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DEFORMED YANGIANS AND INTEGRABLE MODELS

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ABSTRACT

Twisted Hopf algebra $sl_\xi(2)$ gives rise to the deformation of the Yangian $\mathcal{Y}(sl(2))$. The corresponding deformations of the integrable XXX-spin chain and the Gaudin model are discussed.

1. Introduction

The development of the quantum inverse scattering method (QISM) gave rise to the quantum group theory [1–3]. The latter one is influence the QISM as well [4]. Taking account recent results on further deformation of the Yangian algebra $\mathcal{Y}$ [5] and the applications of the Yangian symmetries to variety of integrable models [6] the simplest integrable XXX-model related to the Yangian $\mathcal{Y}(gl(2))$ is discussed in this paper. The deformation related to a simple twist of the Lie algebra $sl(2) : \Delta_\xi = \mathcal{F}\Delta\mathcal{F}^{-1}$, with an appropriate element $\mathcal{F} \in U(sl(2)) \otimes U(sl(2))$ preserves the regularity property of the corresponding R-matrix [7]: $R(0) = \mathcal{P}$, where $\mathcal{P}$ is the permutation map in $\mathbb{C}^2 \otimes \mathbb{C}^2$. This observation enables us to write down a deformed Hamiltonian for the Heisenberg chain of length N (XXX_ξ-model) [8],

$$H = \sum_n (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_n^y + \sigma_n^z \sigma_{n+1}^z + \xi^2 \sigma_n^- \sigma_{n+1}^- + \xi(\sigma_n^- - \sigma_{n+1}^-))$$

where ξ is a deformation parameter, σ_n^x, σ_n^y, σ_n^z are Payli sigma-matrices acting in $\mathbb{C}_n^2$ related to the nth site of the chain and $\sigma_n^- = \frac{1}{2}(\sigma_n^x - i\sigma_n^y)$.

However, it turns out that this Hamiltonian is not Hermitian that creates extra difficulties in the constructing the algebraic Bethe Ansatz (ABA) for this model. Due to the triangular structure of the twist we demonstrate that the spectrum of the transfer matrix and the Bethe equations of the deformed XXX-model are coincide with the usual ones.

The connection of the XXX-spin chain with the representation theory of the $sl(2)$ algebra in the fremework of the QISM was explained in [9]. The deformed XXX_ξ-spin chain has a similar connection with the representation theory of the twisted $sl(2)$.

Let us point out that for the periodic boundary condition the obtained Hamiltonian coincides with appropriate limit of the seven-vertex model Hamiltonian [10]. An interesting application of the Drinfel'd twist depending on the spectral parameter, to the original XXX-model is discussed in the recent paper [11]. The Gaudin model directly related to the Yangian, was studied also in the framework of the QISM [12].

2. Yang–Baxter algebra

Let us consider the Yang–Baxter algebra [7, 13]

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v),$$

for the 2×2 matrix

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},$$

given by the twisting of the Yang solution

$$R(u-v) = F_{21}\left(I - \frac{\eta}{u-v}\mathcal{P}\right)F_{12}^{-1} = R_\xi - \frac{\eta}{u-v}\mathcal{P},$$

with

$$F_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \xi & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\xi & 1 \end{pmatrix}, \quad R_\xi = F_{21}F_{12}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\xi & 1 & 0 & 0 \\ \xi & 0 & 1 & 0 \\ \xi^2 & -\xi & \xi & 1 \end{pmatrix}.$$

This is a reduction to the fundamental representation ρ in $\mathbb{C}^2$ of the universal twist element $\mathcal{F}$, $F_{12} = (\rho \otimes \rho)\mathcal{F}$ and the universal R-matrix $\mathcal{R}$ of the twisted Yangian $\mathcal{Y}_{\eta,\xi}(sl(2))$ [5]. Using the generators h, e of the $sl(2) : [h, e] = -2e$, the universal twist $\mathcal{F}$ is

$$\mathcal{F} = 1 + \xi h \otimes e + \frac{1}{2!}\xi^2 h(h+2) \otimes e^2 + \dots .$$

There are quite a few papers on the non-standard quantum algebra $sl_\xi(2)$ (see e.g. [14, 15] and Refs therein). Introducing the variable $\sigma : 1 - 2\xi e = \exp(-\sigma)$ this twist is $\mathcal{F} = \exp(h \otimes \sigma/2)$.

To study the corresponding Bethe Ansatz we need all 16 commutation relations (CR) of the form

$$\sum t_{ij}(u)t_{kl}(v) = \sum t_{mn}(v)t_{pq}(u)\dots$$

Constructing $T(u) = L_N(u)\dots L_2(u)L_1(u)$ from local L-operators given by the R-matrix (?) itself according to the QISM [7, 11] we write down at the beginning the CR of $A(u)$ and $D(u)$ with $C(v)$, using notation for the R-matrix entries:

$$\alpha(u,v) = 1 + \beta(u,v) = 1 - \frac{\eta}{u-v},$$

$$\begin{aligned}A(u)C(v) = \alpha(u,v)C(v)A(u) - (\beta(u,v)C(u) - \xi A(u))A(v)\\ - \xi D(v)A(u) + (\xi C(v) + \xi^2 D(v))B(u),\end{aligned}$$

$$D(u)C(v) = (\alpha(v,u)C(v) - \xi A(v))D(u) - (\beta(v,u)C(u) - \xi D(u))D(v) + (\xi C(v) + \xi^2 A(v))B(u),$$

$$B(u)C(v) = (C(v) + \xi D(v))B(u) + (\xi B(u) - \beta(u,v)D(u))A(v) + \beta(u,v)D(v)A(u)$$

$$\begin{aligned}\alpha(u,v)C(u)C(v) = (\alpha(u,v)C(v) - \xi D(v))C(u)\\ + (\xi C(v) + \xi^2 D(v))D(u) + (-\xi C(u) - \xi A(u))A(v) + \xi A(u)C(v)\end{aligned}$$

$$\begin{aligned}\alpha(u,v)A(u)A(v) &= (\alpha(u,v)A(v) - \xi B(v))A(u) + (\xi A(v) + \xi^2 B(v))B(u),\\ \alpha(u,v)A(u)B(v) &= B(v)A(u) + (\beta(u,v)A(v) - \xi B(v))B(u),\\ B(u)B(v) &= B(v)B(u),\\ A(u)D(v) &= D(v)A(u) + (\xi A(u) - \beta(u,v)C(u))B(v) + (\beta(u,v)C(v) - \xi D(v))B(u),\\ \alpha(v,u)D(u)B(v) &= B(v)D(u) + (\beta(v,u)D(v) - \xi B(v))B(u),\\ (C(u) + \xi A(u))A(v) &= (\alpha(u,v)A(v) - \xi B(v))C(u) + (\xi A(v) + \xi^2 B(v))D(u) - \beta(u,v)A(u)C(v),\\ B(u)C(v) &= (C(v) + \xi A(v))B(u) + \xi B(u)D(v) + \beta(v,u)(A(v)D(u) - A(u)D(v)),\\ \beta(u,v)B(u)C(v) &= \beta(u,v)B(v)C(u) + (A(v) + \xi B(v))D(u) - (D(u) + \xi B(u))A(v),\\ D(u)B(v) &= \alpha(u,v)B(v)D(u) - \beta(u,v)B(u)D(v) - B(u)B(v),\end{aligned}$$

$$\begin{aligned}(\alpha(u,v)C(u) - \xi A(u))C(v) = (\alpha(u,v)C(v) - \xi D(v))C(u)\\ + (\xi C(v) + \xi^2 D(v))D(u) - (\xi C(u) + \xi^2 A(u))A(v).\end{aligned}$$

The first step of the ABA is the existing of a bare vacuum Ω: (such that)

$$A(u)\Omega = a(u)\Omega, \quad D(u)\Omega = d(u)\Omega, \quad B(u)\Omega = 0.$$

For the twisted XXX-model of spin 1/2 (there is the corresponding state for the XXX_ξ-models of other spin values too)

$$\Omega = \bigotimes_{k=1}^{N} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad a(u) = 1, \quad d(u) = (\alpha(u))^N. \tag{2}$$

Let us analyse the conditions under which the state (vector) $C(v)\Omega$ is an eigenvector of the transfer matrix $t(u)$:

$$t(u) = \operatorname{tr} T(u) = A(u) + D(u), \quad t(u)\Omega = (1 + d(u))\Omega.$$

Using the CR to intertwise $(A(u) + D(u))$ over $C(v)$ and the properties of the bare vacuum $\Omega(2)$ one gets

$$\begin{aligned} t(u)C(v)\Omega &= (1 \cdot \alpha(u,v) + d(u)\alpha(v,u))C(v)\Omega \\ &\ominus (\beta(u,v) + \beta(v,u)d(v))C(u)\Omega \\ &+ \xi(1 - d(u))(1 - d(v))\Omega. \end{aligned}$$

Therefore we can conclude that $C(v)\Omega$ is an eigenvector of the transfer matrix $t(u)$ if $(\alpha(v))^N = 1$. This is the Bethe equation for the one magnon sector. The action of $t(u) = A(u) + D(u)$ on the vector $C(v_1)\ C'(v_2)\ \Omega$ results in the following combination of $C \cdot C \cdot \Omega$, $C\Omega$ and Ω vectors:

$$(\alpha(u,v_1)\alpha(u,v_2) + d(u)\alpha(v_1,u)\alpha(v_2,u))C(v_1)C(v_2)\Omega + \text{“unwanted terms”}.$$

Hence, the eigenvalue of the transfer matrix $t(u)$ is the same as for the XXX-model, although the structure of the “unwanted terms” is more complicated and the standard Bethe equations

$$\left(\frac{v_j - \eta}{v_j}\right)^N = \prod_k{}' \frac{v_k - v_j + \eta}{v_k - v_j - \eta}.$$

are not enough to have

$$\Psi(v_1, \ldots, v_M) = C(v_1) \ldots C(v_M)\Omega$$

as an eigenvector of the transfer matrix: $t(u)\Psi = \Lambda(u, \{v_j\})\Psi$. This fact is related with triangularity of the perturbation of the XXX-model Hamiltonian [10]. One can see this phenomenon by looking even at the Clebsch–Gordan coefficients of the twisted $sl_\xi(2)$ algebra in the original basis h, e, f. Due to the connection of the twisted coproduct with the standard one by the similarity transformation $\Delta_\xi = \mathcal{F}\Delta\mathcal{F}^{-1}$, the CG coefficients of the $sl(2)$ are linear combinations of the usual ones with the elements of triangular matrix $F^{n_1,n_2}_{m_1,m_2}(\pi,\rho)$ of the twist $(\pi \otimes \rho)\mathcal{F}$ in the tensor product of two irreducible representations $V_\pi \otimes V_\rho$ (cf. [15]).

3. Symmetry algebra $sl_\xi(2)$

The local structure of the monodromy matrix $T(u)$ gives rise to the following asymptotic expension:

$$T_N(u) = L_N(u)\dots L_1(u) = \prod_{k=1}^{N} R_{ka}(\xi) + \frac{1}{u}\sum_{k=1}^{N} M_k \mathcal{P}_{ak} M_{N-k-1} + O\left(\frac{1}{u^2}\right),$$

where $M_k = \prod_{m=1}^{k-1} R_{ma}(\xi) = (\Delta^{(k-1)} \otimes id)R(\xi)$. Using for the constant term the notation:

$$T_0 = \begin{pmatrix} E & 0 \\ G & E^{-1} \end{pmatrix} = \prod_{k=1}^{N} R_{ka}(\xi) = (\Delta^{(N-1)} \otimes id)R(\xi),$$

one gets the symmetry algebra $sl_\xi(2)$ of the quantum scattering data

$$\begin{aligned}
EG &= GE - \xi(1-E^2),\\
E\,A(u) &= A(u)E - \xi\,B(u)E,\\
E\,D(u) &= D(u)E + \xi\,EB(u),\\
E\,B(u) &= B(u)E,\\
E\,C(u) &= C(u)E + \xi\,E\,A(u) - \xi\,D(u)E =\\
&= C(u)E + \xi(A(u) - D(u))E - \xi^2 B(u)E;\\
G\,B(u) &= B(u)G - \xi(E\,B(u) + B(u)E^{-1}),\\
G\,A(u) &= A(u)G - \xi(E\,A(u) - A(u)E^{-1} + B(u)G) + \xi^2 B(u)E^{-1},\\
GD(u) &= D(u)G + \xi(ED(u) - D(u)E^{-1} - GB(u)) - \xi^2 B(u)E,\\
GC(u) &= C(u)G + \xi(EC(u) + C(u)E^{-1} - GA(u) - D(u)G) +\\
&\quad + \xi^2(D(u)E^{-1} - EA(u)).
\end{aligned}$$

The element E is the group like one and it commutes with the transfer matrix $t(u)$. The elemnts E and G have simple behavior under coproduct map:

$$\Delta(E) = E \otimes E, \quad \Delta(G) = G \otimes E + E^{-1} \otimes G.$$

However, to extract from $T(u)$ the third generator of the twisted Hopf algebra $sl_\xi(2)$ one has to consider the term $1/u$ in the asymptotic expension of $T(u)$.

The twist structure of the R-matrix (1) leads to the similar structure of the monodromy matrix $T(u)$. Hence, the quantum scattering data of the XXX_ξ-model can be expressed as linear combinations of the quantum scattering data of the XXX-model with coefficients constructed from the twist $\mathcal{F}$.

The twisted R-matrix has the same spectral decomposition as the original one:

$$\mathcal{P}R = F_{12}\mathcal{P}F_{12}^{-1} = P_+ - P_-.$$

The same is true for the R-matrix which depends on the spectral parameter. Hence, applying the fusion procedure for the monodromy matrix [8] one gets relations among the transfer matrices in different representations for the auxiliary space

$$t^{(l+1)}(u) = t^{(l)}(u)t^{(1)}(u-\eta) + (u+\eta)^N t^{(l-1)}(u).$$

Another important ingredient of the QISM is the Baxter difference equation

$$\Lambda(u)Q(u) = Q(u-\eta) + d(u)Q(u+\eta).$$

The appropriate quasiclassical limit $\eta \simeq \xi \to 0$ gives rise to a deformation of the Gaudin model, analyzed also in the framework of the QISM [12]. One can apply to the proposed XXX_ξ-model the functional Bethe Ansatz [16] as well.

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