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TOPOLOGIES ON PRODUCTS AND RATIOS

A. A. IVANOV

Russian Academy of Sciences
Steklov Mathematical Institute
St. Petersburg Department
Fontanka 27, St. Petersburg 191011, Russia

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§1. Preliminaries

1. Basic definitions. We shall consider here different sets X, Y, Z, ... supplied with some topological structures, denoting the corresponding topological spases by the same letters X, Y, Z, ... If we wish to accentuate the fact that on some space, for example X, a (new) topological structure τ is fixed, we shall write X_τ. The same approach we can use for mappings and their graphs. Note, that we consider arbitrary mappings including partial and many-valued mappings, that is mappings for which the images of some points may be empty or contain more then one point.

If $F \subset X \times Y \times Z$ then we shall consider two mappings

$$F : X \times Y \longrightarrow Z$$

and

$$F : X \longrightarrow Y \times Z,$$

where

$$F(x,y) = F_{x,y} = \{z | (x,y,z) \in F\},$$

$$F(x) = F_x = \{(y,z) | (x,y,z) \in F\}.$$

The set F_x is a subset of the set $Y \times Z$ and, therefore, we can consider this set as the graph of the mapping $G(x) = F_x : Y \longrightarrow Z$ (we are forced to use here letter G instead of F to avoid misunderstanding). The set of mappings of Y into Z we shall denote by Y/Z and call ratio of Y and Z. So we have a new mapping $G : X \longrightarrow Y/Z$. There is the one-to-one correspondence between F and G, defined by correlation $F(x,y) = G(x)(y)$.

Let S be a topological space, $\mathcal{A} = (A_\xi | \xi \in \Xi)$ be a directed system of subsets of S and A be some subset of S. We shall say that $\mathcal{A}$ converges to A in S, writing down the fact by $\mathcal{A} \longrightarrow_S A$, if for each point $s \in A$ and neighbourhoods U_s, U_A in S of the point s and the set A, there exists $\xi_0 \in \Xi$ such that $A_\xi \subset U_A$, $A_\xi \cap U_s \neq \emptyset$ for $\xi \geq \xi_0$.

We shall say, that a mapping $m : S' \longrightarrow S$ is continuous, if m transforms any directed system $\mathcal{A}'$ of points of S', convergent in S' to a point a, in the directed system $\mathcal{A} = m(\mathcal{A}')$ of subsets of S, convergent to $m(a)$ in S. Now we shall consider only continuous mappings of Y into Z, denoting by means of Y/Z just the set of all such mappings. Thus we have

$$F : X \times Y \longrightarrow Z,$$

$$G : X \longrightarrow Y/Z,$$

$$F(x,y) = F_{x,y} = \{z | (x,y,z) \in F\} \subset Z,$$

$$G(x)(y) = F(x,y),$$

Now we are going to introduce the main notions of the paper, namely the idea of splitting topological structures and the idea of jointly continuous topological structures, bouth on products and ratios. We shall consider in the corresponding definitions topological spaces X, Y, Z, $X \times Y$, $(X \times Y)_\tau$, Y/Z, $(Y/Z)_\tau$, their role will distinct one. In all these definitions we consider an arbitrary fixed topological space Y. Each topological space $X \times Y$ we consider with some fixed topological structure. Here X can vary, so we can consider a system $\mathcal{X}_Y$ of topological spaces, for which a topological structure on each $X \times Y$ is fixed. Under the circumstances these topological structures on spaces $X \times Y$ have to be additive according to the first argument for any system $\mathcal{X}_Y$. Fixed on each $X \times Y$ topological structure is called here the $\mathcal{X}_Y$-system-topology.

On the other hand, each topological space Y/Z we consider also with some fixed topological structure, but in this case the second argument Z can vary, so we can consider a system $\mathcal{Z}_Y$ of topological spaces, for which a topological structure on each Y/Z is fixed. Under the circumstances these topological structures on spaces Y/Z have to be multiplicative according the second argument for any system $\mathcal{Z}_Y$. Fixed on each Y/Z topological structure is called here the $\mathcal{Z}_Y$-system topology.

Topological structures τ on $(X \times Y)_\tau$ and $(Y/Z)_\tau$ are not fixed. We shall looking for topological structure τ which answers the requirements included in the corresponding definitions. Here are the definitions.

Let $\mathcal{A}$ be a fixed set of topological spaces. A topology τ on Y/Z is called $\mathcal{A}$-splitting topology on Y/Z iff for any $X \in \mathcal{A}$ the continuity of any mapping $F : X \times Y \longrightarrow Z$ implies the continuity of the corresponding mapping $G : X \longrightarrow (Y/Z)_\tau$.

A topology τ on Y/Z is called $\mathcal{A}$-jointly continuous topology on Y/Z iff for any $X \in \mathcal{A}$ the continuity of any mapping $G : X \longrightarrow (Y/Z)_\tau$ implies the continuity of the corresponding mapping $F : X \times Y \longrightarrow Z$.

A topology τ on $X \times Y$ is called $\mathcal{A}$-splitting topology on $X \times Y$ iff for any $Z \in \mathcal{A}$ the continuity of any mapping $F : (X \times Y)_\tau \longrightarrow Z$ implies the continuity of the corresponding mapping $G : X \longrightarrow Y/Z$.

A topology τ on $X \times Y$ is called $\mathcal{A}$-jointly continuous topology on $X \times Y$ iff for any $Z \in \mathcal{A}$ the continuity of any mapping $G : X \longrightarrow Y/Z$ implies the continuity of the corresponding mapping $F : (X \times Y)_\tau \longrightarrow Z$.

2. The history of the problem. The problem of investigation of different function spaces, that is topological structures on sets $C(Y, Z)$ (Z^Y in another terminology, Y/Z in our terminology) of continuous mappings Y into Z attracted the great attention of many mathematicians for a long time. They investigated some individual topological structures on $C(X, Y)$ and some classes of such topological structures using different methods and different constructions. One of such constructions has been proposed in [1]. The construction deals with splitting and jointly continuous topological structures for the case when $X \times Y$ has the usual topological structure of production and X is any topological space. It is interesting to remark, that in [1] an attempt to consider on $X \times Y$ not only the usual structure of product has been realized, but nevertheless the notions of splitting topology and jointly continuous topology were inroduced there for function spaces only. The considrable progress in the theory of splitting and jointly continuous topologies on function spaces was achieved in [2]. In the paper the first space X of product $X \times Y$ is not arbitrary fixed space, but a space from a set $\mathcal{A}$ of topological spaces and so the new notions of $\mathcal{A}$-splitting and $\mathcal{A}$-jointly continuous topological structures on $C(Y, Z)$ was introduced. In particular, we can consider a set $\mathcal{A}$, containing one space only. But both [2] and [1] deals with usual product structure $X \times Y$. Totally new idea has been proposed in [3]. Here we are met with different topological structures not only on Y/Z but also on $X \times Y$, Y/Z and $X \times Y$ can as if mutually interchange their places in definitions. Thus the notions of spitting topological structure and jointly continuous topological structure proves to be possible both for Y/Z and for $X \times Y$. On the bases of these notions a theory of duality between topological structures on products and topological structures on ratios has been created in [3]. In the present paper the whole of this theory extends by introducing in the consideration some new ideas, in particular partial and many-valued mappings.

§2. Topological structures on ratios

1. Basic properties. We introduced the notions of $\mathcal{A}$-splitting topological structures and $\mathcal{A}$-jointly continuous topological structures on Y/Z on condition that the topologies on $X \times Y$

for all X (not necessary for $X \in \mathcal{A}$ only) were fixed additively relative to the first argument. Of course, we can change these topological structures then the corresponding systems of $\mathcal{A}$-splitting topological structures and $\mathcal{A}$-jointly continuous topological structures on Y/Z will change too. We shall have it in mind not fixing in notations. Here are some properties of the notions.

Let $\tau_1,\ \tau_2$ be topological structures on Y/Z and $\tau_1 \subset \tau_2$. If τ_2 is an $\mathcal{A}$-splitting topological structure then τ_1 will be an $\mathcal{A}$-splitting topological structure, if τ_1 is an $\mathcal{A}$-jointly continuous topological structure then τ_2 will be an $\mathcal{A}$-jointly continuous topological structure.

Let $\mathcal{A}_1,\ \mathcal{A}_2$ be families of topological spaces and $\mathcal{A}_1 \subset \mathcal{A}_2$. If τ is an $\mathcal{A}_2$-splitting topological structure on Y/Z or an $\mathcal{A}_2$-jointly continuous topological structure on Y/Z then τ will be an $\mathcal{A}_1$-splitting topological structure on Y/Z or an $\mathcal{A}_1$-jointly continuous topological structure on Y/Z, respectively.

The proofs of these properties are evident. Let us consider now the following important property.

For every family $\mathcal{A}$ of topological spaces there exists the greatest $\mathcal{A}$-splitting topological structure on Y/Z denoted by $\tau(\mathcal{A})$. If we consider a system $\mathcal{X}_Y$ of all topological spaces with system-topologies, we shall use the notation $\tau(\mathcal{X}_Y)$. Sometimes we shall write simply $\tau(\mathcal{X})$, Y being fixed.

The proof of this property is evident too. On the other hand the least $\mathcal{A}$-jointly continuous topological structure on Y/Z not necessary exists. The fact is that in the first case we are looking for the greatest topological structure τ on Y/Z for which each mapping

$$G : X \longrightarrow (Y/Z)_\tau$$

of some fixed system $\mathfrak{G}$ is continuous one. It is the intersection of the maximal topological structures $\tau_G,\ G \in \mathfrak{G}$ for which

$$G : X \longrightarrow (Y/Z)_{\tau_G}$$

is continuous. But in the second case we are looking for the least topological structure τ on Y/Z for which each mapping

$$G : X \longrightarrow (Y/Z)_\tau$$

of some fixed system $\mathfrak{G}'$ is not continuous one. Such topological structure not necessary exists.

The greatest $\mathcal{X}$-splitting topological structures on Y/X, have the following important property.

Theorem. *The topological structures $\tau(\mathcal{X})$ on Y/Z are multiplicative according to the second argument.*

Proof. Let $Z = \prod Z_\xi | \xi \in \Xi$, τ_ξ be the topological structure on Z_ξ, $\tau = \prod \tau_\xi | \xi \in \Xi$, $\tau(\mathcal{X})$ and $\tau_\xi(\mathcal{X})$ are the greatest $\mathcal{X}$-splitting topological structure on Y/Z and, respectively, the greatest $\mathcal{X}$-splitting topological structures on Y/Z_ξ for all $\xi \in \Xi$. We have evidently $Y/Z = \prod Y/Z_\xi | \xi \in \Xi$. Consider the topological structure $\tau = \prod \tau_\xi(\mathcal{X}) | \xi \in \Xi$ on Y/Z. It is easy to prove that $\tau = \tau(\mathcal{X})$, taking into account the fact that

$$F : X \times Y \longrightarrow Z \text{ is continuous} \iff \text{all } F_\xi : X \times Y \longrightarrow Z_\xi \text{ are continuous,}$$

$$G : X \longrightarrow (Y/Z)_\tau \text{ is continuous} \iff \text{all } G_\xi : X \longrightarrow (Y/Z_\xi)_{\tau_\xi} \text{ are continuous.}$$

The theorem is proved.

2. Theorem of equivalence. Following [2] we call two families $\mathcal{A}_1$ and $\mathcal{A}_2$ equivalent ($\mathcal{A}_1 \sim \mathcal{A}_2$) if any topological structure τ on Y/Z is $\mathcal{A}_1$-splitting or $\mathcal{A}_1$-jointly continuous iff τ is $\mathcal{A}_2$-splitting or $\mathcal{A}_2$-jointly continuous, respectively. In [2] the following theorem has been proved. Of course the author supposed that on each $X \times Y$ the usual topological structure of product is fixed, but the proof is suitable also for the general situation.

Theorem. *For every family $\mathcal{A}$ of topological spaces there exists a space $X(\mathcal{A})$ such that $\mathcal{A} \sim \{X(\mathcal{A})\}$.*

Proof. Let T^{sp} be the set of all topological structures on the set Y/Z, which are not $\mathcal{A}$-splitting structures and T^{jc} be the set of all topological structures on the set Y/Z, which are not $\mathcal{A}$-jointly continuous, X_τ^{sp} be an element of $\mathcal{A}$ defined for each $\tau \in T^{sp}$ such that τ is not X_τ^{sp}-splitting, X_τ^{jc} be an element of $\mathcal{A}$ defined for each $\tau \in T^{jc}$ such that τ is not X_τ^{jc}-jointly continuous,

$$\mathcal{A}_0 = \{X_\tau^{sp} | \tau \in T^{sp}\} \cup \{X_\tau^{jc} | \tau \in T^{jc}\}$$

We can suppose, of course, that all topological spaces from $\mathcal{A}_0$ have no common points.

Let $X(\mathcal{A})$ be the free union of all elements of $\mathcal{A}_0$, $F : X(\mathcal{A}) \times Y \longrightarrow Z$ be a mapping, F_X be the restriction of F on $X \times Y$, $X \in \mathcal{A}_0$. It is easy to see that the mapping G_X corresponding to F_X will be the restriction on X of the mapping G corresponding to F. We shall prove that $\mathcal{A} \sim \{X(\mathcal{A})\}$.

Let τ be an $\mathcal{A}$-splitting topological structure on Y/Z and $F : X(\mathcal{A}) \times Y \longrightarrow Z$ be a continuous mapping. Then F_X is the continuous mapping too and, since τ is $\mathcal{A}$-splitting mapping, the mapping G_X will be continuous mapping for every $X \in \mathcal{A}_0$, therefore G will be continuous mapping. It means that τ is $\{X(\mathcal{A}_0\})$-splitting topological structure.

Let now τ be a not $\mathcal{A}$-splitting topological structure on Y/Z, that is $\tau \in T^{sp}$. It means that τ is not X_τ^{sp}-splitting topological structure, that is there exists a mapping $F_{X_\tau^{sp}}$ for which the mapping $G_{X_\tau^{sp}}$ is not continuous one. We can extend $F_{X_\tau^{sp}}$ on $X(\mathcal{A})$ arbitrarily to receive a mapping $F : X(\mathcal{A}) \times Y \longrightarrow Z$. It is easy to see that the corresponding mapping $G : X(\mathcal{A}) \longrightarrow Y/Z$ is not continuous one. It means that τ is not $X(\mathcal{A})$-splitting one.

In the same way we can prove that each topological structure τ on Y/Z is $\mathcal{A}$-jointly continuous iff τ is $X(\mathcal{A})$-jointly continuous. The theorem is proved.

Just proved theorem is of greate importance one. It allows to reduce the problem of investigation of all $\mathcal{A}$-splitting and $\mathcal{A}$-jointly continuous topological structures on Y/Z to the partial problem of investigation of X-splitting and X-jointly continuous topological structures, that is to the problem of special investigation of triplets X, Y, Z.

3. Functoral systems $\mathcal{X}$. We shall call a system $\mathcal{X}$ *functoral system* if for every topological spaces X, X' the continuity of $f : X' \longrightarrow X$ implies the continuity of $f \times 1 : X' \times Y \longrightarrow X \times Y$, $(f \times 1)(x, y) = (f(x), y)$. Here we mean, of course, that on $X' \times Y$ and on $X \times Y$ we consider the $\mathcal{X}$-system-topologies. It can be proved the following statement.

Theorem. *Let $f : X' \longrightarrow X$ be factor-mapping and $\mathcal{X}$ be a functoral system. Then for this system $\tau(X') \subset \tau(X)$.*

Proof. Consider the following mappings for X' and for X.

$$for X'$$

$$F' : X' \times Y \longrightarrow Z,$$

$$G' : X' \longrightarrow (Y/Z)_{\tau(X')}.$$

$$for X$$

$$F : X \times Y \longrightarrow Z,$$

$$G : X \longrightarrow (Y/Z)_{\tau(X)}.$$

It is easy to see, that continuity of mappings F and G implies continuity of mappings $F' = F(f\times 1) : X'\times Y \longrightarrow Z$ and $G' = Gf : X \longrightarrow (Y/Z)_{\tau(X)}$. Let now the mapping F be continuous, then mappings F' and G will be continuous too, the letter means the continuity of mapping $G' = Gf : X \longrightarrow (Y/Z)_{\tau(X)}$. Thus the mappings

$$F' : X'\times Y \longrightarrow Z,\ F' = F(f\times 1),$$

$$G' : X' \longrightarrow (Y/Z)_{\tau(X)},\ G' = Gf,$$

are continuous. Moreover, the continuity of mapping G' implies the continuity of mapping G for any topological structure τ on Y/Z, since f is a factor-mapping, and so $\tau(X') \subset \tau(X)$. The theorem is proved.

§3. Topological structures on products

1. Basic properties. We introduced the notions of $\mathcal{A}$-splitting topological structures and $\mathcal{A}$-jointly continuous topological structures on $X\times Y$ on condition that the topologies on Y/Z for all Z (not necessary for $Z\in\mathcal{A}$ only) were fixed multiplicatively relative to the second argument. Of course we can change these topological structures then the corresponding systems of $\mathcal{A}$-splitting topological structures and $\mathcal{A}$-jointly continuous topological structures on $X\times Y$ will change too. We shall have it in mind not fixing in notations. Here are some properties of the notions.

Let $\tau_1,\ \tau_2$ be topological structures on $X\times Y$ and $\tau_1\subset\tau_2$. If τ_2 is an $\mathcal{A}$-splitting topological structure then τ_1 will be an $\mathcal{A}$-splitting topological structure, if τ_1 is an $\mathcal{A}$-jointly continuous topological structure then τ_2 will be an $\mathcal{A}$-jointly continuous topological structure.

Let $\mathcal{A}_1,\ \mathcal{A}_2$ be families of topological spaces and $\mathcal{A}_1\subset\mathcal{A}_2$. If τ is an $\mathcal{A}_2$-splitting topological structure on $X\times Y$ or an $\mathcal{A}_2$-jointly continuous topological structure on $X\times Y$ then τ will be an $\mathcal{A}_1$-splitting topological structure on $X\times Y$ or an $\mathcal{A}_1$-jointly continuous topological structure on $X\times Y$, respectively.

The proofs of these properties are evident. Let us consider now the following important property.

For every family $\mathcal{A}$ of topological spaces there exists the least $\mathcal{A}$-jointly continuous topological structure on $X\times Y$ denoted by $\tau(\mathcal{A})$.

The proof of this property is evident too. On the other hand the greatest $\mathcal{A}$-jointly continuous topological structure on $X\times Y$ not necessary exists. The fact is that in the first case we are looking for the least topological structure τ on $X\times Y$ for which each mapping

$$F : (X\times Y)_\tau \longrightarrow Z$$

of some fixed system $\mathfrak{F}$ is continuous one. It is the union of the minimal topological structures $\tau_F,\ F\in\mathfrak{F}$ for which

$$F : (X\times Y)_{\tau_F} \longrightarrow Z$$

is continuous. But in the second case we are looking for the greatest topological structure τ on $X\times Y$ for which each mapping

$$F : (X\times Y)_\tau \longrightarrow Z$$

of some fixed system $\mathfrak{F}'$ is not continuous one. Such topological structure not necessary exists.

The least $\mathcal{X}$-jointly continuous topological structures on $X\times Y$, has the following important property.

Theorem. *The topological structures $\tau(\mathcal{Z})$ on $X \times Y$ are additive according to the first argument.*

Proof. Let $X = \bigcup X_\xi | \xi \in \Xi$, τ_ξ be the topological structure on X_ξ, $\tau = \bigvee \tau_\xi | \xi \in \Xi$, $\tau(\mathcal{Z})$ and $\tau_\xi(\mathcal{Z})$ are the least $\mathcal{Z}$-jointly continuous topological structure on $X \times Y$ and, respectively, the least $\mathcal{Z}$-jointly continuous topological structures on $X_\xi \times Y$ for all $\xi \in \Xi$. We have evidently $X \times Y = \bigvee X_\xi \times Y | \xi \in \Xi$. Consider the topological structure $\tau = \bigvee \tau_\xi(\mathcal{Z}) | \xi \in \Xi$ on $X \times Y$. It is easy to prove that $\tau = \tau(\mathcal{Z})$, taking into account the fact that

$$F : (X \times Y)_\tau \longrightarrow Z \text{ is continuous} \iff \text{all } F_\xi : (X_\xi \times Y)_{\tau_\xi} \longrightarrow Z \text{ are continuous,}$$

$$G : X_\xi \longrightarrow Y/Z \text{ is continuous} \iff \text{all } G_\xi : X_\xi \longrightarrow Y/Z \text{ are continuous.}$$

The theorem is proved.

2. Theorem of equivalence. Following [3] we call two families $\mathcal{A}_1$ and $\mathcal{A}_2$ equivalent ($\mathcal{A}_1 \sim \mathcal{A}_2$) if any topological structure τ on $X \times Y$ is $\mathcal{A}_1$-splitting or $\mathcal{A}_1$-jointly continuous iff τ is $\mathcal{A}_2$-splitting or $\mathcal{A}_2$-jointly continuous, respectively. In [3] the following theorem has been proved in the general situation.

Theorem. *For every family $\mathcal{A}$ of topological spaces there exists a space $Z(\mathcal{A})$ such that $\mathcal{A} \sim \{Z(\mathcal{A})\}$.*

Proof. Let T^{sp} be the set of all topological structures on the set $X \times Y$, which are not $\mathcal{A}$-splitting structures and T^{jc} be the set of all topological structures on the set $X \times Y$, which are not $\mathcal{A}$-jointly continuous, Z^{sp}_τ be an element of $\mathcal{A}$ defined for each $\tau \in T^{sp}$ such that τ is not Z^{sp}_τ-splitting, Z^{jc}_τ be an element of $\mathcal{A}$ defined for each $\tau \in T^{jc}$ such that τ is not Z^{jc}_τ-jointly continuous,

$$\mathcal{A}_0 = \{Z^{sp}_\tau | \tau \in T^{sp}\} \cup \{Z^{jc}_\tau | \tau \in T^{jc}\}$$

Let $Z(\mathcal{A})$ be the product of all elements of $\mathcal{A}_0$, $F : X \times Y \longrightarrow Z(\mathcal{A})$ be a mapping, F_Z be the projection of F on Z, $Z \in \mathcal{A}_0$ ($F_Z = pr_Z F$). It is easy to see that the mapping G_Z corresponding to F_Z will be the projection on Y/Z of the mapping G corresponding to F ($G_Z = pr_{Y/Z} G$). We shall prove that $\mathcal{A} \sim \{Z(\mathcal{A})\}$.

Let τ be an $\mathcal{A}$-splitting topological structure on $X \times Y$ and $F : (X \times Y)_\tau \longrightarrow Z(\mathcal{A})$ be a continuous mapping. Then F_Z is the continuous mapping too and, since τ is $\mathcal{A}$-splitting mapping, the mapping G_Z will be continuous mapping for every $Z \in \mathcal{A}_0$, therefore G will be continuous mapping. It means that τ is $\{Z(\mathcal{A}_0\})$-splitting topological structure.

Let now τ be a not $\mathcal{A}$-splitting topological structure on $X \times Y$, that is $\tau \in T^{sp}$. It means that τ is not Z^{sp}_τ-splitting topological structure, that is there exists a mapping $F_{Z^{sp}_\tau}$ for which the mapping $G_{Z^{sp}_\tau}$ is not continuous one. We can draw $F_{Z^{sp}_\tau}$ through $Z(\mathcal{A})$ arbitrarily to receive a mapping $F : X \times Y \longrightarrow Z(\mathcal{A})$. It is easy to see that the corresponding mapping $G : X \longrightarrow Y/Z(\mathcal{A})$ is not continuous one. It means that τ is not $Z(\mathcal{A})$-splitting one.

In the same way we can prove that each topological structure τ on $X \times Y$ is $\mathcal{A}$-jointly continuous iff τ is $Z(\mathcal{A})$-jointly continuous. The theorem is proved.

Just proved theorem is of greate importance one. It allows to reduce the problem of investigation of all $\mathcal{A}$-splitting and $\mathcal{A}$-jointly continuous topological structures on $X \times Y$ to the partial problem of investigation of Z-splitting and Z-jointly continuous topological structures, that is to the problem of special investigation of triplets X, Y, Z.

3. Functoral systems $\mathcal{Z}$. We shall call a system $\mathcal{Z}$ *functoral system* if for every topological spaces Z, Z' the continuity of $f : Z' \longrightarrow Z$ implies the continuity of $1/f : Y/Z' \longrightarrow Y/Z$, $(1/f)(h') = fh'$. Here we mean, of course, that on Y/Z' and on Y/Z we consider the $\mathcal{Z}$-system-topologies. It can be proved the following statement.

Theorem. *Let $f : Z' \longrightarrow Z$ be cofactor-mapping and $\mathcal{Z}$ be a functoral system. Then for this system $\tau(Z') \subset \tau(Z)$.*

Proof. Consider the following mappings for Z' and for Z.

$$forZ'$$

$$F' : (X \times Y)_{\tau(Z')} \longrightarrow Z',$$

$$G' : X \longrightarrow (Y/Z').$$

$$forZ$$

$$F : (X \times Y)_{\tau(Z)} \longrightarrow Z,$$

$$G : X \longrightarrow (Y/Z).$$

It is easy to see, that continuity of mappings F' and G' implies continuity of mappings $F = fF' : X \times Y \longrightarrow Z$ and $G : X \longrightarrow Y/Z$, $G = (1/f)G'$. Let now the mapping G' be continuous, then mappings G and F' will be continuous, the letter means the continuity of mapping $F = fF' : (X \times Y)_{\tau(Z')} \longrightarrow Z$. Thus the mappings

$$F : (X \times Y)_{\tau(Z')} \longrightarrow Z, \ F = fF',$$

$$G : X \longrightarrow Y/Z, \ G = (1/f)G',$$

are continuous. Moreover, the continuity of mapping F implies the continuity of mapping $F\prime$ for any topological structure τ on $X \times Y$, since f is a co factor-mapping, and so $\tau(X') \subset \tau(X)$. The theorem is proved.

§4. Dual topological structures

1. Topological structures on products and ratios. Up to this point we investigated topological structures on ratios of topological spaces and topological structures on products of topological spaces separately. Now we are going to investigate these topological structures (on ratios and on products) within a bound of united theory. We considered in the previous paragraphs systems $\mathcal{X}_Y$ and systems $\mathcal{Z}_Y$. Any system $\mathcal{X}_Y$ consists of all topological spaces with fixed for each X a topological structure on $X \times Y$, the letter being additive according to the first argument. Any systems $\mathcal{Z}_Y$ consists of all topological spaces with fixed for each Z a topological structure on Y/Z, the letter being multiplicative according to the second argument. It was proved, that the greatest $\mathcal{X}_Y$-splitting topological structures $\tau(\mathcal{X}_Y)$ on Y/Z are multiplicative according the second argument. Thus topological spaces Z considered with topological structures $\tau(\mathcal{X}_Y)$ on Y/Z make up a system $\mathcal{Z}_Y$. It was proved too, that the least $\mathcal{Z}_Y$-jointly continuous topological structures $\tau(\mathcal{Z}_Y)$ on $X \times Y$ are additive according the first argument. Thus topological spaces X considered with topological structures $\tau(\mathcal{Z}_Y)$ on $X \times Y$ make up a system $\mathcal{X}_Y$. Two systems $\mathcal{X}_Y$ and $\mathcal{Z}_Y$ are called dual to each other, if topological spaces Z considered with topological structures $\tau(\mathcal{X}_Y)$ on Y/Z make up a system $\mathcal{Z}_Y$ and topological spaces X considered with topological structures $\tau(\mathcal{Z}_Y)$ on $X \times Y$ make up a system $\mathcal{X}_Y$.

2. Topological structures for triplets. We shall consider now triplets $X, \ Y, \ Z$, topological structures on products $X \times Y$, topological structures on ratios Y/Z and mutual relations between them within a bound of united theory. Fixe a triplet $X, \ Y, \ Z$ of topological spaces and consider Z-jointly continuous topological structures on the product $X \times Y$ and X-splitting topological structures on the ratio Y/Z. If τ is a topological structure on Y/Z, then τ^* will mean the least $(Y/Z)_\tau$-jointly continuous structure on $X \times Y$. On the other hand, if τ is a topological structure on $X \times Y$, then τ_* will mean the greatest $(X \times Y)_\tau$-splitting structure on Y/Z.

We can begin with a topological structure τ on Y/Z, to consider the least $(Y/Z)_\tau$-jointly continuous topological structure τ^* on $X \times Y$ and after that to consider the greatest $(X \times Y)_{\tau^*}$-splitting topological structure $(\tau^*)_*$ on Y/Z. It is easy to see that the following relation takes place

$$(\tau^*)_* \subset \tau$$

Really, τ^* is the least topological structure on $X \times Y$ for which some system of mappings $F : (X \times Y)_{\tau^*} \longrightarrow Z$ consists of continuous mappings. They are exactly those mappings which correspond to the continuous mappings $G : X \longrightarrow (Y/Z)_\tau$. But it is possible that this system did not exhaust the set of all continuous mappings $F : (X \times Y)_{\tau^*} \longrightarrow Z$, therefore the set of all continuous mappings $G : X \longrightarrow (Y/Z)_{(\tau^*)_*}$ contains the set of all continuous mappings $G : X \longrightarrow (Y/Z)_\tau$. Hence $(\tau^*)_* \subset \tau$.

In the same way the relation

$$\tau \subset (\tau_*)^*$$

can be proved.

Consider the case $(\tau^*)_* = \tau$. Designate in this case τ^* by $\overline{\tau}$ and τ by $\underline{\tau}$. Then the initial equality will take the following form

$$\overline{\tau}_* = \underline{\tau}$$

On the other hand, in these designations we have the relations $\overline{\tau} = \tau^* = \underline{\tau}^*$, that is

$$\underline{\tau}^* = \overline{\tau}$$

Consider now the case $\tau = (\tau_*)^*$. Designate in this case τ_* by $\underline{\tau}$ and τ by $\overline{\tau}$. Then the initial equality will take the following form

$$\underline{\tau}^* = \overline{\tau}$$

On the other hand, in these designations we have the relations $\underline{\tau} = \tau_* = \overline{\tau}_*$, that is

$$\overline{\tau}_* = \underline{\tau}$$

Thus in both cases considered above we come to the identical relations

$$\underline{\tau}^* = \overline{\tau}, \ \overline{\tau}_* = \underline{\tau}$$

Topological structure $\overline{\tau}$ om $X \times Y$ and topological structure $\underline{\tau}$ on Y/Z for which these relations take place we shal call *dual to each other* . The following statement proves to be true.

Theorem. *For any triplet X, Y, Z of topological spaces there exist dual to each other topological structures $\overline{\tau}$ on $X \times Y$ and $\underline{\tau}$ on Y/Z, respectively.*

Proof. Let τ be a topological structure on Y/Z, then $(\tau^*)_* \subset \tau$. If under the circumstances $(\tau^*)_* = \tau$ then we already have found a dual topological structures τ^* and τ, but if the inclusion is strict, we consider the inclusions $(((\tau^*)_*)^*)_* \subset (\tau^*)_* \subset \tau$ or in the other designations $(\tau_1^*)_* \subset \tau_1 \subset \tau$. If the first inclusion is not strict, that is $(\tau_1^*)_* = \tau_1$, we already have found a dual topological structures $(\tau_1^*)_*$ and τ_1, but if the inclusion is strict, we continue this process, constructing in consecutive order topological structures $\tau, \ \tau_1, \ \ldots, \ \tau_n$. If such sequence proves to be finite, that is $(\tau_n^*)_* = \tau_n$, then we have found again the dual topological structures $(\tau_n^*)_*$ and τ_n. But if the sequence infinite, we consider the topological structure $\tau_\omega = inf\{\tau_n | n \in \mathbb{N}\}$ and so on in correspondence with the principle of transfinite construction. Such process of transfinite construction of strictly decreasing topological structures on fixed set can't have too much power therefore it has to end up with a topological structure $\underline{\tau}$ and we shall find at last dual topological structures $\overline{\tau} = (\underline{\tau})^*$ and $\underline{\tau}$. The theorem is proved.

Remark. As a matter of fact, the proof of the above proved theorem can be made trivial, beginning with the trivial structure τ on Y/Z. Then every mapping $G : X \longrightarrow (Y/Z)_\tau$ is continuous mapping and therefore τ^* is the least topological structure on $X \times Y$ for which all mappings $F : (X \times Y)_{\tau^*} \longrightarrow Z$ are continuous. It is easy to see that $\tau = (\tau^*)_*$ and so τ and τ^* are dual topological structures. Initial proof allows us to begin with any topological structure.

The existence of duality between topological structures on $X \times Y$ and topological structures on Y/Z allows to pass from different problems concerning topological structures on $X \times Y$ to the corresonding problems concerning topological structures on Y/Z and vice versa. Moreover, there is a possibility to use the transition from topological structures τ on $X \times Y$ to $(X \times Y)_\tau$-splitting topological structures τ_* on Y/Z and the transition from topological structures τ on Y/Z to $(Y/Z)_\tau$-jointly continuous topological structures on $X \times Y$. Each of these transitions is monotonously diminishing. Really, let $\tau_1 \subset \tau_2$ are topological structures on $X \times Y$ then the continuity of $F : (X \times Y)_{\tau_1} \longrightarrow Z$ implies the continuity of $(X \times Y)_{\tau_2} \longrightarrow Z$ therefore the continuity of $G : X \longrightarrow (Y/Z)_{(\tau_1)_*}$ implies the continuity of $G : X \longrightarrow (Y/Z)_{(\tau_2)_*}$. But then $(\tau_2)_* \subset (\tau_1)_*$. Let now $\tau_1 \subset \tau_2$ are topological structures on Y/Z then the continuity of $G : X \longrightarrow (Y/Z)_{\tau_2}$ implies the continuity of $X \longrightarrow (Y/Z)_{\tau_1}$ therefore the continuity of $F : (X \times Y)_{(\tau_2)^*} \longrightarrow Z$ implies the continuity of $F : (X \times Y)_{(\tau_1)^*} \longrightarrow Z$. But then $(\tau_2)^* \subset (\tau_1)^*$.

It would be very interesting to consider triplets X, Y, $\mathbb{R}$. Here $F : X \times Y \longrightarrow \mathbb{R}$ and mappings from $Y/\mathbb{R}$ are real functions defined on $X \times Y$ and on Y, respectively. The case $X = Y = Z$ is especially interesting. Here we deal with two topological type structures on the same set X, namely with topological structures on $X \times X$ (such type structures are called bitopological structures) and with topological structures on X/X, which we can consider as topological structures on $P(X \times X)$. Such topological type structures we shall call the ratio structures on X.

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