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SCALING LIMITS IN THE FOURTH PAINLEVÉ TRANSCENDENT

ANDREI A. KAPAEV

Russian Academy of Sciences
Steklov Mathematical Institute
St. Petersburg Department
Fontanka 27, St. Petersburg 191011, Russia

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ABSTRACT

The scaling limits in the fourth Painlevé transcendent are considered from the point of view of the associated with P4 linear equation. The discriminant set for the scaling limits is described. The cases of reductions to the lower Painlevé equations are found as well.

In the present article, we continue the discussion of the asymptotic properties of the fourth Painlevé transcendent [1, 2]. We will consider the so-called scaling limits in the fourth Painlevé equation

$$y_{xx} = \frac{(y_x)^2}{2y} + \frac{3}{2}y^3 + 4xy^2 + 2\big(x^2 - 2\alpha + \frac{\beta}{2}\big)y - \frac{\beta^2}{2y}. \tag{1}$$

It is important for the following that the equation (1) is the equation of isomonodromy deformation for the linear matrix equation $\Psi_\lambda = \mathcal{A}\Psi$ with the matrix

$$\mathcal{A} = \Big(\frac{1}{2}\lambda^3 + \lambda(x + uv) + \frac{\alpha}{\lambda}\Big)\sigma_3 + i\Big(\lambda^2 u + 2xu + u'\Big)\sigma_+ + i\Big(\lambda^2 v + 2xv - v'\Big)\sigma_-, \tag{2}$$

with the square of the eigenvalues

$$\mu^2 = \frac{1}{4}\lambda^6 + x\lambda^4 + (x^2 + \alpha - \beta)\lambda^2 + D + \frac{\alpha^2}{\lambda^2}, \tag{3}$$

where

$$\begin{aligned} D &= 2\alpha(x + uv) - (2xu + u')(2xv - v') \equiv \\ &\equiv \frac{(y')^2}{4y} - \frac{1}{4}y^3 - xy^2 + \big(2\alpha - \frac{\beta}{2} - x^2\big)y + (2\alpha - \beta)x - \frac{\beta^2}{4y}. \end{aligned} \tag{4}$$

The scaling change of variables is given by

$$\begin{aligned} &x = \delta^{-1/2}t_0 + \delta^{1/2}t, \\ &y = \delta^{-1/2}r, \quad u = \delta^{-1/4}p, \quad v = \delta^{-1/4}q, \\ &\alpha = \delta^{-1}\alpha_0 + \alpha_1, \quad \beta = \delta^{-1}\beta_0 + \beta_1, \\ &\lambda = \delta^{-1/4}\zeta, \end{aligned} \tag{5}$$

where t_0, α_0, α_1, β_0 and β_1 are some complex constants, δ is the positive scaling parameter, t is the new independent variable, while p, q, r are the new dependable variables.

After (5), the original P4 equation (1) goes into

$$\begin{aligned} r'' &= \frac{(r')^2}{2r} + \frac{3}{2}r^3 + 4t_0 r^2 + 2\big(t_0^2 - 2\alpha_0 + \frac{\beta_0}{2}\big)r - \frac{\beta_0^2}{2r} + \\ &+ \delta\Big(4tr^2 + \big(-4\alpha_1 + \beta_1 + 4t_0 t\big)r - \frac{\beta_0\beta_1}{r}\Big) + \delta^2\big(2t^2 r - \frac{\beta_1^2}{2r}\big), \end{aligned} \tag{6}$$

while the linear equation associated with P4 reads

$$\begin{aligned} \Psi_\zeta = \delta^{-1}\Big\{ &\Big(\frac{1}{2}\zeta^3 + \zeta(t_0 + \delta t + r) + \frac{\alpha_0 + \delta\alpha_1}{\zeta}\Big)\sigma_3 + \\ &+ i\Big(\zeta^2 p + 2(t_0 + \delta t)p + p'\Big)\sigma_+ + \\ &+ i\Big(\zeta^2 q + 2(t_0 + \delta t)q - q'\Big)\sigma_- \Big\}\Psi, \end{aligned} \tag{7}$$

with the square of the eigenvalues

$$\begin{aligned} \mu^2 &= \frac{1}{4}\zeta^6 + (t_0 + \delta t)\zeta^4 + \\ &\big[(t_0 + \delta t)^2 + \alpha_0 + \delta\alpha_1 - \beta_0 - \delta\beta_1\big]\zeta^2 + A + \frac{(\alpha_0 + \delta\alpha_1)^2}{\zeta^2}, \end{aligned} \tag{8}$$

where

$$A = \frac{(r')^2}{4r} - \frac{1}{4}r^3 - (t_0 + \delta t)r^2 + \\ + \big(2(\alpha_0 + \delta\alpha_1) - \frac{1}{2}(\beta_0 + \delta\beta_1) - (t_0 + \delta t)^2\big)r + \\ + \big(2(\alpha_0 + \delta\alpha_1) - (\beta_0 + \delta\beta_1)\big)(t_0 + \delta t) - \frac{(\beta_0 + \delta\beta_1)^2}{4r}. \tag{9}$$

To have a possibility to interpret the limit $\delta \to +0$ in the isomonodromy sense, i.e. as result of applying of some Bäcklund transformation chain in accord with method of ref. [3], we should assume the reality of the parameters α_0 and β_0 in (5).

The case $\alpha_0 = \beta_0 = 0$ can be described in terms of the asymptotic formulae for $x \to \infty$ listed in refs. [1, 2], so we assume here that at least one of the parameters α_0, β_0 is not trivial.

Moreover, redefining the scaling parameter δ, we can normalize one of the nontrivial real parameters (either α_0, or β_0, or some combination of them) by its absolute value. Thus the set of the scaling changes (5) depends on one complex parameter t_0, which marks the center of the asymptotic region considered, and one real parameter related to the kind of the Bäcklund transformation chain used. In particular, if the chain related to the certain scaling limit contains no transformation like (4c) and (4d) of ref.[2] then $\alpha_0 - \beta_0 = 0$. Similarly, if the chain of Bäcklund transformations does not have any transformations of the type (4a) and (4b) of ref.[2], we should assume that $\alpha_0 = 0$.

To describe the main order of the scaling limits we need to consider the main order of (8)

$$\mu_{as}^2 = \frac{1}{4}\zeta^6 + t_0\zeta^4 + (t_0^2 + \alpha_0 - \beta_0)\zeta^2 + A_0 + \frac{\alpha_0^2}{\zeta^2} = \\ = \frac{1}{4\zeta^2}(\zeta^2 - \zeta_1^2)(\zeta^2 - \zeta_3^2)(\zeta^2 - \zeta_5^2)(\zeta^2 - \zeta_7^2), \\ \zeta_{2k} = -\zeta_{2k-1}, \quad k = 1, 3, 5, 7, \tag{10}$$

with

$$A_0 = \frac{(r')^2}{4r} - \frac{1}{4}r^3 - t_0 r^2 + \big(2\alpha_0 - \frac{1}{2}\beta_0 - t_0^2\big)r + \big(2\alpha_0 - \beta_0\big)t_0 - \frac{\beta_0^2}{4r}. \tag{11}$$

In the situation of general position, the parameters determining the curve $\mu_{as}(\zeta)$ satisfy the following system of Boutroux equations:

$$\Re \oint_{\ell_j} \mu_{as}(\lambda)\, d\lambda = 0, \tag{12}$$

where ℓ_j form the complete system of nonhomologous cycles of the Riemann surface of the curve $\mu_{as}(\lambda)$.

Below, we intend to describe the discriminant set for the scaling limits in the fourth Painlevé equation, points of which correspond to the cases of one or another kind of degeneration of the curve. From the point of view of P4 itself, the mentioned discriminant set separates the regions of the real 3d-space $(\Re t_0, \Im t_0, \alpha_0 - \beta_0)$, in the interior of which the Painlevé transcendent is described asymptotically via elliptical functions; on the faces of the discriminant set, the Painlevé function is given by trigonometrical functions; edges of the discriminant set correspond in the main order about $\delta \to 0$ to algebraic degeneration of P4 which in the next order can be described by use of the first Painlevé transcendent; vertices of the discriminant set appear as the points of the most high degeneration of P4 described by means of the second Painlevé transcendent.

I. Scaling limits for $\alpha_0 = 0$.

In the case $\alpha_0 = 0$

$$\mu_{as}^2 = \frac{1}{4}\zeta^6 + t_0\zeta^4 + \big(t_0^2 - \beta_0\big)\zeta^2 + A_0 = \frac{1}{4}(\zeta^2 - \zeta_1^2)(\zeta^2 - \zeta_3^2)(\zeta^2 - \zeta_5^2),$$
$$\zeta_{2k} = -\zeta_{2k-1}, \quad k = 1, 3, 5, \tag{13}$$

where

$$A_0 = \frac{(r')^2}{4r} - \frac{1}{4}r^3 - t_0 r^2 - \big(\frac{1}{2}\beta_0 + t_0^2\big)r - \beta_0 t_0 - \frac{\beta_0^2}{4r}. \tag{14}$$

Moreover, by use of redefining of the parameter δ, we can normalize β_0 by its absolute value to unit, i.e. we can admit $\beta_0 = \pm 1$.

The node points of the discriminant set correspond to more deeply degenerated curve (13).

1. Completely degenerated curve:

$$\zeta_1^2 = \zeta_3^2 = \zeta_5^2, \quad \mu_{as}^2 = \frac{1}{4}\big(\zeta^2 - \zeta_1^2\big)^3. \tag{15}$$

The degenerated curve (15) appears if and only if

$$\zeta_1^2 = -\frac{4}{3}t_0, \quad \beta_0 = -\frac{t_0^2}{3}, \quad A_0 = \frac{16}{27}t_0^3. \tag{16}$$

Under (16), the Boutroux system (12) is simplified up to the only condition $\beta_0 \in \mathbb{R}$, while the relation (14) takes the form of the first order differential equation

$$(r')^2 = \big(r + \frac{t_0}{3}\big)^3(r + 3t_0), \tag{17}$$

As a matter of fact, the equation (17) is an approximation for the exact one for $r(t)$ (one time integrated (6) and is convenient for getting model equation for the function $r(t)$ in the asymptotic region distinguished by $t_0 = \pm i\sqrt{3\beta_0}$. Indeed, let us make an additional scaling change of variables

$$r = -\frac{t_0}{3} + \delta^{2/5}\rho, \quad t = \delta^{-1/5}\tau. \tag{18}$$

Then the equation (6) goes into

$$\begin{aligned}\rho'' &= 4t_0\rho^2 - \frac{8}{9}t_0^2\tau + \delta^{1/5}\cdot\frac{4}{3}(\alpha_1 - \beta_1)t_0 + \\ &+ \delta^{2/5}\Big(\frac{3}{2t_0}\big(2\rho\rho'' - (\rho')^2\big) + 4t_0\tau\rho - 6\rho^3\Big) + \\ &+ \delta^{3/5}\cdot 2(\beta_1 - 4\alpha_1)\rho - \delta^{4/5}\cdot\frac{3}{2t_0}\big(3\rho^4 + \frac{4}{9}t_0^2\tau^2\big) + \\ &+ \delta\cdot\frac{3}{t_0}(4\alpha_1 - \beta_1)\rho^2 + \delta^{6/5}\cdot\frac{3}{2t_0}\big(\beta_1^2 - 8\rho^3\tau + \frac{8}{3}t_0\tau^2\rho\big) - \\ &- \delta^{8/5}\cdot\frac{6\tau^2\rho^2}{t_0},\end{aligned} \tag{19}$$

which, in the limit $\delta \to 0$, reduces to the first Painlevé equation P1

$$\rho'' = 4t_0\rho^2 - \frac{8}{9}t_0^2\tau. \tag{20}$$

2. Highly degenerated curve (13) is characterized by the conditions

$$\zeta_3 = \zeta_4 = \zeta_5 = \zeta_6 = 0, \quad \mu_{as}^2 = \frac{1}{4}\zeta^4(\zeta^2 - \zeta_1^2), \tag{21}$$

so that

$$\zeta_1^2 = -4t_0, \quad \beta_0 = t_0^2, \quad A_0 = 0, \tag{22}$$

Under (22), the Boutroux system (12) reduces to the condition $\beta_0 \in \mathbb{R}$. The definition of the parameter A_0 (14) goes into asymptotic first order equation for $r(t)$

$$(r')^2 = (r + t_0)^4 \tag{23}$$

which allows us to get the proper additional scaling change of variables to describe the scaling limit in the asymptotic region corresponding to $t_0 = \pm\sqrt{\beta_0}$:

$$r = -t_0 + \delta^{1/3}\rho, \quad t = \delta^{-1/3}\tau. \tag{24}$$

The new function $\rho(\tau)$ satisfies the equation

$$\begin{aligned} &\rho'' = 2\rho^3 - 4t_0\tau\rho + 4\alpha_1 t_0 + \\ &+ \delta^{1/3}\Big(\frac{1}{2t_0}(2\rho\rho'' - (\rho')^2) - \\ &- \frac{3}{2t_0}\rho^4 - 8\alpha_1\rho + 2\beta_1\rho + 8\tau\rho^2 - 2t_0\tau^2\Big) + \\ &+ \delta^{2/3}\Big(\frac{1}{t_0}(4\alpha_1\rho^2 - \beta_1\rho^2 - 4\tau\rho^3) + 4\tau^2\rho\Big) + \delta\frac{1}{2t_0}(\beta_1^2 - 4\rho^2\tau^2), \end{aligned} \tag{25}$$

which in the limit $\delta \to 0$ goes into equation as being equivalent to the second Painlevé equation P2:

$$\rho'' = 2\rho^3 - 4t_0\tau\rho + 4\alpha_1 t_0. \tag{26}$$

3. One can introduce two kinds of the trigonometrical degeneration of the curve (13):
i) The trigonometrical degeneration of the first kind arises when

$$\lambda_3^2 = \lambda_5^2, \quad \mu_{as}^2 = \frac{1}{4}(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_3^2)^2. \tag{27}$$

The restriction (27) implies

$$\begin{aligned} \lambda_1^2 &= -\frac{4}{3}\Big(t_0 \mp \sqrt{t_0^2 + 3\beta_0}\Big), \\ \lambda_3^2 &= \lambda_5^2 = -\frac{4}{3}\Big(t_0 \pm \frac{1}{2}\sqrt{t_0^2 + 3\beta_0}\Big), \\ A_0 &= \frac{16}{27}\Big(t_0 \mp \sqrt{t_0^2 + 3\beta_0}\Big)\Big(t_0 \pm \frac{1}{2}\sqrt{t_0^2 + 3\beta_0}\Big)^2 = \\ &= \frac{4}{27}\Big(t_0^3 - 9t_0\beta_0 \mp (t_0^2 + 3\beta_0)^{3/2}\Big), \end{aligned}$$

while the Boutroux system (12) yields the known condition $\beta_0 \in \mathbb{R}$ and the transcendent equation for the parameter t_0:

$$\begin{aligned} &\Re\Big\{ \lambda_3 \big(\lambda_1^2 + 2\lambda_3^2\big)\sqrt{\lambda_3^2 - \lambda_1^2} - \\ &- \lambda_1^2\big(-\lambda_1^2 + 4\lambda_3^2\big) \ln\Big(\frac{\lambda_3 + \sqrt{\lambda_3^2 - \lambda_1^2}}{\lambda_1}\Big)\Big\} = 0, \end{aligned} \tag{28}$$

or, in terms of t_0, β_0,

$$\begin{aligned} &\Re\Big\{ \frac{1}{2\sqrt{3}}\, t_0 \sqrt{t_0^2 + 3\beta_0 \pm 2t_0\sqrt{t_0^2 + 3\beta_0}} - \\ &- \beta_0 \ln \frac{\sqrt{t_0 \pm \frac{1}{2}\sqrt{t_0^2 + 3\beta_0}} + \sqrt{\pm\frac{3}{2}\sqrt{t_0^2 + 3\beta_0}}}{\sqrt{t_0 \mp \sqrt{t_0^2 + 3\beta_0}}}\Big\} = 0. \end{aligned} \tag{29}$$

ii) Trigonometrical degeneration of another kind appears if

$$\lambda_1^2 = 0, \quad \mu_{as}^2 = \frac{1}{4}\lambda^2\big(\lambda^2 - \lambda_3^2\big)\big(\lambda^2 - \lambda_5^2\big), \tag{30}$$

so that

$$\begin{aligned} \lambda_3^2 &= -2t_0 \pm 2\sqrt{\beta_0}, \\ \lambda_5^2 &= -2t_0 \mp 2\sqrt{\beta_0}, \\ A_0 &= 0. \end{aligned} \tag{31}$$

The system of Boutroux equations (12) gives the condition $\beta_0 \in \mathbb{R}$ and the following transcendent equation for the parameter t_0:

$$\Re\Big\{ t_0\sqrt{t_0^2 - \beta_0} - \beta_0 \ln \frac{t_0 + \sqrt{t_0^2 - \beta_0}}{\sqrt{\beta_0}}\Big\} = 0. \tag{32}$$

II. Scaling limit for $\alpha_0 \neq 0$.

Besides the curve (10), we introduce also the elliptical curve

$$\begin{aligned} \nu^2 &= \frac{1}{16z^2}(z - z_1)(z - z_3)(z - z_5)(z - z_7), \\ z_k &= \zeta_k^2, \quad k = 1, 3, 5, 7, \end{aligned} \tag{33}$$

so that, if $z = \zeta^2$,

$$\int \mu_{as}(\zeta)\, d\zeta = \int \nu(z)\, dz.$$

1. The curves (10), (33) are completely degenerated, i.e., are characterized by the equations

$$z_1 = z_3 = z_5 = z_7, \quad \nu^2 = \frac{1}{16z^2}\big(z - z_1\big)^4, \tag{34}$$

if and only if

$$z_1 = -t_0, \quad t_0^2 = 2(\alpha_0 - \beta_0), \quad A_0 = t_0^3, \quad t_0^4 = 4\alpha_0^2. \tag{35}$$

The second and the fourth of the equations (35) mean

$$\text{either} \quad \text{a) } \alpha_0 - \beta_0 = \alpha_0, \quad \text{i.e.} \quad \beta_0 = 0, \tag{36}$$

$$\text{or} \quad \text{b) } \alpha_0 - \beta_0 = -\alpha_0, \quad \text{i.e.} \quad \beta_0 = 2\alpha_0. \tag{37}$$

The conditions (36) and (37) give the parameter values which allow the complete degeneration to exist for nontrivial α_0.

a) In the case (36) $\beta_0 = 0$, the equation (11) transforms to the asymptotic first order equation for the Painlevé function

$$(r')^2 = r^3\big(r + 4t_0\big), \tag{38}$$

and the additional scaling change of variables

$$r = \delta^{2/3}\rho, \quad t = \delta^{-1/3}\tau, \tag{39}$$

converts the Painlevé equation (6) into

$$\begin{aligned} 2\rho\rho'' - (\rho')^2 &= 8t_0\rho^3 + 8t_0\tau\rho^2 - \beta_1^2 + \\ &+ \delta^{1/3}\big(-4\alpha_1 + \beta_1\big)2\rho^2 + \delta^{2/3}\big(3\rho^4 + 8\tau\rho^3 + 4\tau^2\rho^2\big). \end{aligned} \tag{40}$$

In the limit $\delta \to 0$, the equation for $g(\tau) = \sqrt{\rho(\tau)}$ goes into

$$g'' = 2t_0g^3 + 2t_0\tau g - \frac{\beta_1}{4g^3}. \tag{41}$$

For $\beta_1 = 0$, (41) is the second Painlevé equation P2. The complete variant of (41) and the main order of (40), we will refer as the equations P2'.

b) In the case (37) $\beta_0 = 2\alpha_0$, the equation (11) becomes the first order asymptotic equation for $r(t)$

$$(r')^2 = \big(r + t_0\big)^4, \tag{42}$$

and, after the additional scaling change of variables

$$r = -t_0 + \delta^{1/3}\rho, \quad t = \delta^{-1/3}\tau, \tag{43}$$

the equation (6) reads

$$\begin{aligned} \rho'' &= 2\rho^3 - 4t_0\tau\rho + 2t_0\big(2\alpha_1 - \beta_1\big) + \\ &+ \delta^{1/3}\Big(\frac{1}{2t_0}\big(2\rho\rho'' - (\rho')^2\big) - \\ &- \big(4\alpha_1 - \beta_1\big)2\rho + 8\tau\rho^2 - \frac{3}{2t_0}\rho^4 - 2t_0\tau^2\Big) + \\ &+ \delta\frac{1}{2t_0}\big(\beta_1^2 - 4\tau^2\rho^2\big). \end{aligned} \tag{44}$$

In the asymptotic limit $\delta \to 0$, the equation (44) goes into P2

$$\rho'' = 2\rho^3 - 4t_0\tau\rho + 2t_0\big(2\alpha_1 - \beta_1\big). \tag{45}$$

2. Besides the points of complete degeneration (34) described, the cases of high degeneration make importance:

$$z_3 = z_5 = z_7, \quad \nu^2 = \frac{1}{16z^2}\big(z - z_1\big)\big(z - z_3\big)^3. \tag{46}$$

The relations (46) demand

$$\begin{aligned}
&z_1 = -t_0 \mp \sqrt{3}\sqrt{t_0^2 - 2(\alpha_0 - \beta_0)},\\
&z_3 = -t_0 \pm \frac{1}{\sqrt{3}}\sqrt{t_0^2 - 2(\alpha_0 - \beta_0)},\\
&A_0 = -\frac{1}{4}\,z_3^2\big(3z_1 + z_3\big) =\\
&= 2t_0\big(\alpha_0 - \beta_0\big) \pm \frac{2}{3\sqrt{3}}\big(t_0^2 - 2(\alpha_0 - \beta_0)\big)^{3/2},\\
&\alpha_0^2 = (\alpha_0 - \beta_0)^2 -\\
&-\frac{1}{3}\big(t_0^2 - 2(\alpha_0 - \beta_0)\big)^2 \pm \frac{2t_0}{3\sqrt{3}}\big(t_0^2 - 2(\alpha_0 - \beta_0)\big)^{3/2}.
\end{aligned} \tag{47}$$

In this case, the system of Boutroux equations (12) reduces to the relations $\alpha_0, \beta_0 \in \mathbb{R}$, and the parameter t_0 corresponding to this kind of degenerations and given α_0, β_0, is determined by the last of the equations (47). In the case of general position, it yields eight distinct values of t_0 (four for each branch of the root). Near every such single solution t_0 of the equation (47), the scaling limit can be described via the first Painlevé transcendent.

In some symmetrical cases, the solution of (47) can be simplified:

a) $\beta_0 = 0$. The last equation of the system (47) can be reduced to the following two alternatives: i) $t_0^2 = 2\alpha_0$, that yields the situation of the complete degeneration described above; ii) $2t_0 = \pm\sqrt{3}\sqrt{t_0^2 - 2\alpha_0}$, that gives

$$\begin{aligned}
&t_0 = \pm\sqrt{-6\alpha_0},\\
&z_1 = -3t_0, \quad z_3 = -\frac{1}{3}\,t_0, \quad A_0 = \frac{7}{27}\,t_0^3,
\end{aligned} \tag{48}$$

so the equation (11) gets the form of the following first order asymptotic equation

$$(r')^2 = r\big(r + \frac{4t_0}{3}\big)^3, \tag{49}$$

and the following extra scaling

$$r = -\frac{4t_0}{3} + \delta^{2/5}\rho, \quad t = \delta^{-1/5}\tau, \tag{50}$$

converts the equation (6), in the main order with respect to $\delta \to 0$, to P1:

$$
\begin{aligned}
\rho'' = &-2t_0\rho^2 + \frac{16}{9}\,t_0^2\tau + \delta^{1/5}(4\alpha_1 - \beta_1)\frac{4t_0}{3} + \\
&+ \delta^{2/5}\Big(\frac{3}{8t_0}(2\rho\rho'' - (\rho')^2) + 3\rho^3 - 8t_0\tau\rho\Big) + \\
&+ \delta^{3/5}(-4\alpha_1 + \beta_1)2\rho + \delta^{4/5}\big(9\tau\rho^2 - \frac{9}{8t_0}\rho^4 - \frac{8}{3}t_0\tau^2\big) + \\
&+ \delta(4\alpha_1 - \beta_1)\frac{3}{4t_0}\rho^2 + \delta^{6/5}\big(4\tau^2\rho - \frac{3}{t_0}\tau\rho^3 + \frac{3\beta_1^2}{8t_0}\big) - \\
&- \delta^{8/5}\frac{3}{2t_0}\tau^2\rho^2.
\end{aligned}
\tag{51}
$$

It is obvious that the local model equation is now P1 in the form of

$$
\rho'' = -2t_0\rho^2 + \frac{16}{9}\,t_0^2\tau. \tag{52}
$$

b) $\beta_0 = 2\alpha_0$. The last of the equations (47) yields the following possibilities: i) $t_0^2 = -2\alpha_0$, that gives the case of complete degeneration (see above); ii) $2t_0 = \pm\sqrt{3}\sqrt{t_0^2 + 2\alpha_0}$ that gives

$$
\begin{aligned}
&t_0 = \pm\sqrt{6\alpha_0}, \\
&z_1 = -3t_0, \quad z_3 = -\frac{1}{3}\,t_0, \quad A_0 = \frac{7}{27}\,t_0^3,
\end{aligned}
\tag{53}
$$

so the equation (11) gets the form of the following first order asymptotic equation

$$
(r')^2 = \big(r + \frac{t_0}{3}\big)^3\big(r + 3t_0\big), \tag{54}
$$

and the additional scaling

$$
r = -\frac{t_0}{3} + \delta^{2/5}\rho, \quad t = \delta^{-1/5}\tau, \tag{55}
$$

converts the equation (6) to P1:

$$
\begin{aligned}
\rho'' = &\,4t_0\rho^2 - \frac{8}{9}\,t_0^2\tau + \delta^{1/5}(2\alpha_1 + \beta_1)\frac{2t_0}{3} + \\
&+ \delta^{2/5}\Big(\frac{3}{2t_0}(2\rho\rho'' - (\rho')^2) - 6\rho^3 + 4t_0\tau\rho\Big) + \\
&+ \delta^{3/5}(-4\alpha_1 + \beta_1)2\rho + \delta^{4/5}\big(-\frac{9}{2t_0}\rho^4 - \frac{2}{3}t_0\tau^2\big) + \\
&+ \delta(4\alpha_1 - \beta_1)\frac{3}{t_0}\rho^2 + \delta^{6/5}\big(4\tau^2\rho - \frac{12}{t_0}\tau\rho^3 + \frac{3\beta_1^2}{2t_0}\big) - \\
&- \delta^{8/5}\frac{6}{t_0}\tau^2\rho^2.
\end{aligned}
\tag{56}
$$

It is obvious that the local model equation is now P1 in the form of

$$
\rho'' = 4t_0\rho^2 - \frac{8}{9}\,t_0^2\tau. \tag{57}
$$

c) $\beta_0 = \alpha_0$. Now, the system (47) reads

$$\begin{aligned} z_1 &= -t_0(1 \pm \sqrt{3}), \quad z_3 = -t_0\big(1 \mp \frac{1}{\sqrt{3}}\big), \\ A_0 &= \pm\frac{2}{3\sqrt{3}}\,t_0^3, \quad \alpha_0^2 = \pm\frac{t_0^4}{3\sqrt{3}}(2 \mp \sqrt{3}). \end{aligned} \tag{58}$$

The last of the relations (58) determines eight values of t_0:

$$\begin{aligned} &t_0 = \xi_k = e^{i\frac{\pi}{2}k}\xi_0, \quad \xi_0 = \sqrt[4]{\frac{3\sqrt{3}}{2-\sqrt{3}}}\,\sqrt{\alpha_0}, \\ &t_0 = \zeta_n = e^{i\frac{\pi}{2}n}\zeta_0, \quad \zeta_0 = e^{i\frac{\pi}{4}}\sqrt[4]{\frac{3\sqrt{3}}{2+\sqrt{3}}}\,\sqrt{\alpha_0}, \\ &k, n = 0, 1, 2, 3. \end{aligned} \tag{59}$$

d) $\beta_0 = -2\alpha_0$. Now, the system (47) reads

$$\begin{aligned} &t_0 = \xi_k\sqrt{2} = e^{i\frac{\pi}{2}k}\xi_0\sqrt{2}, \quad t_0 = \zeta_n\sqrt{2} = e^{i\frac{\pi}{2}n}\zeta_0\sqrt{2}, \\ &k, n = 0, 1, 2, 3, \end{aligned} \tag{60}$$

with ξ_0, ζ_0 defined in (59).

3. There are two kinds of trigonometrical degenerations of the fourth Painlevé transcendent in the scaling limit for nontrivial α_0:

i) Double degeneration is characterized by two conditions

$$z_1 = z_5, \quad z_3 = z_7, \quad \nu^2 = \frac{1}{16z^2}\big(z - z_1\big)^2\big(z - z_3\big)^2, \tag{61}$$

so that

$$\begin{aligned} z_1 &= -t_0 + \sqrt{t_0^2 - 2(\alpha_0 - \beta_0)}, \\ z_3 &= -t_0 - \sqrt{t_0^2 - 2(\alpha_0 - \beta_0)}, \\ A_0 &= 2t_0(\alpha_0 - \beta_0), \quad \beta_0(\beta_0 - 2\alpha_0) = 0, \end{aligned} \tag{62}$$

and we should consider two cases $\beta_0 = 0$ and $\beta_0 = 2\alpha_0$ again.

a) In the case β_0=0, the asymptotic relation (11) may be written as

$$(r')^2 = r^2\big((r + 2t_0)^2 - 8\alpha_0\big); \tag{63}$$

b) In the case $\beta_0 = 2\alpha_0$, another asymptotic relation appears:

$$(r')^2 = \Big(r + t_0 + \sqrt{t_0^2 + 2\alpha_0}\Big)^2\Big(r + t_0 - \sqrt{t_0^2 + 2\alpha_0}\Big)^2. \tag{64}$$

The system of the Boutroux equations (12) reduces to the condition $\alpha_0 - \beta_0 \in \mathbb{R}$ and to the transcendent equation

$$\begin{aligned} &\Re\Bigg(t_0\sqrt{t_0^2 - 2(\alpha_0 - \beta_0)} + \\ &+ (\alpha_0 - \beta_0)\ln\frac{t_0 - \sqrt{t_0^2 - 2(\alpha_0 - \beta_0)}}{t_0 + \sqrt{t_0^2 - 2(\alpha_0 - \beta_0)}}\Bigg) = 0. \end{aligned} \tag{65}$$

ii) The ordinary trigonometrical degeneration occurs if

$$z_1 \neq z_3 \neq z_5 = z_7. \tag{66}$$

Hence the branch points z_k satisfy the system

$$\begin{aligned}
&z_1 + z_3 + 2z_5 = -4t_0,\\
&z_1 z_3 + 2z_5(z_1 + z_3) + z_5^2 = 4\big(t_0^2 + \alpha_0 - \beta_0\big),\\
&2z_1 z_3 z_5 + (z_1 + z_3)z_5^2 = -4A_0,\\
&z_1 z_3 z_5^2 = 4\alpha_0^2.
\end{aligned} \tag{67}$$

Besides the usual conditions $\alpha_0 - \beta_0, \alpha_0 \in \mathbb{R}$, the system of Boutroux equations (12) yields a rather complicated transcendent equation

$$\begin{aligned}
&\Re\Big\{\frac{1}{4}(z_1 + z_3 + 2z_5)\sqrt{(z_5 - z_1)(z_5 - z_3)} +\\
&+ \frac{1}{8}\Big((z_1 + z_3)^2 - 4(z_1 z_3 + z_3 z_5 + z_5 z_1)\Big) \times\\
&\times \ln\frac{\sqrt{z_5 - z_3} + \sqrt{z_5 - z_1}}{\sqrt{z_5 - z_3} - \sqrt{z_5 - z_1}} +\\
&+ \sqrt{z_1 z_3 z_5^2}\,\ln\frac{\sqrt{z_1(z_5 - z_3)} + \sqrt{z_3(z_5 - z_1)}}{\sqrt{z_1(z_5 - z_3)} - \sqrt{z_3(z_5 - z_1)}}\Big\} = 0.
\end{aligned} \tag{68}$$

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