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Catching geodesics in Hadamard spaces

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Abstract

For every quasi-isometric map $f : X \to Y$ of Hadamard spaces we define its asymptotic limit s_f which sends the boundary at infinity $\partial_\infty X$ to the cone $C_\infty Y$ over $\partial_\infty Y$ and establish its analytical properties. In the case when X and Y are cocompact rank one spaces with respect to the same discrete isometry group Γ and hence Γ-equivariantly quasi-isometric we give a sufficient condition for s_f to be an equivariant homeomorphism between $\partial_\infty X$ and $\partial_\infty Y$ w.r.t. the standard topologies and biLipschitz homeomorphism w.r.t. Tits metrics. Apart of this condition there is a large number of equivariantly quasi-isometric cocompact Hadamard spaces whose boundaries at infinity are not equivariantly homeomorphic. This answers a question of M. Gromov.

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0. Introduction

This work is motivated by the following questions of M. Gromov [Gr2, sect.6.B_4]. Suppose we have two discrete cocompact actions of a group Γ on two Hadamard spaces X and Y. Are the boundaries at infinity $\partial_\infty X$ and $\partial_\infty Y$ Γ-equivariantly homeomerphic? Are they isometric for the Tits metrics?

The answer to the above questions is known to be positive in the following cases.

(a) X is a hyperbolic (in the sense of [Gr1]) Hadamard space; this easily follows from the Morse quasi-isometric lemma;

(b) X and Y are manifolds, the group Γ is irreducible and has rank at least two [BaEb].

Under the indicated conditions there is a Γ-equivariant quasi-isometry $f : X \to Y$, which induces the required isomorphism $\partial_\infty X \to \partial_\infty Y$ of the boundaries at infinity. Note that in the case (a) any quasi-isometry $f : X \to Y$ induces a homeomorphism $\partial_\infty X \to \partial_\infty Y$ (w.r.t. the standard topologies). Moreover, if $X = Y$ is a quaternionic hyperbolic space $\mathbb{HH}^n$, $n > 1$ or the Cayley hyperbolic plane $\mathbb{C}a\mathbb{H}^2$ then any quasi-isometry $f : X \to Y$ is a bounded pertubation of an isometry [Pa], [GrPa]; the same holds for $X = Y$ be nonflat irreducible symmetric space of noncompact type of rank at least two, or thick irreducible Euclidean building of rank at least two with cocompact affine Weyl group [KL].

We add to this list the following positive result (see Theorem 15.1; all necessary definitions can be found in Chapter 3).

0.1. Theorem. *If the group Γ has rank one in the sense of Ballmann-Brin and the deviation ν_f of the Γ-equivariant quasi-isometry $f : X \to Y$ vanishes at least at one point $z \in \partial_\infty X$, then there exists a Γ-equivariant bijection $\partial_\infty X \to \partial_\infty Y$, which is homeomorphism w.r.t. the standard topologies and biLipschitz homeomorphism for the Tits metrics.*

A cocompact Hadamard space (X, Γ) of rank 1 may contain k-flats with $k \geq 2$, and the Tits geometry of its boundary at infinity $\partial_\infty X$ may be rather nontrivial in contrast to the case (a), although it is not so rigid as in the case (b).

It is known that if a cocompact Hadamard space (X, Γ) is not of rank one in the sense of Ballmann-Brin and X is a manifold or a boundaryless 2-polyhedron then X is isometric to nontrivial metric product, or a symmetric space of rank at least two, or a thick Euclidean building of one of the standard types A_2, B_2, G_2, endowed with its canonical metric [BB]. In other words, most of the cocompact Hadamard spaces have rank one.

In sect. 15.2 we give a number of examples showing that no one condition of theorem 0.1 can be dropped. Therefore, the answers to the cited above questions are in general negative. Moreover, the abundance of counterexamples indicates that the existence of an equivariant homeomorphism $\partial_\infty X \to \partial_\infty Y$ or an isometry for the Tits metrics is an exception rather than a rule.

The main technical result of the paper is that we define for any quasi-isometric map $f : X \to Y$ of Hadamard spaces its *asymptotic limit* $s_f : \partial_\infty X \to C_\infty Y$, where $C_\infty Y$ is the cone over the boundary at infinity $\partial_\infty Y$ (see Chapter 3). For the angle metric α on $\partial_\infty X$ and the Euclidean cone $C_\alpha Y$ over $\partial_\alpha Y$ the asympttotic limit $s_f : \partial_\alpha X \to C_\alpha Y$ satisfies the Hölder condition with exponent 1/2 (see Proposition 12.1). At the same time there are natural quasi-isometric maps (for instance, an embedding $H^3(\lambda) \to H^2 \times H^2$ of the hyperbolic 3-space of curvature $-\lambda^2$ into the product of hyperbolic planes as a horosphere, see 14.2), whose asymptotic limit is not continuous w.r.t. the standard topologies on $\partial_\infty X$, $C_\infty Y$.

The proof of theorem 0.1 as well as the mentioned above examples are based on the properties of the asymptotic limit s_f of a quasi-isometric map. In turn, the definition of s_f and its properties follow from the construction of the curve of virtual centers, which is considered in Chapter 1.

This construction, besides of applications to quasi-isometric maps, has others applications, for example, to the study of fixed points of parabolic isometries. We prove (see Proposition 10.1 and Theorem 10.2) in Chapter 2 the following.

0.2. Proposition. *For any different points z, $z' \in \partial_\infty X$ of the boundary at infinity of a hyperbolic Hadamard space X there exists a geodesic $c : \mathbb{R} \to X$ with $c(\infty) = z$, $c(-\infty) = z'$.*

0.3. Theorem. *Let $\gamma : X \to X$ be a parabolic isometry of a hyperbolic Hadamard space, $d_\gamma : X \to \mathbb{R}$ its displacement function, $d_\gamma(x) = |x\gamma(x)|$. Then $\inf d_\gamma = 0$, there exists a unique fixed point $z \in \partial_\infty X$, $\gamma z = z$, and every horoball centered at z is invariant for γ.*

For Hadamard manifolds these facts are well known [BGS], and its proofs are easy to extend to the case of locally compact Hadamard spaces [Ba]. What is new here is that we do not suppose the Hadamard space X to be locally compact.

The paper is organized as follows. In Chapter 1, we give the construction of the curve of virtual centers. We start with Bruhat-Tits characterization of a Hadamard space. Using it we define the virtual radius and the virtual center of a bounded sequence, the notions which are key to the work. Further, we establish convexity of the virtual radius for a sequence of geodesic segments (Lemma 4.1), which leads to our first result, Theorem 5.1, about virtual geodesic for a sequence of segments. Proposition 0.2 follows from this theorem.

Applications to quasi-isometric maps are based on the notions of curve and second curve of virtual centers of a sequence leaving bounded sets, which we discuss in sect. 6, 7. It is important that the second curve of virtual centers is a geodesic ray (Theorem 7.1), whose speed $v \in [0,1]$ is an invariant of the class of boundedly equivalent sequences (Theorem 8.1). The proofs use estimates (Proposition 8.1.2) implied by Bruhat-Tits characterization. We also use these estimates to establish analytic properties of the asymptotic limit of a quasi-isometric map (Chapter 3) in terms of the drift and the deviation—others invariants of a sequence (sect. 9, 10).

Using results obtained in Chapter 1, we prove in Chapter 2 Proposition 0.2, Theorem 0.1 and give a sufficient condition for a parabolic isometry of X to have a fixed point on $\partial_\infty X$ (Proposition 10.5). This condition is meaningful for Hadamard spaces, which are not locally compact, since there are examples [BH] of parabolic isometries of such spaces without fixed points on $\partial_\infty X$.

In Chapter 3, we define the asymptotic limit s_f of a quasi-isometric map $f : X \to Y$ of Hadamard spaces, establish some its analytical properties (Propositions 12.1, 12.3, Corollary 12.4), introduce the notion of the deviation ν_f of a quasi-isometric map f and prove that s_f is continuous (w.r.t. standard topologies) at any point $z \in \partial_\infty X$, where $\nu_f(z) = 0$ (Theorem 14.1). Several examples of quasi-isometric maps whose asymptotic limits have some pathology are given. The results of this Chapter allow to find the relation between the asymptotic cone $\mathrm{Con}_\infty X$, introduced by Gromov [Gr2], and the Tits geometry on $\partial_\infty X$: there is a retraction $\mathrm{Con}_\infty X \to C_\alpha X$ satisfying the Hölder condition with exponent $1/2$ (Proposition 13.1, cf. [Gr2, sect.3.C_1'']).

At last in sect. 15 equivariant quasi-isometries are considered, Theorem 0.1 is proved and a number examples showing that no one of its conditions can be dropped are given.

Necessary prerequisites on Hadamard spaces can be find in [BGS], [Ba].

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Chapter 1. Curves of virtual centers

We start with well-known characterization of Bruhat-Tits of Hadamard spaces (see [BT], [Ba], [Br]), which can be take as the (most convenient) definition.

A complete metric space X is Hadamard iff for every x, $y \in X$ there is $u \in X$ such that

$$|xy|^2 + 4|zu|^2 \leq 2\big(|xz|^2 + |yz|^2\big) \tag{1}$$

for all $z \in X$.

Choosing $z = x$, y we obtain that u is a midpoint between x and y, in particular, X is a geodesic space.

It seems the power of this definition is yet not fully recognized, and the aim of the work is to demonstrate this.

1. The center and the radius of a bounded set.

For a bounded set B of a metric space X one puts $r(B) = \inf_{x \in X} r(x, B)$, called *the radius* of B, where $r(x, B) = \sup_{y \in B} |xy|$. *The center* of B is a point $c \in X$ such that $r(c, B) = r(B)$. If X is a Hadamard space then for any nonempty bounded $B \subset X$ there is a unique center. This easy follows from (1) (see e.g. [Ba]).

2. The diameter and the radius of a bounded set.

2.1. Proposition. *Let B be a nonempty bounded set in a Hadamard space X, c its center, r its radius. Then for any $b \in X$ we have*

$$r^2(b, B) \geq |bc|^2 + r^2.$$

The claim follows immediately from the next lemma.

2.2. Lemma. *Let B be a bounded set in a Hadamard space X, $\#B \geq 2$, c its center, r its radius, $b \in X$, $b \neq c$ and $0 < \delta < r/2$. Then for every $\epsilon > 0$ there exists $a \in B$ such that $|ac| \geq r - \delta$ and $\angle \bar{a}\,\bar{c}\,\bar{b} > \pi/2 - \epsilon$, where $\bar{a}\,\bar{b}\,\bar{c} \subset \mathbb{R}^2$ is the comparison triangle for abc.*

(A comparison triangle $\bar{a}\,\bar{b}\,\bar{c} \subset \mathbb{R}^2$, by definition, has the same side lengths as $abc \subset X$).

Proof. Assume that this is not the case. Then there exists $\epsilon > 0$ such that for each $a \in B$ with $|ac| \geq r - \delta$ we have

$$\angle \bar{a}\,\bar{c}\,\bar{b} \leq \pi/2 - \epsilon.$$

Pick $c' \in bc$, $|c'c| = t > 0$. Then

$$|ac'| \leq |\bar{a}\,\bar{c}'|,$$

where $\bar{c}' \in \bar{b}\,\bar{c}$, $|\bar{c}'\bar{c}| = t$. Using the Cosine theorem we find that for sufficiently small $t > 0$

$$|\bar{a}\,\bar{c}'| \leq r - \frac{t}{2}\epsilon$$

for all $a \in B$ with $|ac| \geq r - \delta$. On the other hand, if $a \in B$ and $|ac| < r - \delta$, then $|ac'| \leq |ac| + t < r - \delta/2$ for $t < \delta/2$. Thus $r(c', B) < r$ for sufficiently small $t > 0$, which contradicts the definition of r. $\square$

2.3. Corollary. *For every bounded subset B of a Hadamard space X the inequality*

$$r(B) \le \frac{1}{\sqrt{2}} \mathrm{diam}(B)$$

holds.

Proof. Let $c \in X$ be the center of B and r its radius. Given $\delta > 0$ there is $b \in B$ with $|bc| \ge r - \delta$. By lemma 2.2, for every $\epsilon > 0$ there exists $a \in B$ such that $|ac| \ge r - \delta$ and $\angle \overline{a}\,\overline{c}\,\overline{b} \ge \pi/2 - \epsilon$. Then by the cosine inequality we have

$$(\mathrm{diam}\, B)^2 \ge |ab|^2 \ge 2(r-\delta)^2 - \mathrm{O}(\epsilon),$$

which implies the claim since δ, ϵ could be chosen arbitrarily small. $\square$

2.4. Remark. Inequality of Corollary 2.3 is sharp; the equality takes place for an orthonormal basis $B = (e_1, e_2, \dots)$ of the Hilbert space $l^2(\mathbb{N})$.

3. The virtual center and the virtual radius of a bounded sequence.

Let $B = \{y_n \,|\, n \ge 1\}$ be a bounded subset in a Hadamard space X, $B_k = \{y_n \,|\, n \ge k\}$, $r_k = r(B_k)$ the radius of B_k, $c_k \in X$ the center of B_k.

3.1. Lemma. *There exist limits*

$$r_\infty = \lim_k r_k \quad \text{and} \quad c_\infty = \lim_k c_k,$$

and $\limsup_n |c_\infty y_n| = r_\infty$.

Proof. The existence of the limit r_∞ is obvious because the sequence $\{r_k\}$ is nonincreasing.

Let $m \ge k$. For the midpoint $u \in c_k c_m$ we have by (1)

$$|c_k c_m|^2 + 4|uz|^2 \le 2\big(|c_k z|^2 + |c_m z|^2\big)$$

for any $z \in X$. Choosing $z \in B_m$ we obtain

$$|c_k c_m|^2 \le 2\big(r^2(c_k, B_m) + r^2(c_m, B_m)\big) - 4r_m^2 \le 2(r_k^2 - r_m^2),$$

where the inequality $r(c_k, B_m) \le r_k$ follows from $B_m \subset B_k$. Thus $\{c_k\}$ is a Cauchy sequence together with $\{r_k\}$, and there exists the limit $c_\infty = \lim_k c_k$. The last claim of the lemma follows from the first two ones. $\square$

The number $r_\infty = r_\infty(B)$ is called *the virtual radius* of the sequence B, and the point $c_\infty = c_\infty(B) \in X$ its *virtual center*.

4. Convexity of the virtual radius.

Assume that a sequence of geodesic segments $\sigma_k : [0,1] \to X$, $k \ge 1$, with uniformly bounded speeds $0 \le v_k \le v$ has the bounded set $B(0) = \big\{\sigma_k(0) \,|\, k \ge 1\big\}$ of initial points. Then the set $B(t) = \big\{\sigma_k(t) \,|\, k \ge 1\big\}$ is bounded for any $t \in [0,1]$, and hence there exists its virtual radius $r_\infty(t)$ and virtual center $c_\infty(t)$.

4.1. Lemma. *The function $r_\infty : [0,1] \to \mathbb{R}$ is convex.*

Proof. Let $t, t' \in [0,1]$, $2t'' = t+t'$, $h : [t,t'] \to X$ be a geodesic with $h(t) = c_\infty(t)$, $h(t') = c_\infty(t')$, $\epsilon > 0$. By lemma 3.1 there is $n \geq 1$ such that $\big|h(t)\sigma_n(t)\big| \leq r_\infty(t) + \epsilon$, $\big|h(t')\sigma_n(t')\big| \leq r_\infty(t') + \epsilon$. On the other hand by Proposition 2.1 one can assume that

$$\big|h(t'')\sigma_n(t'')\big| \geq r_\infty(t'') - \epsilon.$$

By convexity of distance function we have

$$\begin{aligned}\big|h(t'')\sigma_n(t'')\big| &\leq \frac{1}{2}\big(\big|h(t)\sigma_n(t)\big| + \big|h(t')\sigma_n(t')\big|\big)\\ &\leq \frac{1}{2}\big(r_\infty(t) + r_\infty(t')\big) + \epsilon,\end{aligned}$$

which obviously implies the claim. $\square$

5. The virtual geodesic for a sequence of segments.

5.1. Theorem. *Let X be a Hadamard space. Assume that a sequence of unit speed geodesic segments $\sigma_n : [-l_n, l_n] \to X$ satisfies the conditions*

(i) *$l_n \to \infty$ as $n \to \infty$;*
(ii) *there is $\delta > 0$ such that for each $t \in \mathbb{R}$ the set*

$$B(t) = \big\{\sigma_n(t) \,|\, l_n \geq |t|\big\}$$

has diameter at most δ.

Then there exists a geodesic $c : \mathbb{R} \to X$ such that $\big|c(t)\,\sigma_n(t)\big| \leq \delta$ for all sufficiently large n and all $t \in [-l_n, l_n]$.

Proof. We prove that the map $c : \mathbb{R} \to X$, where $c(t) = c_\infty(t)$ is the virtual center of the sequence $B(t)$, is the required geodesic. By the condition, the function $r_\infty : \mathbb{R} \to X$ of virtual radii of sequences $B(t)$ is uniformly bounded, $r_\infty(t) \leq \delta/\sqrt{2}$, (cf. Corollary 2.3). By lemma 4.1, this function is convex, hence it is constant, $r_\infty(t) = r$ for all $t \in \mathbb{R}$.

Given $t < t'$ let $h : [t,t'] \to X$ be a geodesic with $h(t) = c(t)$, $h(t') = c(t')$. Let $t < t'' < t'$, $\epsilon > 0$. By lemma 3.1 there is $n \geq 1$ such that $\big|h(t)\sigma_n(t)\big|$, $\big|h(t')\sigma_n(t')\big| \leq r+\epsilon$. By Proposition 2.1 one can assume that

$$\big|h(t'')\sigma_n(t'')\big|^2 \geq \big|h(t'')c(t'')\big|^2 + r^2 - \epsilon.$$

By convexity of distance function we have

$$\big|h(t'')\sigma_n(t'')\big| \leq a\big|h(t)\sigma_n(t)\big| + b\big|h(t')\sigma_n(t')\big|,$$

where $a = (t' - t'')/(t' - t)$, $b = (t'' - t)/(t' - t)$. Therefore,

$$r^2 + \big|h(t'')c(t'')\big|^2 - \epsilon \leq (r + \epsilon)^2.$$

Since ϵ is arbitrary, we have $h(t'') = c(t'')$. $\square$

6. The curve of virtual centers of a sequence.

Assume that a sequence $x : \mathbb{N} \to X$ in a Hadamard space X leaves bounded sets, i.e., for any bounded $B \subset X$ we have $x_n \notin B$ for all sufficiently large n. Given $x_0 \in X$ let $\sigma_n : [0, l_n] \to X$ be the unit speed geodesic from $x_0 = \sigma_n(0)$ to x_n, $n \geq 1$. For $t \geq 0$ and $k \in \mathbb{N}$ we put

$$B_k(t) = \{\sigma_n(t) \,|\, n \geq k\}.$$

The set $B_k(t)$ is bounded, $\operatorname{diam} B_k(t) \leq 2t$, and not empty for sufficiently large k. Let $c_x(t)$ be the virtual center of the set $B_k(t)$, $r_x(t)$ its virtual radius. The map $c_x : [0, \infty) \to X$ is called *the curve of virtual centers* (with respect to x_0) of the sequence x. By lemma 4.1 the function r_x is convex. The condition $r_x \equiv 0$ is equivalent for the sequence σ_n to converge to c_x, and hence in this case c_x is a geodesic ray of unit speed. This takes place if the function of virtual radii r_x has a sublinear growth, $\lim_{t\to\infty} r_x(t)/t = 0$, since $r_x(t) \geq 0$ and $r_x(0) = 0$.

We say that a curve $\xi : [0, \infty) \to X$ leaves bounded sets with *a finite speed* if for some $x_0 \in X$ we have

$$\limsup_{t\to\infty} \big|x_0\, \xi(t)\big|/t < \infty.$$

The curve of virtual centers c_x leaves bounded sets with a finite speed, since $\big|x_0\, \sigma_n(t)\big| = t$ and hence, the virtual center $c_x(t)$ of $B_k(t)$ lies in the ball of radius t centered at x_0, $\big|x_0\, c_x(t)\big| \leq t$. Therefore, the curve of virtual centers c_x leaves bounded sets with a finite speed (this includes the possibility $\big|x_0\, c_x(t)\big| \leq C < \infty$ for all $t \geq 0$).

7. Second curve of virtual centers.

Assume that a curve $\xi : [0, \infty) \to X$ in a Hadamard space X leaves bounded sets with a finite speed. Its curve of virtual centers is defined similarly to that of a sequence. The distinction is that the geodesics $\sigma_t : [0, t] \to X$, $\sigma_t(0) = x_0 = \xi(0)$, $\sigma_t(t) = \xi(t)$, do not have, in general, unit speed. Nevertheless, for $s \geq 0$ the set

$$B_t(s) = \{\sigma_\tau(s) \,|\, \tau \geq t\}$$

is bounded because ξ leaves bounded sets with a finite speed, and nonempty for $t \geq s$. Let $c_t(s)$ be the virtual center of $B_t(s)$. The map $c_\xi : [0, \infty) \to X$ is called *the curve of virtual centers* of the curve ξ. Let $\rho_\xi(s)$ be the virtual radius of $B_t(s)$. By lemma 4.1, the function $\rho_\xi : [0, \infty) \to \mathbb{R}$ is convex. If ρ_ξ has a sublinear growth then $\rho_\xi \equiv 0$. In this case the curve of virtual centers c_ξ is a geodesic.

Let $x : \mathbb{N} \to X$ be a sequence in a Hadamard space leaving bounded sets. *Its second curve of virtual centers* is the curve of virtual centers $c_x^2 : [0, \infty) \to X$ of the curve c_x.

7.1. Theorem. *Let $x : \mathbb{N} \to X$ be a sequence in a Hadamard space leaving bounded sets. Then its second curve of virtual centers c_x^2 is a geodesic ray of speed $0 \leq v \leq 1$.*

Let $\xi = c_x$ be the curve of virtual centers of the sequence x with respect to $x_0 = c_x(0)$, r_x the corresponding function of virtual radii, ρ_ξ the function of virtual radii of the curve ξ. We have $\big|x_0\, c_x^2(s)\big| \leq s$ for $s \geq 0$, since $\big|x_0\, \xi(t)\big| \leq t$, $t \geq 0$. Thus for the proof of the theorem it suffices to show that ρ_ξ has a sublinear growth.

7.1.1. Lemma. *Let $h : [t, t'] \to X$ be a geodesic with $h(t) = \xi(t)$, $h(t') = \xi(t')$, where $t' \geq t \geq 0$. Then for $t \leq t'' \leq t'$ we have*

$$\big|h(t'')\xi(t'')\big|^2 \leq \big(a r_x(t) + b r_x(t')\big)^2 - {r_x}^2(t''),$$

where $a=(t'-t'')/(t'-t)$, $b=(t''-t)/(t'-t)$.

Proof. Let $\epsilon>0$. By lemma 3.1 there exists $n\geq 1$ such that $\big|h(t)\,\sigma_n(t)\big|\leq r_x(t)+\epsilon$, $\big|h(t')\,\sigma_n(t')\big|\leq r_x(t')+\epsilon$. Using Proposition 2.1 one can assume that

$$\big|h(t'')\,\sigma_n(t'')\big|^2\geq\big|h(t'')\,\xi(t'')\big|^2+{r_x}^2(t'')-\epsilon.$$

By convextity of distance function we have

$$\big|h(t'')\,\sigma_n(t'')\big|\leq ar_x(t)+br_x(t')+\epsilon.$$

Hence

$${r_x}^2(t'')+\big|h(t'')\xi(t'')\big|^2-\epsilon\leq\big(ar_x(t)+br_x(t')+\epsilon\big)^2.$$

The claim follows, since ϵ was chosen arbitrary. □

Let $\beta_x=\sup_{t>0}r_x(t)/t$. Since

$$r_k(t)\leq\frac{1}{\sqrt{2}}\operatorname{diam}B_k(t)\leq\sqrt{2}t,$$

we have $\beta_x\leq\sqrt{2}$.

7.1.2. Lemma. *For the function ρ_ξ we have*

$$\rho_\xi(s)/s\leq\big({\beta_x}^2-{r_x}^2(s)/s^2\big)^{\frac{1}{2}}$$

for all $s>0$. In particular, ρ_ξ has sublinear growth.

Proof. Let $\sigma_t:[0,t]\to X$ be the geodesic with $\sigma_t(0)=x_0=\xi(0)$, $\sigma_t(t)=\xi(t)$. By lemma 7.1.1 given $0\leq s\leq t$ we have

$$\begin{aligned}\big|\sigma_t(s)\,\xi(s)\big|^2&\leq(s/t)^2{r_x}^2(t)-{r_x}^2(s)\\&\leq s^2\big({\beta_x}^2-{r_x}^2(s)/s^2\big),\end{aligned}$$

which implies the required estimate for ρ_ξ. □

Proof of Theorem 7.1. By lemma 7.1.2 we have $\rho_\xi\equiv 0$, hence the claim. □

8. Stability of second curve of virtual centers of a sequence.

Sequences x, $x':\mathbb{N}\to X$ in a metric space X are called b-*equivalent* if

$$|xx'|=\sup_n|x_nx'_n|<\infty.$$

Second curves of virtual centers of a sequence $x:\mathbb{N}\to X$ in a Hadamard space X are stable under bounded perturbations of x.

8.1. Theorem. *Assume that b-equivalent sequences x, x' : $\mathbb{N}\to X$ in a Hadamard space X leave bounded sets. Then their second curves of virtual centers c_x^2, $c_{x'}^2$ emanating from x_0, $x'_0\in X$ correspondingly are asymptotic geodesics having equal speeds.*

For the proof we need several estimates. Let c_x, $c_{x'}$: $[0,\infty)\to X$ be the curves of virtual centers of sequences x, x', $c_x(0)=x_0$, $c_{x'}(0)=x'_0$, and let r_x, $r_{x'}$: $[0,\infty)\to\mathbb{R}$ be their functions of virtual radii. As usual, let σ_n, σ'_n be the unit speed geodesics from x_0 to x_n and from x'_0 to x'_n correspondingly. For $0\leq t\leq\min\{|x_0\,x_n|,|x'_0\,x'_n|\}$ we have $\big|\sigma_n(t)\,\sigma'_n(t)\big|\leq|x_0\,x'_0|+2t$. For $t\geq 0$ we put

$$d_{x,x'}(t)=\limsup_{n\to\infty}\big|\sigma_n(t)\,\sigma'_n(t)\big|.$$

8.1.1. Lemma. *For each $t \geq 0$ we have*

$$\left|c_x(t)\, c_{x'}(t)\right|^2 \leq \left(r_{x'}(t) + d_{x,x'}(t)\right)^2 - r_x^2(t).$$

Proof. Let $\epsilon > 0$. Then $\left|c_x(t)\, \sigma_n(t)\right| \leq r_x(t) + \epsilon$ for all sufficiently large n by Lemma 3.1. One can assume that $\left|c_{x'}(t)\, \sigma'_n(t)\right| \leq r_{x'}(t) + \epsilon$ as well. Let u be the midpoint of $c_x(t)c_{x'}(t)$. Using Proposition 2.1 one can assume that

$$\left|u\, \sigma_n(t)\right|^2 \geq \frac{1}{4}\left|c_x(t)\, c_{x'}(t)\right|^2 + {r_x}^2 - \epsilon.$$

Since

$$\begin{aligned}\left|c_{x'}(t)\, \sigma_n(t)\right| &\leq \left|c_{x'}(t)\, \sigma'_n(t)\right| + \left|\sigma_n(t)\, \sigma'_n(t)\right| \\ &\leq r_{x'}(t) + d_{x,x'}(t) + 2\epsilon,\end{aligned}$$

we have by inequality (1)

$$\begin{aligned}\left|c_x(t)\, c_{x'}(t)\right|^2 &\leq 2\left|c_x(t)\, \sigma_n(t)\right|^2 + 2\left|c_{x'}(t)\, \sigma_n(t)\right|^2 - 4\left|u\, \sigma_n(t)\right|^2 \\ &\leq 2\left(r_x(t) + \epsilon\right)^2 + 2\left(r_{x'}(t) + d_{x,x'}(t) + 2\epsilon\right)^2 \\ &\quad - \left|c_x(t)\, c_{x'}(t)\right|^2 - 4r_x(t^2) + 4\epsilon.\end{aligned}$$

Hence the claim. ☐

8.1.2. Proposition. *For each $t \geq 0$ we have*

$$\left|c_x(t)\, c_{x'}(t)\right|^2 \leq d_{x,x'}(t)\left(d_{x,x'}(t) + r_x(t) + r_{x'}(t)\right).$$

Proof. Interchanging x and x' in the estimate of Lemma 8.1.1 and adding the obtained inequalities we get the required estimate. ☐

8.1.3. Lemma. *Let $\sigma : [0, l] \to X$, $\sigma' : [0, l'] \to X$ be unit speed geodesics in a Hadamard space X. Then for all $0 \leq t \leq \min\{l, l'\}$ we have*

$$\left|\sigma(t)\, \sigma'(t)\right| \leq \left|\sigma(0)\, \sigma'(0)\right| + 2\left|\sigma(l)\, \sigma'(l')\right|.$$

Proof. One can assume that $l \leq l'$. By convexity of the distance function we have

$$\left|\sigma(t)\, \sigma'(t)\right| \leq \max\left\{\left|\sigma(0)\, \sigma'(0)\right|,\ \left|\sigma(l)\, \sigma'(l)\right|\right\}$$

for all $0 \leq t \leq l$. The claim now follows from the next obvious estimates

$$\left|\sigma(l)\, \sigma'(l)\right| \leq \left|\sigma(l)\, \sigma'(l')\right| + l' - l \text{ and } l' - l \leq \left|\sigma(0)\, \sigma'(0)\right| + \left|\sigma(l)\, \sigma'(l')\right|. \quad \square$$

Proof of theorem 8.1. By the condition and Lemma 8.1.3 we have

$$d_{x,x'}(t) = \limsup_{n\to\infty}\left|\sigma_n(t)\, \sigma'_n(t)\right| \leq |x_0\, x'_0| + 2|x\, x'| < \infty$$

for all $t \geq 0$. Since $\beta = \sup_{t>0} r_x(t)/t$ and $\beta' = \sup_{t>0} r_{x'}(t)/t$ are finite, using Proposition 8.1.2 we obtain that the function $f(t) = \left|c_x(t)\, c_{x'}(t)\right|$ has sublinear growth. Thus the function $g(t) = \left|c_x^2(t)\, c_{x'}^2(t)\right|$ has sublinear growth as well. On the other hand, $g : [0, \infty) \to \mathbb{R}$ is convex since c_x^2, $c_{x'}^2$ are geodesics. This immediately implies all assertions of the theorem. ☐

8.2. Corollary. *Under the condition of Theorem 8.1 we have*

$$\beta_x = \beta_{x'}$$

for $\beta_x = \sup_{t>0} r_x(t)/t$, $\beta_{x'} = \sup_{t>0} r_{x'}(t)/t$.

Proof. Since $\left|r_x(t) - r_{x'}(t)\right| \le d_{x,x'}(t) + \left|c_x(t)\, c_{x'}(t)\right|$ for $t \ge 0$, the function $\left|r_x(t) - r_{x'}(t)\right|$ has a sublinear growth. It remains to notice that $\beta_x = \lim_{t\to\infty} r_x(t)/t$, because the function r_x is convex and $r_x(0) = 0$. $\square$

8.3. Corollary. *Assume that a sequence $x : \mathbb{N} \to X$ in a Hadamard space X leaves bounded sets. Then its second curves of virtual centers c_x^2, ${c'_x}^2$, emanating from x_0, x'_0, are asymptotic geodesics having equal speeds. Furthermore, $\beta_x = \sup_{t>0} r_x(t)/t$ does not depend on the choice of initial point x_0.* $\square$

9. The speed and the drift of a sequence.

Let $x : \mathbb{N} \to X$ be sequence in a Hadamard space leaving bounded sets. Its second circumcenter curve c_x^2 can be, in general, a constant map. We indicate here a condition which ensures that c_x^2 is nondegenerated.

The speed v_x of a sequence x is defined to be the speed of a geodesic c_x^2. By Corollary 8.3, this definition does not depend of the choice of an initial point $x_0 = c_x^2(0)$. By Theorem 7.1 we have $0 \le v_x \le 1$.

We define *the drift* of the sequence x to be $\beta_x = \sup_{t>0} r_x(t)/t$, where r_x is the function of virtual radii of x. Since r_x is convex and $r_x(0) = 0$, we have $\beta_x = \lim_{t\to\infty} r_x(t)/t$. By Corollary 8.3 this definition does not depend on the choice of an initial point x_0.

9.1. Proposition. *Assume that a sequence $x : \mathbb{N} \to X$ in a Hadamard space X leaves bounded sets. Then for its speed v_x and drift β_x we have*

$$v_x + \beta_x \ge 1 \tag{9.1.1}$$

$$v_x^2 + {\beta_x}^2 \le 1. \tag{9.1.2}$$

In particular, $v_x = 0$ iff $\beta_x = 1$ and $v_x = 1$ iff $\beta_x = 0$.

Proof. For the curve of virtual centers $c_x : [0, \infty) \to X$, $c_x(0) = x_0$ we have by the triangle inequality $\left|x_0\, c_x(t)\right| + r_x(t) \ge t$ for all $t \ge 0$. Thus

$$\left|x_0\, c_x(t)\right| \ge (1 - \beta_x)t.$$

By Theorem 7.1, geodesic segments $x_0 c_x(t)$ converge to c_x^2 when $t \to \infty$. Thus $\left|x_0\, c_x^2(s)\right| \ge (1 - \beta_x)s$, which implies $v_x = \left|x_0\, c_x^2(s)\right|/s \ge 1 - \beta_x$ and hence (9.1.1) follows.

By Proposition 2.1 we have

$$t^2 \ge \left|x_0\, c_x(t)\right|^2 + r_x^2(t)$$

for all $t \ge 0$. Thus

$$1 \ge \limsup_{t\to\infty} \left|x_0\, c_x(t)\right|^2/t^2 + \beta_x^2.$$

Since $\left|x_0\, c_x(t)\right|/t = \left|x_0\, \sigma_t(s)\right|/s$ for all $0 \le s \le t$ and geodesic segments $x_0\, \sigma_t(s)$ converge to the segment $x_0\, c_x^2(s)$ of the geodesic c_x^2, we have $\limsup_{t\to\infty} \left|x_0\, c_x(t)\right|^2/t^2 = v_x^2$. $\square$

Chapter 2. Fixed points of parabolic isometries

The results discussed in this chapter are well known for locally compact Hadamard spaces. They are obtained here without assumption of locally compactness.

The boundary at infinity of a geodesic space X is, by definition, the set $\partial_\infty X$ of classes of asymptotic geodesic rays in X, where two rays c, c' are called asymptotic if they lie at finite distance one from each other. The point on $\partial_\infty X$ corresponding to nondegenerated ray $c : [0, \infty) \to X$ we denote by $c(\infty)$.

A geodesic space X is called *hyperbolic* if there is $\delta \geq 0$ such that any side of any geodesic triangle in X lies in the δ-neighborhood of two others. One says in this case that the triangle is δ-thin and X is δ-hyperbolic.

10.1. Proposition. *Let X be a hyperbolic Hadamard space. Then for any z, $z' \in \partial_\infty X$, $z \neq z'$, there exists a geodesic $c : \mathbb{R} \to X$, for which $c(\infty) = z$, $c(-\infty) = z'$.*

Proof. Assume that X is δ-hyperbolic. Let $x \in X$ and let $c \in z$, $c' \in z'$ be unit speed geodesic rays emanating from x. The triangle $xc(n)c'(n)$ is δ-thin for every $n \in \mathbb{N}$, hence for some $\delta' \geq \delta$ there is $\sigma_n(0) \in c(n)c'(n)$ at distance at most δ' from x. Then for a maximal subsegment $\sigma_n(-l_n)\sigma_n(l_n) \subset c(n)c'(n)$ of length $2l_n$ symmetric with respect $\sigma_n(0)$ we have $l_n \to \infty$ when $n \to \infty$. The convexity of the distance function implies $\left|\sigma_n(t)\, c\right| \leq \delta'$ for $0 \leq t \leq l_n$ and $\left|\sigma_n(t)\, c'\right| \leq \delta'$ for $-l_n \leq t \leq 0$. Thus for unit speed geodesics $\sigma_n : [-l_n, l_n] \to X$ all the conditions of Theorem 5.1 are satisfied. Hence, the geodesic $c : \mathbb{R} \to X$, the existence of which is ensured by Theorem 5.1, possesses all required properties. $\square$

For an isometry $\gamma : X \to X$ of a metric space we denote by $d_\gamma : X \to \mathbb{R}$, $d_\gamma = \left|x\gamma(x)\right|$, its displacement function. If X is a Hadamard space then d_γ is convex. An isometry γ is called parabolic if the function d_γ does not have minima on X.

10.2. Theorem. *Let $\gamma : X \to X$ be a parabolic isometry of a hyperbolic Hadamard space. Then $\inf d_\gamma = 0$, there exists a unique fixed point $z \in \partial_\infty X$, $\gamma z = z$, and each horoball centered at z is invariant for γ.*

For the proof we need some known facts about Hadamard spaces, which are collected below together with auxiliary assertions.

10.2.1. It is known (see e.g. [BGS]) that $\inf d_\gamma = \lim_n d_{\gamma^n}(x)/n$ for any $x \in X$, in particular, $\inf d_{\gamma^k} = |k| \inf d_\gamma$.

10.2.2. Lemma. *Let X be a δ-hyperbolic Hadamard space, the points x, y, x', $y' \in X$ satisfy $\angle x'xy$, $\angle y'yx \geq \pi/2$ and $|xy| \geq 2\delta$. Then $|x'y'| \geq |xx'| + |yy'| - 4\delta$.*

Proof. Let $z \in xy$, $z' \in x'y'$ be closest points of segments xy, $x'y'$. By the condition we have $|x'z| \geq |xx'|$, $|zy'| \geq |yy'|$. Thus $|x'y'| = |x'z'| + |z'y'| \geq |x'z| - |zz'| + |y'z| - |zz'| = |xx'| + |yy'| - 2\sigma$, where $\sigma = \min\{|zz'| \,|\, z \in xy, z' \in x'y'\}$. It suffices to show that $\sigma \leq 2\delta$. One can assume that $|x'x|$, $|y'y| > 2\delta$. Then there exists $u \in x'y$ with the distance δ from $x'y'$. Since $x'y$ lies in a δ-neighborhood of $x'y' \cup yy'$ we have $|u\,(yy')| < \delta$. By the condition the distances between points of segments xx' and yy' are at least 2δ and $x'y$ lies also in a δ-neighborhood of $xx' \cup xy$. Hence $|u\,(xy)| < \delta$. Therefore, $\sigma \leq 2\delta$. $\square$

10.2.3. Lemma. *Let $\{A_n\}$ be a nonincreasing (i.e., $A_n \subset A_m$ when $n \geq m$) sequence of closed convex subsets of a Hadamard space X with $\cap_n A_n = \emptyset$. Then each sequence $x : \mathbb{N} \to X$, $x_n \in A_n$, leaves bounded sets.*

Proof. If x does not leave bounded sets then passing to a subsequence one can assume that $B = \{x_n \,|\, n \geq 1\}$ is bounded. Let $B_k = \{x_n \,|\, n \geq k\}$, c_k be the center of B_k. Since A_k is closed and convex, we have $c_k \in A_k$. Thus the virtual center $c = \lim_k c_k$ of the sequence B lies in A_k for every $k \geq 1$. This contradicts the fact that $\cap_n A_n = \emptyset$. $\square$

At last we need the following known fact (see [Ba]).

10.2.4. If an isometry $\gamma : X \to X$ has a fixed point $z \in \partial_\infty X$, $\gamma z = z$, and b_z is a corresponding Busemann function then the function $b_z \circ \gamma - b_z$ is constant on X.

Proof of Therem 10.2. First, we prove the uniqueness. By Proposition 10.1 for any distinct z, $z' \in \partial_\infty X$ there exists a geodesic $c : \mathbb{R} \to X$, $c(\infty) = z$, $c(-\infty) = z'$. The set A_c consisting of points of all geodesics c' parallel to c is closed, convex and splits isometrically as $C \times \mathbb{R}$. It follows from hyperbolicity of X that C is bounded. Hence, if γ possesses two fixed points z, $z' \in \partial_\infty X$ then there exists a γ-invariant geodesic c', and the isometry γ cannot be parabolic. Assume that $\inf d_\gamma \neq 0$. Then using 10.2.1 one can assume that $\inf d_\gamma \geq 2\delta$, since γ^k is parabolic for all $k \neq 0$.

The set $C_t = \{x \in X \,|\, d_\gamma(x) \leq t\}$ is closed, convex, γ-invariant and nonempty for $t > \inf d_\gamma$. Let $x_0 \in X$, $x_n \in C_{t_n}$ be the nearest to x_0 point, where $t_n = \inf d_\gamma + 1/n$. Since $\cap_n C_{t_n} = \emptyset$, by lemma 10.2.3 the sequence $\{x_n\}$ leaves bounded sets. Moreover, we have

$$\angle x_0 x_n \gamma(x_n),\ \angle \gamma(x_0)\gamma(x_n)x_n \geq \pi/2,$$

since $\gamma(x_n) \in C_{t_n}$ is the nearest to $\gamma(x_0)$ point, and $\left|x_n\gamma(x_n)\right| \geq 2\delta$. By lemma 5.5 we have

$$\left|x_0\gamma(x_0)\right| \geq 2|x_0x_n| - 4\delta$$

for all n. This contradicts the fact that $\{x_n\}$ leaves bounded sets. Thus $\inf d_\gamma = 0$.

Let $\epsilon > 0$. One can assume that $d_\gamma(x_0) < \epsilon$. By convexity of C_{t_m} we have $\angle x_0 x_m x_n \geq \pi/2$ for $n > m$. Then it follows from hyperbolicity of X that for every $s > 0$ the points $\sigma_n(s) \in x_0x_n$, $\left|x_0\sigma_n(s)\right| = s$, form a Cauchy sequence, since $|x_0x_n| \to \infty$ when $n \to \infty$. Furthermore, $\left|\sigma_n(s)\,\gamma\circ\sigma_n(s)\right| \leq \left|x_0\gamma(x_0)\right| < \epsilon$ for all $s \geq 0$ and all n. Thus x_0x_n converge to a ray $c \in z \in \partial_\infty X$, and the function d_γ is bounded along c, $d_\gamma \circ c(s) < \epsilon$, $s \geq 0$. Hence, $\gamma z = z$.

Let b_z be a Busemann function of z. According to 10.2.4, the function $b_z \circ \gamma - b_z$ is constant on X. It follows from the uniqueness of z and

$$\left|b_z \circ \gamma(x_0) - b_z(x_0)\right| \leq d_\gamma(x_0) < \epsilon$$

that $b_z \circ \gamma \equiv b_z$ since ϵ is arbitrary. This means that γ leaves invariant every horoball centered at z. $\square$

10.3. Remark. In general case a parabolic isometry $\gamma : X \to X$ might not have fixed points in $X \cup \partial_\infty X$. A corresponding example for the Hilbert space $X = l^2(\mathbb{Z})$ can be found in [BH].

Let X be a metric space, $x : \mathbb{N} \to X$ be a sequence leaving bounded sets. For $x_0 \in X$ we put

$$\nu_x(x_0) = \limsup_{m,n \to \infty} \angle \overline{x}_m \overline{x}_0 \overline{x}_n,$$

where $\overline{x}_0\overline{x}_m\overline{x}_n \subset \mathbb{R}^2$ is the comparison triangle for $x_0x_mx_n \subset X$. For $\overline{\alpha} = \angle \overline{x}_m\overline{x}_0\overline{x}_n$ we have

$$\cos\overline{\alpha} = \frac{1}{2}\left(\frac{|x_0x_m|}{|x_0x_n|} + \frac{|x_0x_n|}{|x_0x_m|} - \frac{|x_nx_m|^2}{|x_0x_m||x_0x_n|}\right),$$

hence, $\nu_x(x_0) = \nu_x$ does not depend on x_0. The number $\nu_x \in [0,\pi]$ is said to be *the deviation* of x.

10.4. Lemma. *Let $x : \mathbb{N} \to X$ be a sequence in a Hadamard space X leaving bounded sets, β_x its drift, ν_x its deviation. Then*

$$\beta_x^2 \le 1 - \cos \nu_x .$$

Proof. Fix $\epsilon > 0$, $t \ge 0$. For the angle $\overline{\alpha}$ at $\overline{x}_0$ of the comparison triangle $\overline{x}_0 \overline{x}_m \overline{x}_n \subset \mathbb{R}^2$ for $x_0 x_m x_n \subset X$ we have

$$\overline{\alpha} \le \nu_x + \epsilon$$

for all sufficiently large m, n. Thus for $\sigma_n(t) \in x_0 x_n$, $\left|x_0\, \sigma_n(t)\right| = t$, we have

$$\left|\sigma_m(t)\, \sigma_n(t)\right|^2 \le 2t^2 (1 - \cos \overline{\alpha}) \le 2t^2 \left(1 - \cos(\nu_x + \epsilon)\right).$$

Then

$$(\operatorname{diam} B_k(t))^2 = \sup_{m,n \ge k} \left|\sigma_m(t) \sigma_n(t)\right|^2 \le 2t^2 \left(1 - \cos(\nu_x + \epsilon)\right).$$

By Proposition 2.3 we have $\beta_x^2 \le 1 - \cos \nu_x$. $\square$

10.5. Proposition. *Let $\gamma : X \to X$ be a parabolic isometry of a Hadamard space such that there exists a sequence $x : \mathbb{N} \to X$ leaving bounded sets with $\nu_x < \pi/2$ such that the sequence $d_\gamma \circ x$ is bounded.*

Then there exists $z \in \partial_\infty X$ fixed by γ.

Proof. For the sequence $x' = \gamma \circ x$ we have $|x\, x'| = d_\gamma \circ x < \infty$, in particular, x' leaves bounded sets. Let $x_0 \in X$, $x_0' = \gamma x_0$. Then by Corollary 8.3 its second curves of virtual centers c_x^2, $c_{x'}^2$ emanating from $x_0 = c_x^2(0)$, $x_0' = c_{x'}^2(0)$ are asymptotic geodesics of speed v_x. It follows from Proposition 9.1 and Lemma 10.4 that $v_x \ge 1 - \beta_x \ge 1 - \sqrt{1 - \cos \nu_x} > 0$. Thus $c_x^2 : [0, \infty) \to X$ is a nondegenerate geodesic ray. For $z = c_x^2(\infty) \in \partial_\infty X$ we have $\gamma z = z$ since $c_{x'}^2 = \gamma \circ c_x^2$ by the construction. $\square$

10.6. Remark. Let $\gamma : X \to X$ be a parabolic isometry of a Hadamard space. Then any sequence $x : \mathbb{N} \to X$ minimizing the displacement function d_γ leaves bounded sets as it follows from Lemma 10.2.3. Let $A_m = \left\{x \in X \,|\, d_\gamma(x) \le \inf d_\gamma + 1/n\right\}$, $x_0 \in X$, $x_n \in A_n$ be the nearest point to x_0. Since $A_n \subset A_m$ for $n > m$, we have $\angle \overline{x}_0 \overline{x}_m \overline{x}_n \ge \pi/2$ by convexity of A_n. Thus $\angle \overline{x}_m \overline{x}_0 \overline{x}_n < \pi/2$ for all sufficiently large m, n. Therefore, under the conditions of Proposition 10.5 there exists a sequence $x : \mathbb{N} \to X$ with bounded displacement $d_\gamma \circ x$ leaving bounded sets and having deviation $\nu_x \le \pi/2$ (cf. Remark 10.3).

Chapter 3. Quasi-isometric maps of Hadamard spaces

A map $f : X \to Y$ of metric spaces is said to be (a, b)-quasi-isometric if

$$\frac{1}{a}|xy| - b \le \left|f(x)f(y)\right| \le a|xy| + b$$

for all x, $y \in X$, where $a \ge 1$, $b \ge 0$. If in this case the image $f(X)$ is cobounded in Y then f is called *quasi-isometry.*

In this chapter, we apply second curves of virtual centers of sequences to the study of quasi-isometric maps of Hadamard spaces.

11. Asymptotic limit of a quasi-isometric map.

Let X be a Hadamard space. For $z,\ z' \in \partial_\infty X$ the angle $\alpha_\infty(z,z') \in [0,\pi]$ is defined by

$$\alpha_\infty(z,z') = \lim_{t\to\infty} \overline{\alpha}(t),$$

where $\overline{\alpha}(t)$ is the angle at the vertex $\overline{x}$ of the comparison triangle $\overline{x}\,\overline{c}(s)\overline{c}'(t) \subset \mathbb{R}^2$ for $xc(t)c'(t) \subset X$, where geodesic rays $c,\ c' : [0,\infty) \to X$ radiating from $x = c(0) = c'(0)$ represent the points z, z' correspondingly. This definition does not depend on the choice of $x \in X$, and the function α_∞ is a complete metric on $\partial_\infty X$ geodesic at distances $< \pi$ and of curvature ≤ 1 (see [Ba], [KL]). The metric α_∞ is called *the angle metric* on $\partial_\infty X$, the corresponding topology on $\partial_\infty X$ is called *the metric topology.* The interior metric Td associated with α_∞ is said to be the Tits metric. Therefore, Td and α_∞ coincide at distances $< \pi$.

We put $\partial_\alpha X = (\partial_\infty X, \alpha_\infty)$, $C_\alpha X$ be the Euclidean cone over $\partial_\alpha X$, SX be the set of classes of b-equivalent sequences in X leaving bounded sets.

(*The Euclidean cone* $CZ = Z \times [0,\infty)/Z \times 0$ over a metric space Z has the metric defined by the Cosine theorem

$$|\overline{x}\,\overline{y}|^2 = s^2 + t^2 - 2st\cos\bigl(\min\{|xy|, \pi\}\bigr),$$

where $\overline{x} = (x,s)$, $\overline{y} = (y,t)$. The point $o = Z \times 0$ is called the vertex of CZ. One can identify Z with unit sphere $\bigl\{(z,1)\,|\,z \in Z\bigr\}$ of the cone CZ. The induced interior metric on $\bigl\{(z,1)\,|\,z \in \partial_\infty X\bigr\} \subset C_\alpha X$ coincides with the metric of $C_\alpha X$ at distances $< \pi$).

There is an embedding $e_X : \partial_\infty X \to SX$, which sends a point $z \in \partial_\infty X$ into the class $e_X(z)$ of the sequence $n \mapsto c(n)$, $n \in \mathbb{N}$, where $c \in z$ is a geodesic ray of unit speed.

For $x \in SX$ we put

$$s_X(x) = \bigl(c_x^2(\infty), v_x\bigr) \in C_\infty X = \partial_\infty X \times [0,\infty)/\partial_\infty X \times 0,$$

where $c_x^2 : [0,\infty) \to X$ is the second curve of virtual centers of a sequence $\overline{x} \in x$, v_x its speed. By Theorem 8.1 this definition of the map $s_X : SX \to C_\infty X$ does not depend on the choice $\overline{x} \in x$.

Let $f : X \to Y$ be a quasi-isometric map of Hadamard spaces. We define its *asymptotic limit* as the map

$$s_f = s_Y \circ \overline{f} \circ e_X : \partial_\infty X \to C_\infty Y,$$

where the maps $e_X : \partial_\infty X \to SX$, $s_Y : SY \to C_\infty Y$ are defined above, and the map $\overline{f} : SX \to SY$ is induced by f, $\overline{f}(x) = f \circ x : \mathbb{N} \to Y$. The following example shows that the asymptotic limit may be a constant map (into the vertex of $C_\infty Y$) even in the case of a quasi-isometry with Lipschitz constant $a > 1$ arbitrary close to 1.

11.1. Example. For every $\epsilon > 0$ there exists a $(1+\epsilon, 1)$-quasi-isometry $F : \mathbb{R}^2 \to \mathbb{R}^2$ for which the asymptotic limit $s_F : \partial_\infty \mathbb{R}^2 \to C_\infty \mathbb{R}^2$ is constant, $s_F(z) = o$ for all $z \in \partial_\infty \mathbb{R}^2$, where o is the vertex of $C_\infty \mathbb{R}^2$.

Proof. Let $S_k = \{z \in \mathbb{C}\,|\,|z| = k\}$, $k \in \mathbb{N}$. The set $A = \cup_{k\in\mathbb{N}} S_k$ is a $(1/2+\delta)$-net in $\mathbb{R}^2 = \mathbb{C}$ for every $\delta > 0$. We define a map $f_k : A \to A$ to be identical on S_n for $1 \le n \le k$ and the rotation of S_n by the angle ϵ/k about 0 for $n > k$ (all rotations considered are in the same direction). Then, obviously, there exists a limit (in the compact open topology)

$$F = \lim_{k\to\infty} f_k \circ \cdots \circ f_1,$$

which defines the required quasi-isometry $\mathbb{R}^2 \to \mathbb{R}^2$. The asymptotic limit s_F is constant, since the series $\sum_k \epsilon/k$ diverges and, hence, corresponding second curves of virtual centers have zero speeds. The Lipschitz constant of F is close to 1, since that series diverges sufficiently slowly. $\square$

12. Analytic properties of the asymptotic limit.

12.1. Proposition. *Let $f : X \to Y$ be an (a,b)-quasi-isometric map of Hadamard spaces. Then for its asymptotic limit $s_f : \partial_\alpha X \to C_\alpha Y$ we have*

$$|s_f(z)\, s_f(z')|^2 \le 2a^2 \sin\frac{|zz'|}{2}\left(2a^2\sin\frac{|zz'|}{2} + \beta_y + \beta_{y'}\right)$$

for all $z,\ z' \in \partial_\alpha X$, where $y = f\circ z$, $y' = f\circ z' \in SY$, β_y, $\beta_{y'}$ are drifts of (classes of) sequences $y,\ y'$ correspondingly.

Proof. Fix $x_0 \in X$. Let $c \in z$, $c' \in z'$ be geodesic rays of unit speed emanating from x_0. Put $\alpha = |z\,z'| = \alpha_\infty(z,z')$. Then for each $n \in \mathbb{N}$ we have

$$|c(n)\, c'(n)| \le 2n\sin\frac{\alpha}{2},$$

and for $y_n = f\circ c(n)$, $y'_n = f\circ c'(n)$ we obtain

$$|y_n\, y'_n| \le a\,|c(n)\, c'(n)| + b \le 2an\sin\frac{\alpha}{2} + b.$$

Let σ_n, σ'_n be geodesics of unit speed from $y_0 = f(x_0)$ to y_n, y'_n correspondingly. To be specific assume that $|y_0 y_n| \le |y_0 y'_n|$. Then $t' = t|y_0 y'_n|/|y_0 y_n| \ge t$ and

$$|\sigma_n(t)\, \sigma'_n(t)| \le |\sigma_n(t)\, \sigma'_n(t')| \le t|y_n y'_n|/|y_0 y_n|$$

for all $0 \le t \le |y_0 y_n|$. Since $|y_0 y_n| \ge n/a - b$, we have

$$|\sigma_n(t)\, \sigma'_n(t)| \le \frac{2an\sin\alpha/2 + b}{n/a - b}t,$$

and, hence (see sect. 8),

$$d_{y,y'}(t) \le 2a^2 t\sin\frac{\alpha}{2}.$$

Then for the curves of virtual centers of sequences $y,\ y'$ we obtain using Proposition 8.1.2

$$|c_y(t)c_{y'}(t)|^2/t^2 \le 2a^2\sin\frac{\alpha}{2}\left(2a^2\sin\frac{\alpha}{2} + \beta_y + \beta_{y'}\right).$$

Let σ_t, $\sigma'_t : [0,t] \to Y$ be geodesics between y_0 and $c_y(t)$, $c_{y'}(t)$ correspondingly. Then

$$|\sigma_t(s)\, \sigma'_t(s)| \le s\,|c_y(t)c_{y'}(t)|\,/t$$

for $0 \le s \le t$ and, therefore, for second curves of virtual centers we have

$$\left|c^2_y(s)c^2_{y'}(s)\right|^2/s^2 \le 2a^2\sin\frac{\alpha}{2}\left(2a^2\sin\frac{\alpha}{2} + \beta_y + \beta_{y'}\right).$$

On the other hand,

$$|s_f(z)\, s_f(z')| = \lim_{s\to\infty}\frac{1}{s}\left|c^2_y(s)c^2_{y'}(s)\right|$$

for all $z,\ z' \in \partial_\alpha X$ by defintion of the metric of the cone $C_\alpha Y$ and of the asymptotic limit s_f. $\square$

The estimates from below for $|s_f(z)\, s_f(z')|$ contains the deviation ν_y and is based on the following lemma, which is a version of Lemma 10.4.

12.2. Lemma. *Assume that a sequence $x:\mathbb{N}\to X$ in a Hadamard space X leaves bounded sets and let $x_0\in X$, $a>0$. For $m\in\mathbb{N}$ pick $t_m\geq 0$ with $|t_m-|x_0x_m||\leq a$. Then*

$$\beta_x^2+\limsup_{m\to\infty}|x_m c_x(t_m)|^2/t_m^2\leq 2(1-\cos\nu_x),$$

where c_x is the curve of virtual centers of x with $c_x(0)=x_0$, β_x, ν_x are the drift and the deviation of x.

Proof. Let $\epsilon>0$. For all sufficiently large m, n we have

$$\angle\overline{x}_m\overline{x}_0\overline{x}_n\leq\nu_x+\epsilon,$$

where $\overline{x}_0\overline{x}_m\overline{x}_n\subset\mathbb{R}^2$ is the comparison triangle for $x_0x_mx_n\subset X$. It follows from this and from the condition that

$$r^2\left(x_m,B_k(t_m)\right)\leq 2t_m^2\left(1-\cos(\nu_x+\epsilon)\right)+O(t_m)$$

for all sufficiently large m, k, where $B_k(t)=\{\sigma_n(t)\,|\,n\geq k\}$, σ_n is the unit speed geodesic from x_0 to x_n. Using Proposition 2.1 and letting $k\to\infty$ we obtain

$$r_x^2(t_m)+|x_m c_x(t_m)|^2\leq 2t_m^2\left(1-\cos(\nu_x+\epsilon)\right)+O(t_m),$$

which implies the required estimate. $\square$

12.3. Proposition. *Let $f:X\to Y$ be an (a,b)-quasi-isometric map of Hadamard spaces. Then for its asymptotic limit $s_f:\partial_\alpha X\to C_\alpha Y$ we have*

$$\left|s_f(z)\,s_f(z')\right|\geq\frac{2}{a^2}\sin\frac{|z\,z'|}{2}-2\sin\frac{\nu_y}{2}-2\sin\frac{\nu_{y'}}{2}$$

for all z, $z'\in\partial_\alpha X$, where $y=f\circ z$, $y'=f\circ z'\in SY$, ν_y, $\nu_{y'}$ are deviations of (classes of) y, y' correspondingly.

Proof. Fix $x_0\in X$. Let $c\in z$, $c'\in z'$ be unit speed geodesic rays emanating from x_0, $y_m=f\circ c(m)$, $y_n'=f\circ c'(n)$, m, $n\in\mathbb{N}$. By the definition of the map s_f and the metric of the cone $C_\alpha Y$ we have $|s_f(z)\,s_f(z')|=\lim_{s\to\infty}\left|c_y^2(s)\,c_{y'}^2(s)\right|/s$, where c_y^2, $c_{y'}^2$ are second curves of virtual centers of sequences y, y' correspondingly starting at $y_0=f(x_0)$. By lemma 7.1.2, the last limit coincides with $\lim_{t\to\infty}|c_y(t)\,c_{y'}(t)|/t$, where c_y, $c_{y'}$ are curves of virtual centers of y, y' with $c_y(0)=y_0=c_{y'}(0)$.

For $t\geq 0$ let $m=m(t)\in\mathbb{N}$ satisfies the condition $\left||y_0\,y_m|-t\right|\leq\left||y_0\,y_k|-t\right|$ for all $k\in\mathbb{N}$. Likewise we define $n=n(t)\in\mathbb{N}$ for the sequence y'. Then, obviously, $\left||y_0\,y_m|-t\right|$, $\left||y_0\,y_n'|-t\right|\leq a$.

By lemma 12.2,

$$\limsup_{t\to\infty}\left|y_m\,c_y(t)\right|/t\leq 2\sin\frac{\nu_y}{2}$$

and, similarly,

$$\limsup_{t\to\infty}\left|y_n'\,c_{y'}(t)\right|/t\leq 2\sin\frac{\nu_{y'}}{2}.$$

Since $\left|c_y(t)\,c_{y'}(t)\right|\geq|y_m\,y_n'|-\left|y_m\,c_y(t)\right|-\left|y_n'\,c_{y'}(t)\right|$, we obtain using the estimates above

$$\left|s_f(z)\,s_f(z')\right|\geq\liminf_{t\to\infty}|y_m\,y_n'|/t-2\sin\frac{\nu_y}{2}-2\sin\frac{\nu_{y'}}{2}.$$

It remains to estimate from below the first term of the right hand side of this inequality. We have $|y_m\, y'_n| \ge \big|c(m)\, c'(n)\big|/a - b$ and $t \le |y_0\, y_m| + a \le am + a + b$, since f is (a,b)-quasi-isometric. Thus assuming without loss of generality that $m \le n$ we obtain

$$\begin{aligned}\liminf_{t\to\infty} |y_m\, y'_n|/t &\ge \frac{1}{a^2} \lim_{m\to\infty} \big|c(m)\, c'(m)\big|/m \\ &= \frac{2}{a^2} \sin \frac{|z\, z'|}{2},\end{aligned}$$

which completes the proof. □

12.4. Corollary. *Let $f : X \to Y$ be a (a,b)-quasi-isometric map of Hadamard spaces. Assume that for $z,\ z' \in \partial_\alpha X$ sequences $y = f \circ z$, $y' = f \circ z'$ have zero deviations, ν_y, $\nu_{y'} = 0$. Then for its asymptotic limit $s_f : \partial_\alpha X \to C_\alpha Y$ we have $s_f(z)$, $s_f(z') \in \partial_\alpha Y$ and*

$$\frac{2}{a^2} \sin \frac{|z\, z'|}{2} \le |s_f(z)\, s_f(z')| \le 2a^2 \sin \frac{|z\, z'|}{2}.$$

Proof. By the condition and Lemma 10.4 we have $\beta_y = \beta_{y'} = 0$, thus by Proposition 9.1 second curves of virtual centers of y, y' are unit speed geodesics, i.e., $s_f(z)$, $s_f(z') \in \partial_\alpha Y$. The estimate from below for $|s_f(z)\, s_f(z')|$ follows from Proposition 12.3. The estimate from above follows from Proposition 12.1. □

13. Relation between asymptotic cone $\mathrm{Con}_\infty X$ and Tits geometry.

Asymptotic cone $\mathrm{Con}_\infty X$ of a metric space X is defined (see [Gr2]) as the set of sequences $x : \mathbb{N} \to X$ with $|x_0 x_n| \le \mathrm{const}_{\,x} \cdot n$ for some $x_0 \in X$, factorized by relation $\mathrm{dist}_{\,\omega} = 0$, where $\mathrm{dist}_{\,\omega}(x, x') = \lim_\omega |x_n x'_n|/n$, ω is a non-principal ulterfilter on $\mathbb{N}$, is a metric on $\mathrm{Con}_\infty X$. This notion has important applications (see e.g. [KL]), absorbing detailed information about asymptotic properties of X. There is an obvious embedding $e_X : \partial_\infty X \to \mathrm{Con}_\infty X$. Using Proposition 12.1 we obtain (cf. [Gr2, 3.C''_1]).

13.1. Proposition. *For any Hadamard space X the map*

$$s_X : \mathrm{Con}_\infty X \to C_\alpha X$$

of the asymptotic cone $\mathrm{Con}_\infty X$ into the Euclidean cone over $\partial_\alpha X$ satisfies the Hölder condition with exponent 1/2, and $e_X \circ s_X : \partial_\alpha X \to \partial_\alpha X$ is identity map. □

14. The deviation of a quasi-isometric map.

We introduce here a characteristic of a quasi-isometric map $f : X \to Y$ of Hadamard spaces called the deviation ν_f, the vanishing of which ensures that the asymptotic limit $s_f : \partial_\infty X \to C_\infty Y$ is continuous w.r.t. standard topologies and $s_f(\partial_\infty X) \subset \partial_\infty Y$.

Let X be a Hadamard space. Recall that sets

$$U_R(z) = \left\{ z' \in \partial_{x_0} X \,\middle|\, \big|z(R)\, z'(R)\big| < 1 \right\},$$

$R > 0$, form a base of neighborhoods of $z \in \partial_\infty X$ in the standard topology on $\partial_\infty X$, where $x_0 \in X$ and $\partial_\infty X$ is canonically identified with the set $\partial_{x_0} X$ of geodesic rays of unit speed radiating from x_0.

Given points y_0, y, y' of a metric space Y, y, $y' \ne y_0$, we put $\nu(y_0, y, y') = \angle \overline{y}\, \overline{y}_0 \overline{y}'$, where $\overline{y}_0 \overline{y}\, \overline{y}' \subset \mathbb{R}^2$ is the comparison triangle for $y_0\, y\, y'$.

The deviation of a quasi-isometric map $f : X \to Y$ at a point $z \in \partial_\infty X$ is defined as

$$\nu_f(z) = \lim_{R\to\infty} \sup_{\substack{z'\in U_R(z)\\ m,n\geq R}} \nu\big(y_0, y'(m), y'(n)\big),$$

where $y_0 = f(x_0)$, $y'(m) = f \circ z'(m)$, $x_0 \in X$. It is not difficult to see that this definition does not depend on the choice of x_0.

It follows from definition that the deviation $\nu_f : \partial_\infty X \to [0, \pi]$ is semi-continuous from above (in the standard topology), i.e., if $z_k \to z \in \partial_\infty X$ then

$$\limsup_k \nu_f(z_k) \leq \nu_f(z).$$

Furthermore, we obviously have $\nu_f(z) \geq \nu_{f\circ z}$ for any $z \in \partial_\infty X$. In particular, if $\nu_f(z) = 0$ for a quasi-isometric map of Hadamard spaces, $z \in \partial_\infty X$, then $s_f(z) \in \partial_\infty Y$ and a geodesic representing $s_f(z') \in C_\infty Y$ has speed as close to 1 as we like for $z' \in \partial_\infty X$ sufficiently close to z in the standard topology. We will see below that the equality $\nu_{f\circ z'} = 0$ even for all $z' \in \partial_\infty X$ is not sufficient for vanishing of ν_f at some $z \in \partial_\infty X$.

14.1. Theorem. *Let $f : X \to Y$ be a quasi-isometric map of Hadamard spaces having zero deviation at a point $z \in \partial_\infty X$, $\nu_f(z) = 0$. Then its asymptotic limit $s_f : \partial_\infty X \to C_\infty Y$ is continuous at z in the standard topologies on $\partial_\infty X$, $C_\infty Y$.*

Proof. Fix $x_0 \in X$ and identify $\partial_\infty X$ with the set $\partial_{x_0} X$ of unit speed geodesic rays radiating from x_0. Likewise, we identify $\partial_\infty Y$ with $\partial_{y_0} Y$, where $y_0 = f(x_0)$. For $w = s_f(z) \in \partial_\infty Y$ (see remark above), $R' > 0$ let

$$U_{R'}(w) = \big\{w' \in \partial_\infty Y \,|\, \big|w'(R')\, w(R')\big| < 1\big\}$$

be a neighborhood of w in $\partial_\infty Y$. For sufficiently large $R > 0$ a geodesic representing $s_f(z')$, $z' \in U_R(z)$, has speed arbitrary close to 1. Thus $s_f(U_R(z))$ lies in an arbitrary small neighborhood of $\partial_\infty Y$ in the cone $C_\infty Y$. In what follows we ignore this distinction and suppose for simplicity that $s_f(U_R(z)) \subset \partial_\infty Y$. It suffices to find $R > 0$ such that $s_f(U_R(z)) \subset U_{R'}(w)$.

Let $0 < \epsilon < (4R')^{-1}$. By the definition of the deviation ν_f and the condition $\nu_f(z) = 0$ there exists $R_0 > 0$ such that for all m, $n \geq R_0$ and all $z' \in U_{R_0}(z)$ we have

$$\angle \overline{y}'_n \overline{y}_0 \overline{y}'_m < \epsilon,$$

where $y'_n = f \circ z'(n)$, $\overline{y}_0 \overline{y}'_m \overline{y}'_n \subset \mathbb{R}^2$ is the comparison triangle for $y_0 y'_m y'_n \subset Y$. Recalling the definition of s_f we can suppose that for $m \geq R_0$

$$\angle \overline{w}'(R')\, \overline{y}_0\, \overline{y}'_m \leq \epsilon$$

holds for all $z' \in U_{R_0}(z)$, where $w' = s_f(z')$.

Fix $m \geq R_0$ and put $R = m$. Recall that $z' \in U_R(z)$ iff $|z(m)\, z'(m)| < 1$. Thus for $y_m = f \circ z(m)$ we have $|y_m\, y'_m| < a + b$ for all $z' \in U_R(z)$ (we assume that f is an (a, b)-quasi-isometric map). Since $|y_0\, y_m|$, $|y_0\, y'_m| \geq m/a - b$, the comparison with $\mathbb{R}^2$ gives $\angle \overline{y}_m \overline{y}_0 \overline{y}'_m \leq \overline{\alpha}$, where $\overline{\alpha}$ is the angle at the vertex of isosceles triangle in $\mathbb{R}^2$ with base $a + b$ and sides $m/a - b$,

$$\overline{\alpha} \leq \frac{\pi}{2} \frac{a(a+b)}{m - ab}.$$

Let $\sigma_m(R') \in y_0 y_m$, $\sigma'_m(R') \in y_0 y'_m$ be points at distance R' from y_0 (we assume that $R_0/a - b > R'$). Then

$$\begin{aligned}|w(R')\, w'(R')| &\leq |w(R')\, \sigma_m(R')| + |\sigma_m(R')\, \sigma'_m(R')| + |\sigma'_m(R')\, w'(R')| \\ &\leq (2\epsilon + \overline{\alpha})R' < 1/2 + \overline{\alpha} R'\end{aligned}$$

for all $z' \in U_R(z)$. When $m > ab + \pi a(a + b)R'$ we obtain $s_f(U_R(z)) \subset U_{R'}(w)$. $\square$

14.2. Examples. (1) If Y is a hyperbolic Hadamard space, then for every quasi-isometric map $f : X \to Y$ we have $\nu_f(z) = 0$ for all $z \in \partial_\infty X$ by the quasi-isometric Morse lemma.

(2) For a (1, b)-quasi-isometric map $f : X \to Y$ we, obviously, have $\nu_f(z) = 0$ for all $z \in \partial_\infty X$.

(3) For the $(1+\epsilon, 1)$-quasi-isometry $F : \mathbb{R}^2 \to \mathbb{R}^2$ of example 11.1 we, obviously, have $\nu_F(z) = \pi$ for all $z \in \partial_\infty \mathbb{R}^2$.

(4) For $\lambda \in \mathbb{R}$, $\lambda \neq 0$, the product

$$(a, n) \cdot (a', n') = (a + a', n + e^{-\lambda a} n')$$

defines the structure of a solvable group Lie $G(\lambda)$ on $\mathbb{R}^2$. The metric $ds^2 = da^2 + e^{2\lambda a} dn^2$ on $G(\lambda)$ is left invariant and has the constant Gaussian curvature $K = -\lambda^2$. With respect to this metric the subgroup $A = \{(a, 0) \,|\, a \in \mathbb{R}\}$ is a geodesic, and the normal subgroup $N = \big[G(\lambda), G(\lambda)\big] = \{(0, n) \,|\, n \in \mathbb{R}\}$ is a horosphere centered at $A(-\infty)$.

This allows to introduce on $X = \mathbb{R} \times H^2$ the structure of a solvable group $\mathbb{R} \times G(1)$ by fixing $z_0 \in \partial_\infty H^2 \subset \partial_\infty X$ and putting $H^2 \equiv A \cdot N$, where $A(-\infty) = z_0$.

For $\alpha \in [-\pi/2, \pi/2]$ the embedding $f_\alpha : G(\lambda) \to \mathbb{R} \times G(1)$,

$$f_\alpha(a, n) = (a \sin \alpha, a \cos \alpha, n),$$

is a homomorphism when $\lambda = \cos \alpha$, and, hence, it is an isometric embedding of a plane of constant curvature $-\cos^2 \alpha$ into $\mathbb{R} \times H^2$. It is not difficult to see that its image $G_\alpha = f_\alpha\big(G(\cos \alpha)\big)$ is a horosphere in $\mathbb{R} \times H^2$. Remarkable, that f_α is a quasi-isometric embedding for $-\pi/2 < \alpha < \pi/2$ (see [Gr2, 3.D_2]). In this case the deviation ν_{f_α} of its asymptotic limit $s_\alpha = s_{f_\alpha} : \partial_\infty G(\lambda) \to \partial_\infty X$ vanishes at any point $z \in \partial_\infty G(\lambda)$ distinct from z_0 and s_α is discontinuous in the standard topologies at z_0 when $\alpha \neq 0$, in particular, $\nu_{f_\alpha}(z_0) \neq 0$ in this case. Note that the deviation $\nu_{f_\alpha \circ z}$ of the sequence $f_\alpha \circ z$ equals zero for any $z \in \partial_\infty G(\lambda)$.

Similar properties possess (nonsingular) horospheres in products $H^2 \times \cdots \times H^2$ of hyperbolic planes.

15. Equivariant quasi-isometries.

Suppose that a group Γ acts by isometries properly discontinuous and cocompactly on a Hadamard space X. In this case, one says that (X, Γ) is a cocompact Hadamard space.

Assume that (X, Γ), (Y, Δ) are cocompact Hadamard spaces. Then any isomorphism $\overline{f} : \Gamma \to \Delta$ induces a quasi-isometry $f : X \to Y$ equivariant with respect to the actions of Γ on X and Δ on Y, i.e., $f(\gamma(x)) = \overline{f}(\gamma) f(x)$ for all $x \in X$, $\gamma \in \Gamma$. In this case we say about equivariant quasi-isometry $f : (X, \Gamma) \to (Y, \Delta)$.

A cocompact Hadamard space (X, Γ) has rank 1 (in the sense of Ballmann-Brin) if there is an axial isometry $\gamma \in \Gamma$ having an axis (invariant geodesic) which does not bound any halfplane in X (by definition, an axial isometry $\gamma : X \to X$ possesses an invariant geodesic acting on it as a shift). According [BB, theorem 5.9] this property is a quasi-isometric invariant of Γ, in particular, if $f : (X, \Gamma) \to (Y, \Delta)$ is an equivariant quasi-isometry then (Y, Δ) has rank 1 as well.

15.1. Theorem. *Let $f : (X, \Gamma) \to (Y, \Delta)$ be an equivariant quasi-isometry of cocompact Hadamard spaces of rank 1, whose deviation ν_f vanishes at least at one point $z \in \partial_\infty X$, $\nu_f(z) = 0$. Then the asymptotic limit s_f is an equivariant homeomorphism $\partial_\infty X \to \partial_\infty Y$ for the standard topologies and biLipschitz homeomorphism $\partial_\alpha X \to \partial_\alpha Y$ for the angle metrics and, hence, for the Tits metrics on $\partial_\infty X$, $\partial_\infty Y$.*

Proof. One can assume that $\partial_\infty X$ contains more than two points, since else the claim is obvious. It follows from [BB, theorem 4.1] that the action of Γ on $\partial_\infty X$ in minimal, i.e., any nonempty, closed, Γ-invariant subset $A \subset \partial_\infty X$ coincides with $\partial_\infty X$. Since the deviation $\nu_f : \partial_\infty X \to [0,\pi]$ is semi-continuous from above and Γ-invariant, it is constant on $\partial_\infty X$. Therefore, $\nu_f \equiv 0$. By Theorem 14.1, asymptotic limit $s_f : \partial_\infty X \to \partial_\infty Y$ is continuous. By the condition, spaces X, Y are locally compact, hence, their boundary at infinity $\partial_\infty X$, $\partial_\infty Y$ are compact in the standard topology. Since the last is Hausdorff, the image $s_f(\partial_\infty X)$ is nonempty, closed, Γ-invariant subset in $\partial_\infty Y$, consequently, $s_f(\partial_\infty X) = \partial_\infty Y$, and s_f is surjective. Injectivity of s_f follows from Proposition 12.3. This proves that $s_f : \partial_\infty X \to \partial_\infty Y$ is an equivariant homeomorphism with respect to the standard topologies. According to Corollary 12.4, $s_f : \partial_\alpha X \to \partial_\alpha Y$ is a biLipschitz homeomorphism w.r.t. the angle metrics. $\square$

We give several examples showing that no one of conditions of Theorem 15.1 can be dropped.

15.2. Examples. (1) Let $X = \mathbb{R} \times H^2$ be the universal cover of the metric product $X/\Gamma = S^1 \times F_g$, where F_g is a closed orientable surface of genus $g \geq 2$ with metric of constant curvature -1. Here $\Gamma = \mathbb{Z} \oplus \Gamma_g$, $\Gamma_g = \pi_1(F_g)$. Put $(Y,\Delta) = (X,\Gamma)$ and define an isomorphism $\overline{f} : \Gamma \to \Delta$ taking $\overline{f}(0,a_1) = (1,a_1)$ and choosing $\overline{f}$ to be identity on the remaining generators $(1,e)$, $(0,a_i)$, $(0,b_j)$ of the group Γ, where $\Gamma_g = \langle a_i, b_i \mid \prod_{i=1}^g [a_i,b_i] \rangle$ is the standard corepresentation, e the unit of Γ_g. The isomorphism $\overline{f}$ induces an equivariant (a,b)-quasi-isometry $f : (X,\Gamma) \to (Y,\Delta)$ with $a = \sqrt{l_0^2 + l_1^2}/l_1$, where l_0 is the length of the factor S^1, l_1 the length of the closed geodesic on F_g representing the conjugate class of $a_1 \in \Gamma_g$.

In this example the deviation $\nu_{f\circ z}$ of a sequence $f \circ z \in SY$, $z \in \partial_\infty X$ takes any value from the interval $[0,\alpha]$, where $\cos(\alpha/2) = 1/a$. At the same time $\nu_{f\circ z} = 0$ for almost all $z \in \partial_\infty X$ (one can see this using symbolic dynamics of Γ_g). The deviation ν_f of the quasi-isometry f vanishes only at the points z_0, $z_1 \in \partial_\infty X$ representing ends of the factor $\mathbb{R}$. The asymptotic limit $s_f : \partial_\infty X \to C_\infty Y$ is discontinuous for the standard topologies at the remaining points and according to Proposition 12.1 satisfies the Hölder condition with exponent $1/2$ for the angle metrics.

This example seems to be somewhat artificial since there is an equivariant isometry $(X,\Gamma) \to (Y,\Delta)$. However slightly modifying it we arrive at the following example, where $\partial_\infty X$, $\partial_\infty Y$ are not equivariantly homeomorphic for any virtual isomorphism $\Gamma \to \Delta$ (i.e., an isomorphism between subgroups of finite index).

(2) Let $X = Y$, $\Gamma = \Delta$ and the action of the group Γ on $X = \mathbb{R} \times H^2$ be as in example (1), l_0 be the length of the factor S^1 in the decomposition $X/\Gamma = S^1 \times F_g$. Fix $c > 0$ with irrational c/l_0 and define an action of the group Δ on $Y = \mathbb{R} \times H^2$ putting $(0,a_1)(t,u) = \big(t + c, a_1(u)\big)$ for the generator $(0,a_1)$ of Δ and $(t,u) \in Y$, and assuming that remaining generators $(1,e)$, $(0,a_i)$, $(0,b_j)$ act on Y so as on X being elements of the group Γ, i.e., $(1,e)(t,u) = (t + l_0, u)$, $(0,a_i)(t,u) = \big(t, a_i(u)\big)$, $i = 2,\dots,g$, $(0,b_j)(t,u) = \big(t, b_j(u)\big)$, $j = 1,\dots,g$ (in the case $c = l_0$ we get the action of Γ on Y via the isomorphism $\overline{f} : \Gamma \to \Delta$ from previous example).

This time for any virtual isomorphism $\overline{f} : \Gamma \to \Delta$ all the properties of example (1) hold. In particular, for the cocompact Hadamard space (X,Γ) and for any $\epsilon > 0$ there exists an equivariant $(1+\epsilon, b)$-quasi-isometry $\overline{f} : \Gamma \to \Delta$ for which the asymptotic limit $s_f : \partial_\infty X \to C_\infty Y$ is discontinuous everywhere on $\partial_\infty X$ excepting two points, and the spaces $\partial_\infty X$, $\partial_\infty Y$ are not equivariantly homeomorphic for any virtual isomorphism $\Gamma \to \Delta$.

In other terms example (2) is based on the following effect. While the space $X/\Gamma = S^1 \times F_g$ is the metric product and, hence, the holonomy of the factor S^1 along any loop in F_g is trivial, the space Y/Δ being diffeomorphic to $S^1 \times F_g$ metrically splits only locally, and there are loops in the base F_g the holonomy of S^1 along which is an irrational multiple of identity.

(3) It is not difficult to construct examples with similar properties for the case when F_g is

an compact oriented surface of genus $g \geq 1$ with metric of curvature -1 and nonempty geodesic boundary; if the boundary has at least three component one can take $g = 0$.

In examples (1)–(3) all considered cocompact spaces are not spaces of rank 1. Using (3) we construct the following example of equivariantly homeomorphic cocompact Hadamard spaces of rank 1 (in the sense of Ballmann-Brin), whose boundary at infinity are not equivariantly homeomorphic.

(4) Let M be a 3-dimensional orientable graph-manifold consisting of two blocks M_0, M_1 diffeomorphic to the product $S^1 \times F$, where F is 2-torus with two holes. The blocks M_0, M_1 are glued by a diffeomorphism $h : \partial M_0 \to \partial M_1$, which switches S^1-generators on each component $\partial M_0^{\pm} = S^1 \times S^1$ of the boundary ∂M_0, $M = M_0 \cup_h M_1$. A metric on M for which the universal cover of each block M_i is isometric to the metric product $\mathbb{R} \times A_i$, where A_i is a convex subset of the hyperbolic plane H^2 bounded by geodesic lines, $i = 0, 1$, is called a *geometrization* of M.

According to [BK] any geometrization of M is characterized by the condition

$$\cos\omega_+ + \cos\omega_- = 0,$$

where $\omega_\pm$ is the angle between (appropriately oriented) S^1-factors of blocks M_0, M_1 on the gluing torus $T^2_\pm$. Therefore, in the case $\omega_+ = \omega_- = \pi/2$ the block M_i is isometric to the metric product $S^1_i \times F_i$, where S^1_i is the circle of length l_i, and the metric on the torus with two holes F_i has constant curvature -1 and the geodesic boundary, each component of which has length l_{i+1}, $i + 1 = 0, 1 \bmod 2$.

The choice $\omega_+ + \phi = \omega_- - \phi = \pi/2$ with $0 < \phi < \pi/2$ implies an effect described at the end of example (2), namely, so geometrized block M_i metrically splits only locally, and there are nontrivial holonomies of the factor S^1, in particular, along loops representing components of ∂F_i. One can assume that the holonomy group $\pi_1(F_i) \to SO(2)$ is generated by boundary components and under an appropriate choice of ϕ it has infinite order.

Now let (X, Γ) be the cocompact Hadamard space, where $\Gamma = \pi_1(M)$, X/Γ be the geometrized manifold M with $\omega_+ = \omega_- = \pi/2$, $\Delta = \Gamma$, Y/Δ be the geometrized manifold M with $\omega_+ + \phi = \omega_- - \phi = \pi/2$, where $0 < \phi < \pi/2$ is choosen as above giving an infinite holonomy group on blocks M_0, M_1.

Then the identity map $\overline{f} : \Gamma \to \Delta$ induces an equivariant quasi-isometry $f : (X, \Gamma) \to (Y, \Delta)$ for which the situation of example (2) takes place on each block M_0, M_1. This time the spaces (X, Γ), (Y, Δ) are of rank 1, thus the asymptotic limit $s_f : \partial_\infty X \to C_\infty Y$ is discontinuous everywhere on $\partial_\infty X$, the deviation $\nu_f = \mathrm{const} \neq 0$ and, moreover, the spaces $\partial_\infty X$, $\partial_\infty Y$ are not equivariantly homeomorphic for any virtual isomorphism $\Gamma \to \Delta$. As above, for any $\epsilon > 0$ one can choose $0 < \phi < \pi/2$ such that f is an $(1 + \epsilon, b)$-quasi-isometry. Furthermore, it is easy to see that $\partial_\alpha X$ and $\partial_\alpha Y$ are not isometric for the angle metrics and Tits metrics.

Results of [BK] shows that this situation is generic for graph-manifolds. It is naturally to think that this is true for the whole class of cocompact Hadamard spaces of rank 1.

Addendum. This work has already been completed when the author was informed that examples similar to those from 15.2 were independently found by B. Kleiner and C. Croke with an additional property that it does not exist any equivariant homeomorphism $\partial_\infty X \to \partial_\infty Y$ (i.e. one does not suppose that a homeomorphism $\partial_\infty X \to \partial_\infty Y$ has to be induced by an equivariant quasi-isometry $X \to Y$). Note that for examples 15.2(1)–(3) the identity map $\partial_\infty X \to \partial_\infty Y = \partial_\infty X$ is an equivariant homeomorphism (of course, not induced by any quasi-isometry $X \to Y$). The author is grateful to B. Kleiner who attracted his attention to this effect.

Let us show that the spaces $\partial_\infty X$, $\partial_\infty Y$ from example 15.2(4) are not equivariantly homeomorphic also in that more strong sense. To this end we need only a part of the Tits geometry on $\partial_\infty X$, $\partial_\infty Y$. Namely, we take some universal cover $X_i \subset X$ of M_i such that $X_0 \cap X_1 = E$ is a 2-flat covering one of the tori, along which the blocks M_0, M_1 are glued. Then $\partial_\infty X_i$ is the suspension over the Cantor set $\partial_\infty A_i$, i.e. the graph with two vertices z_i^0, z_i^1 connected by edges of length π, which one-to-one correspond to points of $\partial_\infty A_i$. There is a countable collection of pairs of edges in $\partial_\infty X_i$, which correspond to the boundary flats of X_i. Since X_i is convex in X, we have $\partial_\infty X_i \subset \partial_\infty X$. The pairs of points z_0^0, z_0^1 and z_1^0, z_1^1 separate each other on the circle $\partial_\infty E \subset \partial_\infty X$, and the condition $\omega_+ = \omega_- = \pi/2$ implies $|z_1^j z_0^0| = |z_1^j z_0^1| = \pi/2$, $j = 0, 1$, in the Tits metric.

Let $\gamma \in \Gamma$ be an element of the form $(1, \overline{\gamma}) \in \pi_1(M_0) \subset \Gamma$, where $\overline{\gamma} \in \pi_1(F)$, $\overline{\gamma} \neq 1$, is a nonperipheral element. The isometry (in the Tits metric) $\gamma : \partial_\infty X \to \partial_\infty X$ leaves invariant $\partial_\infty X_0$ and acts identically on the circle $\partial_\infty E_\gamma \subset \partial_\infty X_0$, which is the boundary of the flat $E_\gamma \subset X_0$ of the form $\mathbb{R} \times l_\gamma$, where $l_\gamma \subset A_0$ is the axis of $\overline{\gamma}$. Its ends $l_\gamma(\pm\infty) \in \partial_\infty X_0$ lie at the distance $\pi/2$ from z_0^j, $j = 0, 1$. Thus the sequences $\{\gamma^n(z_1^0)\}$, $\{\gamma^n(z_1^1)\} \subset \partial_\infty X$ converge in the standard topology to the same point $l_\gamma(\infty)$ as $n \to \infty$.

On the other hand, the condition $\omega_+ + \phi = \omega_- - \phi = \pi/2$ with $0 < \phi < \pi/2$ implies that the corresponding sequences in $\partial_\infty Y$ converge to *distinct* points. Note that the indicated part of the Tits geometry on $\partial_\infty X$, $\partial_\infty Y$ can be reconstructed from the action of groups Γ, Δ as corresponding fixed points sets of appropriate elements from Γ, Δ. Since the groups $\Gamma = \Delta$ act by isometries in the Tits metrics on $\partial_\infty X$, $\partial_\infty Y$, there exists no homeomorphism $\partial_\infty X \to \partial_\infty Y$ conjugating these actions.

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