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TOPOLOGIES ON PRODUCTS AND RATIONS. 2

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§1. Preliminaries

1. Some remarks. This paper is a continuation of the paper [1] "Topologies on products and ratios" (PDMI preprint – 2/1997) and so a first-hand view of that preprint is desirable. We can't list here all definitions and results of [1] and remind only some of them to give a possibility to understand this paper.

Here and there we consider sets $X,\ Y,\ Z,\ \dots$ equipped with some topological structures, denoting the corresponding topological spaces by the same letters $X,\ Y,\ Z,\ \dots$. To accentuate the fact that on some space, for example $X\times Y$, a (new) topological structure τ is fixed, we write $(X\times Y)_\tau$. So we consider in the corresponding definitions topological spaces $X,\ Y,\ Z,\ X\times Y,\ (X\times Y)_\tau,\ Y/Z,\ (Y/Z)_\tau$ (here Y/Z means the set of all continuous mappings of a topological space Y into a topological space Z – ratio of Y and Z). In the situation when for some $F : X\times Y \longrightarrow Z$ and $G : X \longrightarrow Y/Z$ the relation $G(x)(y) = F(x, y)$ takes place the mapping F and G are considered as corresponding to each other. We shall consider corresponding to each other mappings F and G only.

Let $\mathcal{A}$ be a fixed class of topological spaces. In [3] we introduced the following definitions.

A topology τ on Y/Z is called $\mathcal{A}$-splitting topology on Y/Z iff for any $X \in \mathcal{A}$ the continuity of any mapping $F : X \times Y \longrightarrow Z$ implies the continuity of the corresponding mapping $G : X \longrightarrow (Y/Z)_\tau$.

A topology τ on Y/Z is called $\mathcal{A}$-jointly continuous topology on Y/Z iff for any $X \in \mathcal{A}$ the continuity of any mapping $G : X \longrightarrow (Y/Z)_\tau$ implies the continuity of the corresponding mapping $F : X \times Y \longrightarrow Z$.

A topology τ on $X \times Y$ is called $\mathcal{A}$-splitting topology on $X \times Y$ iff for any $Z \in \mathcal{A}$ the continuity of any mapping $F : (X \times Y)_\tau \longrightarrow Z$ implies the continuity of the corresponding mapping $G : X \longrightarrow Y/Z$.

A topology τ on $X \times Y$ is called $\mathcal{A}$-jointly continuous topology on $X \times Y$ iff for any $Z \in \mathcal{A}$ the continuity of any mapping $G : X \longrightarrow Y/Z$ implies the continuity of the corresponding mapping $F : (X \times Y)_\tau \longrightarrow Z$.

In these definitions we mean that Y is some fixed space, on each $X \times Y$ and on each Y/Z, where X and Y vary, passing whole class of topological spaces, are defined some fixed topological structures, depending relative to X additively and relative to Z multiplicatively.

Note that the greatest $\mathcal{A}$-splitting topology on Y/Z and the least $\mathcal{A}$-jointly continuous topology on $X\times Y$ are the most interesting. They exist always while the least $\mathcal{A}$-jointly continuous topology on Y/Z and the greatest $\mathcal{A}$-splitting topology on $X \times Y$ not necessary exist.

We can consider any class $\mathcal{A}$ of topological spaces. In particular we can consider any one space set that is consider any triplet $X,\ Y,\ Z$ of topological spaces. The other extreme case is the class of all topological spaces. In any case we get some correspondence between topological structures on $X \times Y$ and topological structures on Y/Z.

The present paper is devoted to the subsequent development of the theory of such correspondences. Unlike the previous preprint, it is written by two authors. The new author (N. V. Khmylko) prepared §3.

2. Some potentialities of the theory. The problem of investigation of different function spaces, that is topological structures on sets $C(Y, Z)$ (Z^Y in another terminology, Y/Z in our terminology) of continuous mappings Y into Z, attracted the great attention of many mathematicians for a long time. They investigated some individual topological structures on $C(X, Y)$ and some classes of such topological structures using different methods and different constructions. One of such constructions has been proposed in [1]. This construction dealt with splitting and jointly continuous topological structures for the case when $X \times Y$ has the usual topological structure of production and X is any topological space. Our approach, in which can be used any topology on $X \times Y$ while the notions of splitting topology and the notion of jointly continuous topology apply not only to Y/Z but to $X \times Y$ also, extends possibility of application of the method to investigation of function spaces.

Another old problem concerns different spaces of subsets, that is topological structures on sets $PY = \{A | A \subset Y\}$. Sometimes topological spaces $P_{cl}Y$ ($P_{co}Y$) of all closed (compact) subsets of topological spaces Y are considered. The problem of investigation of different spaces of subsets also attracted the great attention of many mathematicians for a long time. They investigated some individual topological structures on PY ($P_{cl}Y$, $P_{co}Y$) and some classes of such topological structures, using different methods and different constructions. Here the following approach to this problem is proposed.

Consider triplet X , Y , Z where Z is two point set $D = \{0,\ 1\}$ with topological structure $\tau_r = \{\emptyset,\ D,\ \{0\}\}$. Then there exists one to one correspondence between the set Y/D of all continuous mappings $f : Y \longrightarrow D$ and the set of all closed subsets of Y ($f \longleftrightarrow f^{-1}(1)$). Thus any topological structure on the set Y/D we can consider as a topological structure on the set $P_{cl}Y$ of all closed subsets of topological space Y. It allows to use various correlations between topological structures on $X \times Y$ and topological structures on Y/D for study interesting correlations between topological structures on $X \times Y$ and topological structures on PY ($P_{cl}Y$, $P_{co}Y$).

§2. The case $Z = D$

1. Bitopological and settopological structures. Consider the case $X = Y$, $Z = D$. Any topological structure on $Y \times Y$ we can consider as a bitopological structure on Y. On the other hand the ratio Y/D we can consider as a space of all closed subsets of Y that is as a settopological space. Thus the theory allows us to investigate some relations between bitopological and settopological structures. In considered situation we have mappings

$$F : Y \times Y \longrightarrow D,\ G : Y \longrightarrow P_{cl}Y$$

where $F(x, y) = 1 \iff y \in G(x)$.

Instead of mappings F and G we can consider the only set $\mathcal{F} = F^{-1}(1) \subset Y \times Y$ and define $G(x) = \{y | (x, y) \in \mathcal{F}\}$ for any $x \in Y$. On the other hand $\mathcal{F} = \cup \mathcal{F}_x | x \in Y$ where $\mathcal{F}_x = \{x\} \times G(x)$.

Let $(Y \times Y)_\tau$ be given. Consider all closed sets $\mathcal{F}$ of $(Y \times Y)_\tau$ and the corresponding subsets $\{G_x | x \in Y\}$ of $P_{cl}Y$. Then τ_* will be the greatest topological structure on $P_{cl}Y$ for which all mappings $G : Y \longrightarrow (P_{cl}Y)_{\tau_*}$ are continuous. We shall say that τ_* is the corresponding to τ topological structure on $P_{cl}Y$.

Let $(P_{cl}Y)_\tau$ be given. Consider all continuous mappings $G : Y \longrightarrow (P_{cl}Y)_\tau$ and the corresponding to them sets $\mathcal{F} = \cup(\{x\} \times G(x)) | x \in Y$. Then τ^* will be the the least topological structure on $Y \times Y$ containing all $\mathcal{F}$. We shall say that τ^* is the corresponding to τ topological structure on $Y \times Y$.

The situation when we have topological structures $\overline{\tau}$ on $Y \times Y$ and $\underline{\tau}$ on $P_{cl}Y$ such that

$$(\overline{\tau})_* = \underline{\tau},$$

$$(\underline{\tau})^* = \overline{\tau}$$

is especially interesting. In the case we shall say that $\overline{\tau}$ and $\underline{\tau}$ are dual to each other.

Consider the following example.

Example. Let Y be a connected topological space, $\overline{\tau}$ consist of all sets $\mathcal{F} = Y \times C$, where C is any closed in Y set, $\underline{\tau}$ be discrete topological structure on $P_{cl}Y$, then $\overline{\tau}$ and $\underline{\tau}$ are dual to each other topological structures.

Proof. If $F : (Y \times Y)_{\overline{\tau}} \longrightarrow D$ is continuous mapping, then $F^{-1}(1) = Y \times C$, where C is a closed in Y set, and so $G(x) = C$ for all $x \in Y$. It means that $G : Y \longrightarrow (P_{cl}Y)_{\underline{\tau}}$ is constant mapping, therefore it is continuous mapping. On the other hand, if $G : Y \longrightarrow (P_{cl}Y)_{\underline{\tau}}$ is a continuous mapping, then G is the constant mapping, since $\underline{\tau}$ is the discrete topological structure and Y is a connected topological space, that is $G(x) = C$ for some fixed C and all $x \in Y$. But it means that $F^{-1}(C) = Y \times C$ and so $F : (Y \times Y)_{\overline{\tau}} \longrightarrow D$ is a continuous mapping.

The following remark shows that the just considered example is, in some sence, one of the extreme cases.

Remark. Let τ be a topological structure on $P_{cl}Y$, then corresponding to τ topological structure τ^* on $Y \times Y$ containes all sets $Y \times C$ where C is any closed set in Y.

Proof. Really, any constant mapping $G : Y \longrightarrow (P_{cl}Y)_\tau$ is a continuous mapping. Since $G(x) = C$ for all $x \in Y$ and some fixed closed in Y set C, the corresponding set $\mathcal{F} = F^{-1}(1) = Y \times C$ and so $Y \times C \in \tau^*$.

The other extreme case is the following one.

Example. Let $\overline{\tau}$ be a topological structure on $Y \times Y$, having closed sub-base consisting of all sets $A \times C$, where $A \subset Y$, C is any closed set of Y (this topological structure is the product of discrete topological structure on Y and the given topological structure of Y), $\underline{\tau}$ be the trivial topological structure on $P_{cl}Y$, then $\overline{\tau}$ and $\underline{\tau}$ are dual to each other topological structures.

Proof. Since $\underline{\tau}$ is trivial topological structure, continuity of $F : (Y \times Y)_{\overline{\tau}} \longrightarrow D$ implies continuity of $G : Y \longrightarrow (P_{cl}Y)_{\underline{\tau}}$. On the other hand, if $G : Y \longrightarrow (P_{cl}Y)_{\underline{\tau}}$ is a mapping, then for any $x \in Y$ the set $G(x) = C_x$ is the closed in Y set. Then the corresponding set $\mathcal{F} = \cup C_x | x \in Y$ will be closed in $(Y \times Y)_{\overline{\tau}}$ and so $F : (Y \times Y)_{\overline{\tau}} \longrightarrow D$ is a continuous mapping.

2. Dual bitopological and settopological structures. We know already that not any bitopological structure on Y is dual to some settopological structure or other on Y. On the other hand we can prove that not any settopological structure on Y is dual to some bitopological structure or other on Y. Consider, in this connection, the following example.

Example. Let $Y = \mathbb{R}$, G_l and G_r be such mappings that $G_l(x) = (-\infty,\ x]$, $G_r(x) = [-x,\ \infty)$ for all $x \in \mathbb{R}$, τ be the maximal topological structure on $P_{cl}\mathbb{R}$ for which the mappings G_l and G_r are continuous. Then τ is not dual to some bitopological structure or other on $\mathbb{R}$.

Proof. Consider the bitopological structure τ^* on $\mathbb{R}$, corresponding to τ. τ^* containes the sets $\mathcal{G}_l = \cup G_l(x) | x \in \mathbb{R}$ and $\mathcal{G}_r = \cup G_r(x) | x \in \mathbb{R}$ therefore it contains the set $\mathcal{G}_l \cap \mathcal{G}_r = \cup G_l(x) \cap G_r(x) | x \in \mathbb{R}$. Thus the mapping $G : \mathbb{R} \longrightarrow P_{cl}\mathbb{R}$ for which $G(x) = [-x,\ x]$ is $(\tau^*)_*$-continuous mappings and therefore $(\tau^*)_* \neq \tau$ since G is not τ-continuous mapping.

The above considered examples demonstrate non-triviality of the dual pairs problem for bitopological and settopological structures. We shall try to solve this problem looking for conditions which must be fulfiled if a considered bitopological structure has a dual settopological structure and, on the contrary, if a considered settopological structure has a dual bitopoogical structure. Any condition we can formulate in two variants using connection between corresponding to each other mappings $F : Y \times Y \longrightarrow D$ and $G : Y \longrightarrow P_{cl}Y$. Any mapping F we can replace by closed set $F^{-1}(1)$, the correspondence between F and G will be replaced by the correspondence between $F^{-1}(1)$ and G, $G(x) = \{y|(x,y) \in F^{-1}(1)\}$ for any $x \in Y$.

Fix some topological space Y, the set $Y \times Y$ and the set $P_{cl}Y$. Fix also some dual to each other topological structures $\overline{\tau}$ and $\underline{\tau}$, and the corresponding topological spaces, which we shall designate, for simplicity, by $Y \times Y$ and $P_{cl}Y$.

Let $F : Y \times Y \longrightarrow D$ and $G : Y \longrightarrow P_{cl}Y$ correspond to each other, that is $F(x,y) = 1 \iff y \in G(x)$, $f : Y \longrightarrow Y$ be a continuous mapping, $G' = Gf$, F' corresponds to G'. Then

$$F'(x,y) = 1 \iff y \in G'(x) \iff y \in G(f(x)) \iff F(f(x),y) = 1$$

Thus

$$F'(x,y) = 1 \iff F(f(x),y) = 1$$

It means that the set $\mathcal{F}' = \{(x,y)|F(f(x),y) = 1\}$ is $\overline{\tau}$-closed set in $Y \times Y$. Thus *the mapping $(f,1) : Y \times Y \longrightarrow Y \times Y$, $(f,1)(x,y) = (f(x),y)$, is continuous for every continuous mapping $f : Y \longrightarrow Y$.*

Such topological structures on $Y \times Y$ we shall call *horizontally invariant.*

We received a necessary condition for topological structure on $Y \times Y$ to be τ^*, in particular to be $\overline{\tau}$. Note that the corresponding to it condition for topological structures on $P_{cl}Y$ is trivial.

Consider now another pair conditions, trivial for $Y \times Y$ and non-trivial for $P_{cl}Y$. As to $Y \times Y$ it is simply the condition for τ to be topological structure. Pass, now, to the topological space $P_{cl}Y$. Its topological structure τ_* is defined by the system $\mathfrak{G}$ of mappings $G : Y \longrightarrow P_{cl}Y$. Each of these mappings is defined by a τ-closed set $\mathcal{F}$, $G(x) = \{y|(x.y) \in \mathcal{F}\}$. Therefore this system of mappings must have the properties corresponding to the known properties of topological structures, namely the following conditions.

$$G_\lambda \in \mathfrak{G}|\lambda \in \Lambda \Longrightarrow \wedge G_\lambda \in \mathfrak{G}, (\wedge G_\lambda)(x) = \cap G_\lambda(x),$$

$$G_i \in \mathfrak{G}|i = 1,2,\ldots,n \Longrightarrow \vee G_i \in \mathfrak{G}, (\vee G_i)(x) = \cup G_i(x).$$

Each system $\mathfrak{G}$ which satisfies these conditions we shall call *topological.*

Pass now to the new, corresponding to each other, conditions which are not trivial for both $Y \times Y$ and $P_{cl}Y$.

Let $\overline{\tau}$ and $\underline{\tau}$ be dual topological structures on $Y \times Y$ and $P_{cl}Y$, accordingly. Topological structure $\underline{\tau}$ can be defined as follows.

C is $\underline{\tau}$-closed $\iff$ $G^{-1}(C)$ is closed set for any continuous mapping $G : Y \longrightarrow P_{cl}Y$. Notice, that in this definition we use duality between $\overline{\tau}$ and $\underline{\tau}$.

It is useful to introduce a general notion of completness for systems of mappings of topological spaces. Let X be a topological space, Y be a set, $\mathfrak{A}$ be a system of mappings $f : X \longrightarrow Y$, τ be the topological structure on Y such that $U \in \tau \iff f^{-1}(U)$ is open set in X for any $f \in \mathfrak{A}$. *The system $\mathfrak{A}$ is called complete if any continuous mapping $f : X \longrightarrow (Y)_\tau$ belongs to $\mathfrak{A}$.*

It is easy to see that the notion of bitopological completness for systems of mappings introduced in [2] is a partial case of just introduced one. Really, we considered there any set $\mathfrak{A}$ of mappings $f : X \longrightarrow X$ of a topological space X into itself and mappings $(f \times 1) : X \longrightarrow X \times X$, $(f \times 1)(x,y) = (f(x),y)$. The set $\mathfrak{A}$ defined a topological structure ζ on $X \times X$ by the set $\{(f \times 1)|f \in \mathfrak{A}\}$ in above described way. The set $\mathfrak{A}$ was called bitopologically complete if continuity of a mapping $f \times 1$ implies $f \in \mathfrak{A}$.

Just so settopological completness can be define. Let $\mathfrak{A}$ be a set of mappings $f : Y \longrightarrow P_{cl}Y$ of a topological space Y into the set $P_{cl}Y$, τ be the topological structure on $P_{cl}Y$ such that

$U \in \tau \iff f^{-1}(U)$ is open set in Y for any $f \in \mathfrak{A}$. *The system $\mathfrak{A}$ is called settopologically complete if any continuous mapping $f: Y \longrightarrow (P_{cl}Y)_\tau$ belongs to $\mathfrak{A}$.*

Thus we can assert that *if a topological structure τ on $P_{cl}Y$ has a dual one then it must be settopologically complete.*

We received a necessary condition for topological structure on $P_{cl}Y$ to be τ_*, in particular to be $\underline{\tau}$. Note that the corresponding to it condition for topological structures on $Y \times Y$ is just as not trivial.

Let τ be a topological structure on $Y \times Y$. Consider subset $\mathcal{A}$ of the set $P_{cl}Y$ such that for any τ-closed set $\mathcal{F}$ the set $\{x|G(x) \in \mathcal{A}\}$ is a closed in Y set, $G(x) = \{y|(x,y) \in \mathcal{F}\}$. If, on the other hand, the set $\{x|G(x) \in \mathcal{A}\}$ is a τ-closed for any such $\mathcal{A}$ and some $\mathcal{F}$ then we shall call the topological structure τ *bitopologically complete* topological structure. It is easy to see that the condition of settopological completness for systems of mappings and the condition of bitopological completness for topological structures correspond to each other.

Now we can formulate the conditions ensuring existence of dual topologies for topological structures on $Y \times Y$ and for topological structures on $P_{cl}Y$.

Theorem. *Topological structure τ on $Y \times Y$ has a dual topological structure on $P_{cl}Y$ iff τ is a horizontally invariant, bitopologically complet topological structure, topological structure τ on $P_{cl}Y$ has a dual topological structure on $Y \times Y$ iff τ is defined by a topological, settopologically complete system of mappings $G: Y \longrightarrow P_{cl}Y$.*

Proof. Let τ be a topological structure on $Y \times Y$, satisfying the conditions of the theorem. Consider topological structure τ_* on $P_{cl}Y$. It is easy to see that τ_*, being a topological structure on $P_{cl}Y$, satisfies the conditions of the theorem therefore the set of all mappings corresponding to the τ-continuous mappings $F: Y \times Y \longrightarrow D$ coincides with the set of all τ_*-continuous mappings $G: Y \longrightarrow P_{cl}Y$. It follows from this that the set of all τ-continuous mappings $F: Y \times Y \longrightarrow D$ coincides with the set of all mappings corresponding to τ_*-continuous mappings $G: Y \longrightarrow P_{cl}Y$. Since τ satisfies the conditions of the theorem it implies $\tau = (\tau_*)^*$. It means that any topological structure on $Y \times Y$, satisfying the conditions of the theorem, has a dual topological structure. In the same way one can prove that any topological structure on $P_{cl}Y$, satisfying the conditions of the theorem, has a dual topological structure.

3. Two special topologies on Y/D. We consider now two special topologies on Y/D, namely the topological sructure of point-wise convergence and compact-open topological structure. As a rool, these topology are different, they coincide for rather exotic spaces only. To understand character of spaces Y for which the compact-open topology on Y/D and topology of point-wise convergence coincide we consider the following examples.

Example. Let $Y = D$ then the compact-open topology on D/D coincides with the topology of the point-wise convergence.

Proof. It is easy to see that D/D consists of three points, namely of $f_{0,0}$, $f_{0,1}$, $f_{1,1}$ where $f_{i,j}$ is the following mapping:

$$f_{i,j}(x) = \begin{cases} i, & \text{for } x = 0 \\ j, & \text{for } x = 1, \end{cases}$$

$i = 0,1$ $j = 0,1$.

The compact-open topological structure is defined by sets $U(C,O)$ where $C = \{0\}, \{1\}, \{0,1\}$ are the compact sets and $O = \{0\},\ \{0,1\}$ are the open sets. So we have six variants for $U(C,O)$.

$$U(\{0\},\{0\}) = \{f_{0,0},\ f_{0,1}\},$$

$$U(\{1\},\{0\}) = \{f_{0,0}\},$$
$$U(\{0,1\},\{0\}) = \{f_{0,0}\},$$
$$U(\{0\},\{0,1\}) = U(\{1\},\{0,1\}) = U(\{0,1\},\{0,1\}) = D/D.$$

Thus compact-open topological structure on D/D consists of four sets: $\emptyset$, D/D, $\{f_{0,0}, f_{0,1}\}$, $\{f_{0,0}\}$ and defined exactly three non-trivial convergences:

$$f_{0,0} \longrightarrow f_{0,1},$$

$$f_{0,0} \longrightarrow f_{1,1},$$

$$f_{0,1} \longrightarrow f_{1,1}.$$

On the other hand, point-wise convergence consists of just the same convergences since on D there is only one non-trivial convergence $0 \longrightarrow 1$. So the compact-open topological structure on D/D coincides with topological structure of point-wise convergence.

Example. Let Y be an infinite set equipped with the topological structure, consisting of complements of finite sets and $\emptyset$. Then the compact-open topology on Y/D and the topology of the point-wise convergence don't coincide. Note, that each point of Y/D is either constant mapping or a mapping f for which $f^{-1}(1)$ is a finite set.

Proof. $U(Y,\{0\})$ is an open set in compact-open topology on Y/D since Y is a compact space. On the other hand, we can prove that the same set $U(Y,\{0\})$ is not open set in topology of point-wise convergence. Really, let $f : Y \longrightarrow D$ such that $f(x) = 0$ for each $x \in Y$ then $f \in U(Y,\{0\})$. Let $(f_n | n \in \mathbb{N})$ be a sequence of mappings such that

$$f_n(x) = \begin{cases} 0, & \text{for } x \neq x_n \\ 1, & \text{for } x = x_n \end{cases}$$

where $x_1,\ x_2, \dots, x_n, \dots$ are different points of Y. Then $f_n \notin U(Y,\{0\})$ for all $n \in \mathbb{N}$ and so $(f_n | n \in \mathbb{N}) \not\longrightarrow f$ in the compact-open topology on Y/D. On the other hand $(f_n | n \in \mathbb{N}) \longrightarrow f$ in the topology of point-wise convergence since $f_n(x) = 0$ for all n if $x \notin \{x_n | n \in \mathbb{N}\}$, $f_n(x_k) = 0$ for all $n \neq k$.

Theorem. *On any Y/D compact-open topological structure contains topological structure of point-wise convergence.*

Proof. Let $(f_\lambda | \lambda \in \Lambda) \longrightarrow f$ in a compact-open topology and $x \in Y$. If $f(x) = 1$ then always $(f_\lambda(x) | \lambda \in \Lambda) \longrightarrow f(x)$ but if $f(x) = 0$ then there exists λ_0 such that $f_\lambda \in U(\{x\},\{0\})$ for $\lambda \geq \lambda_0$, that is $f_\lambda(x) = 0$ for $\lambda \geq \lambda_0$. Thus we proved that any convergence $(f_\lambda | \lambda \in \Lambda) \longrightarrow f$ in compact-open topology implies the convergence $(f_\lambda | \lambda \in \Lambda) \longrightarrow f$ in the topology of point-wise convergence and so compact open topological structure contains topological structure of point-wise convergence.

Corollary. *Compact-open topology on Y/D differs from topology of point-wise convergence iff there exist $(f_\lambda : Y \longrightarrow D | \lambda \in \Lambda)$ and $f : Y \longrightarrow D$ such that $(f_\lambda | \lambda \in \Lambda) \not\longrightarrow f$ in the compact-open topology but $(f_\lambda | \lambda \in \Lambda \longrightarrow f$ in the topology of point-wise convergence.*

Proof. The condition of the corollary is sufficient since failure to coincide for convergences implies failure to coincide for the corresponding topological sructures. On the other hand, this condition is necessary. Really, let compact-open topology on Y/D differs from topology of point-wise convergence, but $(f_\lambda|\lambda \in \Lambda) \not\longrightarrow f$ in the compact-open topology implies $(f_\lambda|\lambda \in \Lambda) \not\longrightarrow f$ in the topology of point-wise convergence. Since, in view of above proved theorem, $(f_\lambda|\lambda \in \Lambda) \longrightarrow f$ in the compact-open topology implies $(f_\lambda|\lambda \in \Lambda) \longrightarrow f$ in the topology of point-wise convergence, the convergences will coincide what contradicts the condition.

Just proved theorem and its corollary one can use for proofs of different theorems coincidence or non-coincidence.

4. Two special duality. For any topological space Y we considered in previous section two special topologies on Y/D, namely the compact-open topology and the topology of point-wise convergence. We considered there some examples of such topologies. Here we shall consider the corresponding to them dual topologies on $Y \times Y$.

Example. The topological sructure on $D \times D$ dual to the compact-open topology on D/D is the usual topology of product.

Proof. We proved earlier that D/D consists of three points $f_{0,0},\ f_{0,1},\ f_{1,1}$, the compact open topology coincides with the topology of point-wise convergence, having two non-trivial open sets: $\{f_{0,0}\}$ and $\{f_{0,0},\ f_{0,1}\}$. So there are three non-constant mapping $G : D \longrightarrow D/D$, namely $G_1,\ G_2,\ G_3$.

$$G_1(x) = \begin{cases} f_{0,0}, & \text{for } x = 0 \\ f_{0,1}, & \text{for } x = 1 \end{cases},$$

$$G_2(x) = \begin{cases} f_{0,0}, & \text{for } x = 0 \\ f_{1,1}, & \text{for } x = 1 \end{cases},$$

$$G_3(x) = \begin{cases} f_{0,1}, & \text{for } x = 0 \\ f_{1,1}, & \text{for } x = 1 \end{cases}.$$

Each of these mappings defines the corresponding open set of the dual topological structure on $D \times D = \{(0,0),\ (0,1),\ (1,0),\ (1,1)\}$.

$$G_1 \longrightarrow \{(0,0),\ (0,1),\ (1,0)\},$$

$$G_2 \longrightarrow \{(0,0),\ (0,1)\},$$

$$G_3 \longrightarrow \{(0,0)\}.$$

Besides, there is an open set which is invers image

$$F^{-1}(0) = \{(0,0),\ (1,0)\}$$

These open sets define usual topology of product $D \times D$. It is easy to prove that this topological structure is dual to the compact-open topology on D/D.

Example. Let Y be an non-countable infinite set equipped with the topological structure, consisting of complements of finite sets and $\emptyset$. Then the topology on $Y \times Y$ dual to the compact-open topology on Y/D is the usual topology of product.

Proof. Prove, at first, that any mapping $G : Y \longrightarrow Y/D$ is continuous mapping of the topological space Y with above described topological structure into the topological space Y/D with compact-open topology iff G is either a constant mapping or $Y \backslash G^{-1}(f_0)$ is a finite set, $f_0(Y) = 0$. Suppose that it is not the fact that is $Y \setminus G^{-1}(f_0)$ is infinite set. Note that if G is not constant mapping then $G^{-1}(f_1)$ is finite set, $f_1(Y) = 1$, since $\{f_1\}$ is closed set and so $G^{-1}(f_1) \neq Y$ is closed set. Thus we must consider only the case when $Y \setminus (G^{-1}(f_1) \cup G^{-1}(f_0))$ is infinite set. Consider a countable subset of this set. This subset consists of some mappings $f_n : Y \longrightarrow D | n \in \mathbb{N}$ such that all $A_n = f_n^{-1}(1)$ are finite sets. Then the union A of all A_n is not more than countable set therefore it doesn't exhaust Y. Thus $G^{-1}(\{f | f^{-1}(1) \subset A\})$ will be a proper infinite closed subset of Y what is impossible.

We proved, in realily, that $F^{-1}(1)$, F-corresponds considered mapping G, is a basic closed set of usual product $Y \times Y$. It is easy to prove that we received dual topology on $Y \times Y$.

In the previous example we presupposed that Y is non-countable space. The following example demonstrates that this condition is essential.

Example. Let $Y = \mathbb{N}$ be the countable infinite sets equipped with the topology consisting of complements of finite sets and $\emptyset$. Then the topology on $Y \times Y$ dual to the compact-open topology on Y/D doesn't coincide with usual topology of product.

Proof. Let $Y = \mathbb{N}$, $\mathbb{N}_k = \{n | n \leq k\}$, $G : \mathbb{N} \longrightarrow \mathbb{N}/D$, $G(k) = f_k$, $f_k^{-1}(1) = \mathbb{N}_k$. Then the mapping G is continuous mapping. Really, any set $G^{-1}(C)$,where C is a basic closed set in $\mathbb{N}/D$, is either finite set or infinit set. In the first case $G^{-1}(C)$ is finite set and so it is closed in $\mathbb{N}$ set. In the second case C contains infinitely many mappings f_{k_i} and since $\cup \mathbb{N}_{k_i} = \mathbb{N}$ then $C = \mathbb{N}/D$ and so $G^{-1}(C) = \mathbb{N}$ is also closed set. But then $\mathfrak{F} = F^{-1}(1) = \{(m, n) | m \leq n\}$, where F corresponds G, will be closed set in the dual to compact-open topology. Since this set is not closed in the usual topology of product these topologies don't coincides.

§3. The case $X = Y = Z = \mathbb{R}$

1. Some dualities. In [3] three variants of duality for the case $X = Y = Z = \mathbb{R}$ were proposed without any proof. These variants can be formulated as follows.

1) $\overline{\tau}$ is the product of discrete topological structure on $\mathbb{R}$ and usual topological structure of the real line, $\underline{\tau}$ is trivial topological structure on $\mathbb{R}/\mathbb{R}$.

2) $\overline{\tau}$ is defined on $\mathbb{R} \times \mathbb{R}$ by a convergency originated by directed systems stationary for a co-ordinate, $\underline{\tau}$ is defined by the point-wise convergency on $\mathbb{R}/\mathbb{R}$.

3) $\overline{\tau}$ is the usual topological structure of the real plane, $\underline{\tau}$ is compact-open topological structure on $\mathbb{R}/\mathbb{R}$.

The proof of duality in the first variant is quite evident. The fact is that both $F : (\mathbb{R} \times \mathbb{R})_{\overline{\tau}} \longrightarrow \mathbb{R}$ and $G : \mathbb{R} \longrightarrow (\mathbb{R}/\mathbb{R})_{\underline{\tau}}$ can be arbitrary permissible mappings.

2. The second variant of duality. The proof of duality in the second variant is the following.

Let F and G be corresponding to each other mappings that is $F(x, y) = G(x)(y)$ for all x, y in $\mathbb{R}$. Let us prove that G is continuous, supposing F to be continuous. Let $(x_\lambda | \lambda \in \Lambda) \longrightarrow x$ in $\mathbb{R}$ with usual topology, $f_\lambda = G(x_\lambda)$, $f = G(x)$, $y_0 \in Y$, $y_\lambda = f_\lambda(y_0)$. Since F is continuous we have $F((x_\lambda, y_0) | \lambda \in \Lambda) \longrightarrow F(x, y_0)$ that is $(G(x_\lambda)(y_0) | \lambda \in \Lambda) \longrightarrow G(x)(y_0)$ or else $(f_\lambda(y_0) | \lambda \in \Lambda) \longrightarrow f(y_0)$. Since y_0 is arbitrary point of Y it means point-wise convergency of directed system $(f_\lambda | \lambda \in \Lambda)$ to f. Thus we proved the continuity of G.

Let us prove now that F is continuous, supposing G to be continuous. If $((x_\lambda, y_0) | \lambda \in \Lambda) \longrightarrow (x, y_0)$ than $(F(x_\lambda, y_0) | \lambda \in \Lambda) = G(x_\lambda)(y_0) \longrightarrow G(x)(y_0) = F(x, y_0)$, in view of continuity of G, but if $((x_0, y_\lambda) | \lambda \in \Lambda) \longrightarrow (x_0, y)$ than $(F(x_0, y_\lambda)) | \lambda \in \Lambda) = G(x_0)(y_\lambda) \longrightarrow G(x_0, y)$ in view of continuity of $G(x_0)$. Thus we proved the continuity of F.

3. The third variant of duality. The proof of duality in the third variant is the following.

Let F and G be corresponding to each other mappings that is $F(x,y) = G(x)(y)$ for all x, y in $\mathbb{R}$. Let us prove that G is continuous, supposing F to be continuous. Assume that it is not the case and so there is a sequence $(x_\lambda|\lambda \in \Lambda) \longrightarrow x$ such that $G(x_\lambda)|\lambda \in \Lambda) \not\longrightarrow G(x)$. It implies, in turn, the existence of a neighbourhood $U(C,O)$ of $G(x)$ such that $G(x_\lambda) \notin U(C,O)|\lambda \in \Lambda_0$ where Λ_0 is a cofinal part of Λ. It means that $G(x_\lambda)(C) \not\subset O$ for $\lambda \in \Lambda_0$ and therefore there exists a directed system $(y_\lambda|\lambda \in \Lambda_0)$ such that $y_\lambda \in C$ but $G(x_\lambda)(y_\lambda) \notin O$. But since C is compact set there exists a convergence $(y_\lambda|\lambda \in \Lambda_1) \longrightarrow y$ where Λ_1 is a cofinal part of Λ_0. Thus the assumption on non-continuity of G leads to the existence of a directed system $((x_\lambda, y_\lambda)|\lambda \in \Lambda_1)$ such that $((x_\lambda, y_\lambda)|\lambda \in \Lambda_1) \longrightarrow (x,y)$ but $(G(x_\lambda)(y_\lambda)|\lambda \in \Lambda_1) \not\longrightarrow G(x)(y)$ or else $(F(x_\lambda, y_\lambda)|\lambda \in \Lambda_1) \not\longrightarrow F(x,y)$ what contradicts continuity of F.

Let us prove now that F is continuous, supposing G to be continuous. Consider any convergence $((x_\lambda, y_\lambda)|\lambda \in \Lambda) \longrightarrow (x,y)$ and convergence $(G(x_\lambda)|\lambda \in \Lambda) \longrightarrow G(x)$, existing for $(x_\lambda|\lambda \in \Lambda) \longrightarrow x$, in view of continuity of G. Consider now the point $F(x,y) = G(x)(y) \in \mathbb{R}$ and any its neighbourhood O. Since $G(x) : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{R}$ is continuous mapping, there exists a compact neighbourhood C of the point y such that $G(x)(C) \subset O$. Now we have a neighbourhood $U(C,O)$ of the point $G(x)$ in the space $\mathbb{R}/\mathbb{R}$ considered with compact-open topology and, in view of continuity of G, there exists $\lambda_0 \in \Lambda$ such that $G(x_\lambda)(C) \subset O$ for all $\lambda \geq \lambda_0$. On the other hand we can think that $y_\lambda \in C$ for $\lambda \geq \lambda_0$. Thus $G(x_\lambda)(y_\lambda) = F(x_\lambda, y_\lambda) \in O$ for $\lambda \geq \lambda_0$. Since O is arbitrary neighbourhood of $F(x,y)$ it means that $F(x_\lambda, y_\lambda)|\lambda \in \Lambda) \longrightarrow F(x,y)$. Thus we proved that F is continuous mapping.

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