PDMI PREPRINT — 11/1996

# Boundary Control and Inverse Problems: One-Dimensional variant of the BC-Method

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Oktober 1996

This is the first paper included in a planned series of articles entitled "Boundary Control and Inverse Problems". It deals with one approach to Inverse Problems (so-called "BC-method") based upon ideas and results of the Boundary Control Theory. One-dimensional variant of the method is demonstrated on the case of classical problem of recovering a density of inhomogeneous string. Both dynamical and spectral statements of the problem are considered. The paper is written so that to be an introduction to the multidimensional BC-method. Bibliography–23.

## §0. Introduction

This is the first paper included in a planned series of articles lumped together as "Boundary Control and Inverse Problems". The series is devoted to one approach to Inverse Problems (IP's) based upon ideas and results of the Boundary Control Theory. It is so-called the BC-method proposed in [1].

It should be noted that the approach was originally created to solve multidimensional problems. Early in development of it, a scheme of the method was rather complicated. Later it was simplified, but, nevertheless, to introduce the method it would be convenient to start from some of its "elementary" variant. This paper plays the role of introduction in the BC-method. It is demonstrated on the case of classical (one-dimensional) problem of recovering a density of inhomogeneous string. Both dynamical and spectral statements of the problem are considered.

The dynamical IP for a string plays a central role. Its statement concerns to a dynamical system with boundary control

$$\begin{cases} \rho u_{tt} - u_{xx} = 0, \quad x > 0, \quad t > 0, \\ u|_{t=0} = u_t|_{t=0} = 0, \\ u|_{x=0} = f. \end{cases}$$

Entering into the wave equation, a density $\rho = \rho(x)$ is assumed to be a smooth positive function. Let $u = u^f(x,t)$ be a solution. The map

$$R\colon f \to u^f_x\big|_{x=0}$$

realizes an "input $\to$ output" correspondence of the system. The IP is to recover a density $\rho$ via the given $R$. We demonstrate how can one solve the problem by the BC-method. It should be noted that 1-dim version of the method is very closely related to the "local approach" belonging to A. S. Blagovestchenskii [2] who first solved the problem in intrinsic dynamical terms.

The spectral IP for a finite string goes back to classical results of M.G. Krein [3–6]. It is connected with the boundary-value problem

$$\begin{cases} \varphi'' + \lambda\rho\varphi = 0, \quad x \in \big(0; x_*\big) \\ \varphi|_{x=0} = \varphi|_{x=x_*} = 0. \end{cases}$$

Let $\big\{\lambda_k\big\}_{k=1}^{\infty}$ be its spectrum, and $\big\{\varphi(\cdot)\big\}_{k=1}^{\infty}$ be the corresponding eigenfunctions normalized by conditions

$$\int\limits_0^{x_*} \rho(x)\varphi_k(x)\varphi_l(x)\,dx = \delta_{kl}; \quad \varphi'_k(0) > 0.$$

The set of pairs $\big\{\lambda_k; \varphi'_k(0)\big\}_{k=1}^{\infty}$ is said to be spectral data of a string. The spectral IP is to recover $\rho$ on $\big(0; x_*\big)$ via given spectral data. This problem is solved by spectral variant of the BC-method [7].

So, we solve the problems solved earlier by M. G. Krein and A. S. Blagovestchenskii. However, in contrast to known approaches the BC-method may be directly extended on multidimensional cases.

By its nature the BC-method is an approach of a complex character. It uses the Control Theory, the System Theory, asymptotic methods in Propagation of Singularities, and the Functional Analysis. However, in our opinion its principle feature is that the method recovers and uses very close and deep relations between the Boundary Control Theory and IP's. To demonstrate these relations is one of main goals of this paper.

## §1. Kinematics

**1.1. Eikonal.** A string is identified with the semiaxis $x \geq 0$ provided with a function (density) $\rho(\cdot) \in C^2_{\rm loc}[0;\infty)$; $\rho(x) > 0$, $x \geq 0$. So, on every compact interval $0 \leq x \leq h$ a density is separated from zero.

Function $\tau(\cdot)$,

$$\tau(x) := \int\limits_0^x \rho^{1/2}(s)\, ds >, \quad x \geq 0,$$

is called an eikonal (time function, optical length). It is a smooth positive function strongly growing along a string; $\tau(0) = 0$. From a physical view-point, its value $\tau(x_0)$ coincides with the time needed for a wave initiated at the endpoint $x = 0$ to pass the interval $[0; x_0]$. An eikonal plays a role of natural coordinate on a string.

An inverse function is denoted $x(\cdot)$; so, $\tau\big(x(\xi)\big) = \xi$ and $x\big(\tau(s)\big) = s$ for any $\xi, s \geq 0$. It may be represented in the form of

$$x(\tau) = \int\limits_0^\tau \rho^{-1/2}\big(x(\xi)\big)\, d\xi, \quad \tau \geq 0.$$

Following from the definitions the relations

$$\tau'(x) = \rho^{1/2}(x), \quad x > 0; \quad x'(\tau) = \rho^{-1/2}\big(x(\tau)\big), \quad \tau > 0, \tag{1.1}$$

are valid.

Below we shall use an increasing family of sets

$$\Omega^\xi := \big\{x > 0 : \tau(x) < \xi\big\} = \big(0; x(\xi)\big),$$

each one having the sense of interval of a string filled by waves up to moment $t = \xi$. The whole of string is filled over the time

$$T_* := \sup_{x \geq 0} \tau(x) \leq \infty.$$

**1.2. Power Moments.** Let us introduce functions (moments)

$$\mu_0(\tau) := \int\limits_0^{x(\tau)} \rho(x)\, dx; \quad \mu_1(\tau) := \int\limits_0^{x(\tau)} x\rho(x)\, dx; \quad \nu(\tau) := \int\limits_0^{x(\tau)} (x_* - x)^2 \rho(x)\, dx, \quad \tau > 0,$$

where $x_* > 0$ is a fixed point on a string. The following representations turn out to be useful for the IP:

$$x(\tau) = \int\limits_0^\tau \frac{d\eta}{\mu_0'(\eta)} = \frac{\mu_1'(\tau)}{\mu_0'(\tau)} = x_* - \Big[\int\limits_0^\tau \frac{d\eta}{\nu'(\eta)} + \frac{1}{x_*}\Big]^{-1}, \quad \tau > 0. \tag{1.2}$$

Let us prove the latter one. Differentiating we have

$$\nu'(\tau) = \big(x_* - x(\tau)\big)^2 \rho\big(x(\tau)\big) x'(\tau) = text(see (1.1)) = \big(x_* - x(\tau)\big)^2 \frac{1}{x'(\tau)},$$

that leads to equation

$$\frac{x'(\tau)}{\big(x_* - x(\tau)\big)^2} = \frac{1}{\nu_h'(\tau)}.$$

An integration gives the needed equality. The other ones are obtained in the same way.

**1.3. Images.** Let $a = a(x)$ be a function given on a string. Function

$$\widetilde{a}(\tau) := \rho^{1/4}\big(x(\tau)\big)\, a\big(x(\tau)\big), \quad \tau \geq 0,$$

is said to be an image of $a(\cdot)$. The (linear) map $I: a \to \widetilde{a}$ is called the image map. Let us point out some of the useful properties:

(i) the inclusion $\operatorname{supp} a \subset \bar{\Omega}^\xi$ is equivalent to $\operatorname{supp} \widetilde{a} \subset [0;\xi]$, $\xi > 0$;

(ii) map $I$ is invertible,

$$\big(I^{-1} f\big)(x) = \rho^{-1/4}(x) f\big(\tau(x)\big), \quad x \geq 0;$$

(iii) let $L$ and $\widetilde{L}$ be the operators acting on the classes $C^2_{\mathrm{loc}}\big[x \geq 0\big]$ and $C^2_{\mathrm{loc}}\big[\tau \geq 0\big]$ respectively,

$$\big(La\big)(x) := \rho^{-1}(x) a''(x); \quad \big(\widetilde{L} f\big)(\tau) := f''(\tau) - q(\tau) f(\tau),$$

where

$$q(\tau) := \frac{1}{4}\frac{\rho''\big(x(\tau)\big)}{\rho^2\big(x(\tau)\big)} - \frac{5}{16}\frac{\big[\rho'\big(x(\tau)\big)\big]^2}{\rho^3\big(x(\tau)\big)},$$

is a continuous function of $\tau \geq 0$. Relation

$$\widetilde{L} I = I L \tag{1.3}$$

may be checked by direct inspection using the change of variables $x \mapsto \tau = \tau(x)$.

In the IP not the image map itself is used, but some of its reduction described here. For fixed $T < T_*$ let us introduce two (real) Hilbert spaces: $\mathcal{H}^T := L_2\big(\Omega^T; \rho\, dx\big)$,

$$(a,b)_{\mathcal{H}^T} = \int\limits_0^{x(\tau)} a(x) b(x) \rho(x)\, dx,$$

and $\widetilde{\mathcal{H}}^T := L_2\big(0; T\big)$,

$$(f,g)_{\widetilde{\mathcal{H}}^T} = \int\limits_0^T f(\tau) g(\tau)\, d\tau.$$

The map $I^T: \mathcal{H}^T \to \widetilde{\mathcal{H}}^T$,

$$\big(I^T a\big)(\tau) = \rho^{1/4}\big(x(\tau)\big)\, a\big(x(\tau)\big), \quad 0 \leq \tau \leq T,$$

is called image operator. It is a unitary operator from $\mathcal{H}^T$ onto $\widetilde{\mathcal{H}}^T$. Indeed, for any $a, b \in \mathcal{H}^T$ and $f \in \widetilde{\mathcal{H}}^T$ we have

$$(a,b)_{\mathcal{H}^T} = \int\limits_0^{x(T)} a(x) b(x) \rho(x)\, dx =$$

$$\int\limits_0^T a\big(x(\tau)\big) b\big(x(\tau)\big) \rho\big(x(\tau)\big) \frac{d\tau}{\rho^{1/2}\big(x(\tau)\big)} = (\widetilde{a}, \widetilde{b})_{\widetilde{\mathcal{H}}^T} = \big(I^T a, I^T b\big)_{\widetilde{\mathcal{H}}^T};$$

$$\Big(\big(I^T\big)^{-1} f\Big)(x) = \Big(\big(I^T\big)^* f\Big)(x) = \rho^{-1/4}(x) f\big(\tau(x)\big), \quad x \in \Omega^T. \tag{1.4}$$

## 2. Dynamics

**2.1. Boundary-Value Initial Problem.** Let us fix some $T < T_*$ and consider the system

Its solution (wave) $u = u^f(x,t)$ describes a vibration of string initiated by a boundary source (control) $f = f(t)$. All of the essential properties of $u^f$ may be extracted from a convenient representation brought out below. To formulate the result it would be useful to accept the following agreement.

**Remark 2.1**. All the functions depending on time $t$ are assumed to be continued by zero on $t < 0$.

**Lemma 2.1.** *For any $f \in C^2[0;T]$, $f(0) = f'(0) = 0$, problem (2.1)–(2.3) has a unique classical solution $u^f$ which may be represented in the form of*

$$u^f(x,t) = \Big(\frac{\rho(x)}{\rho(0)}\Big)^{-1/4} f\big(t-\tau(x)\big) + \int\limits_x^{x(t)} w(x,s) f\big(t-\tau(s)\big)\,ds, \quad x \geq 0, \quad 0 \leq t \leq T, \tag{2.4}$$

*with the piece-wise continuous kernel $w$. This kernel is a twice continuously differentiable function in the domain $\{(x,s) : 0 \leq \tau(x) \leq s \leq T\}$ satisfying $w(0,s) = 0$, $0 \leq s \leq T$; in the complement domain it vanishes identically: $w(x,s) = 0$ for $x(s) < x < \infty$. A possible jump of $w$ may be present only on the characteristic curve $s = \tau(x)$.*

*Sketch of the proof.*

(i) Using (1.3) one can show that function

$$\widetilde{u}^f(\tau,t) := \rho^{-1/4}(0)\big(Iu^f(\cdot,t)\big)(\tau) = \Big(\frac{\rho\big(x(\tau)\big)}{\rho(0)}\Big)^{1/4} u^f\big(x(\tau),t\big)$$

satisfies the system

$$\begin{cases} \widetilde{u}_{tt} - \widetilde{u}_{\tau\tau} + q(\tau)\widetilde{u} = 0, \quad \tau > 0, \quad 0 < t < T, & (2.5)\\ \widetilde{u}\big|_{t=0} = \widetilde{u}_t\big|_{t=0} = 0, & (2.6)\\ \widetilde{u}\big|_{\tau=0} = f. & (2.7) \end{cases}$$

Searching its solution in the form of

$$\widetilde{u}^f(\tau,t) = f(t-\tau) + \int\limits_\tau^t \widetilde{w}(\tau,s) f(t-s)\,ds \tag{2.8}$$

with a kernel $\widetilde{w}$: $\widetilde{w}\big|_{s<\tau} \equiv 0$ the system may be reduced to a classical Goursat problem

$$\begin{cases} \widetilde{w}_{ss} - \widetilde{w}_{\tau\tau} + q(\tau)\widetilde{w} = 0 \quad \text{in } D^T, & (2.9)\\ \widetilde{w}\big|_{\tau=0} = 0, \quad \widetilde{w}\big|_{\tau=s} = -\dfrac{1}{2}\displaystyle\int\limits_0^s q(\eta)\,d\eta, & (2.10) \end{cases}$$

where $D^T := \big\{(\tau,s) : 0 < \tau < s < T\big\}$.

(ii) Problem (2.9), (2.10) is reduced to a Volterra's type equation with the help of the following trick. It is easy to check that function

$$y(\tau, s) := \widetilde{w}(\tau, s) + \frac{1}{2} \int\limits_{\frac{s-\tau}{2}}^{\frac{s+\tau}{2}} q(\eta)\, d\eta \tag{2.11}$$

solves the problem

$$y_{ss} - y_{\tau\tau} = G \quad \text{in } D^T, \tag{2.12}$$

$$y\big|_{\tau=0} = 0, \quad y\big|_{\tau=s} = 0, \tag{2.13}$$

where $G = G(\tau, s) := -q(\tau)y(\tau, s) + q(\tau)F(\tau, s)$, and $F(\tau, s)$ denotes the integral term in (2.11).

Problem (2.12), (2.13) is resolved by the D'Alambert formula:

$$y(\tau, s) = \big(MG\big)(\tau, s) := \iint\limits_{\mathcal{M}^{(\tau,s)}} G(\tau', s')\, d\tau'\, ds', \quad (\tau, s) \in D^T, \tag{2.14}$$

where $\mathcal{M}^{(\tau,s)}$ is the rectangle with vertexes $(\tau, s)$, $(0, s-\tau)$, $(\frac{s-\tau}{2}, \frac{s-\tau}{2})$, $(\frac{s+\tau}{2}, \frac{s+\tau}{2})$. Applying (2.14) to problem (2.12), (2.13) we reduce it to the integral equation

$$y + M(qy) = M(qF) \quad \text{in } D^T \tag{2.15}$$

of Volterra's type which may be analyzed in the space $C^2\big(\bar{D}^T\big)$ by standard methods.

(iii) A solvability of (2.15) leads to a solvability of (2.9), (2.10) that justifies representation (2.8). Returning back on a string by means of map $\rho^{1/4}(0)I^{-1}: \widetilde{u}^f \to u^f$ we obtain (2.4). Equality $w(0, \cdot) = 0$ follows from the condition $\widetilde{w}(0, \cdot) = 0$ in (2.10). The lemma is proved.

**Remark 2.2**. The right-hand side of (2.4) has a meaning for any $f \in L_2(0; T)$. In what follows we consider it as a (generalized) solution $u^f$ of problem (2.1)–(2.3) for controls of this class.

Being a very convenient instrument in direct and inverse problems, representation (2.4), unfortunately, has not an analog in multidimensional case.

**2.2. Properties of waves.**

(i) Along with (2.4) one has $u^f(x, t) \equiv 0$ for $x > x(t)$, that implies the inclusion

$$\operatorname{supp} u^f(\cdot, \xi) \subset \bar{\Omega}^\xi = \big[0; x(\xi)\big] \tag{2.16}$$

being valid for any fixed $t = \xi > 0$. It shows that wave $u^f$ propagates along a string filling the increasing interval $\Omega^\xi$. Its "forward front" $x = x(\xi)$ moves with velocity $x'(\xi) = \rho^{-1/2}\big(x(\xi)\big)$, that motivates a "kinematical" terminology used in sec. 1.

Satisfying $u^f \equiv 0$, $x > x(t)$, any classical, (i.e., twice-differentiable) solution has to vanish on the characteristic curve $t = \tau(x)$ together with the first derivatives. Therefore, in addition to (2.16), the equalities

$$u^f\big(x(\xi), \xi\big) = u^f_x\big(x(\xi), \xi\big) = u^f_t\big(x(\xi), \xi\big) = 0, \quad 0 < \xi < T, \tag{2.16'}$$

**Fig. 1.**

are fulfilled on a forward front of any smooth wave.

(ii) Another important property of waves is known as the "principle of locality." We formulate it omitting the details (see [2]).

Let $\Delta^T := \big\{(x,t)\colon\ x \in \Omega^T,\ \ 0 < t < 2T - \tau(x)\big\}$ be a space-time domain shown on the Fig. 1.

For any $T < T_*$ problem

$$\begin{cases} \rho u_{tt} - u_{xx} = 0 \quad \text{in} \ \ \Delta^T, \\ u|_{t=0} = u_t|_{t=0} = 0, \\ u|_{x=0} = f, \end{cases}$$

turns out to be well-posed. Thus, for given control $f$ a solution $u^f$ in $\Delta^T$ is uniquely determined by values $\rho|_{\Omega^T}$, being independent on behavior of density on $x > x(T)$.

This result has an apparent physical sense being related to a finiteness of velocity of wave propagation.

(iii) An independence of density $\rho$ on time leads to the so-called "stationary state" property. For any $f$ we define a delayed control $f_\xi$,

$$f_\xi(t) := f\big(t - (T - \xi)\big), \quad 0 \le t \le T,$$

($\xi$ be a parameter, $\xi \in [0;T]$), understanding the definition in accordance with Remark 2.1.

Relation

$$u^{f_\xi}(\cdot, T) = u^f(\cdot, \xi) \tag{2.17}$$

is valid for arbitrary $f$. It means that a delay of control leads to the same delay of wave.

**2.3. String as Dynamical System. Control Operator.** Here problem (2.1)–(2.3) is considered in terms adopted in Control Theory. The objects determining a string as a dynamical system (DS) are introduced. The system itself is denoted $\alpha^T$.

The space of controls (inputs) $\mathcal{F}^T := L_2(0;T)$ is said to be *an outer space* of $\alpha^T$.

Wave $u^f(\cdot, T)$ is considered as a state of DS $\alpha^T$. Every state is localized on interval $\Omega^T$ (see (2.16)) and belongs to the space $\mathcal{H}^T := L_2(\Omega^T; \rho\, dx)$ (see Remark 2.2) which is called *an inner space* of the system.

The "input-state" correspondence in $\alpha^T$ is realized by *the control operator* $W^T\colon \mathcal{F}^T \to \mathcal{H}^T$,

$$W^T f := u^f(\cdot, T).$$

This operator resolves problem (2.1)–(2.3). To describe its useful representation we need some auxiliary operators:

integral Volterra operator $K^T: \mathcal{H}^T \to \mathcal{H}^T$,

$$(K^T a)(x) = \int\limits_x^{x(T)} k(x,s)a(s)\,ds, \quad x \in \Omega^T,$$

with the kernel

$$k(x,s) := \left(\frac{\rho(s)}{\rho(0)}\right)^{1/4} w(x,s),$$

where $w$ is taken from (2.4);

operator $Y^T: \mathcal{F}^T \to \mathcal{F}^T$,

$$\big(Y^T f\big)(t) := T - t, \quad 0 < t < T,$$

satisfying $\big(Y^T\big)^{-1} = \big(Y^T\big)^* = Y^T$.

We understand here the image operator as a map from $\mathcal{H}^T$ in $\mathcal{F}^T$,

$$\big(I^T a\big)(t) = \rho^{1/4}\big(x(t)\big)\,a\big(x(t)\big), \quad 0 < t < T.$$

Its adjoint see in (1.4).

In what follows the unit operators are denoted $\mathbb{I}$.

**Lemma 2.2.** *Control operator $W^T$ acts continuously from $\mathcal{F}^T$ in $\mathcal{H}^T$, representation*

$$W^T = \rho^{1/4}(0)\big[\mathbb{I}_{\mathcal{H}^T} + K^T\big]\big(I^T\big)^* Y^T \tag{2.18}$$

*being valid.*

*Proof.* Along with (2.4) one has

$$\begin{aligned}\big(W^T f\big)(x) &= \left(\frac{\rho(x)}{\rho(0)}\right)^{-1/4} f\big(T-\tau(x)\big) + \int\limits_x^{x(T)} w(x,s) f\big(T-\tau(s)\big)\,ds = \\ &= \rho^{1/4}(0)\Big[\rho^{-1/4}(x) f\big(T-\tau(x)\big) + \int\limits_x^{x(T)} k(x,s)\rho^{-1/4}(s) f\big(T-\tau(s)\big)\,ds\Big],\end{aligned} \tag{2.19}$$

that is equivalent to (2.18). The lemma is proved.

In its turn, (2.18) implies the following representation of $\big(W^T\big)^*: \mathcal{H}^T \to \mathcal{F}^T$:

$$\big(W^T\big)^* = \rho^{1/4}(0) Y^T I^T \big[\mathbb{I}_{\mathcal{H}^T} + \big(K^T\big)^*\big];$$

$$((W^T)^* a)(t) = \Big(\rho\big(x(T-t)\big)\rho(0)\Big)^{1/4} a\big(x(T-t)\big) + \int\limits_t^T w^*(\eta,t) a\big(x(T-\eta)\big)\,d\eta, \quad 0 \le t \le T, \tag{2.20}$$

with the smooth kernel

$$w^*(\eta,t) := w\big(x(T-\eta), x(T-t)\big)\left(\frac{\rho\big(x(T-t)\big)}{\rho\big(x(T-\eta)\big)}\right)^{1/4}.$$

A stationary state property of the DS $\alpha^T$ may be expressed as follows. Let $\mathcal{T}^{T,\xi}\colon \mathcal{F}^T \to \mathcal{F}^T$ be the delay operator,

$$(\mathcal{T}^{T,\xi} f)(t) = f_\xi(t) := \begin{cases} 0, & 0 < t < T-\xi, \\ f\big(t-(T-\xi)\big), & T-\xi \le t < T, \quad (0<\xi<T); \end{cases}$$
$$\mathcal{T}^{T,0} := \mathbb{O}_{\mathcal{F}^T}; \quad \mathcal{T}^{T,T} := \mathbb{I}_{\mathcal{F}^T}.$$

Then relation (2.17) may be written in the form of

$$W^T \mathcal{T}^{T,\xi} f = u^f(\cdot,\xi), \quad 0 \le \xi \le T. \tag{2.21}$$

**2.4. Response Operator.** An "input-output" correspondence in DS $\alpha^T$ is realized by *the response (or transfer) operator* $R^T\colon \mathcal{F}^T \to \mathcal{F}^T$, $\operatorname{Dom} R^T = \big\{f \in C^2[0;T]\colon\ f(0) = f'(0) = 0\big\}$,

$$\big(R^T f\big)(t) := u^f_x(0,t), \quad 0 < t < T.$$

From a viewpoint of mechanics, values $u^f_x(0,\cdot)$ describe the forces generated at the endpoint $x=0$ of a string by a wave process which was initiated by control $f$.

**Lemma 2.3.** *For any $f \in \operatorname{Dom} R^T$ representation*

$$(R^T f)(t) = -\rho^{1/2}(0) f'(t) - \frac{\rho'(0)}{4\rho(0)} f(t) + \int_0^t r(t-s) f(s)\, ds, \quad 0<t<T, \tag{2.22}$$

*with kernel $r \in C^1[0;T]$ is valid.*

*Proof.* Differentiating on $x$ in (2.4), one has

$$u^f_x(x,t) = -\left(\frac{\rho(x)}{\rho(0)}\right)^{-1/4} f'\big(t-\tau(x)\big)\tau'(x) - \frac{1}{4}\left(\frac{\rho(x)}{\rho(0)}\right)^{-5/4} \frac{\rho'(x)}{\rho(0)} f\big(t-\tau(x)\big) -$$
$$- w(x,x) f\big(t-\tau(x)\big) + \int_x^{x(t)} w_x(x,s) f\big(t-\tau(s)\big)\, ds.$$

Putting here $x=0$ and taking into account $\tau'(0) = \rho^{1/2}(0)$, $w(0,0)=0$ (see Lemma 2.1), we obtain

$$(R^T f)(t) = u^f_x(0,t) = -\rho^{1/2}(0) f'(t) - \frac{1}{4}\frac{\rho'(0)}{\rho(0)} f(t) + \int_0^{x(t)} w_x(0,s) f\big(t-\tau(s)\big)\, ds, \quad 0 \le t \le T.$$

A change of variables $\eta = t - \tau(s)$, $s = x(t-\eta)$, leads to (2.22) with the kernel

$$r(\xi) = \rho^{-1/2}\big(x(\xi)\big) w_x\big(0, x(\xi)\big), \quad 0 \le \xi \le T.$$

The lemma is proved.

The following notation is similar to Remark 2.2.

**Remark 2.3**. For any $f \in H^1[0;T]$, $f(0)=0$, the right-hand side of (2.22) retains its sense as a function of $\mathcal{F}^T$. In what follows it is put as a definition of $R^T f$ for controls of this Sobolev class.

So, an extended response operator is defined on $\mathrm{Dom}\,(R^T) = \{f \in H^1[0;T]\colon\ f(0)=0\}$. One could easily check its properties brought out below:

(i) adjoint operator $(R^T)^*\colon \mathcal{F}^T \to \mathcal{F}^T$, $\mathrm{Dom}\,(R^T)^* = \{f \in H^1[0;T]\colon\ f(T)=0\}$ is represented in the form of

$$\Big((R^T)^* f\Big)(t) = \rho^{1/2}(0) f'(t) - \frac{\rho'(0)}{4\rho(0)}\, f(t) + \int\limits_t^T r(s-t) f(s)\, ds, \quad 0<t<T; \tag{2.23}$$

(ii) let $\mathcal{T}^{T,\xi}$ be delay operator, and $\mathcal{J}^T$ be the integrating operator in $\mathcal{F}^T$, $(\mathcal{J}^T f)(t)\colon= \int_0^t f(s)\, ds$. The relations

$$\mathcal{T}^{T,\xi} R^T = R^T \mathcal{T}^{T,\xi}, \quad \mathcal{J}^T R^T = R^T \mathcal{J}^T \tag{2.24}$$

are valid on $\mathrm{Dom} R^T$;

(iii) along with (2.22) operator $R^T \mathcal{J}^T$ is represented in the form of

$$(R^T \mathcal{J}^T f)(t) = -\rho^{1/2}(0) f(t) + \int\limits_0^t r_1(t-s) f(s)\, ds, \quad 0<t<T, \tag{2.25}$$

with the kernel

$$r_1(t) = -\frac{\rho'(0)}{4\rho(0)} + \int\limits_0^t r(s)\, ds, \quad t \geq 0;$$

it is a continues operator defined on the whole $\mathcal{F}^T$.

In accordance with principle of locality (see (ii), subsec. 2.2), for any given $f$ the values $u^f_x(0,t)$, $t \in [0;2T]$, are determined by a behavior of density $\rho$ on interval $\Omega^T$. That implies the following important property of a response operator:

(iv) operator $R^{2T}\colon \mathcal{F}^{2T} \to \mathcal{F}^{2T}$, $\mathrm{Dom}\, R^{2T} = \{f \in H^1[0;2T]\colon\ f(0)=0\}$, is determined by values of $\rho|_{\Omega^T}$, being independent of $\rho(x)$, $x > x(T)$.

A local character of dependence of a response operator on a density holds true in multidimensional case.

Representation (2.22) has no natural analog for the case of several space variables. A response operator of multidimensional DS is essentially more complicated object.

**2.5. Connecting Operator.** Operator $C^T\colon \mathcal{F}^T \to \mathcal{F}^T$,

$$C^T\colon= (W^T)^* W^T,$$

is said to be *the connecting operator* of DS $\alpha^T$. Along with the definition one has

$$(C^T f, g)_{\mathcal{F}^T} = (W^T f, W^T g)_{\mathcal{H}^T} = (u^f(\cdot,T), u^g(\cdot,T))_{\mathcal{H}^T}, \tag{2.26}$$

so, operator $C^T$ connects the metrics of outer and inner spaces of DS $\alpha^T$.

A connecting operator plays a central role in the BC-method due to the following remarkable fact. It may be expressed via a response operator in a simple and explicit form. To formulate the result it is convenient to introduce some auxiliary operators, namely, operator of odd continuation of controls $S^T\colon \mathcal{F}^T \to \mathcal{F}^{2T}$,

$$(S^T f)(t) := \begin{cases} f(t), & 0 < t < T, \\ -f(2T-t), & T \le t < 2T; \end{cases}$$

operator $G^{2T}\colon \mathcal{F}^{2T} \to \mathcal{F}^{2T}$,

$$(G^{2T} f)(t) := \frac{1}{2}\big[f(t) - f(2T-t)\big], \quad 0 < t < 2T,$$

selecting an odd part of controls;
operator $N^{2T}\colon \mathcal{F}^{2T} \to \mathcal{F}^T$,

$$N^{2T} f := f\big|_{(0;T)},$$

reducing controls on interval $(0;T)$.

One can easily derive the relation for adjoint operator $\left(S^T\right)^*\colon \mathcal{F}^{2T} \to \mathcal{F}^T$,

$$\left(S^T\right)^* = 2N^{2T}G^{2T}. \tag{2.27}$$

We also use operator $\mathcal{J}^{2T}\colon \mathcal{F}^{2T} \to \mathcal{F}^{2T}$,

$$\left(\mathcal{J}^{2T} f\right)(t) = \int_0^t f(\eta)\, d\eta, \quad 0 < t < 2T.$$

**Theorem 2.1.** *The connecting operator $C^T$ acts continuously in $\mathcal{F}^T$; representations*

$$C^T = -\frac{1}{2}\left(S^T\right)^* R^{2T} \mathcal{J}^{2T} S^T;$$

$$\left(C^T f\right)(t) = \rho^{1/2}(0) f(t) + \int_0^T C^T(t,\eta) f(\eta)\, d\eta, \quad 0 < t < T, \tag{2.28}$$

*with kernel*

$$C^T(t,\eta) = \frac{1}{2} \int_{|t-\eta|}^{2T-t-\eta} r(\xi)\, d\xi,$$

*are valid.*

*Proof.* For arbitrarily chosen controls $f, g \in C_0^\infty[0;T]$ let us denote $f_- := S^T f \in C_0^\infty[0;2T] \subset \mathcal{F}^{2T}$ and introduce function

$$w^{fg}(s,t) := \int_0^\infty u^{f_-}(x,s) u^g(x,t) \rho(x)\, dx, \quad (s,t) \in [0;2T] \times [0;T].$$

In view of (2.16) this integral is really taken on interval $\Omega^T$, equalities

$$u^g\big(x(T),t\big) = u_x^g\big(x(T),t\big) = 0, \quad 0 < t < T, \tag{2.29}$$

being valid at its endpoint $x=x(T)$.

Function $w^{fg}$ satisfies the relations

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial s^2}\right)w^{fg}(t,s)=\int\limits_0^\infty\left[u^{f_-}(x,s)u^g_{tt}(x,t)-u^{f_-}_{ss}(x,s)u^g(x,t)\right]\rho(x)\,dx=$$

$$=\int\limits_0^{x(T)}\left[u^{f_-}(x,s)u^g_{xx}(x,t)-u^{f_-}_{xx}(x,s)u^g(x,t)\right]dx=\ \text{(see (2.29)}\ =$$

$$=-\left[u^{f_-}(0,s)u^g_x(0,t)-u^{f_-}_x(x,s)u^g(0,t)\right]=-f_-(s)\big(R^Tg\big)(t)+\big(R^{2T}f_-\big)(s)g(t)=:F(s,t).$$

Recalling zero Cauchy data for waves (2.2) one obtains the problem

$$\begin{cases} w^{fg}_{tt}-w^{fg}_{ss}=F & \text{in } (0;2T)\times(0;T),\\ w^{fg}|_{t=0}=w^{fg}_t|_{t=0}=0. \end{cases}$$

In triangular domain $\{(s,t)\colon\ 0\le t\le T,\ t\le s\le 2T-t\}$ it may be solved in quadratures by means of D'Alambert formula:

$$w^{fg}(s,t)=\frac12\int\limits_0^t d\eta\int\limits_{s-t+\eta}^{s+t-\eta}d\xi\,F(\xi,\eta).$$

Putting here $s=t=T$, one has

$$w^{fg}(T,T)=\frac12\int\limits_0^T d\eta\int\limits_\eta^{2T-\eta}\left[-f_-(\xi)\big(R^Tg\big)(\eta)+\big(R^{2T}f_-\big)(\xi)g(\eta)\right]d\eta=$$

$$=-\frac12\int\limits_0^T d\eta\,\big(R^Tg\big)(\eta)\int\limits_\eta^{2T-\eta}f_-(\xi)\,d\xi+\int\limits_0^T d\eta\,g(\eta)\frac12\left[\int\limits_0^{2T-\eta}\big(R^{2T}f_-\big)(\xi)\,d\xi-\int\limits_0^\eta\big(R^{2T}f_-\big)(\xi)\,d\xi\right].$$

By virtue of oddness of $f_-$ we have

$$\int\limits_\eta^{2T-\eta}f_-(\xi)\,d\xi\equiv0,\quad 0<\eta<T.$$

Continuing the calculation and taking into account representation

$$\frac12\left[\int\limits_0^{2T-\eta}\big(R^{2T}f_-\big)(\xi)\,d\xi-\int\limits_0^\eta\big(R^{2T}f_-\big)(\xi)\,d\xi\right]=-\Big(N^{2T}G^{2T}\mathcal{J}^{2T}R^{2T}S^T_-f\Big)(\eta),\quad 0<\eta<T,$$

one can obtain

$$w^{fg}(T,T)=-\Big(g,N^{2T}G^{2T}\mathcal{J}^{2T}R^{2T}S^T_-f\Big)\ =\ \text{(see (2.27), (2.24))}\ =$$

$$=\Big(-\frac12\big(S^T\big)^*R^{2T}\mathcal{J}^{2T}S^Tf,g\Big)_{\mathcal{F}^T}.$$

From the other hand, the definition of $w^{fg}$ implies

$$w^{fg}(T.T) = \int\limits_0^\infty u^{f_-}(x,T)u^g(x,T)\rho(x)\,dx =$$
$$= \big(u^f(\cdot,T),u^g(\cdot,T)\big)_{\mathcal{H}^T} = \text{ (see (2.26)) } = \big(C^T f,g\big)_{\mathcal{F}^T}.$$

Comparing two obtained expressions for $w^{fg}(T,T)$, we justify the first representation of (2.28). Substituting (2.22) in it, we can derive the second one. The theorem is proved.

An important possibility to calculate the values of $w^{fg}$ via response operator was first detected by A. S. Blagovestchenskii. The first representation in (2.28) holds true in multidimensional case. It was found in [8].

## 3. Controllability

**3.1. Wave Shaping.** In its general form the question under consideration may be posed as follows. Can one manage a shape of wave on a string? In another words, is it possible to transfer DS $\alpha^T$ from initial zero state to given final state $u^f(\cdot,T)=a(\cdot)$ by means of a proper choice of control $f$?

In view of property (2.16) function $a$, (i.e., prescribed shape of wave) has to be localized on interval $\Omega^T$ filled by waves. So, a natural statement of the problem is the following: *given function $a \in \mathcal{H}^T$ find control $f \in \mathcal{F}^T$ satisfying*

$$u^f(\cdot,T)=a. \tag{3.1}$$

*This sort of problems is a subject of the Boundary Control Theory being a highly developed branch of Mathematical Physics (see, for instance, [9–11]).*

*Problem (3.1) is obviously equivalent to equation*

$$W^T f = a. \tag{3.2}$$

*Its solvability is established by the following result.*

**Lemma 3.1.** *Control operator $W^T$ acts isomorphically from $\mathcal{F}^T$ onto $\mathcal{H}^T$.*

*Proof.* Entering in (2.18), operator $\mathbb{I}_{\mathcal{H}^T}+K^T$ is boundedly invertible in $\mathcal{H}^T$. Operators $\big(I^T\big)^*=\big(I^T\big)^{-1}\colon \mathcal{F}^T\to\mathcal{H}^T$ and $Y^T\colon\mathcal{F}^T\to\mathcal{F}^T$ are isomorphisms. Therefore, operator $W^T$ turns out to be an isomorphism, representation

$$\big(W^T\big)^{-1} = Y^T I^T\big[\mathbb{I}_{\mathcal{H}^T}+K^T\big]^{-1}$$

being valid. The lemma is proved.

**Corollary 1.** *For any $a\in\mathcal{H}^T$, problem (3.1) is uniquely solvable.*

The obtained result may be represented in another terms. In the Control Theory a set of states

$$\mathcal{U}^T := \operatorname{Ran} W^T = \big\{u^f(\cdot,T)\colon\ f\in\mathcal{F}^T\big\}$$

is called a reachable set (at the moment $t=T$). Lemma 4.1 establishes equality

$$\mathcal{U}^T = \mathcal{H}^T. \tag{3.3}$$

Property (3.3) is said to be a complete (exact) controllability of DS $\alpha^T$.

So, a wave shaping problem for a string turns out to be well-posed. It is not true in multidimensional case. The result of Lemma 3.1 is not valid, equality (3.3) being broken. To investigate a multidimensional analog of the problem is essentially more difficult task requiring another instruments.

We conclude this subsection by one more useful consequence of Lemma 3.1.

**Corollary 2.** *Connecting operator $C^T$ is a positive definite isomorphism in $\mathcal{F}^T$, mapping $C[0;T]$ onto $C[0;T]$.*

**3.2. Reachable Sets and Projectors.** For what follows it is convenient to formulate property (3.3) in a slightly more general form and to discuss some of its consequences.

In outer space $\mathcal{F}^T$ let us select an extending family of subspaces of delayed controls

$$\mathcal{F}^{T,\xi} := \mathcal{T}^{T,\xi}\mathcal{F}^T = \left\{f \in \mathcal{F}^T\colon\ f(t) = 0,\ 0 < t < T - \xi\right\}$$

indexed by parameter $\xi \in [0;T]$.

Inner space $\mathcal{H}^T$ contains a corresponding family of reachable sets

$$\mathcal{U}^\xi := W^T\mathcal{F}^{T,\xi} = \text{ (see (2.21)) } = \left\{u^f(\cdot,\xi)\colon\ f \in \mathcal{F}^T\right\}, \quad 0 \le \xi \le T,$$

formed by waves. Let $P^\xi_w$ be operator which acts in $\mathcal{U}^T$ and projects it orthogonally (in $\mathcal{H}^T$-metric) on $\mathcal{U}^\xi$. We call $P^\xi_w$ a wave projector. Let $P^\xi_{w\perp}$ be a complement projector in $\mathcal{U}^T$ on $\mathcal{U}^T \ominus \mathcal{U}^\xi$.

There is the second selected family of subspaces in $\mathcal{H}^T$,

$$\mathcal{H}^\xi := \left\{a \in \mathcal{H}^T\colon\ \operatorname{supp} a \subset \bar{\Omega}^\xi\right\}, \quad \mathcal{H}^0 := \{0\}.$$

The corresponding projectors $P^\xi$ in $\mathcal{H}^T$ on $\mathcal{H}^\xi$, and complement projectors $P^\xi_\perp := \mathbb{I}_{\mathcal{H}^T} - P^\xi$ act by the rule of

$$(P^\xi a)(x) = \begin{cases} a(x), & x \in \Omega^\xi, \\ 0, & x \in \Omega^T \backslash \Omega^\xi, \end{cases}; \qquad (P^\xi_\perp a)(x) = \begin{cases} 0, & x \in \Omega^\xi, \\ a(x), & x \in \Omega^T \backslash \Omega^\xi, \end{cases}$$

cutting off functions on intervals $\big(0; x(\xi)\big)$ and $\big(x(\xi); x(T)\big)$ on a string.

Property (3.3) together with stationary state relation (2.21) implies

$$\mathcal{U}^\xi = \mathcal{H}^\xi; \quad P^\xi_w = P^\xi; \quad P^\xi_{w\perp} = P^\xi_\perp \tag{3.4}$$

for any $\xi \in [0;T]$. Moreover, control operator $W^T$ maps $\mathcal{F}^{T,\xi}$ on $\mathcal{U}^\xi = \mathcal{H}^\xi$ isomorphically.

Reachable sets $\mathcal{U}^\xi$ and wave projectors $P^\xi_w$ are intrinsic objects of DS $\alpha^T$, whereas $\mathcal{H}^\xi$ and $P^\xi$ are something "extrinsic" for it. They turn out to be related due to controllability of a string.

More complicated dynamical systems can loss this property, that leads to a break down of (3.4). As example, the two velocity (vector) system may be mentioned (see [12, 13]).

A structure of reachable sets and properties of wave projectors are the central questions which has to be solved to apply the BC-method.

**3.3. Observability.** It is a traditional matter in the Control Theory to interpret a property of controllability in dual terms. Here we describe such an interpretation of results of the previous subsection.

Boundary-value initial problem

$$\left(\alpha^T\right)^*: = \begin{cases} \rho v_{tt} - v_{xx} = 0, \quad x \in \Omega^T, \quad \tau(x) < t < T, & (3.5) \\ v\big|_{t=T} = 0, \quad v_t\big|_{t=T} = a, & (3.6) \\ v\big|_{x=0} = 0. & (3.7) \end{cases}$$

is said to be dual to problem (2.1)–(2.3), the corresponding DS being denoted $\left(\alpha^T\right)^*$. The known fact is that problem (3.5)–(3.7) is well-posed one. For any $a \in C^2\left(\bar{\Omega}^T\right) \subset \mathcal{H}^T$, $a(0) = a'(0) = 0$, it has a unique classical solution $v = v^a(x,t)$.

The operator $\mathcal{O}^T: \mathcal{H}^T \to \mathcal{F}^T$, $\mathrm{Dom}\,\mathcal{O}^T = \left\{a \in C^2\left(\bar{\Omega}^T\right):\ a(0) = a'(0) = 0\right\}$,

$$\left(\mathcal{O}^T a\right)(t) := v^a_x(0,t), \quad t \in [0;T],$$

is called *the operator of observation* of DS $\left(\alpha^T\right)^*$. From a mechanical viewpoint operator $\mathcal{O}^T$ describes forces generated at fixed endpoint $x = 0$ of a string by a wave process which is initiated by perturbation of velocity. The process is observed in inverted time. The following result explains the term “dual” with respect to system (3.5)–(3.7).

**Lemma 3.2.** *Equality*

$$\mathcal{O}^T = \left(W^T\right)^* \tag{3.8}$$

*is valid on* $\mathrm{Dom}\mathcal{O}^T$.

*Proof.* Let us fix arbitrary $f \in C_0^\infty[0;T] \subset \mathcal{F}^T$ and $a \in \mathrm{Dom}\,\mathcal{O}^T$. Both of the (classical) solutions $u^f$ and $v^a$ are simultaneously defined in domain $\left\{(x,t):\ x \in \Omega^T,\ \tau(x) < t < T\right\}$. Integrating on it, we have

$$\begin{aligned} 0 &= \int_0^T dt \int_0^{x(t)} dx \left[\rho(x) u^f_{tt}(x,t) - u^f_{xx}(x,t)\right] v^a(x,t) = \\ &= \int_0^{x(T)} dx\, \rho(x) \int_{\tau(x)}^T dt\, u^f_{tt}(x,t) v^a(x,t) - \int_0^T dt \int_0^{x(t)} dx\, u^f_{xx}(x,t) v^a(x,t) = \\ &= \text{(see (2.2), (2.3), (2.16)}', \text{(3.6), (3.7))} = \\ &= -\int_0^{x(T)} dx\, \rho(x) u^f(x,T) a(x) + \int_0^T dt\, f(t) v^a_x(0,t) + \\ &+ \int_0^T dt \int_0^{x(t)} dx\, u^f(x,t)\left[\rho(x) v^a_{tt}(x,t) - v^a_{xx}(x,t)\right] = \\ &= -\left(W^T f, a\right)_{\mathcal{H}^T} + \left(f, \mathcal{O}^T a\right)_{\mathcal{F}^T}, \end{aligned}$$

that leads to (3.8). The lemma is proved.

**Corollary 1.** *Operator* $\mathcal{O}^T$ *is extendable from* $\mathrm{Dom}\mathcal{O}^T$ *onto* $\mathcal{H}^T$ *by continuity, the extension being represented in the form of*

$$(\mathcal{O}^T a)(t) = \Big(\rho\big(x(T-t)\big)\rho(0)\Big)^{1/4} a\big(x(T-t)\big) + \int\limits_t^T w^*(\eta, t) a\big(x(T-\eta)\big)\, d\eta, \quad 0 < t < T. \quad (3.9)$$

To prove (3.9) one has just to join (3.8) and (2.20). Next result follows from an elementary analysis of (3.9). It corresponds to a finiteness of velocity of waves $v^a$.

**Corollary 2.** *If* $\mathrm{supp}\, a \subset \bar{\Omega}^T \backslash \Omega^\xi = [x(\xi); x(T)]$ *(for some fixed* $\xi\colon 0 < \xi < T$*), then one has* $\mathrm{supp}\, \mathcal{O}^T a \subset [0; T-\xi]$.

The property of DS $\left(\alpha^T\right)^*$ being dual to a controllability of $\alpha^T$ is given by the following proposition.

**Lemma 3.3.** *Equality*

$$\mathrm{Ker}\, \mathcal{O}^T = \{0\} \quad (3.10)$$

*is valid.*

*Proof.* Along with well-known operator relation one has

$$\mathrm{Ker}\, \mathcal{O}^T = \mathrm{Ker}\, (W^T)^* = \mathcal{H}^T \ominus \mathrm{clos}\, \mathrm{Ran} W^T = \mathcal{H}^T \ominus \mathrm{clos} \mathcal{U}^T = \text{ (see (3.3)) } = \{0\}.$$

The lemma is proved.

Property (3.10) may be interpreted as follows. Any wave $v^a$ initiated by perturbation of velocity $a$, $\mathrm{supp}\, a \subset \Omega^T$, has to manifest itself (to be observed) at the endpoint $x = 0$ during a time $0 < t < T$. In opposite case, if no forces are detected, i.e., $\partial_x v^a(0,t) \equiv 0$, $0 < t < T$, then perturbation $a(\cdot)$ is localized sufficiently far from the endpoint, i.e., out of $\Omega^T$.

A duality "controllability-observability" is a very general feature of dynamical systems (see [9–11]). A multidimensional case is no exception.

## 4. Propagation of Breaks

**4.1. Breaks in DS $\alpha^T$.** The contents of two nearest subsections correspond to a fact which is well-known for physicists. Sharp signals exited in a dynamical system propagate along space-time rays with velocity being inherent in a system.

Let us return to DS $\alpha^T$ and consider a case of discontinuous control $f \in \mathcal{F}^T$ having a jump at the moment $t = T - \xi$ $(0 < \xi < T)$,

$$f(T - \xi + 0) - f(T - \xi - 0) \neq 0.$$

Then, by virtue of (absolute) continuity of the integral term in (2.4), (2.19) one has relation

$$u^f\big(x(\xi) - 0, T\big) - u^f\big(x(\xi) + 0, T\big) = \left(\frac{\rho\big(x(\xi)\big)}{\rho(0)}\right)^{-1/4} \big[f(T - \xi + 0) - f(T - \xi - 0)\big]. \quad (4.1)$$

So, a break of control produces a break of a wave, amplitudes of the breaks being connected by equality (4.1). In particular case of $\xi = T$ one has $f(-0) = 0$ (see Remark 2.1) and $u^f\big(x(T) + 0, T\big) = 0$ (see (2.16)), that leads to equality

$$u^f\big(x(T) - 0, T\big) = \left(\frac{\rho\big(x(T)\big)}{\rho(0)}\right)^{-1/4} f(+0). \quad (4.2)$$

It means that a break of wave moves along a string being localized at "forward front" $x = x(T)$.

**4.2. Breaks in DS $\left(\alpha^T\right)^*$.** A closely similar effect takes place in dual system $\left(\alpha^T\right)^*$.

Let us suppose that function $a(\cdot)$ entering in the Cauchy Data (3.6) has a jump at $x = x(\xi)$. Recalling the definition $\mathcal{O}^T a := v_x^a(0,\cdot)$ and using (3.9) one can derive the relation

$$v_x^a(0, T-\xi-0) - v_x^a(0, T-\xi+0) = \Big(\rho\big(x(\xi)\big)\rho(0)\Big)^{1/4}\big[a\big(x(\xi)+0\big) - a\big(x(\xi)-0\big)\big]. \tag{4.3}$$

It shows that a break of data propagates along a string, reaching its endpoint at the moment $t = T - \xi$ and producing a break of a force.

Having numerous applications, relations of the type of (4.2), (4.3) are usually called "geometrical optics formulas" or "first break equalities". They have direct analogs in multidimensional case.

**4.3. Amplitude Representation of Images.** Here we deduce a relation expressing an image of a function through breaks propagating in system $\left(\alpha^T\right)^*$.

Recall that a family of cutting off projectors $P^\xi$, $P_\perp^\xi$ was introduced in subsec. 3.2.

**Lemma 4.1.** *For any $a \in C[0;T] \subset \mathcal{H}^T$, $\xi \in (0;T)$, relation*

$$\lim_{t \to T-\xi-0} (\mathcal{O}^T P_\perp^\xi a)(t) = \Big(\rho\big(x(\xi)\big)\rho(0)\Big)^{1/4} a\big(x(\xi)\big) \tag{4.4}$$

*is valid.*

*Proof.* Along with definitions one has

$$\operatorname{supp} P_\perp^\xi a \subset \big[x(\xi); x(T)\big]; \quad \big(P_\perp^\xi a\big)\big(x(\xi)+0\big) = a\big(x(\xi)\big). \tag{4.5}$$

By virtue of Lemma 3.2, Corollary 2 it implies

$$\operatorname{supp} \mathcal{O}^T P_\perp^\xi a \subset [0; T-\xi]; \quad (\mathcal{O}^T P_\perp^\xi a)(T-\xi+0) = 0. \tag{4.6}$$

Applying (4.3) we obtain

$$\begin{aligned}(\mathcal{O}^T P_\perp^\xi a)\Big|_{t=T-\xi+0}^{t=T-\xi-0} &= \text{ (see (4.6)) } = (\mathcal{O}^T P_\perp^\xi a)(T-\xi-0) = \\ &= \Big(\rho\big(x(\xi)\big)\rho(0)\Big)^{1/4} (P_\perp^\xi a)\Big|_{x=x(\xi)-0}^{x=x(\xi)+0} = \text{ (see (4.5)) } = \Big(\rho\big(x(\xi)\big)\rho(0)\Big)^{1/4} a\big(x(\xi)\big).\end{aligned}$$

These equalities contain (4.4), that proves the lemma.

Recalling a definition of images (see subsec. 1.3) one can rewrite (4.4) in the form of

$$\lim_{t \to T-\xi-0} (\mathcal{O}^T P_\perp^\xi a)(t) = \rho^{1/4}(0)\widetilde{a}(\xi) = \rho^{1/4}(0)\big(I^T a\big)(\xi), \quad 0 < \xi < T. \tag{4.7}$$

The latter formula represents an image of function as a collection of amplitudes of wave breaks observed on endpoint of a string.

**Fig. 2.**

## 5. Inverse Problem

**5.1. Statement of IP.** A natural statement of dynamical IP is motivated by a local character of dependence of a response operator on density. Recall, that operator $R^{2T}$ is determined by $\rho|_{\Omega^T}$ (see (iii), subsec. 2.4). This property is of clear physical meaning. Indeed, a response of system $\alpha^T$ (i.e., force $u^f_x(0,t)$) generated by action of control $f$ is formed by waves which return back to endpoint $x=0$ being scattered by nonhomogeneities of density (see the arrows on Fig. 2):

The waves reflected from depths $x > x(T)$ reach the endpoint later the moment $t = 2T$. They will not be detected by outer observer measuring values $u^f_x(0,t)$, $0 < t < 2T$. Corresponding to these measurements, operator $R^{2T}$ contains an information about $\rho|_{\Omega^T}$, but it "knows nothing" about $\rho|_{x>x(T)}$. Therefore, it would be natural for outer observer who investigates system $\alpha^T$ through "input–output" relations to pose the problem in the following form:

(IP) *given* $R^{2T}$ *recover* $\rho|_{\Omega^T}$.

A. S. Blagovestchenskii was the first who posed the dynamical IP in such a clear form taking into account the principle of locality (causality). In classical paper [2] he elaborated so-called "local approach", solving the IP and giving conditions of its solvability.

Proposed below the BC-method is closely similar to A. S. Blagovestchenskii's approach, conserving its main feature, i.e., locality. A principle advantage of BC-method is the fact that ideas providing its basis work in a multidimensional case.

**5.2. Relations between Control Problem and Inverse Problem.** To present a heuristic idea of the BC-method let us compare two following situations concerning to DS $\alpha^T$.

(1). Suppose that control (input) $f$ has a shape of unit step, $f(t) \equiv 1$, $0 < t < T$, and let $u^f(\cdot, T)$ be the corresponding wave (state) shown on Fig. 3.

Obviously, a shape of the wave is determined by density $\rho|_{\Omega^T}$. So, in the case of *"standard" input* an information on a density is contained in *a state.*

(2). Suppose that we have succeeded to find the control $f$ producing the wave $u^f(\cdot, T) \equiv 1$ in $\Omega^T$ (see Fig. 4).

Such the control solves Control Problem (3.1) with $a \equiv 1$. So, we have an *input* producing a *"standard" state.* It is quite clear, that, in contrast to the previous case, it is the *input* which contains an information on a density.

A main idea of the BC-method may be formulated as follows. A possibility to control a dynamical system, and to create "standard" states opens a way to transfer an information on its intrinsic structure from inner space to outer one. Obtaining the information, an outer observer can use it to recover the structure.

These considerations are detalized in the following plan of solving of the IP:

**Fig. 3.**

**Fig. 4.**

(i) to demonstrate an existence of controls $f \in \mathcal{F}^T$ producing "standard waves" $u^f(\cdot, T) \in \mathcal{H}^T$;
(ii) to determine such the controls via inverse data (i.e., response operator $R^{2T}$);
(iii) to extract $\rho|_{\Omega^T}$ from these controls.

**5.3. Gelfand-Levitan-Krein Equations.** Realizing the designated plan, we shall operate with standard states having a form of linear functions $l^T \in \mathcal{H}^T$,

$$l^T(x) = \gamma x + \delta, \quad x \in \Omega^T.$$

Let us pose problem (3.1) with a special right-hand side:
*find control $f$ satisfying*

$$u^f(\cdot, T) = l^T. \tag{5.1}$$

A solvability of the problem is guaranteed by results of Lemma 3.1; its solution is

$$f = \left(W^T\right)^{-1} l^T \in \mathcal{F}^T.$$

The following result characterized it in terms of inverse data.

**Theorem 5.1.** *Control $f = \left(W^T\right)^{-1} l^T$ is a unique solution of equation*

$$C^T f = \gamma \varkappa^T - \delta \left(R^T\right)^* \varkappa^T \quad \text{in } \mathcal{F}^T, \tag{5.2}$$

*where $\varkappa^T \in \mathcal{F}^T$, $\varkappa^T(t) := T - t$. The solution belongs to class $C[0; T]$.*

*Proof.* At first let us remark that $\varkappa^T \in \operatorname{Dom}\left(R^T\right)^*$ (see (i), subsec. 2.4) and, therefore, the right hand side of (5.2) is correctly defined.

Choosing arbitrarily a control $g \in C_0^\infty[0;T] \subset \mathcal{F}^T$ one has equalities:

$$\begin{aligned}
\big(C^T f, g\big) &= \big(C^T (W^T)^{-1} l^T, g\big)_{\mathcal{F}^T} = \text{ (see (2.26)) } = \big(l^T, u^g(\cdot, T)\big)_{\mathcal{H}^T} = \\
&= \int\limits_0^{x(T)} l^T(x) u^g(x,T) \rho(x)\, dx = \int\limits_0^{x(T)} dx\, l^T(x) \int\limits_0^T (T-t) \big[\rho(x) u^g_{tt}(x,t)\big]\, dt = \\
&= \int\limits_0^T dt\, (T-t) \int\limits_0^{x(T)} l^T(x) u^g_{xx}(x,t)\, dx = \text{ (see (2.16)}') = \\
&= -\int\limits_0^T dt\, (T-t) \big[l^T(0) u^g_x(0,t) - l^T_x(0) u^g(0,t)\big] = \\
&= -\big(\varkappa^T, \delta R^T g\big)_{\mathcal{F}^T} + \big(\varkappa^T, \gamma g\big)_{\mathcal{F}^T} = \big(\gamma \varkappa^T - \delta (R^T)^* \varkappa^T, g\big)_{\mathcal{F}^T}.
\end{aligned}$$

Comparing the first and the last terms one obtains (5.2).

A solvability of (5.2), as well as uniqueness and continuity of the solution, follows from Corollary 2 of Lemma 3.1. The theorem is proved.

Let us consider two particular cases of equation (5.2):

(1) $\gamma = 0$, $\delta = 1$. Denoting the linear function by $1^T$, $1^T(x) \equiv 1$ in $\Omega^T$ and $q^T := \big(W^T\big)^{-1} 1^T$ we conclude, that control $f = q^T$ which solves problem (3.1) with $a = 1^T$ is a unique solution of equation

$$C^T f = -\big(R^T\big)^* \varkappa^T \quad \text{in } \mathcal{F}^T. \tag{5.3}$$

(2) $\gamma = 1$, $\delta = 0$. In this case the linear function coincides with coordinate function on a string. We denote it $\pi^T(\cdot)$: $\pi^T(x) = x$ in $\Omega^T$. Let us also denote $p^T := \big(W^T\big)^{-1} \pi^T$. Applying Theorem 5.1 one obtains that control $f = p^T$ is a unique solution of equation

$$C^T f = \varkappa^T \quad \text{in } \mathcal{F}^T. \tag{5.4}$$

A significance of equations (5.3), (5.4) for the IP is explained by the following. Possessing operator $R^{2T}: f \to u^f(0,\cdot)|_{(0;2T)}$, outer observer can use them to find controls $q^T$, $p^T$ in $\mathcal{F}^T$. Indeed, reducing measurements on times $0 < t < T$, he can determine operator $R^T: f \to u^f_x(0,\cdot)|_{(0;T)}$ and its adjoint $\big(R^T\big)^*$; operator $C^T$ may be calculated by (2.28). Thus, the observer is able to write the equations and to solve them, obtaining

$$q^T = -\big(C^T\big)^{-1} \big(R^T\big)^* \varkappa^T, \quad p^T = \big(C^T\big)^{-1} \varkappa^T. \tag{5.5}$$

So, items (i), (ii) of the program planned at the end of subsec. 5.2 are realized.

We call (5.3), (5.4) the Gelfand-Levitan-Krein equations (GLK-e) by the following reason. Classical Gelfand-Levitan equations (see [14]) for the case of the wave equation with a potential $u_{tt} - u_{xx} + V(x) u = 0$ may be deduced along with the same "boundary control" considerations as for (5.3), (5.4) (see [15]). In the case of a string with Neumann control $u_x|_{x=0} = f$ equation (5.4) may be transferred into M. Krein's equation [2, 4] by trivial substitution. So, our approach gives the "BC-interpretation" of basic equations known in inverse problems. A main advantage of such the interpretation is that it leads to natural multidimensional analogs of GLK-e [16, 17].

To complete the program (i)–(iii) it only remains for us to find a connection between solutions $q^T$, $p^T$ and parameters of a string. A first way is to use "amplitude relation" (4.2):

$$1 = 1^T\big(x(T)-0\big) = u^{q^T}\big(x(T)-0, T\big) = \left(\frac{\rho\big(x(T)\big)}{\rho(0)}\right)^{-1/4} q^T(+0);$$

$$x(T) = \pi^T\big(x(T)-0\big) = u^{p^T}\big(x(T)-0, T\big) = \left(\frac{\rho\big(x(T)\big)}{\rho(0)}\right)^{-1/4} p^T(+0),$$

that leads to equality

$$x(T) = \frac{p^T(+0)}{q^T(+0)}. \tag{5.6}$$

Another way is to use power moments (see subsec. 1.2):

$$\mu_0(T) = \int\limits_0^{x(T)} \rho(x)\,dx = \big(1^T, 1^T\big)_{\mathcal{H}^T} = \big(W^T q^T, W^T q^T\big)_{\mathcal{H}^T} =$$
$$= \text{ (see (2.26)) } = \big(C^T q^T, q^T\big)_{\mathcal{F}^T} = \text{ (see (5.3)) } = -\big((R^T)^* \varkappa^T, q^T\big)_{\mathcal{F}^T};$$

$$\begin{aligned}\mu_1(T) = \int\limits_0^{x(T)} x\rho(x)\,dx = \big(\pi^T, 1^T\big)_{\mathcal{H}^T} = \big(W^T p^T, W^T q^T\big)_{\mathcal{H}^T} = \\ = \text{ (see (2.26)) } = \big(C^T p^T, q^T\big)_{\mathcal{F}^T} = \text{ (see (5.4)) } = \big(\varkappa^T, q^T\big)_{\mathcal{F}^T}.\end{aligned} \tag{5.7}$$

These relations will be used to recover a density.

**5.4. Family of GLK-e.** As well as the classical theory, the BC-method uses not a single GLK-equation, but a family of them.

Let us consider a set of "reduced" dynamical systems

$$\alpha^\xi = \begin{cases} \rho u_{tt} - u_{xx} = 0, \quad x > 0, \quad 0 < t < \xi; \\ u\big|_{t=0} = u_t\big|_{t=0} = 0; \\ u\big|_{x=0} = f, \end{cases}$$

indexed by parameter $\xi \in (0; T]$. Each of them is described by corresponding spaces and operators, i.e., $\mathcal{F}^\xi := L_2(0;\xi)$, $\mathcal{H}^\xi := L_2(\Omega^\xi; \rho\,dx)$; $W^\xi : \mathcal{F}^\xi \to \mathcal{H}^\xi$, $R^\xi : \mathcal{F}^\xi \to \mathcal{F}^\xi$, $C^\xi = (W^\xi)^* W^\xi$ etc.

Systems $\alpha^\xi$ inherits all the properties of $\alpha^T$. In particular, every $\alpha^\xi$ is controllable.

By apparent reason, operator $R^{2T}$ determines all the operators $R^\xi$, $(R^\xi)^*$, $R^{2\xi}$ for any $\xi < T$. By virtue of (2.28) one has

$$C^\xi = -\frac{1}{2}(S^\xi)^* R^{2\xi} \mathcal{J}^{2\xi} S^\xi, \tag{5.8}$$

so, a family of connecting operators is also determined. Eventually, possessing operator $R^{2T}$, an outer observer has a possibility to use a family of equations

$$C^\xi f = -(R^\xi)^* \varkappa^\xi; \quad C^\xi f = \varkappa^\xi. \tag{5.9}$$

($\varkappa^\xi = \varkappa^\xi(t) := \xi - t$) in outer spaces $\mathcal{F}^\xi$, $0 < \xi \le T$.

Let

$$q^\xi = -(C^\xi)^{-1}(R^\xi)^*\varkappa^\xi, \quad p^\xi = (C^\xi)^{-1}\varkappa^\xi,$$

be the solutions. Replacing $T$ by $\xi$ in (5.7), one has

$$\begin{aligned} \mu_0(\xi) &= -\left((R^\xi)^*\varkappa^\xi, q^\xi\right)_{\mathcal{F}^\xi}; \\ \mu_1(\xi) &= \left(\varkappa^\xi, q^\xi\right)_{\mathcal{F}^\xi}. \end{aligned} \tag{(5.10)}$$

Doing the same in (5.6) and taking into account (1.2), we obtain

$$x(\xi) = \frac{p^\xi(+0)}{q^\xi(+0)} = -\left[\frac{d}{d\xi}\left((R^\xi)^*\varkappa^\xi, q^\xi\right)_{\mathcal{F}^\xi}\right]^{-1} \frac{d}{d\xi}\left(\varkappa^\xi, q^\xi\right)_{\mathcal{F}^\xi}, \quad 0 < \xi \le T. \tag{5.11}$$

There are precisely the relations which permit to solve the Inverse Problem.

**5.5. Solving of IP.** It only remains for us to sum up the consideration of subsec. 5.3, 5.4. The density $\rho$ in $\Omega^T$ may be recovered via given response operator $R^{2T}$ by means of the following procedure:

(i) find operators $R^\xi$, $\left(R^\xi\right)^*$, and $C^\xi$ (see (5.8)) for every $\xi \in (0;T]$;

(ii) solving GLK-equations (5.9), find controls $q^\xi$ and $p^\xi$, $0 < \xi \le T$;

(iii) find function $x(\xi)$, $0 < \xi \le T$, by means of first or second representation (5.11);

(iv) inverting function $x(\cdot)$, find eikonal $\tau = \tau(x)$, $x \in [0; x(T)]$;

(v) recover density $\rho(x) = \left(\tau'(x)\right)^2$, $x \in [0; x(T)]$.

The Inverse Problem is solved.

**5.6. Solvability of the IP.** In addition to procedure (i)–(v) it would be desirable to have a characteristic description of the inverse data providing a solvability of the IP. More exactly, the question is to find necessary and sufficient conditions which guarantee an operator $R$ to be a response operator of some DS $\alpha^T$.

In the case of string with Neumann boundary control such the conditions were found by A. S. Blagovestchenskii in [2]. A closely similar result may be obtained in the frameworks of the BC-method.

**Theorem 5.2.** *Operator* $R: \mathcal{F}^{2T} \to \mathcal{F}^{2T}$, $\operatorname{Dom} R = \left\{f \in H^1[0;T]: \ f(0) = 0\right\}$, *is a response operator of system* (2.1)–(2.3) *with* $\rho \in C^\infty(\bar\Omega^T)$ *iff*

(i) *representation*

$$R = -\gamma\frac{d}{dt} + \delta\mathbb{I}_{\mathcal{F}^{2T}} + \int_0^t r(t-s)(\cdot)(s)\,ds$$

*is valid with some constants* $\gamma > 0$, $\delta$, *and function* $r \in C^\infty[0;2T]$;

(ii) *determined by* $R$ *the operator* $C: \mathcal{F}^T \to \mathcal{F}^T$,

$$C := -\frac{1}{2}(S^T)^* R \mathcal{J}^{2T} S^T$$

*is a positive definite operator, i.e.,* $C \ge \lambda\mathbb{I}_{\mathcal{F}^T}$ *with some constant* $\lambda > 0$.

Omitting the proof, we would like to comment this result. Roughly speaking, to solve the IP means to solve equations (5.3), (5.4). A solvability of GLK-e is guaranteed by condition (ii), that leads to a solvability of the IP itself.

Unfortunately, so clear and completed result as in Theorem 5.2 is not still obtained in multi-dimensional case.

## 6. Visualization of Waves

**6.1. Amplitude Formula.** In sec. 6 some additional opportunities of the BC-method are demonstrated. We shall derive a relation which makes the waves on a string to be visible for outer observer.

Recall a family of subspaces $\mathcal{H}^\xi \subset \mathcal{H}^T$ and corresponding projectors $P^\xi$, $P^\xi_\perp$ which cut off functions (see subsec. 3.2).

**Lemma 6.1.** *Let $f$ be a control of class $C[0;T]$, $u^f(\cdot,T)$ be a corresponding wave. Its image, i.e., function $\widetilde{u}^f(\cdot,T) = I^T u^f(\cdot,T)$ may be represented in the form of*

$$\big(I^T u^f(\cdot,T)\big)(\xi) = \rho^{-1/4}(0) \lim_{t\to T-\xi-0} \big[(W^T)^* P^\xi_\perp W^T f\big](t), \quad 0<\xi<T. \tag{6.1}$$

*Proof.* Recall, that

$$\widetilde{u}^f(\xi,T) = \big(I^T u^f(\cdot,T)\big)(\xi) := \rho^{1/4}\big(x(\xi)\big) u^f\big(x(\xi),T\big), \quad 0\le\xi\le T,$$

(see subsec. 1.3). Substituting $a = u^f(\cdot,T) = W^T f$ in (4.7), one obtains

$$\widetilde{u}^f(\xi,T) = \rho^{-1/4}(0) \lim_{t\to T-\xi-0} \big(\mathcal{O}^T P^\xi_\perp W^T f\big)(t) =$$
$$= \text{ (see (3.8)) } = \rho^{-1/4}(0) \lim_{t\to T-\xi-0} \big[(W^T)^* P^\xi_\perp W^T f\big](t),$$

that proves the lemma.

As well as (4.7), relation (6.1) represents an image of a wave as a collection of wave break amplitudes. That is a reason to call it an "amplitude formula".

**6.2. Operators $\Pi^{T,\xi}$.** Recall, that family of reachable sets $\mathcal{U}^\xi = W^T \mathcal{F}^{T,\xi}$, and corresponding projectors $P^\xi_w$, $P^\xi_{w\perp}$ were introduced in subsec. 3.2.

Entering in the right-hand side of (6.1) the operator may be represented in the form of

$$\begin{aligned} (W^T)^* P^\xi_\perp W^T &= (W^T)^*(\mathbb{I}_{\mathcal{H}^T} - P^\xi) W^T = \\ &= C^T - (W^T)^* P^\xi W^T = \text{ (see (3.4)) } = C^T - (W^T)^* P^\xi_w W^T. \end{aligned} \tag{6.2}$$

A possibility to use the amplitude formula in the IP is opened by the following important fact. Operators $\Pi^{T,\xi}\colon \mathcal{F}^T \to \mathcal{F}^T$,

$$\Pi^{T,\xi} := (W^T)^* P^\xi_w W^T, \quad 0\le\xi\le T,$$

may be expressed via connecting operator $C^T$, and, eventually, via inverse data $R^{2T}$.

Let us introduce operators $X^{T,\xi}\colon \mathcal{F}^T \to \mathcal{F}^T$,

$$(X^{T,\xi} f)(t) := \begin{cases} 0, & 0<t<T-\xi, \\ f(t), & T-\xi\le t<T, \end{cases}$$
$$X^{T,0} := \mathbb{O}_{\mathcal{F}^T}; \quad X^{T,T} := \mathbb{I}_{\mathcal{F}^T},$$

which project in $\mathcal{F}^T$ on $\mathcal{F}^{T,\xi}$, $\xi\in[0;T]$. Denote by $X^{T,\xi}_\#$ the same operator $X^{T,\xi}$ considered as a map from space $\mathcal{F}^T$ into space $\mathcal{F}^{T,\xi}$. Its adjoint $(X^{T,\xi}_\#)^*\colon \mathcal{F}^{T,\xi} \to \mathcal{F}^T$ completes controls by zero on interval $[0;T-\xi]$.

The operator $C^{T,\xi}\colon \mathcal{F}^{T,\xi} \to \mathcal{F}^{T,\xi}$,

$$C^{T,\xi} := X^{T,\xi}_\# C^T (X^{T,\xi}_\#)^*$$

is said to be a block of operator $C^T$ in subspace $\mathcal{F}^{T,\xi}$. It should be noted that $C^{T,\xi}$ may be identified with connecting operator $C^\xi$ of "reduced" system $\alpha^\xi$ (see subsec. 5.4). Together with $C^T$, every block turns out to be a positive definite isomorphism (in $\mathcal{F}^{T,\xi}$). The same is evidently valid for its inverse $(C^{T,\xi})^{-1}$.

**Lemma 6.2.** *For any $\xi \in (0;T]$ representation*

$$\Pi^{T,\xi} = C^T(X^{T,\xi}_{\#})^*(C^{T,\xi})^{-1}X^{T,\xi}_{\#}C^T \tag{6.3}$$

*is valid.*

*Proof.* Let $W^T = E^T|W^T|$ be a polar decomposition of control operator (see [18]), where

$$|W^T| := \left((W^T)^*W^T\right)^{1/2} = (C^T)^{1/2},$$

$E^T$ is a unitary operator acting from $\operatorname{Ran}|W^T| = (C^T)^{1/2}\mathcal{F}^T = \mathcal{F}^T$ onto $\operatorname{Ran} W^T = \mathcal{H}^T$.

Operator $E^T$ connects the subspaces $|W^T|\mathcal{F}^{T,\xi} = (C^T)^{1/2}\mathcal{F}^{T,\xi} \subset \mathcal{F}^T$ and $W^T\mathcal{F}^{T,\xi} =: \mathcal{U}^\xi \subset \mathcal{H}^T$,

$$E^T(C^T)^{1/2}\mathcal{F}^{T,\xi} = \mathcal{U}^\xi. \tag{6.4}$$

Let $Q^{T,\xi}$ be the projector in $\mathcal{F}^T$ on subspace $(C^T)^{1/2}\mathcal{F}^{T,\xi}$; recall, that $P^\xi_w$ denotes the projector in $\mathcal{H}^T$ on $\mathcal{U}^\xi$. Connection (6.4) implies the equality

$$E^TQ^{T,\xi} = P^\xi_wE^T, \tag{6.5}$$

that leads to representation

$$\Pi^{T,\xi} = (E^T|W^T|)^*P^\xi_wE^T|W^T| = |W^T|(E^T)^*P^\xi_wE^T|W^T| = (\text{see } (6.5) = (C^T)^{1/2}Q^{T,\xi}(C^T)^{1/2}. \tag{6.6}$$

Projector $Q^{T,\xi}$ may be represented as follows:

$$Q^{T,\xi} = (C^T)^{1/2}(X^{T,\xi}_{\#})^*(C^{T,\xi})^{-1}X^{T,\xi}_{\#}(C^T)^{1/2}. \tag{6.7}$$

Indeed, as it may be easily checked by direct inspection, the right-hand side of (6.7) satisfies all the characteristic properties of this projector, i.e.,

$$(Q^{T,\xi})^2 = Q^{T,\xi}; \quad (Q^{T,\xi})^* = Q^{T,\xi}; \quad \operatorname{Ran} Q^{T,\xi} = (C^T)^{1/2}\mathcal{F}^{T,\xi}.$$

Substituting (6.7) in (6.6), we obtain (6.3). The lemma is proved.

Analyzing the proof, a reader could note that equality (6.3) owes its origin to "intrinsic" character of wave projectors $P^\xi_w$, as it was mentioned in subsec. 3.2. It explains a reason to replace $P^\xi$ by $P^\xi_w$ in (6.2). So, providing equality $P^\xi_w = P^\xi$, a controllability of string is a very important property for us.

**6.3. Visualization.** Summing up results of subsec. 6.1 and 6.2 we obtain representation

$$\widetilde{u}^f(\xi,T) = \rho^{-1/4}(0)\lim_{t\to T-\xi-0}\left[(C^T-\Pi^{T,\xi})f\right](t), \quad \xi\in[0;T], \tag{6.8}$$

being valid for any $f \in C[0;T]$.

Solving the IP, outer observer can use (6.8) to visualize images of waves. Indeed, operator $R^{2T}$ determines operator $C^T$ (see (2.28)) which, in its turn, determines $\Pi^{T,\xi}$ by (6.3). Unknown constant $\rho(0)$ in (6.8) may be found as follows:

$$\rho(0) = \left[\lim_{t\to+0}(R^{2T}\chi)(t)\right]^2,$$

where $\chi = \chi(t) = t$, $0 < t < 2T$ (see (2.22)).

Using the delay operator (see (2.21)) one could make formula (6.8) to be more expressive:

$$\widetilde{u}^f(\tau,t) = \rho^{-1/4}(0) \lim_{\eta\to T-\tau-0}\Big[(C^T - \Pi^{T,\tau})\mathcal{T}^{T,t}f\Big](\eta), \quad (\tau,t)\in[0;T]\times[0;T].$$

So, outer observer is able to see whole the wave process in system (2.5)–(2.7).

To solve the IP itself, i.e., to recover a density, we can use solutions $q^T$, $p^T$ of GLK-e. Along with their definitions one has

$$\begin{aligned}(C^T - \Pi^{T,\xi})q^T = &\ (\text{see } (5.5)) \ = (C^T - \Pi^{T,\xi})(C^T)^{-1}\big[-(R^T)^*\varkappa^T\big] = \\ = &\ (\text{see } (6.3)) \ = \Big[\mathbb{I}_{\mathcal{F}^T} - C^T(X_{\#}^{T,\xi})^*(C^{T,\xi})^{-1}X_{\#}^{T,\xi}\Big]\big[-(R^T)^*\varkappa^T\big],\end{aligned} \tag{6.9}$$

and, quite analogously,

$$(C^T - \Pi^{T,\xi})p^T = \Big[\mathbb{I}_{\mathcal{F}^T} - C^T(X_{\#}^{T,\xi})^*(C^{T,\xi})^{-1}X_{\#}^{T,\xi}\Big]\varkappa^T. \tag{6.10}$$

From the other hand, we have

$$\rho^{1/4}\big(x(\xi)\big) = \widetilde{1}^T(\xi) = \widetilde{u}^{q^T}(\xi,T) = \ (\text{see } (6.8)) \ = \rho^{-1/4}(0)\lim_{t\to T-\xi-0}\Big[(C^T - \Pi^{T,\xi})q^T\Big](t), \tag{6.11}$$

and, analogously,

$$\rho^{1/4}\big(x(\xi)\big)x(\xi) = \widetilde{\pi}^T(\xi) = \widetilde{u}^{p^T}(\xi,T) = \rho^{-1/4}(0)\lim_{t\to T-\xi-0}\Big[(C^T - \Pi^{T,\xi})p^T\Big](t). \tag{6.12}$$

Calculating right-hand sides of (6.9), (6.10) and substituting them in (6.11), (6.12) one can find functions $x(\xi)$ and $\rho\big(x(\xi)\big)$ on $\xi\in[0;T]$. Obviously, it is enough to recover $\rho|_{\Omega^T}$.

As a commentary, we could say that to solve the IP is to calculate the blocks $(C^{T,\xi})^{-1}$.

**6.4. Amplitude Integral.** An elegant operator construction realizing a wave visualization is described here.

Let $\Xi^N = \{\xi_k\}_{k=0}^N$: $0 = \xi_0 < \xi_1 < \cdots < \xi_n = T$ be a finite partition of time interval $[0;T]$; $r(\Xi^N) := \max(\xi_k - \xi_{k-1})$ be its range. It induces partition $\Theta^N := \{x(\xi_k)\}_{k=0}^N$: $0 = x(\xi_0) < x(\xi_1) < \cdots < x(\xi_N) = x(T)$ of interval $\bar\Omega^T$ on a string having range $r(\Theta^N) = \max\big[x(\xi_k) - x(\xi_{k-1})\big]$. The ranges satisfy the estimate

$$\min_{\bar\Omega^T}\rho^{-1/2} \le \frac{r(\Theta^N)}{r(\Xi^N)} \le \max_{\bar\Omega^T}\rho^{-1/2}, \tag{6.13}$$

which easily follows from (1.1).

Let us compose a Riemannian integral operator sum of the form of

$$S_{\Xi^N} := \sum_{k=1}^N \Delta_k P^\xi W^T \Delta_k X^{T,\xi}, \tag{6.14}$$

where $\Delta_k X^{T,\xi} := X^{T,\xi_k} - X^{T,\xi_{k-1}}$ and $\Delta_k P^\xi := P^{\xi_k} - P^{\xi_{k-1}}$ are projectors:

$$(\Delta_k X^{T,\xi}f)(t) = \begin{cases} f(t), & t\in(T-\xi_k\,;\,T-\xi_{k-1}), \\ 0 & \text{for other} \quad t\in(0;T) \end{cases} \tag{6.15}$$

(see subsec. 6.2);

$$(\Delta_k P^\xi a)(x) = \begin{cases} a(x), & x \in \big(x(\xi_{k-1});\, x(\xi_k)\big), \\ 0 & \text{for other} \quad x \in \Omega^T. \end{cases} \tag{6.16}$$

So, sum $S_{\Xi^N}$ is operator from $\mathcal{F}^T$ in $\mathcal{H}^T$.

We are going to show that refinement of a partition leads to convergence of sums (6.14). To describe the limit let us understand image operator $I^T$ as a map from $\mathcal{H}^T$ in $\mathcal{F}^T$ (see Lemma 2.2). Recall, that its adjoint $(I^T)^* \colon \mathcal{F}^T \to \mathcal{H}^T$ acts by the rule of

$$\big((I^T)^* f\big)(x) = \rho^{-1/4}(x) f\big(\tau(x)\big), \quad x \in \Omega^T,$$

that implies

$$\big(\rho^{1/4}(0)(I^T)^* Y^T f\big)(x) = \left(\frac{\rho(x)}{\rho(0)}\right)^{-1/4} f\big(T - \tau(x)\big), \quad x \in \Omega^T, \tag{6.17}$$

where $Y^T f = f(T - \cdot)$. Note also, that operator $\rho^{1/4}(0)(I^T)^* Y^T$ enters in representation (2.18).

**Lemma 6.3.** *The estimate*

$$\|S_{\Xi^N} - \rho^{1/4}(0)(I^T)^* Y^T\| \leq c r(\Xi^N) \tag{6.18}$$

*is valid with constant $c$ independent on partition.*

*Proof.* Representations (6.15), (6.16) imply

$$\mathbb{I}_{\mathcal{F}^T} = \sum_{k=1}^{N} \Delta_k X^{T,\xi}; \quad \mathbb{I}_{\mathcal{H}^T} = \sum_{k=1}^{N} \Delta_k P^\xi. \tag{6.19}$$

Relation

$$\big[(I^T)^* Y^T\big] \Delta_k X^{T,\xi} = \Delta_k P^\xi \big[(I^T)^* Y^T\big] \tag{6.20}$$

follows from (6.17). Therefore, one has

$$\begin{aligned} \Delta_k P^\xi W^T \Delta_k X^{T,\xi} = \ &\text{(see (2.18), (6.20))} \ = \\ = \rho^{1/4}(0) \Delta_k P^\xi (I^T)^* Y^T &+ \rho^{1/4}(0) \Delta_k P^\xi K^T \Delta_k P^\xi (I^T)^* Y^T. \end{aligned} \tag{6.21}$$

Summarizing in both sides of (6.21) with regards to (6.19), we obtain

$$S_{\Xi^N} = \rho^{1/4}(0)(I^T)^* Y^T + \rho^{1/4}(0) \Big[\sum_{k=1}^{N} \Delta_k P^\xi K^T \Delta_k P^\xi\Big] (I^T)^* Y^T.$$

Since operators $(I^T)^*$ and $Y^T$ are unitary ones (see (1.4)), one has

$$\big\|S_{\Xi^N} - \rho^{1/4}(0)(I^T)^* Y^T\big\| = \rho^{1/4}(0) \Big\|\sum_{k=1}^{N} \Delta_k P^\xi K^T \Delta_k P^\xi\Big\|. \tag{6.22}$$

The latter norm may be estimated as follows:

$$\|\sum_{k=1}^{N}\Delta_k P^\xi K^T\Delta_k P^\xi a\|^2_{\mathcal{H}^T} = \sum_{k=1}^{N}\|\Delta_k P^\xi K^T\Delta_k P^\xi a\|^2_{\mathcal{H}^T} =$$

$$= \sum_{k=1}^{N}\int\limits_{x(\xi_{k-1})}^{x(\xi_k)} dx\,\rho(x)\Big[\int\limits_{x}^{x(\xi_k)} K(x,y)a(y)\,dy\Big]^2 \le$$

$$\le \max_{(x,y)\in\Omega^T\times\Omega^T}\Big[\frac{\rho(x)}{\rho(y)}\,K^2(x,y)\Big]\sum_{k=1}^{N}\big[x(\xi_k)-x(\xi_{k-1})\big]^2\int\limits_{x(\xi_{k-1})}^{x(\xi_k)} a^2(y)\rho(y)\,dy \le$$

$$\le \delta r^2(\Theta^N)\|a\|^2_{\mathcal{H}^T} \le \ \text{(see (6.13))}\ \le\le \delta_1 r^2(\Xi^N)\|a\|^2_{\mathcal{H}^T},$$

that leads to inequality

$$\|\sum_{k=1}^{N}\Delta_k P^\xi K^T\Delta_k P^\xi\| \le \delta_1^{1/2} r(\Xi^N).$$

Using it in (6.22) we obtain (6.18), that proves the lemma.

So, for any sequence of partitions $\Xi^N$, $r(\Xi^N)\to 0$, sums (6.14) tend to the limit

$$\rho^{1/4}(0)(I^T)^*Y^T = \lim S_{\Xi^N} =: \int\limits_0^T dP^\xi\,W^T dX^{T,\xi} \tag{6.23}$$

which is called an amplitude integral. A reason to use this term is explained by approximate equality

$$(\Delta_k P^\xi W^T\Delta_k X^{T,\xi} f)(x) \underset{x\sim x(\xi_k)}{\approx} \Big(\frac{\rho\big(x(\xi_k)\big)}{\rho(0)}\Big)^{-1/4} f(T-\xi_k),$$

where the right-hand side coincides with amplitude of break of wave $W^T\Delta_k X^{T,\xi} f$ on its forward front $x = x(\xi_k)$. So, sum (6.14) is formed by breaks.

On a heuristic level a construction of amplitude integral was proposed in [8] (see also [19]). A spirit of the construction is suggestive of a path integral.

**6.5. Visualization via Amplitude Integral.** Passing on to adjoint operators in (6.23) one can obtain

$$I^T = \rho^{-1/4}(0)Y^T\int\limits_0^T dX^{T,\xi}\,(W^T)^*\,dP^\xi.$$

Therefore, giving a visualization of waves operator $I^T W^T\colon f\to\widetilde{u}^f(\cdot,T)$ may be represented in the form of

$$I^T W^T = \rho^{-1/4}(0)Y^T\int\limits_0^T dX^{T,\xi}\,(W^T)^*\,dP^\xi W^T = \rho^{-1/4}(0)Y^T\int\limits_0^T dX^{T,\xi}\,d\big[(W^T)^* P^\xi W^T\big].$$

Replacing $P^\xi$ by $P^\xi_w$, we obtain

$$I^T W^T = \rho^{-1/4}(0) Y^T \int\limits_0^T dX^{T,\xi}\, d\Pi^{T,\xi}. \tag{6.24}$$

A possibility to express its right-hand side via inverse data (see (6.3)) makes representation (6.24) to be an instrument of solving of the IP. In particular, the visualizing formula may be rewritten as follows:

$$\widetilde{u}^f(\xi, T) = \big(I^T W^T f\big)(\xi) = \rho^{-1/4}(0)\Big[Y^T \int\limits_0^T dX^{T,\xi}\, d\Pi^{T,\xi} f\Big](\xi), \quad \xi \in [0; T].$$

**6.6. IP and Triangular Factorization.** An appearance of operator integral in our considerations is no wonder. The thing is that to visualize the waves is to solve a problem of triangular factorization of the connecting operator.

As it may be easily shown, operator $V^T: \mathcal{F}^T \to \mathcal{F}^T$,

$$V^T := Y^T I^T W^T$$

possesses a triangular property with respect to family of subspaces $\mathcal{F}^{T,\xi} \subset \mathcal{F}^T$, i.e.,

$$V^T \mathcal{F}^{T,\xi} \subset \mathcal{F}^{T,\xi}, \quad 0 \le \xi \le T,$$

(see [20]). From the other hand, connecting operator is represented as follows:

$$C^T := (W^T)^* W^T = (Y^T I^T W^T)^* Y^T I^T W^T = (V^T)^* V^T.$$

It means, that operator $V^T$ provides a triangular factorization of $C^T$ (see [20]).

A classical instrument of factorization is M. Brodskii's operator integral (see [20, 21]). Our construction leads to representation

$$V^T = \text{ (see (6.24)) } = \rho^{-1/4}(0) \int\limits_0^T dX^{T,\xi}\, d\Pi^{T,\xi},$$

being differ from the classical one. Its main advantage is the following. Having a natural dynamical background, an amplitude integral works in multidimensional case. Moreover, it was first proposed in [8] to solve precisely the multidimensional IP.

## 7. Spectral Inverse Problem

**7.1. Spectral Data.** Here we consider a string of finite length located on interval $\Omega := (0; x_*)$, $x_* < \infty$. Its optical length is $T_* = \tau(x_*) < \infty$.

Let us pose the boundary-value problem

$$\begin{cases} \varphi'' + \lambda\rho\varphi = 0 \quad \text{in } \Omega, & (7.1) \\ \varphi\big|_{x=0} = \varphi\big|_{x=x_*} = 0. & (7.2) \end{cases}$$

The problem has a discrete spectrum $\{\lambda_k\}_{k=1}^\infty$, $0 < \lambda_1 < \lambda_2 < \dots$, $\lambda_k \underset{k\to\infty}{\to} \infty$. The corresponding system of eigenfunctions $\{\varphi_k(\cdot)\}_{k=1}^\infty$ forms a basis in space $\mathcal{H} := \mathcal{H}^{T_*} = L_2(\Omega; \rho\, dx)$. Suppose, that they are normalized by conditions

$$(\varphi_k, \varphi_l)_{\mathcal{H}} = \delta_{kl}, \quad \varphi'_k(0) > 0.$$

The set of pairs $\{\lambda_k; \varphi'_k(0)\}_{k=1}^\infty$ is said to be spectral data of a string.

*Spectral IP is to recover $\rho$ in $\Omega$ via given spectral data.*

This statement goes back to classical papers of M.G. Krein.

**7.2. Spectral representation.** Let $U\colon\mathcal{H}\to\hat{\mathcal{H}}=l_2$ be the Fourier Transform corresponding to problem (7.1), (7.2),

$$Ua=\hat{a}=\{a^k\}_{k=1}^{\infty},\quad a^k:=(a,\varphi_k)_{\mathcal{H}},$$

for any $a\in\mathcal{H}$. Space $\hat{\mathcal{H}}$ is said to be a space of spectral representation. Below we describe some objects in $\hat{\mathcal{H}}$ which may be characterized via spectral data.

Dynamical system corresponding to a finite string is of the form

$$\alpha^T:=\begin{cases}\rho u_{tt}-u_{xx}=0,\quad x\in\Omega,\quad 0<t<T, & (7.3)\\ u\big|_{t=0}=u_t\big|_{t=0}=0, u\big|_{t=0}=u_t\big|_{t=0}=0, & (7.4)\\ u\big|_{x=0}=f;\quad u\big|_{x=x_*}=0. & (7.5)\end{cases}$$

Problem (7.3)–(7.5) may be solved by Fourier method. For control $f\in C^2[0;T]$, $f(0)=f'(0)=0$, let us seek the solution $u^f$ in the form of

$$u^f(x,t)=\sum_{k=1}^{\infty}c_k^f(t)\varphi_k(x).$$

The following calculation gives an equation for coefficients:

$$\frac{d^2}{dt^2}c_k^f(t)=\frac{d^2}{dt^2}\big(u^f(\cdot,t),\varphi_k\big)_{\mathcal{H}}=\int\limits_0^{x_*}u_{tt}^f(x,t)\varphi_k(x)\rho(x)\,dx=\int\limits_0^{x_*}u_{xx}^f(x,t)\varphi_k(x)\,dx=$$

$$=\text{ (see (7.2), (7.5)) }=u^f(0,t)\varphi_k'(0)+\int\limits_0^{x_*}u^f(x,t)\varphi_k''(x)\,dx=f(t)\varphi_k'(0)-\lambda_k c_k^f(t).$$

With regards to (7.4) one obtains the problem

$$\begin{cases}[\frac{d^2}{dt^2}+\lambda_k]c_k^f(t)=\varphi'(0)f(t),\quad t>0,\\ c_k^f(0)=\frac{d}{dt}c_k^f(0)=0,\end{cases}$$

which may be solved explicitly:

$$c_k^f(t)=\int\limits_0^t\big[\varphi_k'(0)\lambda_k^{-1/2}\sin\big(\lambda_k^{1/2}(t-s)\big)\big]f(s)\,ds. \tag{7.6}$$

It easy to see, that (7.6) holds true for any $f\in\mathcal{F}^T$. For the final moment one has

$$c_k^f(T)=\big(u^f(\cdot,T),\varphi_k\big)_{\mathcal{H}}=(s_k^T,f)_{\mathcal{F}^T}, \tag{7.7}$$

where $s_k^T=s_k^T(t):=\varphi_k'(0)\lambda_k^{-1/2}\sin\big(\lambda_k^{1/2}(T-t)\big)$.

So, Fourier coefficients of waves are determined by spectral data. Consequently, they determine the operator

$$\hat{W}^T\colon\mathcal{F}^T\to\hat{\mathcal{H}},\quad \hat{W}^T:=UW^T,$$

$$\hat{W}^Tf=\hat{u}^f(\cdot,T)=\{c_k^f(T)\}_{k=1}^{\infty}=\text{ (see (7.7)) }=\big\{(s_k^T,f)_{\mathcal{F}^T}\big\}_{k=1}^{\infty}. \tag{7.8}$$

After $\hat{W}^T$, a family of "spectral images" of reachable sets

$$\hat{\mathcal{U}}^\xi := U\mathcal{U}^\xi = UW^T\mathcal{F}^{T,\xi} = \hat{W}^T\mathcal{F}^{T,\xi}, \quad 0 \le \xi \le T,$$

is also determined by spectral data. Obviously, the same is true for the projector

$$\hat{P}^\xi_w := UP^\xi_w U^{-1}$$

in $\hat{\mathcal{H}}$ on $\hat{\mathcal{U}}^\xi$.

**Remark 7.1.** In situation of spectral IP projector $\hat{P}^\xi_w$ may be constructed efficiently. Let $\{f^\xi_j\}_{j=1}^\infty$ be a system of controls which is complete in $\mathcal{F}^{T,\xi}$:

$$\operatorname{clos}_{\mathcal{F}^T} \operatorname{Lin\,Span}\{f^\xi_j\}_{j=1}^\infty = \mathcal{F}^{T,\xi}.$$

Then $\hat{P}^\xi_w$ may be found as the projector in $\hat{\mathcal{H}}$ on the subspace

$$\operatorname{clos}_{\hat{\mathcal{H}}} \operatorname{Lin\,Span}\{\hat{W}^T f^\xi_j\}_{j=1}^\infty = \hat{\mathcal{U}}^\xi.$$

A controllability of string leads to relations

$$\hat{\mathcal{U}}^\xi\big|_{\xi \ge T_*} = U\mathcal{U}^\xi\big|_{\xi \ge T_*} = U\mathcal{H} = \hat{\mathcal{H}}.$$

Therefore, the value $\xi = T_*$ is selected by condition

$$\hat{P}^\xi_w\big|_{\xi \ge T_*} = \mathbb{I}_{\hat{\mathcal{H}}}. \tag{7.9}$$

Together with $\hat{W}^T$ (see (7.8)), its adjoint $(\hat{W}^T)^*: \hat{\mathcal{H}} \to \mathcal{F}^T$, $(\hat{W}^T)^* = (W^T)^*U^{-1}$ may be expressed via $\{\lambda_k; \varphi'_k(0)\}_{k=1}^\infty$. For any $y \in \hat{\mathcal{H}}$, $y = \{y^k\}_{k=1}^\infty$ one can obtain

$$(\hat{W}^T)^* y = \sum_{k=1}^\infty y^k s^T_k(\cdot), \tag{7.10}$$

the series converging in $\mathcal{F}^T$.

One more important object in $\hat{\mathcal{H}}$ is determined by spectral data. Let $l \in \mathcal{H}$ be a linear function, $l = l(x) = x - x_*$, $x \in \Omega$. Its spectral representation may be found explicitly:

$$\hat{l} = Ul = \{\lambda_k^{-1}\varphi'_k(0)x_*\}_{k=1}^\infty. \tag{7.11}$$

Indeed, integrating by parts, one has

$$l^k = (l, \varphi_k)_{\mathcal{H}} = \int_0^{x_*}(x_* - x)\varphi_k(x)\rho(x)\,dx = -\lambda_k^{-1}\int_0^{x_*}(x_* - x)\varphi''_k(x)\,dx = \lambda_k^{-1}\varphi'_k(0)x_*.$$

At last, possessing $\hat{P}^\xi_w$ and $\hat{l}$ one can calculate the power moment $\nu(\cdot)$ (see subsec. 1.2):

$$\nu(\xi) = \int_0^{x(\xi)}(x_* - x)^2\rho(x)\,dx = \|P^\xi l\|^2_{\mathcal{H}} = \text{ (see (3.4)) } = \|P^\xi_w l\|^2_{\mathcal{H}} = \|\hat{P}^\xi_w \hat{l}\|^2_{\hat{\mathcal{H}}}. \tag{7.12}$$

Recall, that $\nu(\cdot)$ is connected with function $x(\cdot)$ by relation (1.2).

In accordance with (7.9) the value $\xi = T_*$ is selected by condition

$$\nu(\xi) \equiv \text{const} = \|\hat{l}\|^2_{\hat{\mathcal{H}}} \quad \text{for } \xi \ge T_*. \tag{7.13}$$

**7.3. Solving of spectral IP, and Wave Visualization.** Given spectral data $\{\lambda_k; \varphi'_k(0)\}_{k=1}^{\infty}$ one can recover $\rho$ in $\Omega$ by means of the following procedure:

(i) find element $\hat{l} \in \hat{\mathcal{H}} = l_2$ by (7.11);

(ii) for all $\xi \in [0; T]$ construct projectors $\hat{P}_w^\xi$ (see Remark 7.1);

(iii) find function $\nu(\cdot)$ from (7.12), and determine $\xi = T_*$ by (7.13);

(iv) find function $x(\tau)$, $\tau \in [0; T_*]$, by means of the last equality in (1.2), and, then find eikonal $\tau = \tau(x)$, $x \in [0; x_*]$;

(v) recover density $\rho(x) = \big(\tau'(x)\big)^2$, $x \in \Omega$.

The spectral IP is solved.

In addition to procedure (i)–(v), wave visualization may be carried out in the frameworks of the spectral IP. Indeed, entering in (6.8) operator $\Pi^{T,\xi}$ may be represented as follows:

$$\Pi^{T,\xi} = (W^T)^* P_w^\xi W^T = (UW^T)^* U P_w^\xi U^{-1} (UW^T) = (\hat{W}^T)^* \hat{P}_w^\xi \hat{W}^T,$$

i.e., via operators determined by spectral data (see (7.8), Remark 7.1, (7.10)).

The same representation permits to use amplitude integral (see (6.24)). Such a possibility was used in multidimensional spectral problem (see [19]).

## 8. Numerical Testing

The procedures described in subsec. 5.5 and 7.3 may be used as a basis for elaboration of numerical algorithms.

Numerical testing in spectral IP was first realized by A. Kachalov in [7]. T. Sheronova used relations (5.11) for testing in dynamical IP (see [22]).

For some special kinds of function $r(\cdot)$ providing a simple form of kernel $C^T(\cdot,\cdot)$ (see (2.28)) GLK-equations may be solved explicitly. Recently S. He used this possibility for testing (see [23]).

Mentioned results demonstrated an efficiency of algorithms based upon the BC-method. It is a reason to hope for its usefulness for applications.

## Acknowledgments

The work was fulfilled in the frameworks of EC-Russia Collaboration (Contract ESPRIT P 9282 ACTS) and partially supported by the Project INTAS 93 1815.

The author is very gratefull to Prof. V. Isakov for his kind proposal to write this article and a permanent attention to the work.

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