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GENERATING FUNCTION OF CORRELATORS IN THE SL_2 GAUDIN MODEL

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Abstract

For the sl_2 Gaudin model (degenerated quantum integrable XXX spin chain) an exponential generating function of correlators is calculated explicitely. The calculation relies on the Gauss decomposition for the SL_2 loop group. ¿From the generating function a new explicit expression for the correlators is derived from which the determinant formulas for the norms of Bethe eigenfunctions due to Richardson and Gaudin are obtained.

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1. Introduction

The most challenging problem of the theory of quantum integrable systems, after describing the spectrum and eigenfunctions, is finding a reasonably efficient expression for the correlators. Speaking about the systems lying in the realm of Bethe ansatz (R-matrix, quantum inverse scattering) method, two complementary approaches are presently known. One of them is based on studying analyticity and bootstrap equations for factorized S-matrices and formfactors of infinite-volume quantum field models and is useful in the study of short-distance behaviour of correlators (see the review papers [14, 5]). Another one, summarized in the book [6], is, on the contrary, efficient in describing the long-distance asymptotics and is based on meticulous study of the structure of Bethe eigenfunctions in the finite volume.

Whereas the mathematical structures involved with the former approach are now recognized quite satisfactorily (quantum Knizhnik-Zamolodchikov equation, vertex operators, representations of quantum groups) which has resulted in the rapid progress in this subject, this is not the case for the latter approach. After *tour de force* of 1982 by V. Korepin [7], who had managed to overcome the enormous combinatorial difficulties in handling the cumbersome expressions for correlators produced via algebraic Bethe ansatz, very little progress has been achieved in mathematical understanding of his technique. One should mention [8] where Drinfeld's twists were applied to simplify Korepin's calculations, and [10] where an alternative derivation of the determinant formula for the Bethe eigenfunctions based on the semiclassical asymptotics of the solutions to the Knizhnik-Zamolodchikov equation was proposed.

The present paper has grown from an attempt to understand and to simplify Korepin's derivations using a simple toy example. As such, I have chosen the sl_2 Gaudin model [1, 2, 3] (a degenerated case of integrable XXX spin chain). The choice was motivated by the fact that Gaudin model had already played this role once. The determinant formula for the Bethe eigenfunctions of Gaudin model obtained by Gaudin, who, in turn, had benefited from an earlier paper by Richardson [11], allowed to Gaudin, McCoy and Wu to conjecture in [4] the similar formula for the XXX spin chain which was finally proved by Korepin [7].

The intimidating combinatorial complexity of the algebraic expressions for correlators resulting from algebraic Bethe ansatz makes it natural to take advantage of the common combinatorial wisdom which says that using generating functions can help to reduce the complexity drastically. As shown in the present paper, this simple idea works quite effectively in case of the Gaudin model. A simple explicit expression for the exponential generating function of correlators is obtained from the Gauss decomposition for $SL(2)$ loop algebra. From the generating function the formulas for particular correlators are derived quite easily, including the Richardson-Gaudin determinant formula for the norm of Bethe eigenfunction.

The paper is organised as follows. In Section 2 we introduce the sl_2 Gaudin model and present the known results for its eigenfunctions, in particular, the Richardson-Gaudin fomula. In Section 3, we describe a new formula, called Λ-*representation* for correlators which we prove in the next, 4th section using the generating function technique. The idea of the proof is explained first on the elementary finite-dimensional example of sl_2 Lie algebra. The Λ-representation is used in Section 5 to derive the Richardson-Gaudin determinant formula. In the concluding, 6th section we discuss the obtained results and the prospects of their generalisation.

This work has been started during my stay at the Department of Mathematical Sciences, the University of Tokyo, when I was preparing a course of lectures for graduate students on Gaudin model and was continued in Research Institute for Mathematical Studies, Kyoto University. I

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2. Description of the model

The aim of this section is mainly to fix the notation. The results given below are taken from from [3, 13].

Let $\mathfrak{e}$, $\mathfrak{f}$, $\mathfrak{h}$ be the generators of the Lie algebra sl_2:

$$[\mathfrak{e}, \mathfrak{f}] = \mathfrak{h} \qquad [\mathfrak{h}, \mathfrak{e}] = 2\mathfrak{e} \qquad [\mathfrak{h}, \mathfrak{f}] = -2\mathfrak{f}, \tag{2.1}$$

and let a highest weight module labelled with the complex number (highest weight) λ be generated by the vacuum (highest vector) $|0\rangle$ and the relations

$$\mathfrak{e}\,|0\rangle = 0, \qquad \mathfrak{h}\,|0\rangle = \lambda\,|0\rangle\,. \tag{2.2}$$

We shall need also the dual module generated by the dual vacuum $\langle 0|$ and relations

$$\langle 0|\,\mathfrak{f} = 0, \qquad \langle 0|\,\mathfrak{h} = \lambda\,\langle 0|\,, \qquad \langle 0 \mid 0\rangle = 1. \tag{2.3}$$

The corresponding value of the Casimir operator

$$2K = \mathfrak{e}\mathfrak{f} + \mathfrak{f}\mathfrak{e} + \frac{1}{2}\mathfrak{h}^2 \tag{2.4}$$

is then $2K = \lambda(\lambda+2)/2$.

The commuting Hamiltonians Ξ_j of the Gaudin model are defined in the tensor product of D h.w. modules (we label the corresponding generators with the index j and put $|0\rangle := |0\rangle_1 \otimes \cdots \otimes |0\rangle_D$) and are given by the expression

$$\Xi_j = \sum_{k\neq j} \frac{\mathfrak{e}_j\mathfrak{f}_k + \mathfrak{f}_j\mathfrak{e}_k + \frac{1}{2}\mathfrak{h}_j\mathfrak{h}_k}{z_j - z_k}, \qquad j = 1, \dots D \tag{2.5}$$

where z_j are complex constants. It is more convenient, however, to express Ξ_j in terms of a generating function. To this end, consider the one-parameter operator families

$$E(u) = \sum_{j=1}^{D} \frac{\mathfrak{e}_j}{u - z_j}, \qquad F(u) = \sum_{j=1}^{D} \frac{\mathfrak{f}_j}{u - z_j}, \qquad H(u) = \sum_{j=1}^{D} \frac{\mathfrak{h}_j}{u - z_j}, \tag{2.6}$$

and define the generating function $t(u)$ as the result of replacing $\mathfrak{e}$, $\mathfrak{f}$, $\mathfrak{h}$ in the Casimir (2.4) respectively with $E(u)$, $F(u)$ and $H(u)$:

$$2t(u) = E(u)F(u) + F(u)E(u) + \frac{1}{2}H^2(u) \tag{2.7}$$

Using (2.6) and (2.5) one observes that

$$t(u) = \sum_{j=1}^{D} \frac{\Xi_j}{u - z_j} + \frac{\frac{1}{4}\lambda_j(\lambda_j + 2)}{(u - z_j)^2} \tag{2.8}$$

The fundamental property of the operators (2.6) is that they form the highest-weight module over the infinite-dimensional Lie algebra [3, 13]

$$[E(u), E(v)] = [F(u), F(v)] = [H(u), H(v)] = 0, \tag{2.9a}$$

$$[E(u), F(v)] = -\frac{H(u) - H(v)}{u - v}, \tag{2.9b}$$

$$[H(u), E(v)] = -2\frac{E(u) - E(v)}{u - v}, \tag{2.9c}$$

$$[H(u), F(v)] = 2\frac{F(u) - F(v)}{u - v}, \tag{2.9d}$$

characterised by the vacuum $|0\rangle$,

$$E(u)\,|0\rangle = 0, \qquad H(u)\,|0\rangle = \lambda(u)\,|0\rangle\,, \tag{2.10}$$

dual vacuum $\langle 0|$

$$\langle 0|\,F(u) = 0, \qquad \langle 0|\,H(u) = \lambda(u)\,\langle 0|\,, \qquad \langle 0 \mid 0\rangle = 1, \tag{2.11}$$

and the scalar function (highest weight) $\lambda(u)$

$$\lambda(u) = \sum_{j=1}^{D} \frac{\lambda_j}{u - z_j}\,. \tag{2.12}$$

A direct consequence of the relations (2.9) is, in particular, the commutativity of $t(u)$

$$[t(u), t(v)] = 0 \tag{2.13}$$

from which the commutativity of the hamiltonians Ξ_j follows immediately.

The commutation relations (2.9) can be also written down compactly in the so-called *r-matrix* form [3, 12, 13] which we, however, shall not use here.

The theory of Gaudin model is expected to give solution to two main problems. The first one is to determine the eigenvalues and eigenvectors of the commuting Hamiltonians Ξ_j (2.5).

The solution is given in terms of *Bethe vectors* which are defined for any finite set $\mathcal{V}$ of complex numbers as

$$|\mathcal{V}\rangle := \prod_{v\in\mathcal{V}} F(v)\,|0\rangle\,, \qquad \mathcal{V} \subset \mathbb{C}, \qquad |\mathcal{V}| < \infty. \tag{2.14}$$

Theorem 1 *[1, 3, 13] The vector $|\mathcal{V}\rangle$ is a joint eigenvector of commuting operators Ξ_j or, equivalently, of $t(u)$ if and only if the parameters $v \in \mathcal{V}$ satisfy the* Bethe equations

$$\lambda(v) = \sum_{v'\neq v} \frac{2}{v - v'}, \qquad \forall v \in \mathcal{V}. \tag{2.15}$$

The corresponding eigenvalue $\tau(u)$ *of* $t(u)$ *is then*

$$\tau(u) = \frac{1}{4}\widetilde{\lambda}^2(u) - \frac{1}{2}\partial_u\widetilde{\lambda}(u) \tag{2.16}$$

where

$$\widetilde{\lambda}(u) := \lambda(u) - \sum_{v\in\mathcal{V}} \frac{2}{u-v}. \tag{2.17}$$

The spectrum of the model having been determined, the next fundamental problem is to calculate the correlators. The correlators which we are going to study are labelled by three finite sets $\mathcal{U}, \mathcal{V}, \mathcal{W} \subset \mathbb{C}$ and are defined as

$$C(\mathcal{U},\mathcal{W},\mathcal{V}) := \langle\mathcal{U}| \prod_{w\in\mathcal{W}} H(w)\,|\mathcal{V}\rangle \tag{2.18}$$

where $\langle\mathcal{U}|$ is the dual Bethe vector

$$\langle\mathcal{U}| := \langle 0| \prod_{u\in\mathcal{U}} E(u). \tag{2.19}$$

In principle, one could calculate $C(\mathcal{U},\mathcal{W},\mathcal{V})$ moving the operators $H(w)$ and $E(u)$ to the right through $F(v)$ with the use of the formulas

$$F(v)\,|\mathcal{V}\rangle = |\mathcal{V}\cup\{v\}\rangle\,, \tag{2.20a}$$

$$H(w)\,|\mathcal{V}\rangle = \left(\lambda(w) - \sum_{v\in\mathcal{V}}\frac{2}{w-v}\right)|\mathcal{V}\rangle + \sum_{v\in\mathcal{V}}\frac{2}{w-v}\,|\mathcal{V}\setminus\{v\}\cup\{w\}\rangle\,, \tag{2.20b}$$

$$\begin{aligned} E(u)\,|\mathcal{V}\rangle = &\sum_{v\in\mathcal{V}} -\frac{1}{u-v}\left(\lambda(u) - \lambda(v) - \sum_{v'\neq v} 2\left(\frac{1}{u-v'} - \frac{1}{v-v'}\right)\right)|\mathcal{V}\setminus\{v\}\rangle \\ &+ \sum_{\substack{v,v'\in\mathcal{V}\\ v\neq v'}} -\frac{2}{(u-v)(u-v')}\,|\mathcal{V}\setminus\{v,v'\}\cup\{u\}\rangle \end{aligned} \tag{2.20c}$$

which follow from the relations (2.9) and (2.10). For example

$$C(\{u\},\emptyset,\{v\}) = -\frac{\lambda(u)-\lambda(v)}{u-v}, \qquad u\neq v. \tag{2.21}$$

This strategy was adopted in [7] where the formulas similar to (2.20) were used in a more complicated case of XXX spin chain. Such straightforward approach leads, however, to extremely tedious calculations, and it is the aim of the present paper to simplify the derivation using the method of generating functions. Still, one simple observation can be made right now: the correlator $C(\mathcal{U},\mathcal{W},\mathcal{V})$ is a polynomial in $\lambda(u)$, $\lambda(v)$, $\lambda(w)$ with the coeffitients rational in $u\in\mathcal{U}$, $v\in\mathcal{V}$, $w\in\mathcal{W}$.

It is important to stress, following [7], that for the derivation of the formulas for $C(\mathcal{U},\mathcal{W},\mathcal{V})$ the concrete form (2.12) of the function $\lambda(u)$ is not important, it can be considered as a formal parameter. All what matters is analyticity of $\lambda(u)$ at the points of $\mathcal{U}\cup\mathcal{V}\cup\mathcal{W}$ and the relations (2.9), (2.10) and (2.11).

We conclude this list of known results with an important determinant formula for the norm of the Bethe eigenfunction.

Theorem 2 *Let $\mathcal{V}=\{v_1,\ldots,v_N\}$ and v_j satisfy the Bethe equations (2.15). Then*

$$\langle\mathcal{V}\mid\mathcal{V}\rangle=(-1)^N\det\mathcal{M} \tag{2.22}$$

where the matrix $\mathcal{M}$ is given by

$$\mathcal{M}_{jj}=\partial_{v_j}\lambda(v_j)+\sum_{k\neq j}\frac{2}{(v_j-v_k)^2}, \tag{2.23a}$$

$$\mathcal{M}_{jk}=-\frac{2}{(v_j-v_k)^2},\qquad j\neq k \tag{2.23b}$$

The formula (2.22) was derived first by Richardson [11] for a model which can now be recognized as a very degenerate case of Gaudin model. In [2] Gaudin mentioned that the norms for Gaudin model are given by the same formula but had not explained how to generalise Richardson's derivation. To make up the deficiency we give a complete proof of (2.22) in section 5.

3. Λ-representation for correlators

In this section we give a new explicit formula, which we call Λ-representation, for the correlator $C(\mathcal{U},\mathcal{W},\mathcal{V})$. Throughout this paper we suppose, unless otherwise stated, that the sets $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ do not pairwise intersect.

¿From (2.20) it follows immediately that $C(\mathcal{U},\mathcal{W},\mathcal{V})$ is a polynomial in $\lambda(x)$, $x\in\mathcal{U}\cup\mathcal{V}\cup\mathcal{W}$ with the coefficients rational in x. However, the structure of $C(\mathcal{U},\mathcal{W},\mathcal{V})$ can be simplified after introducing a proper notation.

For any finite set $\mathcal{X}\subset\mathbb{C}$ define the function

$$\Lambda(\mathcal{X}):=\sum_{x\in\mathcal{X}}\frac{\lambda(x)}{\prod\limits_{y\neq x}(x-y)}. \tag{3.1}$$

For example,

$$\Lambda(\{x\})=\lambda(x),\qquad \Lambda(\{x_1,x_2\})=\frac{\lambda(x_1)-\lambda(x_2)}{x_1-x_2}. \tag{3.2}$$

Note that, supposing $\lambda(u)$ to be analytic inside a contour Γ circumscribing counterclockwise the set $\mathcal{X}$, one can rewrite (3.1) as the contour integral

$$\Lambda(\mathcal{X})=\frac{1}{2\pi i}\int_\Gamma\frac{\lambda(z)dz}{\prod\limits_{x\in\mathcal{X}}(z-x)},\qquad \mathcal{X}\subset\mathbb{C}_+\,. \tag{3.3}$$

Our aim is to express $C(\mathcal{U},\mathcal{W},\mathcal{V})$ as a polynomial in Λ.

Let $|\cdot|$ be the cardinality of a set, and suppose

$$|\mathcal{U}| = |\mathcal{V}| = N, \qquad |\mathcal{W}| = M \tag{3.4}$$

(obviously, $C(\mathcal{U},\mathcal{W},\mathcal{V})$=0 if $|\mathcal{U}| \neq |\mathcal{V}|$). Let us define a *coordinated partition* $\mathcal{P}$ of the sets $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ as a set of triplets $P = (\mathcal{U}_P, \mathcal{V}_P, \mathcal{W}_P)$, $\mathcal{U} \supset \mathcal{U}_P \neq \emptyset$, $\mathcal{V} \supset \mathcal{V}_P \neq \emptyset$, $\mathcal{W} \supset \mathcal{W}_P$ (note that $\mathcal{W}_P$, in contrast with $\mathcal{U}_P$ and $\mathcal{V}_P$, is allowed to be empty) such that

$$\begin{gathered}
\forall P, P' \in \mathcal{P}: \qquad |\mathcal{U}_P| = |\mathcal{V}_P| > 0, \qquad |\mathcal{W}_P| \geq 0, \\
P \neq P' \qquad \Longrightarrow \qquad \mathcal{U}_P \cap \mathcal{U}_{P'} = \emptyset, \quad \mathcal{V}_P \cap \mathcal{V}_{P'} = \emptyset, \quad \mathcal{W}_P \cap \mathcal{W}_{P'} = \emptyset, \\
\bigcup_{P\in\mathcal{P}} \mathcal{U}_P = \mathcal{U}, \qquad \bigcup_{P\in\mathcal{P}} \mathcal{V}_P = \mathcal{V}, \qquad \bigcup_{P\in\mathcal{P}} \mathcal{W}_P \subset \mathcal{W}, \\
\sum_{P\in\mathcal{P}} |\mathcal{U}_P| = |\mathcal{U}| = \sum_{P\in\mathcal{P}} |\mathcal{V}_P| = |\mathcal{V}| = N, \qquad \sum_{P\in\mathcal{P}} |\mathcal{W}_P| \leq |\mathcal{W}| = M.
\end{gathered}$$

Theorem 3 *The expression for the correlator $C(\mathcal{U},\mathcal{W},\mathcal{V})$ is given by the formula*

$$\begin{aligned}
&C(\mathcal{U},\mathcal{W},\mathcal{V}) \\
&\quad = (-1)^N \sum_{\mathcal{P}} \left(\prod_{P\in\mathcal{P}} |\mathcal{U}_P|!\,(|\mathcal{U}_P|-1)!\,(2\,|\mathcal{U}_P|)^{|\mathcal{W}_P|}\,\Lambda(\mathcal{U}_P \cup \mathcal{V}_P \cup \mathcal{W}_P) \right) \\
&\qquad\qquad \times \left(\prod_{w\in\overline{\mathcal{W}}_{\mathcal{P}}} \lambda(w) \right)
\end{aligned} \tag{3.5}$$

where

$$\overline{\mathcal{W}}_{\mathcal{P}} := \mathcal{W} \setminus \bigcup_{P\in\mathcal{P}} \mathcal{W}_P. \tag{3.6}$$

For example,

$$C(\{u\},\{w\},\{v\}) = -2\Lambda(\{u,w,v\}) - \Lambda(\{u,v\})\Lambda(\{w\}).$$

It is remarkable that $C(\mathcal{U},\mathcal{W},\mathcal{V})$ is a polynomial in Λ with *integer* coefficients, that is all the rationality is hidden in the definition (3.1) of Λ.

To describe the norms of Bethe eigenfunctions we shall need a particular case of (3.5) when $\mathcal{W} = \emptyset$:

$$\langle \mathcal{U} \mid \mathcal{V} \rangle = (-1)^N \sum_{\mathcal{P}} \prod_{P\in\mathcal{P}} |\mathcal{U}_P|!\,(|\mathcal{U}_P|-1)!\,\Lambda(\mathcal{U}_P \cup \mathcal{V}_P) \tag{3.7}$$

where $\mathcal{P}$ runs over the sets of pairs $P = (\mathcal{U}_P, \mathcal{V}_P)$, $\mathcal{U} \supset \mathcal{U}_P \neq \emptyset$, $\mathcal{V} \supset \mathcal{V}_P \neq \emptyset$, such that

$$\begin{gathered}
\forall P \in \mathcal{P}: \qquad |\mathcal{U}_P| = |\mathcal{V}_P|, \\
P, P' \in \mathcal{P}, \quad P \neq P' \qquad \Longrightarrow \qquad \mathcal{U}_P \cap \mathcal{U}_{P'} = \emptyset, \quad \mathcal{V}_P \cap \mathcal{V}_{P'} = \emptyset,
\end{gathered}$$

$$\bigcup_{P\in\mathcal{P}} \mathcal{U}_P = \mathcal{U}, \qquad \bigcup_{P\in\mathcal{P}} \mathcal{V}_P = \mathcal{V},$$

$$\sum_{P\in\mathcal{P}} |\mathcal{U}_P| = |\mathcal{U}| = \sum_{P\in\mathcal{P}} |\mathcal{V}_P| = |\mathcal{V}| = N.$$

For example,

$$\langle 0 \mid 0\rangle = 1, \qquad \langle u \mid v\rangle = -\Lambda(\{u,v\}), \tag{3.8a}$$

$$\langle u_1u_2 \mid v_1v_2\rangle = \Lambda(\{u_1,v_1\})\Lambda(\{u_2,v_2\}) + \Lambda(\{u_1,v_2\})\Lambda(\{u_2,v_1\})$$
$$+2\Lambda(\{u_1,u_2,v_1,v_2\}). \tag{3.8b}$$

Using the integral formula (3.3) for $\Lambda(\mathcal{X})$ it is easy to extend by analyticity the formula (3.5) for $C(\mathcal{U},\mathcal{W},\mathcal{V})$ to the case when the sets $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$ intersect or even have multiple points (being thus not sets but divisors). We leave this exercise to the reader.

4. Proof of the Theorem 3

As mentioned in the Introduction, we are going to circumvent the combinatorial difficulties of the direct approach based on the identities (2.20) by calculating a generating function of correlators $C(\mathcal{U},\mathcal{W},\mathcal{V})$.

Let us illustrate the idea on an elementary finite-dimensional example of the sl_2 Lie algebra (2.1). The analog of $C(\mathcal{U},\mathcal{W},\mathcal{V})$ is the expression $c(k,j) := \langle 0|\, \mathfrak{e}^k\mathfrak{h}^j\mathfrak{f}^k\, |0\rangle$ which can easily be estimated as

$$c(k,j) = (-k)_k(-\lambda)_k(\lambda-2k)^j = (-1)^k\, k!\, (-\lambda)_k(\lambda-2k)^j \tag{4.1}$$

where the Pochhammer symbol is defined as

$$(a)_k := a(a+1)\dots(a+k-1). \tag{4.2}$$

On the other hand, let us introduce the group elements

$$\mathcal{E}_\varepsilon := \mathrm{e}^{\varepsilon\mathfrak{e}}, \qquad \mathcal{H}_\eta := \mathrm{e}^{\eta\mathfrak{h}}, \qquad \mathcal{F}_\varphi := \mathrm{e}^{\varphi\mathfrak{f}}, \tag{4.3}$$

parametrized by coordinates ε, η, φ and consider the exponential generating function

$$\langle 0|\, \mathcal{E}_\varepsilon\mathcal{H}_\eta\mathcal{F}_\varphi\, |0\rangle = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{\varepsilon^k\eta^j\varphi^k}{k!^2\, j!} c(k,j). \tag{4.4}$$

Notice now that if we manage to change the order of factors

$$\mathcal{E}_\varepsilon\mathcal{H}_\eta\mathcal{F}_\varphi = \mathcal{F}_{\varphi'}\mathcal{H}_{\eta'}\mathcal{E}_{\varepsilon'}, \tag{4.5}$$

then the generating function is easily calculated:

$$\langle 0|\, \mathcal{F}_{\varphi'}\mathcal{H}_{\eta'}\mathcal{E}_{\varepsilon'}\, |0\rangle = \langle 0|\, \mathcal{H}_{\eta'}\, |0\rangle = \mathrm{e}^{\eta'\lambda}. \tag{4.6}$$

To establish the identity (4.5) it is sufficient to consider the fundamental (2-dimensional) representation of Lie algebra sl_2

$$\mathfrak{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathfrak{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathfrak{f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{4.7}$$

and the corresponding Lie group SL_2

$$\mathcal{E}_\varepsilon = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}, \qquad \mathcal{H}_\eta = \begin{pmatrix} \mathrm{e}^\eta & 0 \\ 0 & \mathrm{e}^{-\eta} \end{pmatrix}, \qquad \mathcal{F}_\varphi = \begin{pmatrix} 1 & 0 \\ \varphi & 1 \end{pmatrix}. \tag{4.8}$$

The identity (4.5) expresses then equivalence of left and right Gauss decompositions in SL_2. Multiplying 2×2 matrices we find from (4.5) the values of the primed parameters:

$$\mathrm{e}^{\eta'} = \mathrm{e}^\eta + \varepsilon\varphi \mathrm{e}^{-\eta}, \qquad \varepsilon' = \frac{\varepsilon \mathrm{e}^{-\eta}}{\mathrm{e}^\eta + \varepsilon\varphi\mathrm{e}^{-\eta}}, \qquad \varphi' = \frac{\varphi \mathrm{e}^{-\eta}}{\mathrm{e}^\eta + \varepsilon\varphi\mathrm{e}^{-\eta}}, \tag{4.9}$$

whence

$$\langle 0| \, \mathcal{E}_\varepsilon \mathcal{H}_\eta \mathcal{F}_\varphi \, |0\rangle = (\mathrm{e}^\eta + \varepsilon\varphi\mathrm{e}^{-\eta})^\lambda = \mathrm{e}^{\lambda\eta}(1 + \varepsilon\varphi\mathrm{e}^{-2\eta})^\lambda. \tag{4.10}$$

Applying the binomial expansion formula

$$(1-z)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} \, z^k$$

to (4.10) one recovers the result (4.1).

The same Gauss decomposition trick works also in the case of the infinite-dimensional Lie algebra (2.9).

Let $\mathbb{C}_+$ be a simply connected bounded open domain on the complex plain having smooth boundary Γ which is homeomorphic to a circle S^1. Let $\mathbb{C}_-$ be the complement domain: $\mathbb{C}_- = \mathbb{C} \setminus (\mathbb{C}_+ \cup \Gamma)$. Suppose that the contour Γ is oriented counterclockwise. Suppose also that $\mathcal{U}, \mathcal{V}, \mathcal{W} \subset \mathbb{C}_+$ and that $\lambda(z)$ is analytical in $\mathbb{C}_+$.

Let $\varepsilon(x)$, $\varphi(x)$, $\eta(x)$ be some smooth functions on Γ. Define the generators of the Lie algebra $\mathcal{A}$ as

$$E_\varepsilon = \frac{1}{2\pi i} \int_\Gamma \varepsilon(x) E(x), \quad H_\eta = \frac{1}{2\pi i} \int_\Gamma \eta(x) H(x), \quad F_\varphi = \frac{1}{2\pi i} \int_\Gamma \varphi(x) F(x), \tag{4.11}$$

To describe the commutation relations between them we need to introduce the decomposition

$$\psi(x) = \psi_+(x) + \psi_-(x)$$

of any function $\psi(x)$ on the contour Γ into the parts $\psi_\pm(z)$ analytical respectively in $z \in \mathbb{C}_\pm$. It is supposed that $\psi_-(z) \to 0$ as $z \to \infty$. The projections are given by the Cauchy integral

$$\psi_\pm(z) := \pm\frac{1}{2\pi i} \int_\Gamma \frac{\psi(x)dx}{x-z}, \qquad x \in \Gamma, \quad z \in \mathbb{C}_\pm \, .$$

Note the formula for the Hilbert transform defined as the singular integral

$$\frac{1}{2\pi i} \int_\Gamma \frac{\psi(y)dy}{x-y} = \frac{\psi_+(x) - \psi_-(x)}{2}, \qquad x, y \in \Gamma$$

regularized in the sense of the principal value.

Now we can write down the commutation relations for the generators (4.11) as

$$[E_{\varepsilon^{(1)}}, E_{\varepsilon^{(2)}}] = [H_{\eta^{(1)}}, H_{\eta^{(2)}}] = [F_{\varphi^{(1)}}, F_{\varphi^{(2)}}] = 0, \tag{4.12a}$$

$$[E_\varepsilon, F_\varphi] = H_{\varepsilon_+\varphi_+ - \varepsilon_-\varphi_-}\ , \tag{4.12b}$$

$$[H_\eta, E_\varepsilon] = 2E_{\eta_+\varepsilon_+ - \eta_-\varepsilon_-}\ , \tag{4.12c}$$

$$[H_\eta, F_\varphi] = -2F_{\eta_+\varphi_+ - \eta_-\varphi_-}\ , \tag{4.12d}$$

The above formulas suggest the decomposition, see [12], of $\mathcal{A}$ into two mutually commutative subalgebras $\mathcal{A} = \mathcal{A}_+ + \mathcal{A}_-$ generated, respectively, by $E_{\varepsilon_\pm}$, $H_{\eta_\pm}$, $F_{\varphi_\pm}$:

$$E_\varepsilon = E_{\varepsilon_+} + E_{\varepsilon_-}, \qquad H_\eta = H_{\eta_+} + H_{\eta_-}, \qquad F_\varphi = F_{\varphi_+} + F_{\varphi_-}.$$

The commutation relations for the subalgebras $\mathcal{A}_\pm$ are those for the sl_2 loop Lie algebra

$$[E_{\varepsilon^{(1)}_\pm}, E_{\varepsilon^{(2)}_\pm}] = [H_{\eta^{(1)}_\pm}, H_{\eta^{(2)}_\pm}] = [F_{\varphi^{(1)}_\pm}, F_{\varphi^{(2)}_\pm}] = 0, \tag{4.13a}$$

$$[E_{\varepsilon_\pm}, F_{\varphi_\pm}] = \pm H_{\varepsilon_\pm\varphi_\pm}, \quad [H_{\eta_\pm}, E_{\varepsilon_\pm}] = \pm 2E_{\eta_\pm\varepsilon_\pm}, \quad [H_{\eta_\pm}, F_{\varphi_\pm}] = \mp 2F_{\eta_\pm\varphi_\pm}, \tag{4.13b}$$

and allow the fundamental representation as 2×2 matrix-valued functions of $x \in \Gamma$

$$E_{\varepsilon_\pm} = \pm\varepsilon_\pm(x)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H_{\eta_\pm} = \pm\eta_\pm(x)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F_{\varphi_\pm} = \pm\varphi_\pm(x)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

with the pointwise commutator.

Now everything is ready to reproduce the same Gauss decomposition trick which we saw working in the finite-dimensional case.

Introduce the group elements

$$\mathcal{E}_{\varepsilon_\pm} := \exp E_{\varepsilon_\pm}, \qquad \mathcal{H}_{\eta_\pm} := \exp H_{\eta_\pm}, \qquad \mathcal{F}_{\varphi_\pm} := \exp F_{\varphi_\pm},$$

having the fundamental representation

$$\mathcal{E}_{\varepsilon_\pm} = \begin{pmatrix} 1 & \pm\varepsilon_\pm(x) \\ 0 & 1 \end{pmatrix}, \quad \mathcal{H}_{\eta_\pm} = \begin{pmatrix} \mathrm{e}^{\pm\eta_\pm(x)} & 0 \\ 0 & \mathrm{e}^{\mp\eta_\pm(x)} \end{pmatrix}, \quad \mathcal{F}_{\varphi_\pm} = \begin{pmatrix} 1 & 0 \\ \pm\varphi_\pm(x) & 1 \end{pmatrix}.$$

The identity

$$\mathcal{E}_{\varepsilon_\pm}\mathcal{H}_{\eta_\pm}\mathcal{F}_{\varphi_\pm} = \mathcal{F}_{\varphi'_\pm}\mathcal{H}_{\eta'_\pm}\mathcal{E}_{\varepsilon'_\pm} \tag{4.14}$$

is established by the same calculation as in the finite-dimensional case

$$\mathrm{e}^{\pm\eta'_\pm} = \mathrm{e}^{\pm\eta_\pm} + \varepsilon_\pm\varphi_\pm\mathrm{e}^{\mp\eta_\pm}, \tag{4.15}$$

$$\varepsilon'_\pm = \frac{\varepsilon_\pm\mathrm{e}^{\mp\eta_\pm}}{\mathrm{e}^{\pm\eta_\pm} + \varepsilon_\pm\varphi_\pm\mathrm{e}^{\mp\eta_\pm}}, \qquad \varphi'_\pm = \frac{\varphi_\pm\mathrm{e}^{\mp\eta_\pm}}{\mathrm{e}^{\pm\eta_\pm} + \varepsilon_\pm\varphi_\pm\mathrm{e}^{\mp\eta_\pm}}, \tag{4.16}$$

one should rememeber only that now $\varepsilon_\pm$, etc. depend on parameter $x \in \Gamma$.

The formula for the generating function of correlators follows then immediately:

$$\begin{aligned}\langle \mathcal{E}_\varepsilon \mathcal{H}_\eta \mathcal{F}_\varphi \rangle &= \left\langle \mathcal{E}_{\varepsilon_+} \mathcal{H}_{\eta_+} \mathcal{F}_{\varphi_+} \right\rangle \left\langle \mathcal{E}_{\varepsilon_-} \mathcal{H}_{\eta_-} \mathcal{F}_{\varphi_-} \right\rangle \\ &= \left\langle \mathcal{F}_{\varphi'_+} \mathcal{H}_{\eta'_+} \mathcal{E}_{\varepsilon'_+} \right\rangle \left\langle \mathcal{F}_{\varphi'_-} \mathcal{H}_{\eta'_-} \mathcal{E}_{\varepsilon'_-} \right\rangle \\ &= \left\langle \mathcal{H}_{\eta'_+} \right\rangle \left\langle \mathcal{H}_{\eta'_-} \right\rangle = \exp \frac{1}{2\pi i} \int_\Gamma \lambda(z) \eta'(z) dz \end{aligned} \tag{4.17}$$

where

$$\eta' = \eta'_+ + \eta'_-, \qquad \eta'_\pm = \pm \ln(\mathrm{e}^{\pm\eta_\pm} + \mathrm{e}^{\mp\eta_\pm} \varepsilon_\pm \varphi_\pm). \tag{4.18}$$

Now we can prove Theorem 3. Put

$$\varepsilon(z) := \sum_{u \in \mathcal{U}} \frac{\varepsilon_u}{z-u}, \qquad \eta(z) := \sum_{w \in \mathcal{W}} \frac{\eta_w}{z-w}, \qquad \varphi(z) := \sum_{v \in \mathcal{V}} \frac{\varphi_v}{z-v}, \tag{4.19}$$

where $\varepsilon_u, \eta_w, \varphi_v \in \mathbb{C}$. Then, since by assumption $\mathcal{U} \subset \mathbb{C}_+$,

$$E_\varepsilon = \exp \frac{1}{2\pi i} \int_\Gamma \sum_{u \in \mathcal{U}} \frac{\varepsilon_u E(z) dz}{z-u} = \exp \sum_{u \in \mathcal{U}} \varepsilon_u E(u), \tag{4.20}$$

and similarly for H_η, F_φ. The correlator $C(\mathcal{U}, \mathcal{W}, \mathcal{V})$ is given then by the coefficient at

$$\left(\prod_{u \in \mathcal{U}} \varepsilon_u \right) \left(\prod_{v \in \mathcal{V}} \varphi_v \right) \left(\prod_{w \in \mathcal{W}} \eta_w \right)$$

in the expansion of the generating function (4.17) in powers of ε_u, η_w, φ_v.

To perform the expansion, note first that

$$\varepsilon_+(z) = 0, \qquad \varepsilon_-(z) = \varepsilon(z) \tag{4.21}$$

and similarly for $\varphi(z)$, $\eta(z)$. Respectively, $\eta'_+ = 0$ and

$$\begin{aligned} \eta' = \eta'_- &= -\ln(\mathrm{e}^{-\eta} + \varepsilon\varphi \mathrm{e}^\eta) = \eta - \ln(1 + \varepsilon\varphi \mathrm{e}^{2\eta}) \\ &= \eta - \sum_{k=1}^\infty \sum_{j=0}^\infty \frac{(-1)^{k-1}}{k} \frac{(2k)^j}{j!} \varepsilon^k \varphi^k \eta^j, \end{aligned} \tag{4.22}$$

where we omitted the argument z of $\varepsilon(z)$, $\eta(z)$, $\varphi(z)$. As it was explained above, we are interested only in the terms linear in ε_u, φ_v, η_w. For example, the contribution of ε^k is

$$k! \sum_{\substack{\mathcal{U}' \subset \mathcal{U} \\ |\mathcal{U}'| = k}} \prod_{u \in \mathcal{U}'} \frac{\varepsilon_u}{z-u}. \tag{4.23}$$

Respectively, the relevant terms in $\eta'(z)$ are

$$\begin{aligned} &\sum_{w \in \mathcal{W}} \frac{\eta_w}{z-w} + \sum_{\substack{\mathcal{U}' \subset \mathcal{U} \\ |\mathcal{U}'| = |\mathcal{V}'|}} \sum_{\substack{\mathcal{V}' \subset \mathcal{V} \\ \mathcal{V}' \neq \emptyset}} \sum_{\mathcal{W}' \subset \mathcal{W}} (-1)^{|\mathcal{U}'|} \, |\mathcal{U}'|! \, (|\mathcal{U}'| - 1)! \, (2\, |\mathcal{U}'|)^{|\mathcal{W}'|} \\ &\qquad \times \prod_{u \in \mathcal{U}'} \frac{\varepsilon_u}{z-u} \prod_{v \in \mathcal{V}'} \frac{\varphi_v}{z-v} \prod_{w \in \mathcal{W}'} \frac{\eta_w}{z-w}. \end{aligned} \tag{4.24}$$

Integrating the result with $\lambda(z)$ and using the equality (3.3) we obtain finally for the relevant contribution to $(1/2\pi i)\int \lambda(z)\eta'(z)dz$:

$$\sum_{w\in\mathcal{W}} \eta_w \lambda(w) \quad + \quad \sum_{\substack{\mathcal{U}'\subset\mathcal{U}\\ |\mathcal{U}'|=|\mathcal{V}'|}} \sum_{\substack{\mathcal{V}'\subset\mathcal{V}\\ \mathcal{V}'\neq\emptyset}} \sum_{\mathcal{W}'\subset\mathcal{W}} (-1)^{|\mathcal{U}'|} \, |\mathcal{U}'|! \, (|\mathcal{U}'|-1)! \, (2\,|\mathcal{U}'|)^{|\mathcal{W}'|}$$

$$\times \left(\prod_{u\in\mathcal{U}'} \varepsilon_u\right) \left(\prod_{v\in\mathcal{V}'} \varphi_v\right) \left(\prod_{w\in\mathcal{W}'} \eta_w\right) \Lambda(\mathcal{U}'\cup\mathcal{V}'\cup\mathcal{W}') \tag{4.25}$$

Exponentiating (4.25) and retaining the term containing the factor $\prod \varepsilon_u \varphi_v \eta_w$ we finally obtain (3.5).

5. Richardson's determinant formula

For the rest of this paper we consider only the case $\mathcal{W}=\emptyset$. To prove the Theorem 2, we derive first a recurrence relation for the correlators $C(\mathcal{U},\emptyset,\mathcal{V}) = \langle \mathcal{U} \mid \mathcal{V}\rangle$.

Note the following property of the function Λ

$$\Lambda(\mathcal{X}\cup\mathcal{Y}) = \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} \frac{\Lambda(\{x,y\})}{\prod_{x'\neq x}(x-x') \prod_{y'\neq y}(y-y')} \tag{5.1}$$

which holds for $\mathcal{X}\cap\mathcal{Y}=\emptyset$ and can be deduced easily from the definition (3.1) of Λ.

Using (5.1) one can rewrite the Λ-representation (3.7) for $\langle \mathcal{U} \mid \mathcal{V}\rangle$ as a polynomial in the two-point correlators $\Lambda(\{u,v\})$ with the coefficients rational in u, v. Let us denote the polynomial in question as $\mathcal{C}(\mathcal{U},\mathcal{V};\ell)$ considering $\ell(u,v)\equiv\Lambda(\{u,v\})$ as a functional parameter:

$$\langle \mathcal{U} \mid \mathcal{V}\rangle = \mathcal{C}(\mathcal{U},\mathcal{V};\Lambda(\{u,v\})).$$

Richardson [11] has found an alternative description of the function $\mathcal{C}(\mathcal{U},\mathcal{V};\ell)$.

Theorem 4 *The function $\mathcal{C}(\mathcal{U},\mathcal{V};\ell)$ possesses the following properties:*

1. *$\mathcal{C}(\mathcal{U},\mathcal{V};\ell)$ is a polynomial in $\ell(u,v)$, $u\in\mathcal{U}$, $v\in\mathcal{V}$.*

2. *Each of the $\ell(u,v)$ enters $\mathcal{C}(\mathcal{U},\mathcal{V};\ell)$ linearly.*

3. *The polynomial $\mathcal{C}(\mathcal{U},\mathcal{V};\ell)$ has no free term, that is*

$$\mathcal{C}(\mathcal{U},\mathcal{V};0)=0.$$

4. *Recurrence relation. Let $\widetilde{\mathcal{U}}=\mathcal{U}\cup\{\tilde{u}\}$ and $\widetilde{\mathcal{V}}=\mathcal{V}\cup\{\tilde{v}\}$. Then the coefficient at $\ell(\tilde{u},\tilde{v})$ in $\mathcal{C}(\widetilde{\mathcal{U}},\widetilde{\mathcal{V}};\ell)$ is equal to $-\mathcal{C}(\mathcal{U},\mathcal{V};\tilde{\ell})$ where*

$$\tilde{\ell}(u,v) := \ell(u,v) + \frac{2}{(u-\tilde{u})(v-\tilde{v})}. \tag{5.2}$$

5. *Initial condition*

$$\mathcal{C}(\{u\},\{v\};\ell) = \ell(u,v).$$

Obviously, the properties 1–5, if true, characterize the function $\mathcal{C}(\mathcal{U},\mathcal{V};\ell)$ uniquely.

Richardson himself has proved the above theorem by a straightforward calculation for a very degenerate case of Gaudin model. Our proof, valid for general Gaudin model, is based on the Λ-representation for $\langle\mathcal{U}\mid\mathcal{V}\rangle$ and is much simpler due to using the machinery of generating functions.

The statements 1, 2 and 3 of the Theorem follow directly from the Λ-representation (3.7) and (5.1), the statement 5 follows from (2.21) and (3.2). The remaining statement 4 (recurrence relation), in principle, can also be derived directly from the Λ-representation which leads, however, to lengthy combinatorial calculations. As in case of the theorem 3, using generating functions again allows to simplify the derivation, .

Putting $\mathcal{W}=\emptyset$ in (4.25) one obtains the relevant (that is linear in ε_u, φ_v) contribution to $(1/2\pi i)\int\lambda(z)\eta'(z)dz$:

$$\sum_{\substack{\mathcal{U}'\subset\mathcal{U}\\|\mathcal{U}'|=|\mathcal{V}'|}}\sum_{\substack{\mathcal{V}'\subset\mathcal{V}\\\mathcal{V}'\neq\emptyset}}(-1)^{|\mathcal{U}'|}\,|\mathcal{U}'|!\,(|\mathcal{U}'|-1)!\left(\prod_{u\in\mathcal{U}'}\varepsilon_u\right)\left(\prod_{v\in\mathcal{V}'}\varphi_v\right)\Lambda(\mathcal{U}'\cup\mathcal{V}'). \tag{5.3}$$

Let us find, first, how the substitution $\ell\mapsto\widetilde{\ell}$ (5.2) can be expressed in terms of the expansion (5.3). ¿From (5.1) it follows that replacing all $\Lambda(\{u,v\})$ with (5.2) is equivalent to replacing all $\Lambda(\mathcal{U}'\cup\mathcal{V}')$ in (5.3) with

$$\Lambda(\mathcal{U}'\cup\mathcal{V}')+\sum_{u\in\mathcal{U}'}\sum_{v\in\mathcal{V}'}\frac{2}{(u-\widetilde{u})(v-\widetilde{v})}\left(\prod_{u'\neq u}\frac{1}{u-u'}\right)\left(\prod_{v'\neq v}\frac{1}{v-v'}\right). \tag{5.4}$$

Substituting (5.4) into (5.3) we conclude, similarly to the derivation of (4.24), that the terms added to $(1/2\pi i)\int\lambda(z)\eta'(z)dz$ coincide exactly with the expansion of

$$-2\ln\left(1+\varepsilon(\widetilde{u})\varphi(\widetilde{v})\right)$$

where $\varepsilon(z)$ and $\varphi(z)$ are given by (4.19).

Exponentiating, one obtains that the substitution $\ell\mapsto\widetilde{\ell}$ is equivalent to multiplying the generating function $\langle\mathcal{E}_\varepsilon\mathcal{F}_\varphi\rangle$ by the factor

$$\exp\left\{-2\ln\left(1+\varepsilon(\widetilde{u})\varphi(\widetilde{v})\right)\right\}=\left(1+\varepsilon(\widetilde{u})\varphi(\widetilde{v})\right)^{-2}. \tag{5.5}$$

Using the binomial series

$$(1+t)^{-2}=\sum_{k=0}^{\infty}(-1)^k(k+1)t^k$$

one can expand the expression (5.5) in powers of ε_u, φ_v (see again the derivation of (4.24) from (4.22)) and retain the relevant terms obtaining finally

$$\sum_{\substack{\mathcal{U}'\subset\mathcal{U}\\|\mathcal{U}'|=|\mathcal{V}'|}}\sum_{\mathcal{V}'\subset\mathcal{V}}(-1)^{|\mathcal{U}'|+1}\,(|\mathcal{U}'|+1)!\,|\mathcal{U}'|!\left(\prod_{u\in\mathcal{U}'}\frac{\varepsilon_u}{\widetilde{u}-u}\right)\left(\prod_{v\in\mathcal{V}'}\frac{\varphi_v}{\widetilde{v}-v}\right). \tag{5.6}$$

Now we return to the expression (5.3) for the relevant part of $(1/2\pi i)\int\lambda(z)\eta'(z)dz$ and replace the sets $\mathcal{U}$ by $\widetilde{\mathcal{U}}$ and $\mathcal{V}$ by $\widetilde{\mathcal{V}}$. In the resulting expansion

$$\sum_{\substack{\mathcal{U}'\subset\widetilde{\mathcal{U}}\\ |\mathcal{U}'|=|\mathcal{V}'|}}\sum_{\substack{\mathcal{V}'\subset\widetilde{\mathcal{V}}\\ \mathcal{V}'\neq\emptyset}}(-1)^{|\mathcal{U}'|}\,|\mathcal{U}'|!\,(|\mathcal{U}'|-1)!\left(\prod_{u\in\mathcal{U}'}\varepsilon_u\right)\left(\prod_{v\in\mathcal{V}'}\varphi_v\right)\Lambda(\mathcal{U}'\cup\mathcal{V}') \tag{5.7}$$

four types of terms can be distinguished.

1. The terms containing no $\widetilde{u}$, $\widetilde{v}$ reproduce obviously the unperturbed expansion (5.3).
2. The terms containing $\widetilde{u}$ but not $\widetilde{v}$.
3. The terms containing $\widetilde{v}$ but not $\widetilde{u}$.
4. The terms containing $\widetilde{u}\in\mathcal{U}'=\mathcal{U}''\cup\{\widetilde{u}\}$ and $\widetilde{v}\in\mathcal{V}'=\mathcal{V}''\cup\{\widetilde{v}\}$

$$\varepsilon_{\widetilde{u}}\varphi_{\widetilde{v}}\sum_{\substack{\mathcal{U}''\subset\mathcal{U}\\ |\mathcal{U}''|=|\mathcal{V}''|}}\sum_{\mathcal{V}''\subset\mathcal{V}}(-1)^{|\mathcal{U}''|+1}\,(|\mathcal{U}''|+1)!\,|\mathcal{U}''|!\left(\prod_{u\in\mathcal{U}''}\varepsilon_u\right)\left(\prod_{v\in\mathcal{V}''}\varphi_v\right)\Lambda(\mathcal{U}''\cup\mathcal{V}''\cup\{\widetilde{u},\widetilde{v}\}) \tag{5.8}$$

(note that here $\mathcal{U}''$ and $\mathcal{V}''$ are allowed to be empty.

Obviously, only the terms of type 4 produce $\Lambda(\{\widetilde{u},\widetilde{v}\})$ via the expansion (5.1). Noticing that the coefficient at $\Lambda(\{\widetilde{u},\widetilde{v}\})$ in the expansion (5.1) of $\Lambda(\mathcal{U}'\cup\mathcal{V}'\cup\{\widetilde{u},\widetilde{v}\})$ equals

$$\prod_{u\in\widetilde{\mathcal{U}}}(\widetilde{u}-u)^{-1}\prod_{v\in\mathcal{V}'}(\widetilde{v}-v)^{-1}$$

and exponentiating (5.8) we obtain that the coefficient at $\varepsilon_{\widetilde{u}}\varphi_{\widetilde{v}}\Lambda(\{\widetilde{u},\widetilde{v}\})$ is given exactly by the expression (5.6) which proves the theorem 4.

The recurrence relation allows to give for $\mathcal{C}(\mathcal{U},\mathcal{V};\ell)$ a completely different representation.

Theorem 5 *(Richardson) Let us fix an ordering of the sets* $\mathcal{U}=\{u_1,\dots,u_N\}$ *and* $\mathcal{V}=\{v_1,\dots,v_N\}$. *The function* $\mathcal{C}(\mathcal{U},\mathcal{V};\ell)$ *can be expressed as the sum of* $N!$ *determinants*

$$(-1)^N\mathcal{C}(\mathcal{U},\mathcal{V};\ell)=\sum_{\sigma\in S_N}\det\mathcal{M}^\sigma \tag{5.9}$$

where the sum is taken over all permutations σ *of indices* $\{1,\dots,N\}$, *and the matrices* $\mathcal{M}^\sigma$ *are defined as*

$$\mathcal{M}^\sigma_{jj}:=\ell(u_j,v_{\sigma_j})+2\sum_{j'\neq j}(u_j-u_{j'})^{-1}(v_{\sigma_j}-v_{\sigma_{j'}})^{-1} \tag{5.10}$$

$$\mathcal{M}^\sigma_{jk}:=-2(u_j-u_k)^{-1}(v_{\sigma_j}-v_{\sigma_k})^{-1},\qquad j\neq k \tag{5.11}$$

Proof. It is easy to verify (see [11]) that the right-hand-side of (5.9) satisfies the same conditions 1–4 of the Theorem 1 which determine uniquely $\mathcal{C}(\mathcal{U},\mathcal{V};\ell)$. The theorem 2 is obtained now as a simple corollary. Indeed, if $\mathcal{U}=\mathcal{V}$ and the parameters v_k satisfy Bethe equations (2.15) then in the sum (5.9) only the term corresponding to the identical permutation is non-zero, and the formula for the norm of Bethe function simplifies to (2.22). For the details see again [11].

For a completely different proof of the formula (2.22) based on the semiclassical expansion of the solutions to the Knizhnik-Zamolodchikov equation, see [10].

6. Discussion

On a simple example of the Gaudin model we have demonstrated that the calculation of polynomial correlators simplifies drastically when one makes use of exponential generating functions. An intriguing and so far open question is if a similar simplification is possible in case of XXX or XXZ magnetic chains, when the Lie algebra (2.9) is replaced by the yangian $\mathcal{Y}[sl_2]$ or, respectively, quantum Lie algebra $\mathcal{U}_q[\widehat{sl_2}]$. It is natural to expect that the complicated formulas for polynomial correlators due to Korepin [6, 7] can be somehow simplified by appropriate choice of generating functions. It would be also interesting to find in this case an analog of our Λ-representation and to find an interpretation of Korepin's 'dual fields' in terms of generating functions.

As a simple introductory exercise, one can consider the finite-dimensional quantum Lie algebra $\mathcal{U}_q[sl_2]$ and reproduce the derivation given in the beginning of the section 4. The necessary formulas concerning the Gauss decomposition for $\mathcal{U}_q[sl_2]$ can be found in [9]. The main new feature in comparison to the Lie algebraic case is that the parameters of expansion ε, η and φ become noncommuting.

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