M}_v$ is isometric to the metric product $A_v \times \mathbb{R}$, where A_v is a convex subset of the hyperbolic plane H^2 bounded by infinite geodesics. Therefore, the boundary ∂M_v of each geometrized block M_v is totally geodesic and every its component is a flat torus. Notice that a geometrized block is not necessarily isometric to the metric product of a surface F_v with a hyperbolic metric by a circle. It may happen that there is a nontrivial holonomy of the circle S^1 along some (noncontractible) loops in the base F_v. Moreover, such situation is typical and as exception a geometrized block has a metric splitting globally.

A graph-manifold M admits *a geometrization* if on every its block M_v a geometric structure of type $H^2 \times \mathbb{R}$ can be chosen in such a way that all gluing maps be isometries.

Each geometrization defines on M a C^1-smooth metric, which is real analytic inside of each block. That metric has nonpositive curvature (the sectional one inside of the blocks and in the sense of Alexandrov-Busemann everywhere). A graph-manifold M admits a geometrization iff on M there exists a C^∞-smooth Riemannian metric of nonpositive sectional curvature.

The labeled graph $\{\Gamma; B; K\}$ of a graph-manifold M contains an information about the geometrizability of M. Here $B = \{b_w = b_{-w} \in \mathbb{N} \mid w \in W\}$ is the collection of intersection indices of M, $K = \{k_v \in \mathbb{Q} \mid v \in V\}$ is the charge vector. To simplify the statement we restrict to the case when $K > 0$, i.e. $k_v > 0$ for all $v \in V$.

Let J_B be the quadratic form on $\mathbb{R}^V$ given by

$$(A.2) \qquad (J_B\mathcal{L}, \mathcal{L}) = \sum_{w \in W} \frac{1}{b_w} l_{\partial_- w} l_{\partial_+ w},$$

where $\mathcal{L} = \{l_v \mid v \in V\} \in \mathbb{R}^V$.

Let $\operatorname{diag} K$ denote the quadratic form on $\mathbb{R}^V$ given by

$$(A.3) \qquad (\operatorname{diag} K\mathcal{L}, \mathcal{L}) = \sum_{v \in V} k_v l_v^2.$$

A.4. Theorem. *A graph-manifold M with positive charge vector K admits no geometrization if and only if the form*

$$H_M = \operatorname{diag} K - J_B$$

is positive semidefinite.

In the general case the criterion is similar (see [BK2, Theorem 0.6]), only the definition of the form H_M is more complicated.

In other words, this result gives a necessary and sufficient condition for the compatibility equation (0.1) to possess an isometric state.

For instance, if $k_v = 0$ for all $v \in V$, then M admits a geometrization.

Coming back to the case when the graph of M is the dipole, we obtain $k_0k_1b^2 = -ad$, and the condition $0 \leq k_0k_1b^2 < 1$ of geometrizability is fulfilled only if $k_0k_1 = 0$, because $a, d \in \mathbb{Z}$. Then $k_0 = k_1 = 0$, i.e. $a = d = 0$ and $b = c = \pm 1$. Therefore, if such M admits a nonpositively curved metric, then the angle between the fibers $\omega = \pi/2$ for any such metric.

Toral sum and the decomposition principle

For graph-manifolds M', M'' their toral sum is defined as follows. Let $U' \subset M'_{v'}$ and $U'' \subset M''_{v''}$ be some saturated regular neighborhoods of fibers of blocks $M'_{v'} \subset M'$ and $M''_{v''} \subset M''$. Gluing the manifolds $M' \setminus U'$ and $M'' \setminus U''$ by the homeomorphism $T' \to T''$ of their boundaries induced by some fiber homeomorphism $U' \to U''$, we obtain a toral sum $M = M' + M''$, which is also a graph-manifold. The result may depend on the choice of blocks and, also, on the homotopy type of the gluing map. Assume that some orientations of the manifolds M', M'' are fixed. Then the choice of a Waldhausen basis (z', f') for M' is equivalent to the choice of a splitting $M'_{v'} = F'_{v'} \times S^1$ with oriented factors for every block of M'. Let (z'', f'') be a Waldhausen basis for M''. Then the toral sum $M = M' + M''$ along the vertices v', v'' that are identified to give the vertex v is defined by the condition $M_v = F_v \times S^1$, where F_v is the oriented connected sum $F'_{v'} \# F''_{v''}$ of the surfaces corresponding to the vertices v', v''. Furthermore, we have $k_v = k_{v'} + k_{v''}$ for the charges of corresponding blocks of the manifolds M, M', M''. Namely, it is this property on which the decomposition principle from § 5 is based. As an application of that principle we immediately obtain.

A.5. Example. (1) Assume that the graph Γ of a manifold M is a circle with odd number of edges and $k_v = 0$ for all $v \in V$. Then for any geometrization of M all the angles between the fibers of adjacent blocks are $\omega_w = \pi/2$.

This follows from the fact that the labeled graph Γ admits only the decomposition into dipoles with zero charges.

(2) Assume that the graph Γ of M is a circle with even number of edges and $k_v = 0$ for all $v \in V$. Then M admits geometrizations with some angles between the fibers of adjacent blocks ω_w arbitrary close to zero. This is true for all the angles, if all the intersection indices are equal to each other.

Appendix B

Here we derive the Lagrange-Euler equations for the action

$$S(\nabla, \xi) = \mathrm{YM}(\nabla) + E_\nabla(\xi) - m^2\|\xi\|^2.$$

The computations are done in an invariant form avoiding as much as possible a choice of coordinates. The fee is necessity to deal with $\mathcal{A}$-modules $\hat{\mathcal{E}}$, $\mathcal{E} \otimes \Omega^k_D(\mathcal{A}) \otimes \hat{\mathcal{E}}$.

We do not assume that the algebra $\mathcal{A}$ is commutative. We assume, however, that for the spectral triples $\{\mathcal{A}, \mathcal{H}, ds\}$ we consider the following condition

$$[ds^2, \pi(a)] = 0 \qquad (B.1)$$

holds for every $a \in \mathcal{A}$. Notice that this condition is fulfilled automatically for the spectral triples used above.

The conjugate module $\hat{\mathcal{E}}$ to $\mathcal{E}$

Let $\hat{\mathcal{E}}$ be the left $\mathcal{A}$-module of $\mathcal{A}$ linear homomorphisms $\mathcal{E} \to \mathcal{A}$, i.e. for $\zeta \in \hat{\mathcal{E}}$, $\eta \in \mathcal{E}$ and $a \in \mathcal{A}$ we have

$$\zeta(\eta a) = \zeta(\eta)a,$$

and the left action of the algebra $\mathcal{A}$ is defined by $(a\zeta)(\eta) = a\zeta(\eta)$.

There is a standard hermitian structure $(\eta, \xi) = \xi^*\eta$ on $\mathcal{E}$. Thus one can define the mutually inverse canonical isomophisms $\wedge : \mathcal{E} \to \hat{\mathcal{E}}$, $\vee : \hat{\mathcal{E}} \to \mathcal{E}$, where

$$(\wedge\xi)(\eta) = (\eta, \xi), \quad \vee\zeta = e\,(\zeta(e))^*,$$

for ξ, $\eta \in \mathcal{E}$, $\zeta \in \hat{\mathcal{E}}$. The definition of $\vee$ does not depend on the choice of a basis $e \in \mathcal{U}$, since for $e' = eu^*$ we have

$$e'\,(\zeta(e'))^* = eu^*\,(\zeta(e)u^*)^* = e\,(\zeta(e))^*.$$

It easy follows from definitions that $\vee \circ \wedge = \mathrm{id}_{\mathcal{E}}$, $\wedge \circ \vee = \mathrm{id}_{\hat{\mathcal{E}}}$. Furtermore, for bases $e \in \mathcal{E}$, $\wedge e \in \hat{\mathcal{E}}$ we have $\wedge e(e) = e^*e = 1$.

For a section $\zeta \in \hat{\mathcal{E}}$ its coordinate $b = b(\zeta, \wedge e) \in \mathcal{A}$ with respect to the basis $\wedge e$ is defined by the condition $\zeta = b \cdot \wedge e$. The gauge group $\mathcal{U}$ acts on coordinates of ζ as follows. If $e' = eu^*$, then $\wedge e' = u \cdot \wedge e$, and for $\zeta = b' \cdot \wedge e'$ we have $b' = bu^*$.

For a hermitiam structure Λ on $\mathcal{E}$ the corresponding hermitian Λ-structure on $\hat{\mathcal{E}}$ is defined by the condition that the isomorphism $\vee : \hat{\mathcal{E}} \to \mathcal{E}$ is an isometry of that structures,

$$(\zeta, \kappa)_\Lambda := (\vee\kappa, \vee\zeta)_\Lambda$$

for ζ, $\kappa \in \hat{\mathcal{E}}$. Therefore, $(\wedge\xi, \wedge\eta)_\Lambda = (\vee \circ \wedge\eta, \vee \circ \wedge\xi)_\Lambda = (\eta, \xi)_\Lambda$ for ξ, $\eta \in \mathcal{E}$. Whereas for a, $b \in \mathcal{A}$, ξ, $\eta \in \mathcal{E}$ holds

$$(\xi a, \eta b)_\Lambda = b^*(\xi, \eta)_\Lambda\, a,$$

for ζ, $\kappa \in \hat{\mathcal{E}}$ we have

$$(a\zeta, b\kappa)_\Lambda = a(\zeta, \kappa)_\Lambda\, b^*,$$

because $\vee(a\zeta) = \vee\zeta a^*$.

Furthermore, if $\xi = ea \in \mathcal{E}$, then the element $\wedge\xi \in \hat{\mathcal{E}}$ has coordinates $\wedge\xi = a^* \wedge e$.

Hermitian metric on $\Omega_D^k(\mathcal{A})$

For $k \geq 0$ and ω, $\omega' \in \Omega_D^k(\mathcal{A})$ we put

$$\langle \omega, \omega' \rangle = \mathrm{Tr}\,({\omega'}^*\omega).$$

It follows from the properties of the trace Tr that

$$\begin{aligned} \langle \omega', \omega \rangle &= \langle {\omega'}^*, \omega^* \rangle = \overline{\langle \omega, \omega' \rangle}; \\ \langle a\omega, \omega' \rangle &= \langle \omega, a^*\omega' \rangle; \\ \langle \omega b, \omega' \rangle &= \langle \omega, \omega' b^* \rangle \end{aligned}$$

for a, $b \in \mathcal{A}$.

Hermitiam metric on $\mathcal{E} \otimes_{\mathcal{A}} \Omega_D^k(\mathcal{A})$

We assume that a hermitian structure Λ on $\mathcal{E}$ is fixed. For $\xi \otimes \omega$, $\xi' \otimes \omega' \in \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^k(\mathcal{A})$ we put

$$\langle \xi \otimes \omega, \xi' \otimes \omega' \rangle = \langle \omega, (\xi, \xi')_\Lambda \omega' \rangle,$$

and extend it by linearity on $\mathcal{E}\otimes_{\mathcal{A}}\Omega^k_D(\mathcal{A})$. It easy follows from the properties of the hermitian metric on $\Omega^k_D(\mathcal{A})$ that

$$\langle \xi'\otimes\omega', \xi\otimes\omega\rangle = \overline{\langle \xi\otimes\omega, \xi'\otimes\omega'\rangle}.$$

Hermitian metric on $\mathcal{E}\otimes\Omega^k_D(\mathcal{A})\otimes\hat{\mathcal{E}}$

The space of the endomorphisms $\mathrm{End}_{\mathcal{A}}\left(\mathcal{E},\mathcal{E}\otimes_{\mathcal{A}}\Omega^k_D(\mathcal{A})\right)$ of right $\mathcal{A}$-modules $\mathcal{E}$ and $\mathcal{E}\otimes_{\mathcal{A}}\Omega^k_D(\mathcal{A})$ can be identified with vector space (over $\mathbb{C}$) $\mathcal{E}\otimes\Omega^k_D(\mathcal{A})\otimes\hat{\mathcal{E}}$ (the tensor products are over $\mathcal{A}$). For $\xi\otimes\omega\otimes\zeta\in\mathcal{E}\otimes\Omega^k_D(\mathcal{A})\otimes\hat{\mathcal{E}}$ and $\eta\in\mathcal{E}$ we have

$$(\xi\otimes\omega\otimes\zeta)(\eta) = \xi\otimes\omega\cdot\zeta(\eta).$$

We introduce a Λ-hermitian metric on $\mathcal{E}\otimes\Omega^k_D(\mathcal{A})\otimes\hat{\mathcal{E}}$ putting

$$\begin{aligned}\langle \xi\otimes\omega\otimes\zeta, \xi'\otimes\omega'\otimes\zeta'\rangle :&= \langle \xi\otimes\omega(\zeta,\zeta')_\Lambda, \xi'\otimes\omega'\rangle\\ &= \langle \omega(\zeta,\zeta')_\Lambda, (\xi,\xi')_\Lambda\omega'\rangle.\end{aligned}$$

In particular, for a basis $e\in\mathcal{U}$ we have

$$\langle e\otimes\omega\otimes\wedge e, e\otimes\omega'\otimes\wedge e\rangle = \langle \omega(\wedge e,\wedge e)_\Lambda, (e,e)_\Lambda\omega'\rangle = \langle \omega\lambda, \lambda\omega'\rangle.$$

B.2. Lemma. *Assume that the element $\hat{\xi}\in\hat{\mathcal{E}}$ is conjugated with $\xi\in\mathcal{E}$ with respect to the hermitian metrics on $\mathcal{E}\otimes_{\mathcal{A}}\Omega^k_D(\mathcal{A})$, $\mathcal{E}\otimes\Omega^k_D(\mathcal{A})\otimes\hat{\mathcal{E}}$, i.e.*

$$\langle \overline{\omega}, \gamma(\xi)\rangle = \langle \overline{\omega}\otimes\hat{\xi}, \gamma\rangle$$

for any $\overline{\omega}\in\mathcal{E}\otimes_{\mathcal{A}}\Omega^k_D(\mathcal{A})$, $\gamma\in\mathcal{E}\otimes\Omega^k_D(\mathcal{A})\otimes\hat{\mathcal{E}}$. Then for a basis $e\in\mathcal{U}$ we have

$$\hat{a} = a^*\lambda^{-1},$$

where $\xi = ea$, $\hat{\xi} = \hat{a}\cdot\wedge e$, $\lambda = \lambda(\Lambda, e)$.

Proof. One can assume that $\overline{\omega} = e\otimes\omega$, $\gamma = e\otimes\sigma\otimes\wedge e$. Then

$$\langle \overline{\omega}, \gamma(\xi)\rangle = \langle e\otimes\omega, e\otimes\sigma\cdot\wedge e(ea)\rangle = \langle \omega, (e,e)_\Lambda\sigma a\rangle = \langle \omega a^*, \lambda\sigma\rangle.$$

On the other hand,

$$\begin{aligned}\langle \overline{\omega}\otimes\hat{\xi}, \gamma\rangle &= \langle e\otimes\omega\otimes\hat{a}\cdot\wedge e, e\otimes\sigma\otimes\wedge e\rangle = \langle \omega(\hat{a}\cdot\wedge e,\wedge e)_\Lambda, \lambda\sigma\rangle\\ &= \langle \omega\hat{a}\lambda, \lambda\sigma\rangle.\end{aligned}$$

Thus $a^* = \hat{a}\lambda$. $\square$

Notice also that the expression $a^*\lambda^{-1}\wedge e$ does not depend on the choice of a basis: for $e' = eu^*$ we have $\wedge e' = u\cdot\wedge e$, $a' = ua$, ${a'}^* = a^*u^*$, $\lambda' = u\lambda u^*$, ${\lambda'}^{-1} = u\lambda^{-1}u^*$ and

$${a'}^*{\lambda'}^{-1}\wedge e' = a^*u^*u\lambda^{-1}u^*u\wedge e = u^*\lambda^{-1}\wedge e.$$

The Lagrange-Euler equations

We compute the variation of the action $S = S(\nabla, \xi)$ under a variation $\nabla \mapsto \nabla + \delta\nabla$ of a Λ-hermitian connection ∇ on $\mathcal{E}$ and a variation $\xi \mapsto \xi + \delta\xi$ of a section $\xi \in \mathcal{E}$, keeping only linear in $\delta\nabla$, $\delta\xi$ terms. We always assume that the connection $\nabla' = \nabla + \delta\nabla$ is Λ-hermitian. To state the result we need a number of definitions.

(1) For $\xi \in \mathcal{E}$ the element $\hat{\xi} \in \hat{\mathcal{E}}$ is defined by the condition

$$\langle \omega, \gamma(\xi) \rangle = \langle \omega \otimes \hat{\xi}, \gamma \rangle$$

for all $\omega \in \mathcal{E} \otimes_{\mathcal{A}} \Omega^k_D(\mathcal{A})$, $\gamma \in \mathcal{E} \otimes \Omega^k_D(\mathcal{A}) \otimes \hat{\mathcal{E}}$, see Lemma B.2.

(2) The connection $\hat{\nabla} : \hat{\mathcal{E}} \to \Omega^1_D(\mathcal{A}) \otimes \hat{\mathcal{E}}$ on $\hat{\mathcal{E}}$ is defined by ∇ by the condition $\hat{\nabla}\hat{e} = \Phi^* \otimes \hat{e}$, where $\nabla e = e \otimes \Phi$, $\hat{e} = \lambda^{-1} \cdot \wedge e$. It is easy to see that this definition does not depend on the choice of $e \in \mathcal{U}$.

(3) Now we define the current J as an element of the space $\mathcal{E} \otimes \Omega^1_D(\mathcal{A}) \otimes \hat{\mathcal{E}}$,

$$J = J(\nabla, \xi) := \nabla\xi \otimes \hat{\xi} - \xi \otimes \hat{\nabla}\hat{\xi}.$$

(4) Let $\nabla^* : \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A}) \to \mathcal{E}$ be the conjugate to ∇ operator, i.e. $\langle \nabla^*\omega, \eta \rangle = \langle \omega, \nabla\eta \rangle$ for every $\omega \in \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A})$, $\eta \in \mathcal{E}$. *The generalized Laplacian* is then the operator $\nabla^*\nabla : \mathcal{E} \to \mathcal{E}$.

(5) Let $d^* : \Omega^*_D(\mathcal{A}) \to \Omega^*_D(\mathcal{A})$ be the operator conjugated to the exterior differential $d : \Omega^*_D(\mathcal{A}) \to \Omega^*_D(\mathcal{A})$, $\langle d^*\sigma, \omega \rangle = \langle \sigma, d\omega \rangle$ for all σ, $\omega \in \Omega^*_D(\mathcal{A})$.

The operator $d^*_\nabla : \mathcal{E} \otimes \Omega^2_D(\mathcal{A}) \otimes \hat{\mathcal{E}} \to \mathcal{E} \otimes \Omega^1_D(\mathcal{A}) \otimes \hat{\mathcal{E}}$ is defined by $d^*_\nabla(e \otimes \sigma \otimes \hat{e}) = e \otimes (\omega - \omega^*) \otimes \hat{e}$, where

$$\omega = d^*\sigma + \sigma\Phi^* - \Phi\sigma.$$

It will be shown that this definition does not depend on the choice of $e \in \mathcal{U}$.

B.3. Theorem. *The Lagrange-Euler equations for the action $S = S(\nabla, \xi)$ have the form*

$$d^*_\nabla \nabla^2 = -J \qquad (B.3.1)$$

$$\nabla^*\nabla\xi = m^2\xi. \qquad (B.3.2)$$

The operator d^*

Before to start the proof of Theorem B.3, we first describe the operator d^* and show that the operator d^*_∇ is well defined.

B.4. Lemma. *For $\omega \in \Omega^1_D(\mathcal{A})$ we have*

$$d^*\omega = -i[D, \omega],$$

where recall $D = ds^{-1}$ is the Dirac operator. In particular, $(d^*\omega)^* = d^*\omega^*$.

Proof. Let $a \in \mathcal{A}$. Then

$$\begin{aligned} \langle d^*\omega, a \rangle &= \langle \omega, da \rangle = \mathrm{Tr}\ ((da)^*\omega) \\ &= \mathrm{Tr}\ (-i(a^*D - Da^*)\omega) \\ &= -\mathrm{Tr}\ (a^*i(D\omega - \omega D)) \\ &= \langle -i[D, \omega], a \rangle. \quad \square \end{aligned}$$

B.5. Lemma. *For $\omega \in \Omega^1_D(\mathcal{A})$, $\sigma \in \Omega^2_D(\mathcal{A})$ we have*

$$d\omega = i(D\omega + \omega D) \in \Omega^2_D(\mathcal{A}); \quad d^*\sigma = -i(D\sigma + \sigma D) \in \Omega^1_D(\mathcal{A}).$$

Proof. The second equality, obviously, follows from the first one. In turn, the first equality easily follows from definition (2.3) of the exterior differential and condition (B.1). $\square$

B.6. Corollary. *For $a \in \mathcal{A}$, $\sigma \in \Omega^2_D(\mathcal{A})$*

$$\begin{aligned} d^*(a\sigma) &= -da \cdot \sigma + ad^*\sigma; \\ d^*(\sigma a) &= d^*\sigma \cdot a + \sigma da. \quad \square \end{aligned}$$

Let us show now that the operator d^*_∇ is well defined. Choosing another basis $e' = eu^*$, we have $\Phi' = udu^* + u\Phi u^*$, $\lambda' = u\lambda u^*$, $\hat{e'} = u\hat{e}$, and for $\sigma' \in \Omega^2_D(\mathcal{A})$ with $e' \otimes \sigma' \otimes \hat{e'} = e \otimes \sigma \otimes \hat{e}$ we obtain $\sigma' = u\sigma u^*$. Now a straighforward calculation with Corollary B.6 and the relation $u^*u = uu^* = 1$ gives

$$\omega' = u\omega u^*$$

for $\omega = d^*\sigma + \sigma\Phi^* - \Phi\sigma$ and $\omega' = d^*\sigma' + \sigma'\Phi'^* - \Phi'\sigma'$. This gives the required equality

$$e' \otimes (\omega' - \omega'^*) \otimes \hat{e'} = e \otimes (\omega - \omega^*) \otimes \hat{e}.$$

The energy variation $\delta E_\nabla(\xi)$

An operator $\delta\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A})$ is $\mathcal{A}$-linear being the difference of two differentiations. Thus it can be considered as an element of the space $\mathcal{E} \otimes \Omega^1_D(\mathcal{A}) \otimes \hat{\mathcal{E}}$. Furthermore, for a basis $e \in \mathcal{U}$ we have $\delta\nabla = e \otimes \delta\Phi \otimes (\wedge e)$, where $\delta\nabla(e) = e \otimes \delta\Phi$.

B.7. Lemma. *For $\xi \in \mathcal{E}$ we have*

$$\langle \delta\nabla, \nabla\xi \otimes \hat{\xi} \rangle = -\langle \xi \otimes \hat{\nabla}\hat{\xi}, \delta\nabla \rangle.$$

Proof. Since the connections ∇ and $\nabla' = \nabla + \delta\nabla$ are Λ-hermitian, the equality $\lambda \cdot \delta\Phi + (\delta\Phi)^*\lambda = 0$ holds. Thus representing $\xi = ea$, we have $\nabla\xi = e \otimes (da + \Phi a)$, $\hat{\xi} = a^*\hat{e}$ (see Lemma B.2) and

$$\begin{aligned} \langle \delta\nabla, \nabla\xi \otimes \hat{\xi} \rangle &= \langle e \otimes \delta\Phi \otimes (\wedge e), e \otimes (da + \Phi a) \otimes a^*\hat{e} \rangle \\ &= \langle \delta\Phi, \lambda(da + \Phi a)a^* \rangle \\ &= \langle a(da + \Phi a)^*\lambda, (\delta\Phi)^* \rangle \\ &= \langle a(da + \Phi a)^*, (\delta\Phi)^*\lambda \rangle \\ &= -\langle a(da + \Phi a)^*, \lambda \cdot \delta\Phi \rangle \\ &= -\langle e \otimes a(da + \Phi a)^* \otimes \hat{e}, e \otimes \delta\Phi \otimes \wedge e \rangle \\ &= -\langle \xi \otimes \hat{\nabla}\hat{\xi}, \delta\nabla \rangle. \quad \square \end{aligned}$$

B.8. Lemma. *For $e \otimes \omega \in \Omega^1_D(\mathcal{A})$, $\xi = ea \in \mathcal{E}$ we have*

$$\nabla^*(e \otimes \omega) = e \cdot \lambda^{-1} \left(d^*(\lambda\omega) + \Phi^* \lambda\omega \right);$$
$$\nabla^* \nabla \xi = e \cdot \lambda^{-1} \left(d^* \left(\lambda(da + \Phi a) \right) + \Phi^* \lambda(da + \Phi a) \right),$$

where $\nabla e = e \otimes \Phi$.

Proof. The second equality follows from the first one, because $\nabla \xi = e \otimes (da + \Phi a)$. To prove the first equality, we take $\eta \in \mathcal{E}$, $\eta = e \cdot b$ and consider

$$\begin{aligned} \langle \nabla^*(e \otimes \omega), \eta \rangle &= \langle e \otimes \omega, \nabla \eta \rangle = \langle e \otimes \omega, e \otimes (db + \Phi b) \rangle \\ &= \langle \omega, \lambda(db + \Phi b) \rangle = \langle \lambda\omega, db \rangle + \langle \lambda\omega, \Phi b \rangle \\ &= \langle d^*(\lambda\omega) + \Phi^* \lambda\omega, b \rangle \\ &= \langle \lambda^{-1} \left(d^*(\lambda\omega) + \Phi^* \lambda\omega \right), (e, e)_\Lambda b \rangle \\ &= \langle e \cdot \lambda^{-1} \left(d^*(\lambda\omega) + \Phi^* \lambda\omega \right), \eta \rangle. \quad \square \end{aligned}$$

For a Λ-hermitian connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A})$ we define two, in general, different connections $\hat{\nabla}, \overline{\nabla} : \hat{\mathcal{E}} \to \Omega^1_D(\mathcal{A}) \otimes \hat{\mathcal{E}}$ on $\hat{\mathcal{E}}$, putting for a basis $e \in \mathcal{U}$

$$\hat{\nabla} \hat{e} = \Phi^* \otimes \hat{e} \quad \text{and} \quad \overline{\nabla}(\wedge e) = \Phi^* \otimes \wedge e,$$

where $\nabla e = e \otimes \Phi$, $\hat{e} = \lambda^{-1} \cdot \wedge e$. Since ∇ is Λ-hermitian, it follows easily that $(\hat{\nabla} - \overline{\nabla})(\wedge e) = \lambda(\Phi + \Phi^*)\lambda^{-1} \otimes \wedge e$. Thus for an unitary connection ∇, i.e. $\Phi + \Phi^* = 0$, we have $\overline{\nabla} = \hat{\nabla}$.

Given $\xi = e \cdot a \in \mathcal{E}$, one has $\wedge \xi = a^*(\wedge e)$. Hence $\overline{\nabla}(\wedge \xi) = (da^* + a^* \Phi^*) \otimes \wedge e$. From this, as in Lemma B.8, we obtain (leaving details to the reader)

B.9. Lemma. *For $\omega \otimes \wedge e \in \Omega^1_D(\mathcal{A}) \otimes \hat{\mathcal{E}}$ and $\wedge \xi = a^*(\wedge e) \in \hat{\mathcal{E}}$ we have*

$$\overline{\nabla}^*(\omega \otimes \wedge e) = \left(d^*(\omega\lambda) + \omega\lambda\Phi \right) \hat{e};$$
$$\overline{\nabla}^* \overline{\nabla}(\wedge \xi) = \left(d^* \left((da^* + a^* \Phi^*)\lambda \right) + (da^* + a^* \Phi^*)\lambda\Phi \right) \hat{e}. \quad \square$$

Using Lemmas B.8 and B.9, a straightforward computation gives

B.10. Corollary. *For ξ, $\delta\xi \in \mathcal{E}$ we have*

$$\langle \delta\xi, \nabla^* \nabla \xi \rangle = \langle \overline{\nabla}^* \overline{\nabla}(\wedge \xi), \wedge \delta\xi \rangle. \quad \square$$

Let $\delta E_\nabla(\xi)$ be the linear in $\delta\nabla$, $\delta\xi$ part of the difference $E_{\nabla + \delta\nabla}(\xi + \delta\xi) - E_\nabla(\xi)$.

B.11. Lemma. *The enery variation has the form*

$$\delta E_\nabla(\xi) = \langle J, \delta\nabla \rangle + \langle \nabla^* \nabla \xi, \delta\xi \rangle + \langle \overline{\nabla}^* \overline{\nabla}(\wedge \xi), \wedge \delta\xi \rangle,$$

where $J = \nabla \xi \otimes \hat{\xi} - \xi \otimes \hat{\nabla} \hat{\xi}$.

Proof. We have

$$E_{\nabla + \delta\nabla}(\xi + \delta\xi) - E_\nabla(\xi) = \langle (\nabla + \delta\nabla)(\xi + \delta\xi), (\nabla + \delta\nabla)(\xi + \delta\xi) \rangle - \langle \nabla \xi, \nabla \xi \rangle.$$

Thus using Lemma B.7 and Corollary B.10, we obtain

$$\begin{aligned}\delta E_\nabla(\xi) &= \langle \nabla\xi, \nabla(\delta\xi)\rangle + \langle \nabla(\delta\xi), \nabla\xi\rangle + \langle \nabla\xi, \delta\nabla(\xi)\rangle + \langle \delta\nabla(\xi), \nabla\xi\rangle \\ &= \langle \nabla^*\nabla\xi, \delta\xi\rangle + \langle \delta\xi, \nabla^*\nabla\xi\rangle + \langle \nabla\xi\otimes\hat{\xi}, \delta\nabla\rangle + \langle \delta\nabla, \nabla\xi\otimes\hat{\xi}\rangle \\ &= \langle J, \delta\nabla\rangle + \langle \nabla^*\nabla\xi, \delta\xi\rangle + \langle \overline{\nabla}^*\overline{\nabla}(\wedge\xi), \wedge\delta\xi\rangle. \quad \square\end{aligned}$$

The variation of the Yang-Mills action YM(∇)

A straightforward computation shows that the operator $T := \nabla\circ\delta\nabla + \delta\nabla\circ\nabla : \mathcal{E} \to \mathcal{E}\otimes_{\mathcal{A}}\Omega^2_D(\mathcal{A})$ is $\mathcal{A}$-linear, thus it can be considered as an element of the space $\mathcal{E}\otimes\Omega^2_D(\mathcal{A})\otimes\hat{\mathcal{E}}$. Furthermore,

$$T = e\otimes(d(\delta\Phi) + \Phi\cdot\delta\Phi + \delta\Phi\cdot\Phi)\otimes\wedge e,$$

where $\nabla e = e\otimes\Phi$, $\delta\nabla(e) = e\otimes\delta\Phi$ for Φ, $\delta\Phi\in\Omega^1_D(\mathcal{A})$.

B.12. Lemma. *Let $\theta\in\Omega^2_D(\mathcal{A})$ be the curvature operator of the connection ∇, $\nabla^2 e = e\otimes\theta$. Then*

$$\langle \nabla^2, T\rangle = \langle e\otimes\omega\otimes\hat{e}, \delta\nabla\rangle,$$

where $\omega = d^*(\theta\lambda) + \theta\lambda\Phi^* - \Phi\theta\lambda$, $\nabla e = e\otimes\Phi$.

Proof. Representing the curvature ∇^2 as $\nabla^2 = e\otimes\theta\otimes\wedge e$, we obtain

$$\langle \nabla^2, T\rangle = \langle \theta\lambda, \lambda(d(\delta\Phi) + \Phi\cdot\delta\Phi + \delta\Phi\cdot\Phi)\rangle.$$

Since the connection ∇ is Λ-hermitian, we have

$$\lambda\cdot d(\delta\Phi) = d(\lambda\cdot\delta\Phi) - (\lambda\Phi + \Phi^*\lambda)\cdot\delta\Phi.$$

Thus

$$\begin{aligned}\langle \nabla^2, T\rangle &= \langle d^*(\theta\lambda) - \lambda^{-1}(\lambda\Phi+\Phi^*\lambda)^*\theta\lambda, \lambda\cdot\delta\Phi\rangle \\ &\quad + \langle \Phi^*\lambda\theta\lambda, \delta\Phi\rangle + \langle \theta\lambda\Phi^*, \lambda\cdot\delta\Phi\rangle \\ &= \langle d^*(\theta\lambda) - \lambda^{-1}(\lambda\Phi+\Phi^*\lambda)\theta\lambda + \lambda^{-1}\Phi^*\lambda\theta\lambda + \theta\lambda\Phi^*, \lambda\cdot\delta\Phi\rangle \\ &= \langle \omega, \lambda\cdot\delta\Phi\rangle = \langle e\otimes\omega\otimes\hat{e}, \delta\nabla\rangle. \quad \square\end{aligned}$$

Using as in the proof of Lemma B.7 the equality $\lambda\cdot\delta\Phi + (\delta\Phi)^*\lambda = 0$, a similar computation gives (we leave details to the reader)

B.13. Lemma. *Under the conditions of Lemma B.12 we have*

$$\langle T, \nabla^2\rangle = -\langle e\otimes\omega^*\otimes\hat{e}, \delta\nabla\rangle. \quad \square$$

Let δYM(∇) be the linear in $\delta\nabla$ part of the difference YM$(\nabla+\delta\nabla)$ − YM(∇).

B.14. Lemma. *For the variation of the Yang-Mills action we have*

$$\delta YM(\nabla) = \langle d^*_\nabla\nabla^2, \delta\nabla\rangle.$$

Proof. Since $\nabla^2 = e\otimes\theta\otimes\wedge e = e\otimes\theta\lambda\otimes\hat{e}$, applying Lemmas B.12, B.13, we obtain

$$\begin{aligned}\delta\mathrm{YM}(\nabla) &= \langle \nabla^2, T\rangle + \langle T, \nabla^2\rangle \\ &= \langle e\otimes(\omega-\omega^*)\otimes\hat{e}, \delta\nabla\rangle \\ &= \langle d^*_\nabla\nabla^2, \delta\nabla\rangle,\end{aligned}$$

where $\omega = d^*(\theta\lambda) + \theta\lambda\Phi^* - \Phi\theta\lambda$. $\square$

B.15. Corollary. *For the variation of the action S we have*

$$\begin{aligned}\delta S(\nabla,\xi) &:= \delta YM(\nabla) + \delta E_\nabla(\xi) - m^2\delta\|\xi\|^2\\ &= \langle d^*_\nabla \nabla^2 + J, \delta\nabla\rangle\\ &+ \langle \nabla^*\nabla\xi - m^2\xi, \delta\xi\rangle\\ &+ \langle \overline{\nabla}^*\overline{\nabla}\wedge\xi - m^2\wedge\xi, \wedge\delta\xi\rangle.\end{aligned}$$

Proof. This follows from Lemmas B.11, B.14 and that

$$\delta\|\xi\|^2 = \langle\xi,\delta\xi\rangle + \langle\delta\xi,\xi\rangle = \langle\xi,\delta\xi\rangle + \langle\wedge\xi,\wedge\delta\xi\rangle. \quad \square$$

Proof of Theorem B.3. Since for $\delta\xi = e\cdot\delta a$, $\wedge\delta\xi = (\delta a)^*\wedge e$ we have $\delta a = \mathrm{Re}\,(\delta a) + i\cdot\mathrm{Im}\,(\delta a)$, $(\delta a)^* = \mathrm{Re}\,(\delta a) - i\cdot\mathrm{Im}\,(\delta a)$, one can assume that the variations $\delta\xi$, $\wedge\delta\xi$ are independent. Thus the Lagrange-Euler equations for the action S have the form

$$\begin{aligned} d^*_\nabla\nabla^2 &= -J;\\ \nabla^*\nabla\xi &= m^2\xi;\\ \overline{\nabla}^*\overline{\nabla}\wedge\xi &= m^2\wedge\xi.\end{aligned}$$

Using that for $\omega\in\Omega^1_D(\mathcal{A})$ one has $(d^*\omega)^* = d^*\omega^*$ (see Lemma B.4), we obtain from Lemmas B.8 and B.9 that the third equation is implied by the second one. $\square$

Choosing a basis $e\in\mathcal{U}$ and representing $\nabla e = e\otimes\Phi$, $\xi = ea$, $\lambda = \lambda(\Lambda, e)$, $\hat{\xi} = a^*\hat{e}$, we obtain

$$\begin{aligned} J &= \nabla\xi\otimes\hat{\xi} - \xi\otimes\hat{\nabla}\hat{\xi}\\ &= e\otimes((da+\Phi a)a^* - a(da+\Phi a)^*)\otimes\hat{e}.\end{aligned}$$

Hence, it follows from Lemmas B.8, B.14 that the Lagrange-Euler equations (B.3.1), (B.3.2) in the coordinates have the form

$$(B.16.1)\qquad \omega - \omega^* = a(da+\Phi a)^* - (da+\Phi a)a^*$$

$$(B.16.2)\qquad d^*\left(\lambda(da+\Phi a)\right) + \Phi^*\lambda(da+\Phi a) = m^2\lambda a,$$

where $\omega = d^*(\theta\lambda) + \theta\lambda\Phi^* - \Phi\theta\lambda$.

For the dipole case we let $p = (1,0)$, $q = 1-p\in\mathcal{A}$ be projectors. Then $a = \xi_0 p + \xi_1 q$, $\lambda = \lambda_0 p + \lambda_1 q$, $\Phi = -i\Delta s^2(\phi_{01}pdp + \phi_{10}qdp)$, $\theta = i(\phi_{10} - \phi_{01} + i\phi_{10}\phi_{01}\Delta s^2)\cdot 1$, where ξ_0, ξ_1, ϕ_{01}, $\phi_{10}\in\mathbb{C}$, λ_0, $\lambda_1 > 0$. Straightforward calculations with using the Λ-hermitian condition (3.2.3) and Lemmas B.4, B.5 for finding of the exterior differential d^* show that the equation (B.16.1) takes the form (3.2.1) and (B.16.2) the form (3.2.2). One should use on some step that the operator θ is selfadjoint due to condition (3.2.3), $\theta^* = \theta$.

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