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HIGHER CORRECTIONS TO THE MASS CURRENT IN THE WEAKLY INHOMOGENEOUS ^{3}He – A

C. MALYSHEV

Russian Academy of Sciences
Steklov Mathematical Institute
St. Petersburg Department
Fontanka 27, St. Petersburg 191011, Russia
E-mail: malyshev@pdmi.ras.ru

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ABSTRACT

Two new general representations (the series and the integral) for the mass current $\vec{j}$ in weakly inhomogeneous superfluid A-phase of Helium–3 are obtained near zero of temperature by solving the Dyson–Gorkov equation. These representations result in additional correcting contribution to the standard leading expression for $\vec{j}$ which is of first order in gradients of the orbital angular momentum vector $\hat{l}$. A self-contained integral formula for the total supplementary term is deduced, and, provided the London limit holds, the procedure is advanced to expand it at $T = 0$ asymptotically by the Laplace method in powers of gradients of $\hat{l}$. Three special static orientations of $\hat{l}$ with respect to its curl are considered to calculate the higher correcting terms up to third order. Coefficients at the quadratic terms are estimated numerically, new cubic contributions are found which contain the logarithm of the London parameter. Bibliography–30.

§1. Introduction

Superfluidity of Helium–3 is firmly in focus of intensive theoretical and experimental studies [1]. Though considerable attention is paid last years to such problems as quantized vorticity and interfaces [1–4], the weakly inhomogeneous A-phase of Helium–3 ($^3\mathrm{He}-\mathrm{A}$) also can be a subject for theoretical investigation. This phase originates due to p-wave spin triplet BCS–pairing [5, 6], and it demonstrates rather unconventional behaviour [7]. Its peculiarity can be seen, for instance, from the mass current $\vec{j}$ which is acknowledged to be of first order in gradients [5–7]:

$$\vec{j}_0 \; = \; \rho\,\vec{v}_s \, + \, \frac{1}{4m}\,\mathrm{rot}\,(\rho\,\widehat{l}\,) \, + \, \vec{j}_{an}, \qquad\qquad (T=0) \tag{1}$$

$$\vec{j}_{an} \; = \; - \, \frac{1}{2m}\,\mathrm{C}_0\,\widehat{l}\,(\widehat{l}\,\cdot\,\mathrm{rot}\,\widehat{l}\,)$$

where the first two terms are habitual for a nodes–free p-wave superfluid, while the famous anomalous $\vec{j}_{an}$ witnesses highly unusual manifestations of the ground state of $^3\mathrm{He}-\mathrm{A}$. The point is that $\vec{j}_{an}$ is caused by nodes existence in the gap on the Fermi surface for real $^3\mathrm{He}-\mathrm{A}$ [7, 2]. In (1) ρ is the liquid density, m is the atom mass, $\vec{v}_s$ is the superfluid velocity, $\widehat{l}$ is the weakly inhomogeneous orbital angular momentum vector (hat implies a unit vector) and $\mathrm{C}_0 \approx \rho$. The Eq.(1) has been deduced by many authors in different approaches: by solving the Gorkov [8–10] or the matrix kinetic equations [11], as well as in [12] directly with the use of the ground state wave function.

Though the Eq.(1) and the corresponding physical picture have been broadly discussed [13–21], there are indications in [8, 12, 20] that higher corrections to (1) might occur thus causing difficulties of the superfluid hydrodynamics at $T=0$. In the course of a phenomenological consideration of the free energy of $^3\mathrm{He}-\mathrm{A}$ Volovik and Mineev have found one of these corrections in the form $\chi_{\mathrm{orb}}\,D\,\widehat{l}_a\vec{\partial}\,\widehat{l}_a$, where $D=\partial_t+\vec{v}_s\cdot\vec{\partial}$ [13]. Trying to check on their own the presence of $\vec{j}_{an}$ in (1), Combescot and Dombre have developed a microscopic calculation [10] which has allowed to claim at $T=0$ the quadratic correction $|\,\widehat{l}\times\mathrm{rot}\,\widehat{l}\,|(\mathrm{rot}\,\widehat{l}\,)_{\perp}$ in the current perpendicular to $\widehat{l}$, and the terms $|\,\widehat{l}\times\mathrm{rot}\,\widehat{l}\,|\left(\vec{v}_s-(1/4m)\mathrm{rot}\,\widehat{l}\,\right)_{||}$ and $|\,\widehat{l}\times\mathrm{rot}\,\widehat{l}\,|(\partial_1\widehat{l}_2+\partial_2\widehat{l}_1)$ in the current parallel to $\widehat{l}$.

Though [10] has eventually been aimed at the case $T=0$ and the Gorkov equation has been solved exactly after linearization of the order parameter, an intermediate high temperature approximation to obtain a manageable formula for the ξ-integrated Green function has not been avoided. The difference between the approximate and exact formulas has been considered as responsible for the second order corrections at $T=0$. However, due to the strategy adopted in [10], any chance to be accurate with $T\to 0$ and to benefit from the exact solution of the governing equation has been lost. Consequently, the coefficients at the quadratic corrections and some other formulas have become artificially complicated, e.g. requiring additional efforts to state that they are finite and nonzero. As to the higher contributions at $T=0$, the regular expansion procedure has not been convincingly formulated (see discussion in [23]).

Since unambiguous procedure of asymptotic expansion of $\vec{j}$ which would imply a more deep knowledge of the ground state of $^3\mathrm{He}-\mathrm{A}$ seems a meaningful technical problem which deserves resolution, a way to overcome mathematical drawbacks of [10] has been advanced in [22, 23]. That is another method to solve the Dyson–Gorkov equation has been chosen which gave immediately new representations for the fermionic Green functions and thus for $\vec{j}$. It is well known in mathematical physics that Green function of a Sturm–Liouville operator can be written either as integral or as series in eigen–functions. The second possibility is meant here. After the subsequent paper [24] it became clear that this way looks properly because the new representations for $\vec{j}$ admit $T \to 0$ accurately and can rigorously be studied by the Laplace method (steapest descent) provided the characteristic length in the texture is much longer then the coherence lenght ξ_0 (the London limit). As the result, it is possible to deduce corrections to (1) systematically in the form of asymptotic series in powers of gradients of $\widehat{l}$.

This paper completes [22–24] and is organized as follows. Sec. 2 contains the outline of the problem which is the same as in [10] (i.e. the approximations and notations). Sec. 3 is concerned with the solution of the ordinary non-homogeneous differential equation related to the Dyson–Gorkov equation and with the calculation of the mass current in the form of series. Sec. 4 contains the deduction of the integral representations for that series as well as various limiting cases for the correspondingly realized $\vec{j}$: lowest "gradient" limit and zero temperature limit. Three particular cases of mutual orientation of $\operatorname{rot}\widehat{l}$ and $\widehat{l}$ are considered in the Sec. 5 to obtain corrections to $\vec{j}_0$ (1) at $T = 0$ up to third order. Apart from the quadratic terms predicted in [10], new cubic contributions are found which contain the logarithm of the London parameter. The numerical coefficients at the second order terms are provided. Discussion in the Sec. 6 concludes the paper. The present investigation supports rigorously the early results of [10, 13] and would give the basis for any systematical microscopic approach to corrections to (1).

§2. The problem

Since the main task is to calculate the mass current $\vec{j}$ by means of normal Green function, let us start with the standard Dyson–Gorkov equation:

$$\partial_\tau g(\vec{k}, \vec{k}') - \int \mathrm{d}^3 k'' H(\vec{k}, \vec{k}'')\, g(\vec{k}'', \vec{k}') = (2\pi)^3 \delta^{(3)}(k - k')\, \delta(\tau - \tau'). \tag{2}$$

Here τ is "imaginary" time, $g(\vec{k}, \vec{k}')$ is the 2×2 matrix of normal and anomalous two–point Green functions and $H(\vec{k}, \vec{k}'')$ has the form:

$$H(\vec{k}, \vec{k}'') = \begin{pmatrix} \xi_{k''} \delta^{(3)}(k - k'') & (2\pi)^{-3} \Delta(\vec{k}, \vec{k}'') \\ (2\pi)^{-3} \Delta^*(\vec{k}'', \vec{k}) & -\xi_{k''} \delta^{(3)}(k - k'') \end{pmatrix},$$

where $\xi_k \equiv (k^2 - k_F^2)/2m$, k_F is the Fermi momentum and $\Delta(\vec{k}, \vec{k}'')$ is the order parameter corresponding to ${}^3\mathrm{He} - \mathrm{A}$. We shall calculate the current $\vec{j}$ by the formula

$$\vec{j} = \beta^{-1} \sum_\omega (2\pi)^{-3} \int \mathrm{d}^3 k\, \vec{k}\, g_{11}\,. \tag{3}$$

As far as the Refs. [22–24] have been conceived as technical improvement of the Ref.[10], the framework (i.e. approximations, notations) turns out to be unaffected here, and one should be referred to [10] for certain details.

It is appropriate to consider (2), (3) in the mixed coordinate-momentum representation [12, 25, 26]:

$$H(\vec{k}, \vec{r}) = (2\pi)^{-3} \int \mathrm{d}^3 q\, H(\vec{k} + \vec{q}/2, \quad \vec{k} - \vec{q}/2)\, e^{i\, \vec{q}\cdot\vec{r}},$$

$$g_{\vec{k}}(\vec{r}) = (2\pi)^{-3} \int \mathrm{d}^3 q\, g(\vec{k} + \vec{q}, \vec{k})\, e^{i\, \vec{q}\cdot\vec{r}}, \tag{4}$$

where $\vec{r} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$ is the center of mass coordinate and the momentum $\vec{k}$ is conjugated to $\vec{r}_1 - \vec{r}_2$. For instance, it is not difficult to check that the relation

$$(2\pi)^{-3} \int \mathrm{d}^3 k''\, \Delta(\vec{k}, \vec{k}'')\, g(\vec{k}'', \vec{k}') \;=\; \int \mathrm{d}^3 r\, e^{i\,\vec{r}\cdot(\vec{k}'-\vec{k})} \left[\Delta(\vec{k} - i\vec{\partial}_r - \frac{i}{2}\vec{\partial}_y\,,\, \vec{y})\, g_{\vec{k}'}(\vec{r})\right]\Big|_{\vec{y}=\vec{r}}$$

holds by (4). Applying (4) to (2) we set in the lowest in δ/E_F order:

$$\xi_{(k-i\partial)} \;\approx\; \xi_k - i\,m^{-1}\,\vec{k}\cdot\vec{\partial},$$

$$\Delta\left(\vec{k} - i\vec{\partial}_r - \frac{i}{2}\vec{\partial}_y\,,\,\vec{y}\right)\Big|_{\vec{y}=\vec{r}} \;\approx\; \Delta(\vec{k}, \vec{r}) \;=\; \delta(\widehat{k}\cdot\widehat{\Delta}_1(\vec{r}) + i\widehat{k}\cdot\widehat{\Delta}_2(\vec{r}))\,, \tag{5}$$

where δ is the gap amplitude, $\widehat{k}$ is unit reciprocal vector and the orbital momentum vector is given by $\widehat{\Delta}_1 \times \widehat{\Delta}_2 = \widehat{l}$. The Eqs.(5) read that (2) can be written in the mixed representation in the form:

$$i\omega\, g - \begin{pmatrix} \xi - ic_F\widehat{k}\cdot\vec{\partial} & \Delta(\vec{k}, \vec{r}) \\ \Delta^*(\vec{k}, \vec{r}) & -\xi + ic_F\widehat{k}\cdot\vec{\partial} \end{pmatrix} g \;=\; \mathbb{1}\,, \tag{6}$$

where $g \equiv g_{\vec{k}}(\vec{r})$, $\xi \approx c_F(k - k_F)$, c_F is the Fermi velocity, and ω is fermionic Matsubara frequency.

Thus we have obtained the approximate Eq.(6) which can be nicely treated as 1-dimensional because the spatial differentiation is along the directions labeled by $\widehat{k}$. Indeed, in [26] a gradient expansion method is used which is applicable to study dynamics of spatially inhomogeneous systems provided inhomogeneities are slow compared to the relevant length scales. As the result, a separation of 3-dimensional problem into a collection of 1-dimensional local subsystems occurs. Proofs useful for justification of our approach can be picked up from [26].

As far as we are interested in $\vec{j}$ in arbitrary point, say, $\mathcal{O}$, let us define the spherical coordinates ρ, θ, ϕ centered at $\mathcal{O}$ and let us linearize the slowly varying order parameter as follows:

$$\Delta(\vec{k}, \vec{r}) \;\approx\; \Delta(\widehat{k}, \rho = 0) + \alpha\rho \;\equiv\; \alpha(\rho_0 + \rho) + i\Delta, \tag{7}$$

$$\Delta \;\equiv\; \mathrm{Im}\;\Delta(\widehat{k}, \rho = 0)\,.$$

As the physical result in $\mathcal{O}$ is assumed to be independent on the choice of the point, it is given by $\vec{r} \to \mathcal{O}$ in final formulas for any $\vec{r}$. Therefore, we shall solve (6) at $\vec{r} = \rho\widehat{k}$ so that $\widehat{k}\cdot\vec{\partial}$ is simply $\partial/\partial\rho$ and put $\rho = 0$ in the result [10].

To be precise, we shall consider our problem for the coherence length ξ_0 much smaller than a length of the orbital vector $\widehat{l}$ variation:

$$\xi_0 \;=\; \frac{c_F}{\delta} \;\ll\; |\vec{\partial}\otimes\widehat{l}|^{-1}\,,$$

or

$$\frac{1}{\chi^2} \;\equiv\; \xi_0|\vec{\partial}\otimes\widehat{l}| \;\ll\; 1 \tag{8}$$

(the London limit). The parameter α (7) depends on the angle variables, vector's $\vec{v}_s$ components and first derivatives of $\widehat{l}$ taken in $\mathcal{O}$, and it will be written explicitly in the Sec. 5. From (8) it is seen that the condition $\alpha\rho \lesssim \delta$ ensuring the linearization (7) implies $\rho/\xi_0 \lesssim \chi^2$, and holds better provided χ^2 is greater.

Changing the variable $\mathrm{x} = (\alpha/c_{_F})^{1/2}(\rho + \rho_0)$ and eliminating ξ from the L.H.S. in (6) one gets:

$$(i\omega + \mathcal{H})\, G \;=\; e^{i\mathrm{x}\xi(\alpha c_{_F})^{-1/2}}\, \mathbb{I}\,, \tag{9}$$

where

$$\mathcal{H} \;=\; i\sqrt{\alpha c_{_F}}\,\sigma_3 \frac{\mathrm{d}}{\mathrm{dx}} \;-\; \sqrt{\alpha c_{_F}}\,\sigma_1 \mathrm{x} + \Delta\sigma_2 \tag{10}$$

is the Hamiltonian and σ_i are the Pauli matrices. In this case (3) becomes

$$\vec{j} = k_{_F}^3 (8\pi^3 c_{_F})^{-1} \int \mathrm{d}\Omega\, \widehat{k} \left(\beta^{-1} \sum_{\omega} \mathcal{J} \right), \tag{11}$$

where $\mathcal{J}$ is the ξ-integrated Green function:

$$\mathcal{J}(\mathrm{x}) = \int \mathrm{d}\xi e^{-i\mathrm{x}\xi(\alpha c_{_F})^{-1/2}}\, G_{11}(\mathrm{x}), \tag{12}$$

and $G(\mathrm{x})$ is to be determined from (9).

§3. Solution of the non-homogeneous-equation in form of series

To solve (9) let us take $G(\mathrm{x})$ in the form

$$G = \sqrt{2}\,\mathrm{u} \begin{pmatrix} h_1 & h_2 \\ f_1 & f_2 \end{pmatrix}, \quad \mathrm{u} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \tag{13}$$

where $h_{1,2} \equiv h_{1,2}(\mathrm{x})$ and $f_{1,2} \equiv f_{1,2}(\mathrm{x})$ are now to be determined. Adjoint action $\mathrm{u}^{-1}\sigma_1\mathrm{u} = \sigma_2$ (cycl. perm.) of the unitary matrix u on the Pauli matrices transforms the operator $\mathcal{H}$ (10) to $\mathcal{H}_{em}$:

$$\mathrm{u}^{-1}\, \mathcal{H}\, \mathrm{u} \;=\; \mathcal{H}_{em}, \quad \mathcal{H}_{em} \;=\; \begin{pmatrix} \Delta & i\sqrt{\alpha c_{_F}}\, a^- \\ -i\sqrt{\alpha c_{_F}}\, a^+ & -\Delta \end{pmatrix}, \tag{14}$$

where $a^{\pm} = \mathrm{x} \mp \mathrm{d}/\mathrm{d}x$. The operator $\mathcal{H}_{em}$ reminds the Hamiltonian of spinning electron in constant homogeneous magnetic field. It is straightforward to obtain its eigen-values E_0, $\pm E_n$ and eigen-functions $\widehat{\Psi}_0$, $\widehat{\Psi}_n^{\pm}$ $(n \geq 1)$ [22]:

$$\widehat{\Psi}_0 \;=\; \begin{pmatrix} 0 \\ \psi_0(\mathrm{x}) \end{pmatrix}, \quad E_0 = -\Delta,$$

$$\widehat{\Psi}_n^{(s)} \;=\; \frac{1}{\sqrt{2E_n}} \begin{pmatrix} \sqrt{E_n + s\Delta}\;\, \psi_{n-1}(\mathrm{x}) \\ -is\sqrt{E_n - s\Delta}\;\, \psi_n(\mathrm{x}) \end{pmatrix}, \quad sE_n, \tag{15}$$

where $s = \pm$, $E_n = \sqrt{\Delta^2 + 2\alpha c_{_F} n}$ and $\psi_n(\mathrm{x})$ are the Hermite functions.

Let us use (14) to go from (9) to the equation

$$(i\omega + \mathcal{H}_{em}) \begin{pmatrix} h \\ f \end{pmatrix} \;=\; \delta(\mathrm{x} - x')\, e^{i\mathrm{x}\xi(\alpha c_{_F})^{-1/2}} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \tag{16}$$

to calculate G_{11}. The Dirac δ-function is placed in the R.H.S. of (16), the unknown h, f depend now on x, x' and the required matrix element is given by

$$G_{11}(\mathrm{x}) = \int \mathrm{d}x'(h(\mathrm{x}, x') \,+\, f(\mathrm{x}, x'))\,. \tag{17}$$

To solve (16) it is natural to expand $\begin{pmatrix} h \\ f \end{pmatrix}$ in the functions (15):

$$\begin{pmatrix} h \\ f \end{pmatrix}(\mathrm{x}, x') \;=\; B(\mathrm{x}')\,\widehat{\Psi}_0(\mathrm{x}) \;+\; \sum_{s=\pm}\sum_{n=1}^{\infty} b_n^{(s)}(\mathrm{x}')\,\widehat{\Psi}_n^{(s)}(\mathrm{x}) \tag{18}$$

We calculate $B(\mathrm{x}')$, $b_n^{(s)}(\mathrm{x}')$ using orthogonality of the vectors (15) [22, 24], and from (12), (17), (18) obtain $\mathcal{J}$:

$$\mathcal{J} \;=\; \pi\sqrt{\alpha c_{_F}}\left[\frac{<\widehat{\Psi}_0, \widehat{\Psi}_0>}{i\omega + E_0} \;+\; \sum_{s=\pm}\sum_{n=1}^{\infty}\frac{<\widehat{\Psi}_n^{(s)}, \widehat{\Psi}_n^{(s)}>}{i\omega + sE_n}\right], \tag{19}$$

where $<\cdot,\cdot>$ stands for Hermitian scalar product. The representation (19) for the ξ-integrated Green function is alternative to that which has been found in [10] (the Eq.(34)) as a quadratic combination of parabolic cylinder functions.

Now summation over ω is straightforward [27] and one gets:

$$\beta^{-1}\sum_{\omega}\mathcal{J} =$$

$$= \pi\sqrt{\alpha c_{_F}}\left[n(E_0)\psi_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(\psi_{n-1}^2 + \psi_n^2\right) + \frac{\Delta}{2}\sum_{n=1}^{\infty}\left(\psi_{n-1}^2 - \psi_n^2\right)\frac{\tanh(\beta E_n/2)}{E_n}\right], \tag{20}$$

where $n(\epsilon)$ is the Fermi weight. Inserting (20) to $\vec{j}$ (11) we obtain the required general representation for the mass current near zero temperature [23, 24]. Due to the explicit dependence on β, the Eq.(20) admits $T \to 0$ as well: one has to replace $n(E_0)$ by the Heavyside function $\theta(E_0)$ and $\tanh(\beta E_n/2)$ by 1. It can be argued that the second term in (20) is irrelevant with regard to the angle integration and therefore $\vec{j}$ acquires the final form as the series:

$$\vec{j} = k_{_F}^3(8\pi^2 c_{_F})^{-1}\int \mathrm{d}\Omega\,\widehat{k}\sqrt{\alpha c_{_F}}\left[n(E_0)\psi_0^2 \;+\; \frac{\Delta}{2}\sum_{n=1}^{\infty}\left(\psi_{n-1}^2 - \psi_n^2\right)\frac{\tanh(\beta E_n/2)}{E_n}\right]. \tag{21}$$

§4. Integral representations and the IR limiting cases

4.1. Integral representations.

Practically, it is more convenient to be concerned with an integral representation equivalent to (21). Such representation has been found in [23] at zero temperature, and here we shall deduce it for $\beta^{-1}\sum_{\omega}\mathcal{J}$ and $\vec{j}$ at $T \neq 0$ [24]. These representations will admit special limiting cases.

First of all, one has to rearrange (19) [22]:

$$\mathcal{J} \;=\; -\pi\sqrt{\alpha c_{_F}}\sum_{n=0}^{\infty}\psi_n^2\left(\frac{i\omega - \Delta}{\omega^2 + E_{n+1}^2} \;+\; \frac{i\omega + \Delta}{\omega^2 + E_n^2}\right). \tag{22}$$

By the formula

$$\beta^{-1}\sum_{\omega} i\omega(\omega^2 + \epsilon^2)^{-1} = -1/2$$

we see that the odd in ω part of (22) is responsible for the second term in (20) and it is enough to consider only the "even" part of (22):

$$\mathcal{J}_e \;=\; \frac{\pi}{2}\,\frac{\Delta}{\sqrt{\alpha c_{_F}}}\,\sum_{n=0}^{\infty}\psi_n^2\;\left[(|\lambda|^2+n+1)^{-1}-(|\lambda|^2+n)^{-1}\right], \tag{23}$$

where $|\lambda|^2 \equiv (\omega^2+\Delta^2)(2\alpha c_{_F})^{-1}$.

By the formula (AI.2) (*APPENDIX I*) the series (23) can be expressed as the integral

$$\mathcal{J}_e \;=\; -\Delta\left(\frac{\pi}{\alpha c_{_F}}\right)^{1/2}\int\limits_0^{\infty}\mathrm{d}t\,(\tanh t)^{1/2}\,e^{-\mathrm{x}^2\tanh t-2|\lambda|^2\,t}\,. \tag{24}$$

Thus one can go further:

$$\beta^{-1}\sum_{\omega}\mathcal{J}_e \;=\; -\Delta\left(\frac{\pi}{\alpha c_{_F}}\right)^{1/2}\int\limits_0^{\infty}\mathrm{d}t\,(\tanh t)^{1/2}\times\big(T\,\vartheta_2(0,\,i\tau)\big)\,e^{-\mathrm{x}^2\tanh t-(\Delta^2/\alpha c_{_F})\,t}, \tag{25}$$

where the elliptic theta function ϑ_2 [28] implies the series

$$\sum_{m=0}^{\infty}a^{(m+\frac{1}{2})^2} = \frac{1}{2}\vartheta_2(0,i\tau),$$

$\tau = (-1/\pi)\log a$, and $a = \exp(-4\pi^2T^2t/\alpha c_{_F})$. Changing the integration variable $t \longmapsto \kappa t$, $\kappa = \alpha c_F(\beta/2)^2$, one can rewrite (25) more suitably for studying the limiting cases below:

$$\beta^{-1}\sum_{\omega}\mathcal{J}_e \;=\; -\Delta\,\frac{\kappa^{1/2}}{2}\int\limits_0^{\infty}\mathrm{d}t\,\left(\frac{\tanh(\kappa t)}{t}\right)^{1/2}\times\widetilde{\Theta}(t)\,e^{-\mathrm{x}^2\tanh(\kappa t)-(\Delta\beta/2)^2\,t}, \tag{26}$$

where $\widetilde{\Theta}(t) = (\pi t)^{1/2}\,\vartheta_2(0,\,i\,\pi\,t)$.

The Eqs. (25) and (26) are just to be substituted to (11) to get the general integral representations for $\vec{j}$ near zero temperature. These representations are very convenient in calculating higher corrections to (1). Before to proceed to it in the Sec. 5, let us consider some particular limiting cases for (25), (26).

4.2. Limiting cases.

Let us represent the situation by the following "commutative diagram":

$$\begin{array}{ccc} \vec{j}=\vec{j}\big|_{\beta^{-1}\sum\mathcal{J}=(26)} & \xrightarrow{\;1\;} & \mathrm{I} \\ {\scriptstyle 4}\big\downarrow & & \big\downarrow{\scriptstyle 2} \\ \mathrm{II} & \xrightarrow{\;3\;} & (1), \end{array} \tag{27}$$

where the horizontal arrows $1,3$ mean $T\to 0$ and the vertical ones $2,4$ – the lowest "gradient" approximation.

To begin with, the usage of the limit

$$\lim_{\tau\to 0} \vartheta_2(0, i\tau) \;=\; \tau^{-1/2}$$

[28] in (25) allows $\vec{j}$ to be written at $T = 0$ as follows (the arrow *1*):

$$\vec{j} \;=\; -\,3\rho\,(8\pi c_F)^{-1} \int \mathrm{d}\Omega\, \widehat{k}\, \Delta\, F\,(\mathrm{x}^2,\, \Delta^2/\alpha\, c_F)\,, \tag{28}$$

where $\rho = k_{_F}^3/3\pi^2$ (two spin projections are taken into account), and the function $F(p,q)$ is given by

$$F(p,\,q) \;=\; \int\limits_0^\infty \mathrm{d}t\, \left(\frac{\tanh t}{t}\right)^{1/2} e^{-p\tanh t - q\,t}\,, \qquad q\,>\,0\,.$$

To perform the lowest "gradient" approximation (the arrow *2*), we replace tanh t by t in $F(p,\,q)$ (see (8) and (AII.6)) and (28) takes the form:

$$\vec{j} = -3\rho\,(8\pi c_F)^{-1} \int \mathrm{d}\Omega\, \widehat{k}\Delta \int\limits_0^\infty \mathrm{d}t\, e^{-\left(\mathrm{x}^2 + \frac{\Delta^2}{\alpha c_F}\right)t} = 3\rho\,(8\pi)^{-1} \int \mathrm{d}\Omega\, \widehat{k}\, \frac{\alpha\Delta}{\alpha c_F \mathrm{x}^2 + \Delta^2} \tag{29}$$

(compare with the Eq.(44) in [10]). To understand (29), let us recall the representation for $\vec{j}_0$ which has been discussed in [19, 20]:

$$\vec{j}_0 \;=\; 3\rho\,(8\pi)^{-1} \int \mathrm{d}\Omega\, \widehat{k} \left((\widehat{k}\cdot\vec{\partial})\arctan\left(\frac{\widehat{k}\cdot\widehat{\Delta}_2}{\widehat{k}\cdot\widehat{\Delta}_1}\right)\right)\,, \tag{30}$$

and which results in (1) with $\widehat{l} = \widehat{\Delta}_1 \times \widehat{\Delta}_2$ and $(v_s)_i = 2^{-1}\,\widehat{\Delta}_1\cdot\partial_i\widehat{\Delta}_2$. Using

$$\widehat{k}\cdot\vec{\partial} \;=\; \sqrt{\frac{\alpha}{c_F}}\,\frac{\mathrm{d}}{\mathrm{dx}}\,, \qquad \frac{\widehat{k}\cdot\widehat{\Delta}_2}{\widehat{k}\cdot\widehat{\Delta}_1} \;=\; \frac{\Delta}{\mathrm{x}\sqrt{\alpha c_F}}\,,$$

it is easy to check coincidence of (29) and (30). So without any special gauge for α it is seen that (1) is the lowest London limit approximation to (28).

To consider the steps *4*$\to$ *3*, one should replace $\tanh(\kappa t)$ by κt in (26) (the arrow *4*) due to steepest descent validity at $\Delta \neq 0$ for large β :

$$\beta^{-1}\sum_\omega \mathcal{J}_e \;=\; -\,\frac{\alpha c_F \Delta}{8}\,\beta^2 \int\limits_0^\infty \mathrm{d}t\, \widetilde{\Theta}(t)\, e^{-(|\Delta|\beta/2)^2\,t}. \tag{31}$$

The R.H.S. of (31) is the Laplace transform of $\widetilde{\Theta}(t)$, and can be expressed through the so-called Yosida function Y [29]:

$$a^2 \int\limits_0^\infty \mathrm{d}t\, \widetilde{\Theta}(t)\, e^{-a^2 t} = 1 - \int\limits_0^\infty \frac{dy}{\cosh^2\sqrt{y^2 + a^2}} \equiv 1 - \mathrm{Y}(a). \tag{32}$$

To check (32) it is enough to integrate its L.H.S. as the series and to re-express the answer by the Poisson summation formula, whereas $\cosh^{-2}(\mathrm{y})$ in the R.H.S. has to be expanded first in $\exp(-2\mathrm{y})$ and then integrated [24]. Using (32) one obtains:

$$\vec{j} \;=\; -3\rho\,(8\pi)^{-1} \int \mathrm{d}\Omega\, \widehat{k}\, \frac{\alpha\Delta}{|\Delta|^2} \left(1 - \mathrm{Y}\left(\frac{|\Delta|\beta}{2}\right)\right) \tag{33}$$

(the point *II* on (27)). The Eq.(33) is just the leading $\vec{j}_0$ "dressed" by thermal corrections which has been found by Cross in 1975 [8]. At $T = 0$ (the arrow 3) $\mathrm{Y}(\infty) = 0$ and we recover (1).

§5. Explicit calculations

This section is devoted to the main problem of the present paper. That is it will be concerned with asymptotical expansion of $\vec{j}$ (28) in order to deduce the London limit corrections to (1). At fixed $\hat{k}$ the overall phase of the order parameter $\Delta(\hat{k}, \vec{r})$ can always be changed to make α (7) a real positive. Thus the Eq.(7) can be thought of as

$$\exp(-i\psi)\,\Delta(\hat{k}, \vec{r}) \,\equiv\, \Delta_0 \,+\, \alpha\,\rho\,, \tag{34}$$

where

$$\alpha \,=\, \delta\,\mathcal{M}\,\exp(i(\pi/2-\psi))\,. \tag{35}$$

$$\Delta_0 \,=\, \delta\,\sin\theta\,\exp(i(\phi-\psi))\,.$$

In (34), (35) the phase ψ is to be adjusted while $\mathcal{M}$ is given by

$$\begin{aligned}\mathcal{M} \,=\, &-\cos^2\theta\,\partial_3\hat{l}_2 \,+\, 2m\,\sin^2\theta\,e^{i\phi}\,(v_1\cos\phi + v_2\sin\phi)+\\ &+\sin\theta\cos\theta\,e^{i\phi}\,(2mv_3 + \frac{i}{2}\,div\,\hat{l} - \frac{1}{2}\,\hat{l}\cdot\mathrm{rot}\,\hat{l})+\\ &+\frac{1}{2}\,\sin\theta\cos\theta\,e^{-i\phi}\,(-\partial_1\hat{l}_2 - \partial_2\hat{l}_1 \,+\, i(\partial_1\hat{l}_1 - \partial_2\hat{l}_2))\,,\end{aligned} \tag{36}$$

where $\vec{v}\equiv\vec{v}_s$ (*APPENDIX II*). Without loss of generality $\hat{\Delta}_2(\mathcal{O})$ can be chosen along $\hat{l}\times\mathrm{rot}\,\hat{l}$ so that $\partial_3\hat{l}_1$, $\partial_3\hat{l}_3$ become zero and thus $div\,\hat{l} = \partial_1\hat{l}_1 + \partial_2\hat{l}_2$ [10]. Besides, $\hat{l}\cdot\mathrm{rot}\,\hat{l}$ is $\partial_1\hat{l}_2 - \partial_2\hat{l}_1$ once the third axis is chosen along $\hat{l}(\mathcal{O})$. Moreover, $\partial_1\hat{l}_1, \partial_2\hat{l}_2$ can be excluded from the consideration [10]. Therefore, apart of v_1 and v_2, only $\partial_1\hat{l}_2 + \partial_2\hat{l}_1$, $2mv_3-$ $(1/2)\,\hat{l}\cdot\mathrm{rot}\,\hat{l}$, and $\partial_3\hat{l}_2 = \mathrm{rot}\,\hat{l}\times\hat{l}$ are the relevant gradient combinations. Besides, no difference is expected once $\mathcal{M}$ is considered as dependent separately either on $\partial_1\hat{l}_2 + \partial_2\hat{l}_1$ or $2mv_3-$ $(1/2)\,\hat{l}\cdot\mathrm{rot}\,\hat{l}$.

To make calculations manageable it is appropriate to put a part of gradients in $\mathcal{M}$ equal to zero so to consider the dependence of $\vec{j}$ on the remaining ones. Clearly, it is not necessary to enumerate all the possible cases, but it is enough to point out rather characteristic combinations. To this end, let us take $\mathcal{M}$ in the following reduced form:

$$\mathcal{M} = -\partial_3\hat{l}_2\,\cos^2\theta + (2mv - \partial_1\hat{l}_2)\,\sin\theta\cos\theta\,e^{i\phi} \equiv -\frac{1}{\xi_0\chi_1^2}\cos^2\theta + \frac{1}{\xi_0\chi_2^2}\sin\theta\,\cos\theta\,e^{i\phi} \tag{37}$$

(at $\partial_1\hat{l}_2 + \partial_2\hat{l}_1 = 0$, $v\equiv v_3$). Once ψ is obtained explicitly so that $\alpha\in I\!R^+$ we get:

$$\alpha c_F \,=\, \delta^2\,\sin^2\theta\;Q^{-1/2}\,,\quad x_0^2 \,=\, \left(\frac{\sin\phi}{\chi_1^2\,\tan^2\theta}\right)^2\;Q^{3/2}\,,$$

$$\frac{\Delta}{\delta} \,=\, \cos\theta\,Q^{1/2}\,\left(\frac{1}{\chi_1^2}\,\frac{\cos\phi}{\tan\theta} \,-\, \frac{1}{\chi_2^2}\right)\,, \tag{38}$$

where

$$\frac{1}{Q} \,=\, \left(\frac{1}{\chi_2^2}\,\frac{\sin\phi}{\tan\theta}\right)^2 \,+\, \left(\frac{1}{\chi_2^2}\,\frac{\cos\phi}{\tan\theta} \,-\, \frac{1}{\chi_1^2}\,\frac{1}{\tan^2\theta}\right)^2\,.$$

A convenience is apparent after [24] to integrate by parts in $F(p,q)$ in (28) so that

$$F(\mathrm{x}^2,\,\Delta^2/\alpha c_F) \;=\; Q^{-1/2}\,\Big(1\,+\,\Phi(\mathrm{x}^2,\,Q^{1/2})\Big) \tag{39}$$

with

$$\Phi(\mathrm{x}^2,\,Q^{1/2}) \;=\; \int\limits_0^\infty e^{-\,Q^{1/2}\,t}\left(\sqrt{\frac{\tanh t}{t}}\;e^{\mathrm{x}^2(t-\tanh t)}\right)'\,\mathrm{d}t\,. \tag{40}$$

We have used in (39), (40) the following circumstance. According to the Eq.(34), the order parameter in $\mathcal{O}$ is Δ_0 and the square of its modulus has the simple form:

$$|\Delta_0|^2\;\equiv\;|\Delta|^2\;=\;\Delta^2\,+\,\alpha c_F\mathrm{x}^2\;=\;\delta^2\sin^2\theta$$

by the Eqs.(7) and (35). Now, from (38) it is seen that $Q^{1/2}$ in (39), (40) is just $|\Delta|^2/\alpha c_F$. The first term in (39) is responsible for $\vec{j}_0$ (see (29)), while the second one – for the total correcting contribution:

$$\vec{j}_{\rm corr}\;=\;-\,3\rho\,(8\pi)^{-1}\int \mathrm{d}\Omega\,\widehat{k}\,\frac{\alpha\Delta}{|\Delta|^2}\,\Phi\,(\mathrm{x}^2,\,Q^{1/2})\,.$$

In what follows we shall investigate (41) with $\mathcal{M}$ (37):

$$\vec{j}_{\rm corr}\;=\;3\rho\,(8\pi\xi_0)^{-1}\int \mathrm{d}\Omega\,\widehat{k}\,\cos\theta\left(\frac{1}{\chi_2^2}\;-\;\frac{1}{\chi_1^2}\frac{\cos\phi}{\tan\theta}\right)\,\Phi\,\left(\mathrm{x}^2\;=\;\mathrm{x}_0^2\,,\,Q^{1/2}\right)\,. \tag{42}$$

Varying $\chi_1,\ \chi_2$ in (42) the following three cases (*Examples 1, 2,* and *3*) can be obtained. Fixing χ_2 (or χ_1) $\gg 1$ and tending χ_1 (or χ_2) to infinity we shall get *Example 1* (or *Example 2*). Taking $\chi_1=\chi_2=\chi\gg 1$ we shall arrive to *Example 3*. Each time our attention will be called to the quadratic and cubic contributions to $\vec{j}_{\rm corr}$, i.e. to the terms proportional to $(\xi_0\chi^4)^{-1}$ and $(\xi_0\chi^6)^{-1}$. Clearly, the case *1* corresponds to $\mathrm{rot}\,\widehat{l}$ parallel to $\widehat{l}$ and the case *2* – to $\mathrm{rot}\,\widehat{l}$ perpendicular to $\widehat{l}$. Therefore, the case *1* implies all the three contributions in (1), while the second one corresponds only to the pure orbital content of (1).

5.1.Example 1: rot $\widehat{l}$ is parallel to $\widehat{l}$.

Here only $2mv-\partial_1\widehat{l}_2\neq 0$ in $\mathcal{M}$ (37). As far as we deduce from (38) that

$$Q^{1/2}\;=\;\chi^2\,|\tan\theta|\,,\quad \mathrm{x}_0\;=\;0\,,$$

where $(\xi_0\chi^2)^{-1}\equiv 2mv-\partial_1\widehat{l}_2\;>\;0$, the Eq.(42) reads only the third component, say, j to be nonzero now:

$$j\;=\;\frac{3\rho}{2}\,\frac{1}{\xi_0\chi^2}\int\limits_0^\infty \mathcal{F}(\chi^2u)\,\frac{u\,\mathrm{d}u}{(u^2+1)^{5/2}}\,, \tag{43}$$

where

$$\mathcal{F}(\chi^2u)\;=\;\int\limits_0^\infty e^{-\chi^2ut}\left(\sqrt{\frac{\tanh t}{t}}\right)'\,\mathrm{d}t\,,$$

and $u=|\tan\theta|$. Concrete form of $\mathcal{F}$ is not of importance for us. It is enough to know that $\mathcal{F}(s)\to\mathrm{const}$ as $s\to 0$, and

$$\mathcal{F}(\chi^2u)\;=\;\frac{\mathrm{a}}{(\chi^2u)^2}\,+\,\frac{\mathrm{b}}{(\chi^2u)^4}\,+\,\dots,\quad \chi^2u\gg 1\,, \tag{44}$$

where a$= -1/3$.

To do the estimation let us break the integral over u into two parts:

$$\int_0^\infty \mathcal{F}(\chi^2 u)\,\frac{u\,\mathrm{d}u}{(u^2+1)^{5/2}} = U_1 + U_2\,, \tag{45}$$

where

$$U_2 = \int_1^\infty \mathcal{F}(\chi^2 u)\,\frac{u\,\mathrm{d}u}{(u^2+1)^{5/2}} \approx \frac{\mathrm{a}}{\chi^4}\int_1^\infty (u^2+1)^{-5/2}\,\frac{\mathrm{d}u}{u}\,,$$

because $\chi^2 u \gg 1$ is valid, and we are interested in contributions of total degree in χ not less than -6. As to U_1,

$$U_1 = \frac{1}{\chi^4}\int_0^{\chi^2} \mathcal{F}(u)\,\frac{u\,\mathrm{d}u}{(1+u^2/\chi^4)^{5/2}}\,,$$

its denominator can be expanded in powers of u^2/χ^4 so that U_1 will acquire the form of series where each term is given by the appropriate integral. The upper bound χ^2 of all these integrals can be extended to infinity provided the integral is convergent, either some regularization by counter-term is needed. It is easy to see from (44) that $\int \mathcal{F} u\,\mathrm{d}u$ means only the logarithmic divergency, $\int \mathcal{F} u^3 \mathrm{d}u$ – both the quadratic and logarithmic ones and so on.

Let us do the first subtraction writing $U_1 = \mathrm{X} + \mathrm{Y}$, where

$$\mathrm{X} = \frac{1}{\chi^4}\int_0^{\chi^2} \mathcal{F}(u)\left(\left(1+\frac{u^2}{\chi^4}\right)^{-5/2} - 1\right) u\,\mathrm{d}u\,, \quad \mathrm{Y} = \frac{1}{\chi^4}\int_0^{\chi^2} \mathcal{F}(u)\,u\,\mathrm{d}u\,. \tag{46}$$

Once the integral Y is divergent logarithmically at $\chi^2 \to \infty$, it can be represented approximately:

$$\mathrm{Y} \approx \frac{a}{\chi^4}\log\chi^2 + \frac{1}{\chi^4}\int_0^\infty \left(u\,\mathcal{F}(u) - \frac{\mathrm{a}}{u+1}\right)\mathrm{d}u\,.$$

Let us turn to X (46). The asymptotic (44) tells us that X is divergent at $\chi \to \infty$, and the whole (44) is needed for regularization. It is not difficult to realize that the total contribution of the order χ^{-4} appears as that counter-term which results from X once $\mathcal{F}$ is replaced by a/u^2. Therefore,

$$U_2 + \mathrm{X} = \frac{\mathrm{a}}{\chi^4}\left(-\frac{4}{3} + \log 2\right).$$

Therefore at $\mathrm{a} = -1/3$

$$U_1 + U_2 = \frac{1}{3\chi^4}\left(\frac{4}{3} + \int_0^\infty \left(3u\,\mathcal{F}(u) + \frac{1}{u+1}\right)\mathrm{d}u + \log\frac{1}{2\chi^2}\right). \tag{47}$$

Finally, the use of the Eqs.(43), (45) and (47) enables the third component of $\vec{j}$ to be completly written as follows:

$$j_3 = \frac{\rho}{2}\,\frac{1}{\xi_0\chi^2}\left(1 + \frac{1}{\chi^4}\log\frac{\mathcal{B}}{\chi^2}\right), \tag{48}$$

where

$$\log 2\mathcal{B} \;=\; \frac{4}{3} \;+\; \int\limits_0^\infty \left(3u\,\mathcal{F}(u) \;+\; \frac{1}{u+1}\right) \mathrm{d}u\,.$$

All the quadratic corrections predicted in [10] are zero in the present gauge, and the lowest one turns out to be cubic with the logarithm of the London parameter.

5.2.Example 2: rot $\widehat{l}$ is perpendicular to $\widehat{l}$. In this case the Eqs.(38) result in

$$Q^{1/2} \;=\; (\chi \tan\theta)^2\,, \qquad \mathrm{x}_0 \;=\; -\chi|\tan\theta|\sin\phi\,,$$

where $(\xi_0\chi^2)^{-1} \;\equiv\; \partial_3\widehat{l}_2 \;>\; 0$. From (42) the components 2 and 3 of $\vec{j}_{\mathrm{corr}}$ are zero, whereas the first one acquires the form:

$$j \;=\; -\,\frac{3\rho}{4}\,\frac{1}{\xi_0\chi^2}\int\limits_0^\infty \mathcal{F}(\chi^2u^2)\,\frac{u\,\mathrm{d}u}{(u^2+1)^{5/2}}\,, \tag{49}$$

where

$$\mathcal{F}(\chi^2u^2) \;=\; \int\limits_0^\infty e^{-\chi^2u^2t}\left(\sqrt{\frac{\tanh t}{t}}\;{}_1F_1(\,\frac{1}{2};\,2;\,\chi^2u^2(t-\tanh t))\right)'\mathrm{d}t\,,$$

$u \;=\; |\tan\theta|$ and the formula

$$\int\limits_0^1 \mathrm{d}v\,\sqrt{1-v^2}\,e^{v^2p} \;=\; \frac{\pi}{4}\,{}_1F_1\left(\frac{1}{2},\,2;\,p\right), \tag{50}$$

$$p \;=\; u^2(t-\tanh \mathrm{t}\,) \;\geq\; 0$$

is used to re-express the integration over $v \;=\; \sin\phi$. In (50) ${}_1F_1$ is the Kummer function [28]. The relevant analytical properties of $\mathcal{F}$ are the following: $\mathcal{F}(0) = \mathrm{const}$ and asymptotically

$$\mathcal{F}(\chi^2u^2) \;=\; \frac{\mathrm{a}}{(\chi u)^4} \;+\; \frac{\mathrm{b}}{(\chi u)^8} \;+\ldots\,, \qquad \chi^2u^2 \gg 1\,,$$

where $\mathrm{a} = 1/6$.

Again let us represent the integral

$$\int\limits_0^\infty \mathcal{F}(\chi^2u^2)\,\frac{u\,\mathrm{d}u}{(u^2+1)^{5/2}}\,,$$

as the sum of U_1 and U_2 so that

$$U_2 \;=\; \int\limits_1^\infty \mathcal{F}(\chi^2u^2)\,\frac{u\,\mathrm{d}u}{(u^2+1)^{5/2}} \;\approx\; \frac{\mathrm{a}}{\chi^4}\int\limits_1^\infty (u^2+1)^{-5/2}\,\frac{\mathrm{d}u}{u^3}\,.$$

Now two first subtractions are needed to estimate $U_1 = \mathrm{X} + \mathrm{Y} + \mathrm{Z}$, where

$$\mathrm{Z} = \frac{1}{\chi^2}\int\limits_0^{\chi} \mathcal{F}(u^2)\, u \mathrm{d}u\,, \qquad \mathrm{Y} = -\frac{5}{2}\frac{1}{\chi^4}\int\limits_0^{\chi} \mathcal{F}(u^2)\, u^3 \mathrm{d}u\,,$$

$$\mathrm{X} = \frac{1}{\chi^2}\int\limits_0^{\chi} \mathcal{F}(u^2)\left(\left(1+\frac{u^2}{\chi^2}\right)^{-5/2} - 1 + \frac{5}{2}\frac{u^2}{\chi^2}\right) u\,\mathrm{d}u\,.$$

Clearly, Z is convergent at large χ and approximately

$$\mathrm{Z} \approx \frac{1}{\chi^2}\int\limits_0^{\infty} \mathcal{F}(u^2)\, u\,\mathrm{d}u - \frac{\mathrm{a}}{2\,\chi^4}\,.$$

Further, a single counter-term is required for Y:

$$\mathrm{Y} \approx -\frac{5}{2}\frac{1}{\chi^4}\int\limits_0^{\infty}\left(u^3\,\mathcal{F}(u^2) - \frac{a}{u+1}\right)\mathrm{d}u - \frac{5\,\mathrm{a}}{2}\,\frac{\log\chi}{\chi^4}\,.$$

Now let us consider X. Here the series in the brackets begins with the term proportional to $(u/\chi)^4$, and a single regulator is needed. The next term will require two and so on. The total contribution of the order χ^{-4} is given once $\mathcal{F}$ is replaced by a/u^4 in X. The net result reads:

$$U_2 + \mathrm{X} = \frac{\mathrm{a}}{\chi^4}\left(\frac{37}{12} - \frac{5}{2}\log 2\right)\,,$$

and, therefore,

$$U_1 + U_2 = \frac{1}{\chi^2}\int\limits_0^{\infty} \mathcal{F}(u^2) u \mathrm{d}u + \frac{\mathrm{a}}{\chi^4}\left(\frac{31}{12} + \frac{5}{2}\log\frac{1}{2\chi}\right) - \frac{5}{2\chi^4}\int\limits_0^{\infty}\left(u^3\mathcal{F}(u^2) - \frac{\mathrm{a}}{u+1}\right)\mathrm{d}u.$$

As the final result, the non-zero part of $\vec{j}$ is:

$$j_1 = -\frac{\rho}{4}\frac{1}{\xi_0\chi^2}\left(1 + \frac{\mathcal{A}}{\chi^2} + \frac{5}{8}\frac{1}{\chi^4}\log\frac{\mathcal{B}}{\chi^2}\right)\,, \tag{51}$$

where

$$\mathcal{A} = 3\int\limits_0^{\infty} \mathcal{F}(u^2)\, u\,\mathrm{d}u \approx -2\times 10^{-1}\,, \tag{52}$$

$$\log 4\mathcal{B} = \frac{31}{15} - 12\int\limits_0^{\infty}\left(u^3\mathcal{F}(u^2) - \frac{1}{6(u+1)}\right)\mathrm{d}u\,.$$

In this case there are two corrections, and the lowest is of the type $(\mathrm{rot}\,\widehat{l})_{\perp}|\widehat{l}\times\mathrm{rot}\,\widehat{l}|$ found in [10] for the current perpendicular to $\widehat{l}$ (one should be referred to the formula (53) in [10]). The coefficient $\mathcal{A}$ (52) has been estimated numerically in [23]. The next term is the new cubic one and it includes the logarithm of the London parameter.

5.3. Example 3.

In this case we shall take into account the whole (37) which would imply appearance of the quadratic corrections of the type $|\widehat{l}\times \mathrm{rot}\,\widehat{l}|(\vec{v}-(1/4m)\mathrm{rot}\widehat{l})_{||}$ and $|\widehat{l}\times \mathrm{rot}\,\widehat{l}|(\partial_1\widehat{l}_2+\partial_2\widehat{l}_1)$ [10]. However, we put here $\chi_1 = \chi_2$ for simplicity, and so the answer expected would demonstrate such corrections only in principle. To this end we shall investigate the third component $j_{\mathrm{corr},3}$ which is along $\widehat{l}$. We obtain from (42):

$$j \,=\, \frac{3\rho}{2\pi}\,\frac{1}{\xi_0\chi^2}\iint\limits_{\Pi}\frac{\mathrm{d}u\mathrm{d}v}{\sqrt{(u^2+1)^5\,(1-v^2)}}\left((u\,-\,v)\mathcal{F}_-\,+\,(u\,+\,v)\mathcal{F}_+\right), \tag{53}$$

$$\mathcal{F}_{\mp} \,=\, \int\limits_0^\infty \exp\left(-t\,(\chi u)^2\,\mathcal{Q}^{1/2}\right)$$
$$\times\left(\sqrt{\frac{\tanh t}{t}}\,\exp\left((t\,-\,\tanh t)(\chi\,u)^2(1-v^2)\,\mathcal{Q}^{3/2}\right)\right)'\mathrm{d}t\,, \tag{54}$$

where $\mathcal{Q}^{-1}$ stands for

$$\mathcal{Q}_{\mp}^{-1} \,=\, 1\,+\,u^2\,\mp\,2\,u\,v\,=\,1\,-\,v^2\,+\,(u\mp v)^2\,,$$

the domain $\prod$ is given by $\{(u,v):\ u\in[0,\infty[,\, v\in[0,1]\}$, and $u=\tan\theta,\ v=\cos\phi$.

The function $\mathcal{M}$ is still rather complicated and so the present consideration becomes less elegant then the two previous. The estimations we are interested in will be obtained without providing the asymptotic integral formulas for the coefficients. Besides, we shall assume that not only χ but $\log\chi$ also is large (logarithmic accuracy), and thus only the logarithmic terms will be kept in the third order. Before to proceed, it is useful to estimate the expression Φ (40) in general situation. By steepest descent we get:

$$\Phi\,(\mathrm{x}^2, Q^{1/2})\,\simeq\,-\frac{1}{3}\,Q^{-1}\,+\,2\mathrm{x}^2\,Q^{-3/2}\,, \tag{55}$$

at $Q^{1/2}\,=\,|\Delta|^2/\alpha c_F \gtrsim 1$ (i.e. either $\Delta^2/\alpha c_F$ or x^2 must be $\gtrsim 1$, and it is forbidden to tend $\Delta^2/\alpha c_F$ to zero). In the opposite case $Q^{1/2}\ll 1$

$$\Phi\,(\mathrm{x}^2,\,Q^{1/2})\,\simeq\,-1\,+\,\pi^{1/2}\,\mathrm{x}^2\,\left(\frac{\alpha c_F}{\Delta^2}\right)^{1/2}\,. \tag{56}$$

First of all, let us consider the contribution to j (53) which is due to $u\in[1,\infty[$. Here the function $\mathcal{F}$ can be expanded by steepest descent because $(\chi u)^2\mathcal{Q}^{1/2}\gg 1$. This expansion will begin with the third order term $\mathrm{const}\times(\xi_0\chi^6)^{-1}$ which is not of interest for us. So, in what follows we shall take $0\le u\le 1$ in j.

Now let us consider the domain $0\le u\le 1/\chi$. Approximately we put:

$$\frac{3\rho}{2\pi}\,\frac{1}{\xi_0\chi^2}\int\limits_0^{1/\chi}\mathrm{d}u\int\limits_0^1\frac{\mathrm{d}v}{(1-v^2)^{1/2}}\left((u\,-\,v)\mathcal{F}_-\,+\,(u\,+\,v)\mathcal{F}_+\right). \tag{57}$$

In this case $\Delta^2/\alpha c_F \simeq (\chi u)^2\,(u \mp v)^2$ and the Eq.(56) should be used as far as $\Delta^2/\alpha c_F$ is $\lesssim 1$ and can become zero. So we obtain

$$\mathcal{F}_{\mp} \simeq -1 + \pi^{1/2}\,\frac{\chi u\,(1-v^2)}{|u \mp v|}\,. \tag{58}$$

The part of j which is due to the first term in (58) looks as follows:

$$-\frac{3\rho}{\pi}\,\frac{1}{\xi_0\chi^2}\int_0^{1/\chi} u\,\mathrm{d}u \int_0^1 \frac{\mathrm{d}v}{(1-v^2)^{1/2}} = -\frac{3\rho}{4}\,\frac{1}{\xi_0\chi^4}\,. \tag{59}$$

The second term in (58) does not contribute at $v \geq 1/\chi$ because $\mathrm{sign}(u-v) = -1$, and therefore

$$\frac{3\rho}{2\sqrt{\pi}}\,\frac{1}{\xi_0\chi}\int_0^{1/\chi} u\,\mathrm{d}u \int_0^{1/\chi} \mathrm{d}v\,(sign(u\,-\,v)\,+\,1) = \frac{\rho}{\sqrt{\pi}}\,\frac{1}{\xi_0\chi^4}\,. \tag{60}$$

Then, the total contribution below $u = 1/\chi$ is:

$$j' = \frac{\rho}{2}\,\frac{1}{\xi_0\chi^4}\left(\frac{2}{\sqrt{\pi}} - \frac{3}{2}\right)\,. \tag{61}$$

At last, let us consider the rectangle $\{(u,v) : 1/\chi \leq u \leq 1,\ v \in [0,1]\}$. Here $\mathcal{F}_+$ can safely be expanded by the Laplace method. As to $\mathcal{F}_-$, the parameter $\Delta^2/\alpha c_F$ can become zero inside the strip $|u - v| \leq 1/\chi$ along the diagonale $u = v$. In this case $\mathcal{F}_-$ should be estimated as $1/\mathrm{x}^4 \sim 1/\chi^4$. So there is no interesting for us contribution as the strip's width is $2/\chi$. Outside the strip $\mathcal{F}_-$ can be expressed by (55) and only the boundary $u = 1/\chi$ contributes the corrections we are looking for. So, we get with $\mathcal{F}_+$ and $\mathcal{F}_-$:

$$-\frac{\rho}{\pi}\,\frac{1}{\xi_0\chi^6}\int_{1/\chi}^1 \frac{\mathrm{d}u}{u^3}\,\frac{1}{(u^2+1)^{5/2}}\int_0^1 \frac{\mathrm{d}v}{(1-v^2)^{1/2}}\,(u^2\,+\,8\,v^2\,-\,5)\,.$$

The last equation results in the following terms:

$$\frac{\rho}{4}\,\frac{1}{\xi_0\chi^4} + \frac{7\rho}{8}\,\frac{1}{\xi_0\chi^6}\,\log\frac{1}{\chi^2}\,. \tag{62}$$

With regard to j' (61) the total quadratic correction acquires the form:

$$\frac{\rho}{2}\,\frac{1}{\xi_0\chi^4}\left(\frac{2}{\sqrt{\pi}} - 1\right) \approx \frac{\rho}{2}\,\frac{\mathcal{A}}{\xi_0\chi^4}\,, \quad \mathcal{A} \approx 13 \times 10^{-2}\,. \tag{63}$$

§6. Discussion

The present paper is concerned with the two main problems: to calculate the mass current $\vec{j}$ in weakly inhomogeneous $^3\mathrm{He}-\mathrm{A}$ using thermal Green functions and to obtain its asymptotic expansions at $T=0$ provided the London limit holds. Two main assumptions are of importance for our approach: the static order parameter can be linearized due to slowness of its spatial variation, and only those first order differentiations are retained in the chosen mixed coordinate-momentum representation which are due to the kinetic energy of the BCS-Hamiltonian. Using slowness of the orbital vector texture we reduce the three initial dimensions to the one-dimensional situation so that the resulting operator $\mathcal{H}_{em}$ has the simple form of the Hamiltonian of the Landau problem. Therefore, one can solve the governing Dyson–Gorkov equation exactly: just using the eigen-functions of $\mathcal{H}_{em}$ [22–24]. Thus a collection of exact formulas both for the Green function and $\vec{j}$ appears which opens the possibility to derive systematically the higher gradient corrections to the dominant expression $\vec{j}_0$ (1). The present paper completes the preceding ones [22–24] which have been aimed at a more thorough resolution of the mathematical difficulties of the Ref.[10]. Our approach unravels the situation and provides a correct procedure to find the structure and the order of magnitude of higher contributions to (1). The given approach is manifestly advantageous because the Laplace method is highly appropriate in the London limit.

Mathematically, we are mainly concerned with the ξ-integrated and then ω-summated normal Green function which results in two representations for $\vec{j}$: the series and the integral. The integral one seems to be more attractive as far as it allows to obtain a self-contained expression (41) for the net correcting contribution. The last can satisfactory be studied by steepest descent due to the London limit holds, i.e. it can be expanded in powers of the orbital vector gradients. Particular limiting cases (zero temperature limit following the limit of lowest order in gradients, and *vice versa*) confirm the correctness of our manipulations. The known expression (1) arises as the lowest approximation without resorting to specific gauges for the order parameter. The appropriate "commutative" diagram is considered in the Sec. 4.2.

Three special orientations of $\mathrm{rot}\,\widehat{l}$ are considered in the Sec.5 to deduce the correcting terms explicitly: $\mathrm{rot}\,\widehat{l}$ is parallel (*Example 1*), and perpendicular (*Example 2*) to $\widehat{l}$, while the third *Example 3* implies an intermediate orientation of $\mathrm{rot}\,\widehat{l}$ with respect to $\widehat{l}$. Corrections are considered up to third order in gradients of $\widehat{l}$ and they are not only of pure polynomial type. Namely, new cubic corrections are found which contain the logarithm of the London parameter. By comparison with [10], we provide the numerical coefficients at the second order terms. The nodes vicinities $0 \le \theta$ (or $\pi-\theta$) $\lesssim 1/\chi$ on the Fermi sphere give the dominant contribution to the numerical coefficients at the pure power terms.

In the first case only the third order logarithmic correction is present. In the second case both the lowest ones appear: the quadratic and the cubic. As it is clear from the analysis [10], all the quadratic corrections should be proportional to $|\widehat{l}\times\mathrm{rot}\,\widehat{l}|$, and that immediately explains, why the quadratic term is absent in *Example 1*. We have specified the *Example 2* so that $\vec{j}_{\mathrm{corr}}$ is orthogonal to $\widehat{l}$. Thus the second order term in (51) would correspond to that which has been written in [10] in the form $(\mathrm{rot}\,\widehat{l})_\perp|\widehat{l}\times\mathrm{rot}\,\widehat{l}|$, and the corresponding numerical coefficient is $\mathcal{A}$ (52) [23]. *Example 3* also results in the both lowest corrections: the quadratic (63) and the cubic (62). As to the quadratic correction along $\widehat{l}$, the answer of [10] reads:

$$\rho\,\xi_0\,|\widehat{l}\times\mathrm{rot}\,\widehat{l}|\,\Big(A\,(v_3\,-\,\widehat{l}\cdot\mathrm{rot}\,\widehat{l}/4m)\,+\,B\,(\partial_1\widehat{l}_2\,+\,\partial_2\widehat{l}_1)\Big)\;.$$

The term (63) above is just to be compared with the last expression at $A\neq0$, $B=0$. It is clear from (36) that the result obtained by us is also applicable to establish the contribution at $A = 0$, $B \neq 0$.

Moreover, our situation is more rich than in [10] because the logarithmic corrections are demonstrated. In this respect, let us recall the correction found in [13]. As it is seen from (36), the components of the superfluid velocity $\vec{v}$ and gradients of $\widehat{l}$ enter equally to the parameter α, and therefore $\chi_{\rm orb} D\,\widehat{l}_a \vec{\partial}\,\widehat{l}_a$, where $\chi_{\rm orb}$ is logarithmically large [9], should be treated as the logarithmic third order term, and it would correspond to the third order contribution in (62).

To conclude, the investigation presented which is based on thermal Green functions demonstrates various corrections of second and third order to the mass current (1) at $T = 0$ provided the London limit condition is imposed. The higher corrections predicted early in [10, 13] can systematically be deduced from our general formulas. The representations obtained for $\vec{j}$ and $\vec{j}_{\rm corr}$ would serve a basis for further investigations.

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Appendix I

The following Mehler formula is valid:

$$\sum_{n=0}^{\infty} a^n \psi_n^2(y) \,=\, \frac{1}{\sqrt{\pi(1-a^2)}} \exp\left(-\,\frac{1-a}{1+a}\, y^2\right), \qquad |a| \,<\, 1\,, \tag{AI.1}$$

where $\psi_n(y)$ are the Chebyshev–Hermite functions (see generating functions of Hermite polynomials in [28]). Using

$$(n\,+\,q)^{-1} \,=\, \int\limits_0^\infty {\rm d}t\, e^{-t(n+q)}\,, \qquad n \,\geq\, 0\,,\, q \,>\, 0\,,$$

and (AI.1), we obtain:

$$\sum_{n=0}^{\infty} \frac{\psi_n^2(y)}{n+q} \,=\, \frac{1}{\sqrt{2\pi}} \int\limits_0^\infty \frac{{\rm d}t}{\sqrt{\sinh t}} \exp\left(\left(\frac{1}{2} - q\right) t \,-\, y^2 \tanh(t/2)\right) \tag{AI.2}$$

[22].

Appendix II

Here we obtain α (35) as in [10]. To this end let us rewrite

$$\Delta(\widehat{k}, \vec{r}) \,=\, \delta(\widehat{k} \cdot \widehat{\Delta}_1(\vec{r}) + i\widehat{k} \cdot \widehat{\Delta}_2(\vec{r})) \tag{AII.1}$$

with the help of

$$\widehat{\Delta}_\alpha(\vec{r}) \approx \widehat{\Delta}_0(\mathcal{O}) + \vec{r}\cdot\vec{\partial}\,\widehat{\Delta}_\alpha(\mathcal{O}), \qquad (\alpha = 1, 2), \tag{AII.2}$$

where the derivatives $\partial_i\widehat{\Delta}_\alpha(\mathcal{O})$ are linear functions of $\widehat{\Delta}_\alpha(\mathcal{O})$:

$$\partial_i\widehat{\Delta}_\alpha(\mathcal{O}) \;=\; \vec{\omega}^{\,i} \times \widehat{\Delta}_\alpha(\mathcal{O}). \tag{AII.3}$$

The Eq. (AII.3) acquires the form of the Mermin–Ho relation [30] provided the identifications

$$\omega_3^i \;=\; -2mv_i\,, \qquad \omega_2^i \;=\; \partial_i\widehat{l}_1\,, \qquad \omega_1^i \;=\; -\partial_i\widehat{l}_2 \tag{AII.4}$$

are made. Substituting (AII.2)–(AII.4) to (AII.1) one gets

$$\Delta(\widehat{k},\vec{r}) \;=\; \delta(\widehat{k}_1 \,+\, i\widehat{k}_2) \,+\, \rho\, i\delta\, \Big(2mv_p\widehat{k}_p(\widehat{k}_1 + i\widehat{k}_2) - \widehat{k}_3\widehat{k}_p(\partial_p\widehat{l}_2 - i\partial_p\widehat{l}_1)\Big) \;\equiv$$

$$\equiv\; \delta\, \sin\theta\, e^{i\phi} \,+\, \rho\, \Big[\delta\,(...)\; e^{i(\pi/2-\psi)}\Big]\; e^{i\psi} \tag{AII.5}$$

Eventually, the square brackets in (AII.5) are denoted as α, and the factor $e^{i\psi}$ is to make it a real positive fixing therefore $\widehat{\Delta}_1$, $\widehat{\Delta}_2$ in the plane perpendicular to $\widehat{l}$. From (AII.5) it is seen that αc_F is a linear form of gradients which can be written formally as

$$\alpha c_F \;=\; \delta^2\xi_0 \sum\,(gradients) \;=\; \frac{\delta^2}{\chi^2}\sum \frac{gradients}{|gradients|}. \tag{AII.6}$$

By (8) we consider $\alpha c_F/\delta^2$ as small parameter.

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