PDMI PREPRINT — 16/1996

Real functions in weighted Hardy spaces

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December 1996

For a given function $\varphi \in H^p$ on the unit circle, $1 \leqslant p \leqslant \infty$, let us consider the intersection $I_\varphi = \frac{H^p}{\varphi} \cap \frac{\overline{H^p}}{\bar\varphi}$. Obviously, every constant function belongs to I_φ. We discuss the question, when is $I_\varphi = \{\mathrm{const}\}$?

It is easy to show that if φ is not an outer function, then $I_\varphi \neq \{\mathrm{const}\}$. If φ is outer, then let us consider the weight $w = |\varphi|^p$ and the space $L^p(wdm)$, where m is the normalized Lebesgue measure. Then the closed linear span of the set $\{z^n\}_{n\geqslant 0}$ coincides with H^p/φ, and the closed linear span of the set $\{\bar z^n\}_{n\leqslant 0}$ coincides with $\overline{H^p}/\bar\varphi$. It is well known that the existence of a bounded projection onto H^p/φ along $\overline{zH^p}/\bar\varphi$ is equivalent to the Muckenhoupt condition A_p. Obviously, it is sufficient but not necessary for these two spaces to have zero intersection. The latter property is equivalent to our question, whether $I_\varphi = \{\mathrm{const}\}$.

Also, let us remark that for $p = 2$ our problem is equivalent to the famous problem of description of the exposed points of the unit ball of H^1.

If $f \in I_\varphi$, then clearly $\bar f \in I_\varphi$, and hence $\mathrm{Re} f$, $\mathrm{Im} f \in I_\varphi$.

Lemma. *Let $\varphi \in H^p$. Then $I_\varphi \neq \{\mathrm{const}\}$ if and only if there exists a function $\psi \in H^\infty$ such that $\varphi/\psi \in H^p$, $\theta \overset{\mathrm{def}}{=} \psi/\bar\psi \in H^\infty$. In this case, $\frac{1-\theta}{\psi} \in I_\varphi$.*

Proof. In connection with the problem of exposed points in H^1, Helson and Sarason presented a result that the problem has a positive answer if and only if there exist two inner functions u and v such that $u - v$ is a "divisor" of a certain function. Our lemma is a modification of that result, and we shall present a proof in our situation. Let us note that for $\psi = u - v$ we have $\psi/\bar\psi = -uv \in H^\infty$.

Let us assume that φ is an outer function and $I_\varphi \neq \{\mathrm{const}\}$. We want to show that there exist two inner functions u and v such that $\frac{\varphi}{u-v} \in H^p$. Let us take a nonconstant real function $f \in I_\varphi$. Then $\frac{f+i}{f-i}$ is a unimodular function of bounded characteristic (i.e., it can be represented as a ratio of two bounded analytic functions.) Therefore, there exist relatively prime inner functions u and v such that $\frac{f+i}{f-i} = \frac{u}{v}$. This implies $f = i\frac{u+v}{u-v}$. Since $f \in H^p/\varphi$, φ is outer, and the functions u, v are relatively prime, it easily follows that $u - v$ is an outer function, and it suffices to show that $\frac{\varphi}{u-v} \in L^p$. Let $f = ig/\varphi$ with $g \in H^p$. The functions $u + v$ and $u - v$ cannot be small simultaneously (their sum $2u$ has absolute value 2). Therefore, since $\varphi, g \in L^p$, the relation $\frac{\varphi}{u-v} = \frac{g}{u+v}$ easily implies $\frac{\varphi}{u-v} \in L^p$.

If φ is not an outer function, then its inner factor can be taken as ψ.

If there exists a function ψ with the properties required, then one can easily show that $\frac{1-\theta}{\psi} \in I_\varphi$. The Lemma is proved.

Remark. The function θ in the Lemma can be chosen so that $\theta(0) = 0$. Indeed, if we construct a function φ_*, for which $\varphi/\psi_* \in H^p$, $\theta_* = \psi_*/\bar\psi_* \in H^\infty$, then we can take $\psi = \psi_*/(1 - \overline{\theta_*(0)}\theta_*)$. Since $1 - \overline{\theta_*(0)}\theta_*$ is an invertible function, we have $\psi \in H^p$; moreover, $\varphi/\psi \in H^p$, and for $\theta = \frac{\psi}{\bar\psi} = \frac{\theta_* - \theta_*(0)}{1-\overline{\theta_*(0)}\theta_*}$ we have $\theta(0) = 0$. In what follows, we shall assume that $\theta(0) = 0$.

Our main result is the following.

Theorem. *$I_\varphi \neq \{\mathrm{const}\}$ if and only if there exists a nonzero real finite signed measure ν (this means that it can be represented as the difference of two positive finite measures) singular with respect to the Lebesgue measure and such that*

$$\left(\int \frac{\xi + z}{\xi - z} d\nu(\xi)\right)^{-1} \in I_\varphi.$$

Proof. It suffices to show that if $\psi \in H^\infty$, $\psi(0)$ is real, $\theta = \psi/\bar{\psi} \in H^\infty$, $\theta(0) = 0$, then $\frac{\psi}{1-\theta}$ can be represented as $\int \frac{\xi+z}{\xi-z} d\nu(\xi)$.

Let μ be the singular probability measure on the circle determined by the relation

$$\frac{1+\theta(z)}{1-\theta(z)} = \int \frac{\xi+z}{\xi-z} d\mu(\xi).$$

Let us introduce a space $K_\theta^\infty = \{h \in H^\infty : \bar{\theta}h \in \overline{zH^\infty}\}$.

In [P] it is shown that functions from some classes, including K_θ^∞, have nontangential boundary values μ-almost everywhere (a.e.), and every function $h \in K_\theta^\infty$ can be restored by its boundary values $h(\xi)$ by the following formula:

$$h(z) = (1-\theta(z)) \left(\int \frac{h(\xi)\xi d\mu(\xi)}{\xi - z} \right), \quad |z| < 1. \tag{$\spadesuit$}$$

The function $h = \psi + \psi(0)(1-\theta)$ belongs to K_θ^∞. Since $\theta = 1$ μ-*a.e.* and $\psi/\bar{\psi} = \theta$, we obtain that ψ is real μ-a.e. Define ν, $d\nu = \frac{\psi d\mu}{2}$. The formula ($\spadesuit$) can be rewritten as

$$\psi(z) + \psi(0)(1-\theta(z)) = (1-\theta(z)) \int \frac{2\xi d\nu(\xi)}{\xi - z},$$

whence

$$\frac{\psi(z)}{1-\theta(z)} = \int \frac{2\xi d\nu(\xi)}{\xi - z} - \psi(0).$$

Taking $z = 0$, we obtain $\psi(0) = \int d\nu$, and therefore

$$\frac{\psi(z)}{1-\theta(z)} = \int \left(\frac{2\xi}{\xi - z} - 1 \right) d\nu(\xi) = \int \frac{\xi + z}{\xi - z} d\nu(\xi).$$

The Theorem is proved.

Some years ago Sarason conjectured (in terms of the problem of exposed points in H^1) that $I_\varphi \neq \{\text{const}\}$ if and only if there exists an inner function θ such that $\frac{\varphi}{1-\theta} \in H^p$. This is equivalent to the fact that ν in our Theorem can be chosen positive. This conjecture was disproved, and this means that in our Theorem in general ν cannot be chosen positive.

The results of this article may be naturally replaced onto the real line. Then we will obtain a formulation of our main theorem in terms of real functions.

This work was partially supported by RFFI, Grant No. 95-01-00482.

Reference

[P] A. G. Poltoratskiĭ, *Boundary behavior of pseudocontinuable functions*, Algebra i Analiz **5** (1993), no. 2, 189–210 (Russian); transl. into English: St.Petersburg Math. J. **5** (1994), no. 2, 389–406.