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ALGEBRAS IN PLANCHEREL DUALITY AND C-ALGEBRAS

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ABSTRACT

In this paper we discuss algebras in Plancherel duality, i.e. a special class of pairs of semisimple finite-dimensional algebras with involution being in a nondegenerate duality as vector spaces. This class arised more than twenty years ago as a generalization of the Krein-Tanaka duality and Hopf algebras. We present new axiomatics of algebras in Plancherel duality according to the properties of the corresponding pairing. It is proved that the pairing is a Plancherel one iff it is positive, homogeneous and isometric.

It turns out that the above class provides a natural framework for the algebraic approach to combinatorics connected with the notion of C-algebra. For an arbitrary C-algebra (possibly non-commutative) a positivity condition generalizing the Krein condition in commutative case, is defined. We show that the class of positive C-algebras includes those arising in algebraic combinatorics from association schemes (possibly non-commutative). It is proved that the category of positive C-algebras is equivalent to the category of pairs of algebras in Plancherel duality one of which being commutative.

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1 Introduction

In this paper the connection between C-algebras representing one of the main concepts of algebraic combinatorics and an important class of bialgebras, namely algebras in Plancherel duality is established. Besides we present new axiomatics of this class.

The association schemes investigated in algebraic combinatorics, result in matrix algebras having a linear basis with special properties. This basis actually determines a commutative multiplication (more correctly to say comultiplication) and consequently a bialgebra which is a vector space with multiplication and comultiplication consistent to each other. A natural question arises: how to describe this structure by using abstract algebraic terms? The advantage of such a description consists in the opportunity of constructing (and subsequently applying) the general representation theory. The natural category arising in this way can include objects which have not the matrix representations mentioned above and then the problem of the existence of combinatorial objects (such as association schemes or block designs) transforms into the problem of searching special representations of appropriate algebraic systems.

The attempts of such constructions have resulted in the concept of C-algebra (see [1] and section 3 of this paper). However, as it is found out in the paper, this concept is not quite adequate in algebraic combinatorics: the axioms of C-algebra do not include the natural condition of *positivity* which all combinatorial algebras have. It should be noted that while for commutative C-algebras this additional condition (the positivity of the structure constants and the Krein constants) is easily written (see [1]), in general case the situation is more complicated.

In 1972 one of the authors (see [7]), generalizing the duality theory of Krein - Tanaka, Hopf and others, proposed a new, weaker than in the definition of Hopf algebras, coordination of two multiplications in the dual spaces or, in other language multiplication and comultiplication, namely, the concept of *algebras in positive duality.* S. V. Kerov in [5] studied this concept and, in particular, its finite-dimensional case (which is basically necessary for combinatorics). The analogue of the duality theory for groups with invariant measure led naturally to the concept of algebras in Plancherel duality or Plancherel triples discussed in detail in [5] (see section 2). Just this concept has proved to be more adequate to the objects arising in algebraic combinatorics. There are many examples of such triples arising from groups and algebras.

In this paper we propose a more natural approach to the Plancherel duality concept. It is based on analyzing the properties of bilinear forms setting two given semisimple $*$-algebras into nondegenerate duality. It turns out that the pairing is a Plancherel one iff it is positive, homogeneous and isometric (see section 2). The isometry property is in fact equivalent to holding some orthogonality relations in the paired algebras, whereas the homogeneity means that the unity element of each of the algebras determines a one-dimensional $*$-representation of the other. We intend to define and study a nonhomogeneous Plancherel duality (where the homogeneity condition is replaced by a weaker one) separately. A special case of this situation is given by general cellular algebras (non necessarily homogeneous). The main result of the paper consists of the following statement.

Theorem 1. *The categories of positive C-algebras and pairs of algebras in Plancherel duality, one of which (for instance, the second) being commutative, are equivalent.*

In this connection we notice that the axiomatics of algebras in Plancherel duality does not require the commutativity of one of the algebras. There are a few interesting examples of twice non-commutative case such as finite quantum groups, and the study of this case seems to be rather attractive: it can be viewed as the quantum deformation of classical combinatorics.

One of the main problems of algebraic combinatorics consists in the characterization of C-algebras arising from association schemes. The positivity hypothesis in theorem 1 gives a necessary condition of the existence. In fact, the adjacency algebra of a association scheme is a special case of a homogeneous cellular algebra (see section 3). We associate to such an algebra $A \subset \mathrm{Mat}_n(\mathbf{C})$ the triple $(A, B, \langle , \rangle)$ where B is the $*$-algebra coinciding with A as a linear space and equipped with coordinate-wise multiplication with respect to the standard basis of A and complex conjugation as involution. The bilinear form $\langle , \rangle$ is defined by the formula $\langle a, b\rangle = \frac{1}{n}\,\mathrm{Tr}(ab^T)$. The proof of the following statement immediately follows from theorem 1 and proposition 3.6 below.

Theorem 2. *The triple $(A, B, \langle , \rangle)$ is a Plancherel one for any homogeneous cellular algebra A.*

One more application of discussed objects concerns multivalued involutory groups (see [2]).

A preliminary version of this paper was published in [3].

2 Algebras in Plancherel duality

2.1. We will consider triples $(A, B, \langle , \rangle)$ where A, B are semisimple finite-dimensional $*$-algebras over $\mathbf{C}$ with units 1_A and 1_B and

$$\langle , \rangle : A \times B \to \mathbf{C} \tag{1}$$

is a bilinear form setting them into a nondegenerate duality as vector spaces over $\mathbf{C}$. We also assume that

$$\langle 1_A, 1_B\rangle = 1. \tag{2}$$

This assumption is not restrictive, since it is easy to see that under the hypothesis of subsection 2.3 we have $\langle 1_A, 1_B\rangle \neq 0$ (after taking into account that of 2.2 even $\langle 1_A, 1_B\rangle > 0$). Under an isomorphism of triples we mean a pair of isomorphisms between the corresponding algebras preserving the bilinear forms. The elements of the algebras A and B will be denoted by $x, y, z, \ldots$ and $f, g, h, \ldots$ respectively.

Let $\widehat{A}$ and $\widehat{B}$ be the dual spaces for A and B respectively. Then the pairing (1) being a non-degenerate one, induces linear isomorphisms from $\widehat{B}$ to A and from $\widehat{A}$ to B. The image of a functional χ in both cases is denoted by $\widehat{\chi}$.

Using the duality between the algebras A and B one can transfer structures from A to B and vice versa. For example, one can consider coinvolutions η_A, η_B in A and B as operations conjugated to the involutions in B and A:

$$\langle \eta_A(x), f\rangle = \overline{\langle x, f^*\rangle}, \qquad \langle x, \eta_B(f)\rangle = \overline{\langle x^*, f\rangle}, \quad x \in A,\ f \in B. \tag{3}$$

It is easy to see, that η_A and η_B are semilinear 1-1 maps.

Below we define algebras in Plancherel duality (see [5]). Our approach based on studying the properties of the pairing differs mainly in language. The correspondence between the languages will be discussed in subsection 2.6.

2.2. Positive pairing. The pairing (1) is called *positive* if the cones

$$K_A = \{x \in A : \ \langle x, f\rangle \geq 0\ \forall f \in B^+\},$$

$$K_B = \{f \in B : \ \langle x, f\rangle \geq 0\ \forall x \in A^+\}$$

are closed under the multiplications and the involutions in the algebras A and B respectively. Here A^+ (resp. B^+) is the cone in A (resp. B) spanned by the elements of the form xx^*, $x \in A$ (resp. ff^*, $f \in B$). In this case the semilinear maps η_A and η_B defined by (3) are $*$-isomorphisms of the algebras A and B respectively (see [5]).

If one considers the multiplications as linear maps of algebras

$$\triangle_A : A \otimes A \to A, \quad x \otimes y \mapsto xy,$$

$$\triangle_B : B \otimes B \to A, \quad f \otimes g \mapsto fg,$$

then after going over to conjugate maps we obtain the maps

$$\nabla_B : B \to B \otimes B, \qquad \nabla_A : A \to A \otimes A,$$

so-called comultiplications. In our case (of positive pairing) ∇_A and ∇_B are positive in the sense that

$$\nabla_A A^+ \subset A^+ \otimes A^+, \qquad \nabla_B B^+ \subset B^+ \otimes B^+.$$

In the theory of Hopf's algebras and bialgebras one of the comultiplications (and then automatically the second one) is by definition a $*$-homomorphism of algebras (see [6]). Since a $*$-homomorphism is a positive map, the axiomatics of positive triples is a natural weakening of the axiomatics of Hopf's algebras. This topic will be considered in detail separately.

2.3. Homogeneous pairing. The pairing (1) is called *homogeneous* if the sets

$$I_A = \{x \in A : \langle x, 1_B \rangle = 0\},$$

$$I_B = \{f \in B : \langle 1_A, f \rangle = 0\}$$

are two-sided ideals in the algebras A and B respectively. In this case (see (2)) the maps $x \mapsto \langle x, 1_B \rangle$ and $f \mapsto \langle 1_A, f \rangle$ are $*$-homomorphisms from the algebras A and B to $\mathbf{C}$. Being two-sided ideals of codimension one, I_A and I_B define some primitive central idempotents $m_A \in A$ and $m_B \in B$ for which

$$x m_A = m_A x = \langle x, 1_B \rangle m_A, \quad x \in A, \tag{4}$$

$$f m_B = m_B f = \langle 1_A, f \rangle m_B, \quad f \in B \tag{5}$$

and in particular $\langle m_A, 1_B \rangle = \langle 1_A, m_B \rangle = 1$. In terms of the maps ∇_A and ∇_B the homogeneity is equivalent to the equalities

$$\nabla_A(1_A) = 1_A \otimes 1_A, \qquad \nabla_B(1_B) = 1_B \otimes 1_B.$$

Formulas (4) and (5) show that the elements m_A and m_B can be viewed as a kind of normalized invariant measures in A and B respectively. For example, let $A = \mathbf{C}[G]$ be the group algebra of a finite group G and let $B = C(G)$ be the algebra of complex functions on G with pointwise multiplication. Let us set a duality by the formula

$$\langle x, f \rangle = \sum_{g \in G} x(g) f(g), \quad x \in A, \ f \in B.$$

In this case

$$m_A = \frac{1}{\#G} \sum_{g \in G} g, \qquad m_B = \frac{1}{\#G} \sum_{\pi \in G^\wedge} (\dim \pi) \chi_\pi$$

are the Haar measure on G and the Plancherel measure on $G^\wedge$ respectively where χ_π is the character of an irreducible complex representation π of the group G and $\dim \pi$ is its dimension.

2.4. A linear functional $\chi : A \to \mathbf{C}$ is called a *trace* (feasible trace in [4]) on A if $\chi(xy) = \chi(yx)$ for all $x, y \in A$. In this case

$$\chi = \sum_{\pi} m(\pi, \chi)\chi_{\pi}, \quad \chi_{\pi}(x) = \mathrm{Tr}(\pi(x)) \tag{6}$$

where $m(\pi, \chi) \in \mathbf{C}$ and π runs over a full system of pairwise nonequivalent irreducible representations of the algebra A. Such a χ defines a sesquilinear form

$$(x, y)_{\chi} = \chi(xy^*), \quad x, y \in A \tag{7}$$

such that $(uxv, uyv)_{\chi} = (x, y)$ for all $u, v \in A$ with $uu^* = vv^* = 1_A$. Conversely, any sesquilinear form having this property coincides with $(\,,\,)_{\chi}$ for a uniquely determined trace χ. We note that the form (7) is conjugate symmetric iff $m(\pi, \chi)$ is a real number for all π (equivalently, $\chi(x^*) = \overline{\chi(x)}$ for all x) and positive definite iff $m(\pi, \chi) > 0$ for all π (equivalently, $\chi(xx^*) > 0$ for all $x \neq 0$). In the latter case we say that χ is *positive.*

Let now χ_A and χ_B be positive traces on the algebras A and B paired by (1). Then A and B can be viewed as Hilbert spaces with scalar products $(\,,\,)_A = (\,,\,)_{\chi_A}$ and $(\,,\,)_B = (\,,\,)_{\chi_B}$. Since pairing (1) is nondegenerate one, there are Fourier transforms $F_A : A \to B$ and $F_B : B \to A$ defined by

$$\langle y, F_A(x)\rangle = (x, y^*)_A, \quad y \in A,$$
$$\langle F_B(f), g\rangle = (f, g^*)_B, \quad g \in B.$$

It follows from the definition that

$$F_A(1_A) = \widehat{\chi}_A, \quad F_B(1_B) = \widehat{\chi}_B \tag{8}$$

where the elements $\widehat{\chi}_A \in B$ and $\widehat{\chi}_B \in A$ correspond to χ_A and χ_B under identifying $\widehat{A}$ with B and $\widehat{B}$ with A (see subsection 2.1).

2.5. Isometric pairing. We say that the pairing (1) is *isometric* with respect to positive traces χ_A and χ_B if the maps F_A and F_B are multiples of isometries between the Hilbert spaces $(A, (\,,\,)_A)$ and $(B, (\,,\,)_B)$. The pairing is called isometric if it is isometric with respect to some positive traces.

A straightforward computation shows that the isometry property with respect to χ_A and χ_B implies that

$$F_B F_A = M(\eta_A^*)^{-1}, \quad F_A F_B = M(\eta_B^*)^{-1} \tag{9}$$

where $\eta_A^*(x) = \eta_A(x)^*$, $\eta_B^*(f) = \eta_B(f)^*$ (see (3)) and

$$M = \|F_A\|^2 = \|F_B\|^2 = \langle \widehat{\chi}_B \eta_A^*(1_A), \widehat{\chi}_A\rangle = \langle \widehat{\chi}_B, \widehat{\chi}_A \eta_B^*(1_B)\rangle \tag{10}$$

($\|\ \|$ denotes the operator norm). Besides it follows from (9) and (8) that η_A and η_B are isometries with

$$F_A(\widehat{\chi}_B) = M\eta_B(1_B), \quad F_B(\widehat{\chi}_A) = M\eta_A(1_A). \tag{11}$$

Let $\pi : B \to \mathrm{Mat}_{n_\pi}(\mathbf{C})$ be an irreducible $*$-representation of the algebra B. Denote by f_{ij}^{π} the element of B corresponding to a matrix unit e_{ij} of the algebra $\mathrm{Mat}_{n_\pi}(\mathbf{C})$, $i, j = 1, \dots, n_\pi$. The linear base $\{f_{ij}^{\pi}\}_{i,j,\pi}$ of the algebra B defines the dual base in $\widehat{B}$ and so a linear base $\{x_{ij}^{\pi}\}_{i,j,\pi}$ of A. It follows from the definitions of subsection 2.4 that

$$(f_{i,j}^{\pi}, f_{k,l}^{\rho})_B = \delta_{\pi,\rho} m(\pi, \chi_B)\,\mathrm{Tr}(f_{i,j}^{\pi} f_{k,l}^{\rho}) = \delta_{\pi,\rho}\delta_{i,k}\delta_{j,l} m(\pi, \chi_B)$$

and

$$F_B(f^{\pi}_{i,j}) = m(\pi,\chi_B)x^{\pi}_{j,i}.$$

So the formula $(F_B(f), F_B(g))_A = \|F_B\|^2(f,g)_B,\ f,g\in B$ is equivalent to the following orthogonality relations in the algebra A (see (10)):

$$(x^{\pi}_{ij}, x^{\rho}_{kl})_A = \delta_{\pi,\rho}\delta_{i,k}\delta_{j,l}\frac{M}{m(\pi,\chi_B)}.$$

Similar orthogonality relations are easily written also for the algebra B.

2.6. Plancherel duality. Now we collect the properties of the pairing (1) described in subsections 2.2, 2.3 and 2.5.

Definition 2.1 *We say that two semisimple $*$-algebras A and B paired by (1) are in (homogeneous) Plancherel duality, if the pairing is positive, homogeneous and isometric. In this case the pairing $\langle,\rangle$ and the triple $(A,B,\langle,\rangle)$ are called Plancherel ones.*

Let us consider a Plancherel triple $(A,B,\langle,\rangle)$. Then there exist traces χ_A and χ_B with respect to which the pairing is isometric. Without loss of generality we assume that

$$\chi_A(1_A) = \chi_B(1_B) = 1. \tag{12}$$

The following statement describes the relationship between the main invariants of the triple and in fact establishes the uniqueness of such χ_A and χ_B.

Lemma 2.2 *The following statements hold:*

(1) $m_A = \widehat{\chi}_B,\ m_B = \widehat{\chi}_A$.

(2) $\|F_A\|^2 = \|F_B\|^2 = \langle m_A, m_B\rangle$, *in particular* $\langle m_A, m_B\rangle > 0$.

(3) $F_A(1_A) = m_B,\ F_B(1_B) = m_A$ *and* $F_A(m_A) = M1_B,\ F_B(m_B) = M1_A$ *where* $M = \langle m_A, m_B\rangle$.

Proof. Since $\langle m_A, 1_B\rangle = 1 = \chi_B(1_B)$ (see subsection 2.3 and (12)), we conclude that $m(\chi_B) = m_A + x$ for some $x\in I_A$. By (9) and the fact that $\eta_B(1_B) = 1_B$ (see subsection 2.2) we have

$$M\langle x^*, 1_B\rangle = \langle x^* F_A F_B(\eta^*_B(1_B))\rangle = \langle x^* F_A F_B(1_B)\rangle.$$

Further by (8) and the definitions of F_A and m_A

$$\langle x^*, F_A F_B(1_B)\rangle = \langle x^*, F_A(\widehat{\chi}_B)\rangle = (\widehat{\chi}_B, x)_A = (m_A + x, x)_A = (x,x)_A.$$

Since $x^*\in I_A$, we see that $(x,x)_A = 0$, whence $x=0$. So $\widehat{\chi}_B = m_A$. Similarly, $\widehat{\chi}_A = m_B$, which proves the first statement. This together with (10) implies the second statement. Finally, the last statement follows from the first one and equalities (8), (11).■

If algebras A and B are in Plancherel duality, then the natural identifying the spaces A, B, $\widehat{A}$, $\widehat{B}$ produces a Hilbert space with two $*$-algebra structures. Moreover the corresponding scalar product is invariant with respect to the left and the right actions of both unitar groups induced by these structures. The unity with respect to one of these structures is a multiple of a primitive central idempotent with respect to the other. The simplex of characters in one of the algebras is closed under the multiplication and the involution in the other. Such a scheme enables us to see the connections with algebraic combinatorics which will be considered in section 3.

2.7. Let us compare the notions considered above and the notions of algebras in positive and Plancherel duality introduced and studied in [7, 5].

Algebras A and B are in a (homogeneous) *positive duality* with respect to $\langle , \rangle$ ($(A, B, \langle , \rangle)$ is a (homogeneous) positive triple), if the sets

$$P_A = \{x \in A : \ \langle x, 1_B \rangle = 1, \quad \langle x, f \rangle \geq 0 \ \forall f \in B^+\}$$

$$P_B = \{f \in B : \ \langle 1_A, f \rangle = 1, \quad \langle x, f \rangle \geq 0 \ \forall x \in A^+\}$$

are closed under the multiplications and the involutions in the algebras A and B respectively. It is easy to see that such a situation arises if the pairing is positive and homogeneous. Conversely, if $(A, B, \langle , \rangle)$ is a positive triple, then the pairing $\langle , \rangle$ is obviously positive. Moreover, if $x, y \in I_A$ with $1_A + x, 1_A + y \in A^+$, then

$$(1_A + x)(1_A + y) = 1_A + x + y + xy \in P_A.$$

This shows that $xy, (1_A + x)y, x(1_A + y) \in I_A$, so I_A is a two-sided ideal of A. Similarly, I_B is a two-sided ideal of B, i.e. the pairing is homogeneous.

Let now $(A, B, \langle , \rangle)$ be a positive triple. According to (4) and (5) the elements m_A and m_B are zeros of the semigroups P_A and P_B. In paper [5] assuming that they are *central* and *nondegenerate* the notion of algebras in Plansherel duality was defined. It coincides with that of subsection 2.6 up to the language. Indeed, the fact that each zero is central means that

$$\langle m_A, fg \rangle = \langle m_A, gf \rangle, \quad f, g \in B,$$

$$\langle xy, m_B \rangle = \langle yx, m_B \rangle, \quad x, y \in A.$$

In this case there are decompositions

$$m_A = \sum_{\pi} m_A(\pi) \widehat{\chi}_\pi, \quad m_B = \sum_{\pi} m_B(\pi) \widehat{\chi}_\pi \tag{13}$$

where π runs over full systems of pairwise nonequivalent irreducible representations of the algebras A and B respectively (see (6) and subsection 2.1). It is the same thing to say that $m_A = \widehat{\chi}_B$, $m_B = \widehat{\chi}_A$ for some traces on B and A satisfying (12). Also $m_A(\pi)$ and $m_B(\pi)$ coincide with the coefficients $m(\pi, \chi_B)$ and $m(\pi, \chi_A)$ from (6). Besides, the nondegeneracy of the zeros means that $m_A(\pi)$ and $m_B(\pi)$ in (13) are positive reals. (As it is noted in [5], the nondegeneracy of any of the zeros m_A and m_B implies that of the other and is equivalent to the inequality $\langle m_A, m_B \rangle > 0$.) This is equivalent to the fact that χ_A and χ_B are positive traces. Ultimately, holding the isometry property with respect to χ_A and χ_B for the positive triple $(A, B, \langle , \rangle)$ just expresses the fact that A and B are in Plancherel duality in the sense of [5].

The following easy statement which will be of use in section 4 was proven in [5].

Proposition 2.3 *Let $(A, B, \langle , \rangle)$ be a positive triple with nondegenerate central zeros and commutative B. The following conditions are equivalent:*

(1) *The triple $(A, B, \langle , \rangle)$ is a Plancherel one.*

(2) *The orthogonality relations in the algebra A are satisfied:*

$$(x_p, x_q)_A = \frac{M}{m_A(p)} \delta_{p,q} \tag{14}$$

where $x_p = \widehat{\chi}_\pi, x_q = \widehat{\chi}_\rho, m_A(p) = m_A(\pi)$ with π, ρ being irreducible representations of the algebra B corresponding to its primitive central idempotents p, q, and $M = \langle m_A, m_B \rangle$.

Proof. It follows from the hypothesis that m_A and m_B define by duality some positive traces χ_B and χ_A satisfying (12). Then (14) is equivalent to the fact that the map F_B is isometric (see 2.5). Since the triple is positive, the involutory map η_A is an isomorphism of the algebra A (see 2.2) and so an isometry of the Hilbert space $(A, (\ ,\ )_A)$. Hence by (9) the map F_A is also isometric and so $(A, B, \langle , \rangle)$ is a Plancherel triple.■

3 Positive C-algebras

Let A be an associative algebra over $\mathbf{C}$ with the unit 1, semilinear involutory anti-automorphism $*$ and a special linear basis $\mathcal{R}$. The structure constants of A with respect to the basis $\mathcal{R}$ are defined by $r \cdot s = \sum_{t\in\mathcal{R}} c_{r,s}^t \cdot t$ where $r, s \in \mathcal{R}$.

Definition 3.1 *The pair $(A, \mathcal{R})$ is called a C-algebra* [1]*, if the following conditions are satisfied:*

(C1) $1 \in \mathcal{R}$ *and* $\mathcal{R}^* = \mathcal{R}$.

(C2) *The structure constants $c_{r,s}^t$ of the algebra A with respect to the basis $\mathcal{R}$ are real numbers.*

(C3) *For all $r, s \in \mathcal{R}$ the equality $c_{r,s}^1 = d(r)\delta_{r^*,s}$ holds where $d(r) > 0$ and δ is the Kronecker symbol.*

(C4) *A linear functional on A, defined by the map $r \mapsto d(r)$, $r \in \mathcal{R}$, is a one-dimensional $*$-representation of the algebra A. In particular, $d(r) = d(r^*)$ for all $r \in \mathcal{R}$.*

Remark 3.2 *In algebraic combinatorics (see [1]) the concept of C-algebra is used basically for commutative algebras.*

Under an isomorphism of C-algebras we mean a $*$-isomorphism of the corresponding algebras, preserving bases.

Let us define in A a coordinate-wise multiplication $\circ$ with respect to the basis $\mathcal{R}$ by

$$r \circ s = \delta_{r,s} r, \qquad r, s \in \mathcal{R}. \tag{15}$$

Obviously this turns the linear space A into a commutative associative algebra over $\mathbf{C}$ with the unit $J = \sum_{r\in\mathcal{R}} r$.

Definition 3.3 *A C-algebra $\mathcal{A} = (A, \mathcal{R})$ is called positive, if*

(P1) *all structure constants $c_{r,s}^t$ of the algebra A with respect to the basis $\mathcal{R}$ are non-negative reals,*

(P2) *the cone $A^+ = \{aa^* :\ a \in A\}$ of non-negative elements of the algebra A is closed with respect to the multiplication $\circ$.*

Remark 3.4 *The following example shows, that condition (P2) does not follow from (P1). In the commutative C-algebra $(A, \{1, r, s\})$ with the multiplication*

$$r^2 = m \cdot 1 + (m-1) \cdot s$$

$$s^2 = m \cdot 1 + r + (m-2) \cdot s$$

[1]Unfortunately, the term C-algebra is extremely unsuccessful, since it can be mixed with the terms C^*-algebra and $\mathbf{C}$-algebra.

$$rs = sr = (m-1)\cdot r + s$$

and the involutioin $r^* = r$, $s^* = s$ condition (P1) is obviously satisfied for all $m \geq 2$. On the other hand, condition (P2) is violated for each $m > 2$. (Probably, such examples were known earlier.)

Indeed, it can be shown by a direct calculation that there exists a primitive idempotent p of the algebra A such that the coefficient at p in the decomposition of $p \circ p$ with respect to the primitive idempotent basis of A is a negative number. It follows that $p \circ p \notin A^+$ (see proposition 3.5).

Condition (P1) actually means that the cone of non-negative elements with respect to the basis $\mathcal{R}$ (i.e. octant) is closed under the multiplication in A. Condition (P2) is exactly a dual one. The idea of defining the positive duality in [7] consisted just in using this symmetry.

Checking (P1) is trivial whereas checking (P2) for a non-commutative C-algebra is not an easy problem. It should be noted that in commutative case this condition is easily rewritten in terms of the Krein constants. We recall that the Krein constants of a commutative C-algebra $\mathcal{A} = (A, \mathcal{R})$ are just its dual structure constants. More precisely, the Krein constants $c^p_{p_1,p_2}$, $p_1, p_2, p \in \mathrm{Spec}(A)$, are defined by

$$p_1 \circ p_2 = \frac{1}{n} \sum_{p \in \mathrm{Spec}(A)} c^p_{p_1,p_2} p,$$

where $n = \sum_{r \in \mathcal{R}} d(r)$ and $\mathrm{Spec}(A)$ is the set of primitive central idempotents of the algebra A.

Proposition 3.5 *The positiveness of a commutative C-algebra is equivalent to non-negativity of its structure constants and Krein constants.*

Proof. Let $\mathcal{A} = (A, \mathcal{R})$ be a commutative C-algebra. Then it is easy to see, that

$$A^+ = \{ \sum_{p \in \mathrm{Spec}(A)} \alpha_p p : \ \alpha_p \geq 0 \}.$$

However, by the definition of the Krein constants the non-negativity of them means that A^+ is closed with respect to the multiplication $\circ$.■

For C-algebras arising from commutative association schemes the Krein constants are non-negative (Krein's condition), which gives a necessary existence condition for such schemes. Condition (P2) actually generalizes Krein's condition to non-commutative case (see proposition 3.6 below). Numerous examples of such algebras arise from group considerations: those are the group algebra of a finite group and some its subalgebras (see [1, 7, 5]).

Nontrivial examples of positive C-algebras come from *cellular (coherent) algebras*, introduced in [8] (and independently in [4]) as subalgebras of $\mathrm{Mat}_n(\mathbf{C})$ closed under the Hadamard (componentwise) multiplication, the Hermitian conjugation and containing the all one matrix J_n. The centralizer algebras of permutation groups and the adjacency algebras of coherent configurations are examples of cellular algebras. Each cellular algebra $A \subset \mathrm{Mat}_n(\mathbf{C})$ has a uniquely determined linear basis $\mathcal{R}(A)$ consisting of {0,1}-matrices summing to J_n and closed under transponing. This algebra is called *homogeneous*, if the basis $\mathcal{R}(A)$ contains the identity matrix. Examples of homogeneous cellular algebras are the centralizer algebras of transitive permutation groups and the Bose-Mesner algebras of association schemes (possibly non-commutative).

If A is a homogeneous cellular algebra, the pair $(A, \mathcal{R}(A))$ is obviously a C-algebra (possibly non-commutative) for which the numbers $d(r)$, $r \in \mathcal{R}(A)$ are positive integers. On the other hand, such a C-algebra $(A, \mathcal{R})$ together with an exact $*$-representation

$$A \to \mathrm{Mat}_n(\mathbf{C}), \qquad n = \sum_{r \in \mathcal{R}} d(r),$$

for which the multiplication $\circ$ in A (see (15)) goes to the Hadamard multiplication of matrices, can be viewed as a homogeneous cellular algebra.

Proposition 3.6 *If A is a homogeneous cellular algebra, then the C-algebra $(A, \mathcal{R}(A))$ is positive.*

Proof. It is clear that the structure constants of $(A, \mathcal{R}(A))$ are non-negative integers. On the other hand, it is easy to see, that the cone A^+ in this case consists of non-negative definite Hermitian matrices contained in A. Therefore condition (P2) follows from a well known fact, that the Hadamard product of non-negative definite Hermitian matrices is a non-negative definite Hermitian matrix (see, for example, lemma 3.9 in [1]).∎

4 Proof of theorem 1

4.1. Let $\mathcal{A} = (A, \mathcal{R})$ be a positive C-algebra. Let us introduce in the linear space $B = A$ the structure of a commutative $*$-algebra with the multiplication $\circ$ (see (15)) and the conjugation $r^* = r$, $r \in \mathcal{R}$. Thus $\mathrm{Spec}(B) = \mathcal{R}$. Let us define a pairing between A and B by

$$\langle r, s\rangle_{\mathcal{R}} = \langle r, s\rangle = \delta_{r,s} d(r), \qquad r, s \in \mathcal{R}.$$

We will show that the triple $\mathcal{T}(\mathcal{A}) = (A, B, \langle , \rangle)$ is a Plancherel one.

It is easy to see, that

$$P_A = \{x = \sum_{r \in \mathcal{R}} \alpha_r r :\ \alpha_r \geq 0,\ d(x) = 1\}.$$

Therefore the fact that the set P_A is closed under the multiplication and the conjugation in the algebra A follows from the non-negativity of its structure constants with respect to $\mathcal{R}$ (conditions (P1) and (C4)). On the other hand,

$$P_B = \{x \in A^+ :\ \alpha_1(x) = 1\},$$

where $\alpha_1(x)$ is the coefficient at 1_A in the decomposition of x with respect to $\mathcal{R}$. So condition (P2) implies that the set P_B is closed under the multiplication and the conjugation in the algebra B. Thus $\mathcal{T}(\mathcal{A})$ is a positive triple.

Let us consider the zeros m_A and m_B of the algebras A and B. It is not difficult to check that

$$m_A = (\sum_{r \in \mathcal{R}} r)/n, \qquad m_B = 1_A,$$

where $n = \sum_{r \in \mathcal{R}} d(r) = 1/M$. The centrality of m_A is obvious by the commutativity of the algebra B, while that of m_B follows from conditions (C3) and (C4). It follows from the same conditions that

$$m_A(r) = \langle m_A, r\rangle = d(r)/n > 0, \qquad r \in \mathrm{Spec}(B) = \mathcal{R}.$$

This proves the nondegeneracy of m_A. To complete the proof of the Plancherelity it suffices by proposition 2.3 to check the orthogonality relations (14). In our case they have the form

$$(r/d(r), s/d(s))_A = \delta_{r,s}/d(r),$$

which coincides with (C3) by the definition of $(,)_A$.

4.2. Conversely, let $\mathcal{T} = (A, B, \langle , \rangle)$ be a Plancherel triple for which the algebra B is commutative. Set

$$\mathcal{R} = \{r(p) \in A : \ r(p) = (m_A(p)/M)x_p, \ p \in \mathrm{Spec}(B)\},$$

where $m_A(p)$, M and x_p are defined in proposition 2.3. We will prove, that the pair $\mathcal{A}(\mathcal{T}) = (A, \mathcal{R})$ is a positive C-algebra. Note that $\mathcal{R}$ is a linear basis of the algebra A by virtue of the nondegeneracy of m_A. Below for $r = r(p)$ we set $d(r) = m_A(p)/M$.

Since $1_A = r(m_B)$ (see the third statement of lemma 2.2), $1_A \in \mathcal{R}$. The closure of $\mathcal{R}$ under $*$ follows from the fact that the map η_B defined in (3) is a $*$-isomorphism of the algebra B (see [5]). This proves (C1). Further, it follows from the orthogonality relations (14) that

$$(r, s)_A = d(r)\delta_{r,s}, \qquad r, s \in \mathcal{R}.$$

By the definition of the scalar product $(\, , \,)_A$ we have

$$c^1_{r,s} = \langle rs, m_B\rangle = (r, s^*)_A = d(r)\delta_{r,s^*},$$

which implies (C3) since $d(r) > 0$ due to the nondegeneracy of m_A. The last formula shows that

$$c^t_{r,s} = \langle rst^*, m_B\rangle/d(t), \qquad r, s, t \in \mathcal{R}.$$

Note that by the commutativity of B we have

$$B^+ = \{ \sum_{p\in\mathrm{Spec}(B)} \alpha_p p : \ \alpha_p \geq 0\}.$$

¿From here by the definition of $\mathcal{R}$ we conclude that $\mathcal{R} \subset K_A$. Thus, condition (C2) and, moreover, the non-negativity of $c^t_{r,s}$, follow from the closure of the cone K_A under the multiplication and the conjugation together with the fact that $m_B \in B^+$. The property (C4) follows from the formula

$$d(r) = \langle r, p\rangle = \langle r, 1_B\rangle,$$

where $r = r(p)$, $p \in \mathrm{Spec}(B)$, and the fact that the map $x \mapsto \langle x, 1_B\rangle$ is a $*$-character of the algebra A (see subsection 2.3).

To prove the positivity of the C-algebra $\mathcal{A}(\mathcal{T})$ we note that condition (P1) was actually checked at the proof of condition (C2). On the other hand, it is easy to see, that the map $\mathrm{Spec}(B) \to \mathcal{R}$, $p \mapsto r(p)$ is extended to a linear isomorphism of B on A, transferring the multiplication in B to the multiplication $\circ$ in A, for which K_B goes to A^+. So condition (P2) follows from the closure of the cone K_B under the multiplication in B.

4.3. Let us check that the maps $\mathcal{T} \to \mathcal{A}(\mathcal{T})$ and $\mathcal{A} \to \mathcal{T}(\mathcal{A})$ set an equivalence of the corresponding categories. Clearly, $\mathcal{A}(\mathcal{T}(\mathcal{A})) = \mathcal{A}$. On the other hand, let $\mathcal{T} = (A, B, \langle , \rangle)$ be a Plancherel triple with a commutative algebra B. Then the first component of the Plancherel triple $\mathcal{T}(\mathcal{A}(\mathcal{T}))$ coincides with the algebra A, while the second one is a linear space A with the commutative multiplication $\circ$ (see (15)). The isomorphism between these triples is given by the identical map of the algebra A and the linear isomorphism of B on A defined at the end of subsection 4.2. The theorem is proved.

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