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Ergodic Theory on SL(n), Diophantine Approximations and Anomalies in the Lattice Point Problem

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In the present paper we study the appearance of anomalously small, particularly logarithmically small, errors in the lattice point problem for polyhedra in $\mathbb{R}^n$. First examples of such anomalies were given in the author's papers [Skr 1–3]. Here we show that the lattice point counting for a polyhedron can be reduced to simultaneous diophantine approximations for system of linear forms associated with the set of faces of the polyhedron. Next we interpret these diophantine approximations in terms of certain flows on the homogeneous space of n-dimensional unimodular lattices $\mathcal{L}_n = \mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$. Thereby we can apply the methods of ergodic theory to study the behavior of error for a polyhedron. One of our main results, given on this way, can be formulated as follows.

Let tP be the dilatation of a given polyhedron $P \subset \mathbb{R}^n$ by a factor $t > 0$, let $N(tP,\Gamma)$ denote the number of points of an unimodular lattice $\Gamma \in \mathcal{L}_n$ lying inside tP. Then for almost all $\Gamma \in \mathcal{L}_n$ (with respect to the unique invariant probabilistic measure on the homogeneous space $\mathcal{L}_n$) one has the following asymptotic formula as $t \to \infty$

$$N(tP,\Gamma) = t^n \operatorname{vol} P + 0\left((\log t)^{n-1+\varepsilon}\right)$$

with arbitrary small $\varepsilon > 0$.

The present paper consists of Introduction (§ 1–2), Chapter I (§ 3–6), Chapter II (§ 7–12) and two Appendices (§ 13–14).

In Introduction we discuss (§ 1) known results on anomalously small errors (Theorems 1.1–1.2) and state (§ 2) main results of the present paper (Theorems 2.1–2.3).

In Chapter I we give prrofs of the main results in the base of a new general bound for the number of lattice points in a polyhedron (Theorem 6.1 in § 6). This general bound involves certain diophantine and ergodic characteristics, which are defined and studied in § 3–5.

In Chapter II we prove Theorem 6.1 (§ 12). Here we express the error for a polyhedron via Fourie series of a special shape (§ 11). The bounds for these special Fourie series are given in § 9–10 on the base of suitable multidimensional modification of continued fractions, that are called dyadic minima of a lattice (§ 7–8).

Appendices contain necessary facts on invariant integration over the spaces of lattices (§ 13) and auxillary bounds for Fourier integrals (§ 14).

More detailed organization of the paper is given in the following.

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§ 1. Logarithmically small errors in the lattice point problem

Let $\mathbb{R}^n = \{X : X = {}^{\mathrm{T}}(x_1, \dots, x_n),\ x_j \in \mathbb{R},\ 1 \le j \le n\}$ be the n-dimensional Euclidean space of the column vectors. Henceforward we denote by ${}^{\mathrm{T}}(\cdot)$ the transposition of the matrix $(\cdot)$. The standard metric and inner product in $\mathbb{R}^n$ are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ correspondingly.

Let $\Gamma \subset \mathbb{R}^n$ be a lattice, i.e., a discrete subgroup of the group of translation of $\mathbb{R}^n$ with conpact fundamental set $\mathcal{F}(\Gamma) = \mathbb{R}^n/\Gamma$. We denote by $\det\Gamma$ the determinant of Γ, $\det\Gamma = \operatorname{vol}\mathbb{R}^n/\Gamma$. A lattice Γ is called to be unimodular if $\det\Gamma = 1$. Let $\mathbb{Z}^n$ be the integer coordinate lattice. We have

$$\Gamma = \alpha\mathbb{Z}^n, \tag{1.1}$$

where the matrix $\alpha \in GL(n,\mathbb{R})$ is determined by the lattice Γ up to the right-action of the discret subgroup $\mathrm{SL}(n,\mathbb{Z}) \subset \mathrm{SL}(n,\mathbb{R})$. In the case of the unimodular lattice Γ the matrix $\alpha \in \mathrm{SL}(n,\mathbb{R})$. Thereby the set of unimodular lattice in $\mathbb{R}^n$ can be identified with the following homogeneous space

$$\mathcal{L}_n = \mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z}). \tag{1.2}$$

The space $\mathcal{L}_n$ admits the unique $\mathrm{SL}(n,\mathbb{R})$-invariant (probabilistic) measure $\mu_{\mathcal{L}}$ normalized by the condition

$$\mu_{\mathcal{L}}(\mathcal{L}_n) = 1. \tag{1.3}$$

Let $\Gamma^{\perp}$ be the lattice dual to Γ. Recall that $\Gamma^{\perp}$ consists of all vectors $\gamma^{\perp}$ such that the inner product $\langle\gamma^{\perp},\gamma\rangle$ belongs to $\mathbb{Z}$ for each $\gamma \in \Gamma$. Moreover, if Γ is given by (1.1), then

$$\Gamma^{\perp} = {}^{\mathrm{T}}\alpha^{-1}\mathbb{Z}^n. \tag{1.4}$$

For details we refer the reader to the books [Cas 1] by Cassels and [*GL*] by Gruber and Lekkerkerker. Theory of homogeneous spaces $\mathcal{L}_n$ is given in details in the book [R, Chp. 10] by Raghunathan.

Let $\mathcal{O} \subset \mathbb{R}^n$ be a compact region with a piecewise smooth boundary, $\operatorname{vol}\mathcal{O}$ the volume of $\mathcal{O}$, $t\mathcal{O}$ the dilatation of $\mathcal{O}$ by a factor $t \ge 1$, $t\mathcal{O} + X$ the translation of $t\mathcal{O}$ by a vector $X \in \mathbb{R}^n$. Let

$$N(\mathcal{O},\Gamma) = \operatorname{card}(\mathcal{O}\cap\Gamma) = \sum_{\gamma\in\Gamma} X(\mathcal{O},\gamma) \tag{1.5}$$

be the number of points of the lattice Γ lying inside the region $\mathcal{O}$. We denote by $X(\mathcal{O},X)$, $X \in \mathbb{R}^n$, the characteristic function of $\mathcal{O}$.

We define numbers $\mathcal{R}(t\mathcal{O}+X,\Gamma)$ and $R(t\mathcal{O},\Gamma)$ by setting

$$N(t\mathcal{O}+X,\Gamma) = \frac{\operatorname{vol}\mathcal{O}}{\det\Gamma}t^n + \mathcal{R}(t\mathcal{O}+X,\Gamma) \tag{1.6}$$

and

$$R(t\mathcal{O},\Gamma) = \sup_{X\in\mathbb{R}^n} \mid \mathcal{R}(t\mathcal{O}+X,\Gamma) \mid . \tag{1.7}$$

For each matrix $\alpha \in GL(n,\mathbb{R})$ one has the following relations

$$N(t\alpha\mathcal{O},\Gamma) = N(t\mathcal{O},\alpha^{-1}\Gamma), \quad R(t\alpha\mathcal{O},\Gamma) = R(t\mathcal{O},\alpha^{-1}\Gamma). \tag{1.8}$$

For unimodular lattices $\Gamma \in \mathcal{L}_n$ the relation (1.7) takes the form

$$N(t\mathcal{O}+X,\Gamma) = \operatorname{vol}\mathcal{O}\, t^n + \mathcal{R}(t\mathcal{O}+X,\Gamma). \tag{1.9}$$

The classical lattice point problem deals with behavior of the errors $\mathcal{R}(t\mathcal{O}+X,\Gamma)$ and $R(t\mathcal{O},\Gamma)$ as $t \to \infty$. For details we refer to [GL, Sect. IV] and the book [Kr] by Kratzel dedicated to this problem.

For any region $\mathcal{O} \subset \mathbb{R}^n$ and lattice $\Gamma \subset \mathbb{R}^n$ one has the bound

$$R(t\mathcal{O},\Gamma) = 0(t^{n-1}). \tag{1.10}$$

Thereby the relations (1.6), (1.9) are asymptotic formulas as $t \to \infty$. Moreover, the bound (1.10) is best possible on the set of all $\mathcal{O}$ and Γ. Indeed, if $P \subset \mathbb{R}^n$ is a compact polyhedron and, at least, for one its $n-1$ dimensional face the parallel hyperplane contains $n-1$ liner independent vectors of Γ, then, obviously, one has the following omega-bound

$$R(tP,\Gamma) = \Omega(t^{n-1}). \tag{1.11}$$

This proves laso that for any compact polyhedron $P \subset \mathbb{R}^n$ the set of lattices Γ leading to the bound (1.11) is dense in the space of lattices $\mathcal{L}_n$.

The bound (1.10) can be essentially improved for particular region $\mathcal{O}$. For example, if $\mathcal{O}$ is a strongly convex region with non zero Gaussian curvature on the boundary $\partial\mathcal{O}$, then one has the following bounds of Hlawka [H]

$$R(t\mathcal{O},\Gamma) = 0\left(t^{n-1-\frac{n-1}{n+1}}\right), \quad R(t\mathcal{O},\Gamma) = \Omega\left(t^{\frac{n-1}{2}}\right). \tag{1.12}$$

Bounds of the shape (1.12) are quite typical in the lattice point problem (cf. [GL], [Kr]).

We give the following Definition: the error $R(t\mathcal{O},\Gamma)$ is called to be *anomalously small* if one has the bound

$$R(t\mathcal{O},\Gamma) = 0(t^{\varepsilon}) \tag{1.13}$$

with arbitrary small $\varepsilon > 0$.

First examples of the anomalously small errors were given in the author's papers [Skr 1–3]. Moreover, it was shown in these papers that errors in the lattice point problem can be logarithmically small. Now we wish to formulate some of the main resulty given in [Skr 1–3]. First, we consider the case of two dimensions.

Theorem 1.1 ([Skr 1–2]). *Let $P \subset \mathbb{R}^2$ be a compact polygon with h sides, $\theta_1,\ldots,\theta_h$ angles between the sides and the first coordinate axis, $\xi_1 = \operatorname{tang}\alpha_1,\ldots,\sigma_h = \operatorname{tang}\alpha_h$. Then the following assertion are valid.*

1. *If all numbers $\xi_1, \dots, \xi_h$ are badly approximable,*[*)] *then one has the bound*

$$R(tP, \mathbb{Z}^2) = 0(\log t). \tag{1.14}$$

2. *For almost all $\xi_1, \dots, \xi_h \in \mathbb{R}$ (in the sense of the Lebesgue measure) one has the bound*

$$R(tP, \mathbb{Z}^2) = 0\left((\log t)^{1+\varepsilon}\right) \tag{1.15}$$

with arbitrary small $\varepsilon > 0$.

3. *If all numbers $\xi_1, \dots, \xi_h$ are algebraic irrationalities, then one has the bound*

$$R(tP, \mathbb{Z}^2) = 0(t^{\varepsilon}) \tag{1.16}$$

with arbitrary small $\varepsilon > 0$.

In arbitrary dimension $n \geq 2$ we have the following result. Let

$$Nm\Gamma = \inf\{|Nm\gamma| : \gamma \in \Gamma \setminus \{0\}\} \tag{1.17}$$

be the homogeneous minimum of the lattice $\Gamma \subset \mathbb{R}^n$ with respect to the norm form

$$NmX = \prod_{j=1}^{n} x_j, \quad X = {}^{\mathrm{T}}(x_1, \dots, x_n) \in \mathbb{R}^n. \tag{1.18}$$

We recall that a lattice Γ is called to be *admissible* if $Nm\Gamma > 0$ (cf.[Cas 1], [GL], [Skr 1,3]). The set of all admissible lattices is dense in $\mathcal{L}_n$, but its invariant measure $\mu_{\mathcal{L}}$ is equal to zero (cf. § 3 below).

Theorem 1.2 ([Skr 3]). *Let $\Pi \subset \mathbb{R}^n$ be a parallelepiped with edges parallel to the coordinate axes, and let $\Gamma \in \mathcal{L}_n$ be an admissible lattice. Then one has the bound*

$$R(Pi, \Gamma) = 0\left((\log t)^{n-1}\right). \tag{1.19}$$

Moreover, the bound (1.19) *is valid uniformly with respect to the idicated parallelepipeds Π of bounded volume.*

A new proof of the bound (1.19) will be given in the present paper in § 6.

In two dimensions the bound (1.19) follows with the help of (1.8) from the bound (1.14) written for a suitable parallelogram, because a lattice $\Gamma \subset \mathbb{R}^2$,

$$\Gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbb{Z}^2 \tag{1.20}$$

is admissible if and only if both numbers $\frac{a}{b}$ and $\frac{c}{d}$ are badly approximable (cf. [Cas 1], [GL]).

The comparison of the bounds (1.14) and (1.19) with the omega-bound (1.11) shows an extremely strong instability of logarithmically small errors upon the choice of the lattice.

[*)] Recall, cf [Sch], that a number $\xi \in \mathbb{R}$ is called to be badly approximable if the inequality $q|m - \xi q| > c(\xi)$ with a constant $c(\xi) > 0$ is valid for all integrals m and $q > 0$. For example, quadratic irrationalities are badly approximable.

The well-known relationship between algebraic number fields and lattices in Euclidean space give broad classes of admissible lattices for all dimensions $n \geq 2$ (cf. [Cas 1], [GL]). Let k be a totally real algebraic number field of degree $n = [k : \mathbb{Q}]$, and let σ be the canonical embedding of k in $\mathbb{R}^n$,

$$\sigma : k \ni \xi \to \sigma(\xi) = ^{\mathrm{T}}(\sigma_1(\xi), \dots, \sigma_n(\xi)) \in \mathbb{R}^n, \tag{1.21}$$

where $\{\sigma_j\}_1^n$ are n distinct embeddings of k into the field $\mathbb{R}$ of real numbers. It is easy to check that $Nm\sigma(\xi)$ is the norm in the field k. Therefore, under the embedding σ, a full $\mathbb{Z}$-module $m \subset k$ corresponds to an admissible lattice $\Gamma_m = \sigma(m)$. In particular, the ring of algebraic integers $D \subset k$ corresponds to the admissible lattice

$$\Gamma_D = \sigma(D) \tag{1.22}$$

with $\det \Gamma_D = |d_k|^{\frac{1}{2}}$, where d_k is the discriminant of k over $\mathbb{Q}$.

The following application of Theorem 1.2 can be mentioned. We set

$$H_k(t) = \operatorname{card} \{\xi \in D \subset k \, : \, h_k(\xi) \leq t\} , \tag{1.23}$$

where

$$h_k(\xi) = \max_j \mid \sigma_j(\xi) \mid \tag{1.24}$$

is the height on k. The study of asymptotic behavior of $H_k(t)$ as $t \to \infty$ is a known task in algebraic number theory, cf. Everest [E].

Comparing (1.22)–(1.24) with (1.5) we get

$$H_k(t) = N(2tK^n, \Gamma_D), \tag{1.25}$$

where $K^n = [-\frac{1}{2}, \frac{1}{2}]^n \subset \mathbb{R}$ is the unit cube centered at the origin with edges parallel to the coordinate axes. Since the lattice (1.22) is admissible, we derive from Theorem 1.2 and the formula (1.25) that for any totally real algebraic number field k of degree $n = [k : \mathbb{Q}]$ one has the asymptotics

$$H_k(t) = \frac{2^n}{|d_k|^{\frac{1}{2}}} t^n + O\left((\log t)^{n-1}\right). \tag{1.26}$$

Known methods of number theory based on Dirichlet series lead to the asymptotics (1.26) with the bound for the error of the order $O(t^{n-2+\varepsilon})$ with arbitrary small $\varepsilon > 0$ (cf. [E, Eq. (1.7)]), which is close to our bound (1.26) only for real quadratic fields where $n = 2$.

For arbitrary algebraic number field of degree $n = [k : \mathbb{Q}]$ with s real and q pairs of conlex embeddings one has the following asymptotics of the quantity (1.23)

$$H_k(t) = \frac{2^s (2\pi)^q}{|d_k|^{\frac{1}{2}}} + O\left(t^{\frac{q}{n(2+q)}} (\log t)^{s+q+\varepsilon}\right). \tag{1.27}$$

with arbitrary small $\varepsilon > 0$. This bound of the error in (1.27) follows from results of Nikichine and the author [Nskr 1, Thm 2.2], NSkr 2, Thms 1 and 2]. Methods of the papers [NSkr 1–2] give also essential improvements of bounds for the errors in other asymptotic problems in theory of algebraic number fields.

Applications of logarithmically small errors in uniform distributions, numerical mathematics, and spectral theory are given [Skr 1, 3, 5]. For the completeness of references we mention the papers [Ra] by Randol, [T] by Tarnapolska–Weiss, and [By] by Bykovskii where logarithmically small bounds are given for averaged errors with respect to rotations (cf. [Ra], T]) and translations (cf. [By]). In a recent paper Beck [Be] gave 'metrical' results on logarithmically small discrepancies of lattice point distributions for special classes of lattices. It would be very interesting to compare the methods of [Be] with our approach to the problem of logarithmically small errors.

§ 2. Statement of main results

A careful consideration of the assertion given above in Theorem 1.1 and 1.2 leads to the following questions:

– Are there regions in dimensions $n > 2$ besides parallelepipeds with logarithmically small errors?

– In particular, are there for any polyhedron in $\mathbb{R}^n$ lattices giving logarithmically small errors?

– How rich in the sense of the invariant measure $\mu_{\mathcal{L}}$ is the set of such lattices?

In the present paper we will consider there questions.

For the sake of convenience we give the following Definition: a positive monotone increasing function $\varphi(t)$, $t \in \mathbb{R}$, is called to be *a majorant* if one has

$$\varphi(t) = 0(\exp at), \qquad t \to \infty \tag{2.1}$$

with some $0 \leq a < \infty$ and

$$\int_1^\infty \frac{dt}{\varphi(t)} < \infty. \tag{2.2}$$

It should be noted that in this definition only condition (2.2) is essential while (2.1) is accepted only for simplicity.

For example, the positive monotene increasing functions $\varphi_k(t)$, $k = 1, 2, \dots$, are majorants if they have the following asymptotic behavior as $t \to \infty$

$$\varphi_0(t) \sim t^{1+\varepsilon}, \qquad \varphi_1(t) \sim t(\log t)^{1+\varepsilon}, \tag{2.3}$$

and generally,

$$\varphi_k(t) \sim t \left(\prod_{j=1}^{k-1} \log_j t \right) (\log_k t)^{1+\varepsilon}, \qquad k = 1, 2, \dots \tag{2.4}$$

where $\log_1 t = \log t$, $\log_{j+1} t = \log(\log_j t)$, $j = 1, 2, \dots$, and $\varepsilon > 0$ is arbitrary small.

By definition (cf. Berger [Ber]) a convex polyhedron in $\mathbb{R}^n$ is the intersection of a finite number of half-spaces, an arbitrary polyhedron is defined as the union of a finite number of convex ones.

Our main result can be stated as follows.

Theorem 2.1. *Let $P \subset \mathbb{R}^n$ be arbitrary compact polyhedron, then for almost all lattices $\Gamma \in \mathcal{L}_n$ (in the sense of the invariant measure $\mu_c L$) one has the bound*

$$R(tP, \Gamma) = O\left((\log t)^{n-1} \varphi(\log \log t)\right) \tag{2.5}$$

with arbitrary majorant $\varphi(\cdot)$. In particular, for almost all $\Gamma \in \mathcal{L}_n$ one has the bound

$$R(tP, \Gamma) = O\left((\log t)^{n-1+\varepsilon}\right) \tag{2.6}$$

with arbitrary small $\varepsilon > 0$.

Theorem 2.1 shows that the appearance of logarithmically small errors for a polyhedron is quite typical from 'metrical' point of view. In connection with this, it should be noted that Theorem 2.1 implies that for every polyhedron $P \subset \mathbb{R}^n$ the measure $\mu_{\mathcal{L}}$ of the dense set of lattices $\Gamma \in \mathcal{L}_n$ giving the omega-bound (1.11) is equal to zero. It should also be noted that there are some reasons to expect that the bounds (2.5) and (2.6) are best possible.

We can consider submanifolds of lattices in $\mathcal{L}_n$ leading to logarithmically small errors. The existence of a number of such submanifolds follows at once from Theorem 2.1 and elementary arguments of measure theory. Here we want to give a result of another kind.

Theorem 2.2. *Let $P \subset \mathbb{R}^n$ be arbitrary compact polyhedron and let $\Gamma \in \mathcal{L}_n$ be arbitrary lattice, then for almost orthogonal matrices $U \in \mathrm{SO}(n)$ (in the sense of the Haar measure on the group $\mathrm{SO}(n)$) one has the bound*

$$R(tP,\, U\Gamma) = R(tU^{-1}P,\, \Gamma) = O\left((\log t)^{2(n-1)}\varphi(\log\log t)\right) \tag{2.7}$$

with arbitrary majorant $\varphi(\cdot)$. In particular, for almost all $U \in \mathrm{SO}(n)$ one has the bound

$$R(tP,\, U\Gamma) = R(tU^{-1}P,\, \Gamma) = O\left((\log t)^{2(n-1)+\varepsilon}\right) \tag{2.8}$$

with arbitrary small $\varepsilon > 0$.

In connection with Theorems 2.1–2.2 the following question naturally arises: how to construct in explicit form the lattices leading to anomalously small errors for a given polyhedron $P \subset \mathbb{R}^n$? Up to now there was only one result of such kind given in our Theorem 1.2. In the present paper we give new result concerning this problem.

Let $P_k \subset \mathbb{R}^n$ be a compact polyhedron, such that all its vertices have coordinates belonging to a real algebraic number field k of finite degree $[k : \mathbb{Q}] < \infty$. It should be noted that each combinatorial type of polyhedra in $\mathbb{R}^n$ can be realized as a polyhedron of the shape P_k over a suitable field k, however in dimensions $n > 2$ all real algebraic number fields k are needed to realize all combinatorial types, cf. [BLSWZ, Chp. 8]. Let

$$k = k_0 \subset k_1 \subset \ldots \subset k_n \tag{2.9}$$

be a tower of real algebraic number fields of finite degrees, satisfying the conditions

$$[k_{j+1} \,:\, k_j] \geq n+1, \qquad j = 0, \ldots, n-1. \tag{2.10}$$

Because of the conditions (2.10) we can choose in each field k_j, $1 \leq j \leq n$, n numbers ξ_{ij}, $1 \leq i \leq n$, such that the numbers $1, \xi_{1j}, \ldots, \xi_{nj}$ are linearly independent over the field $k_{j-1} \subset k_j$. We introduce the following matrix $A_k \in \mathrm{SL}(n, k)$,

$$A_k = c(\xi_{ij}), \qquad 1 \leq i, j \leq n, \tag{2.11}$$

where the constant c defined by the condition $\det A_k = 1$. The nondegeneracy of the matrix (2.110 will be proved in § 5, see formulas (5.19)–(5.21) and Lemma 5.3 below.

Now we define dual lattices Ω_k and $\Omega_k^{+} \in \mathcal{L}_n$ by setting (cf. (1.4) and (2.11))

$$\Omega_k^{\perp} = A_k \mathbb{Z}^n, \qquad \Omega_k = {}^{\mathrm{T}}A_k^{-1}\mathbb{Z}^n \tag{2.12}$$

Theorem 2.3. *With the above notations one has the bound*

$$R(t\, P_k,\, \Omega_k) = O(t^{\varepsilon}) \tag{2.13}$$

with arbitrarily small $\varepsilon > 0$.

Lattices leading to anomalously small errors for a polyhedron P_k can be constructed in a number of different ways. For example, one has the bound similar to (2.13)

$$R\left(t\, P_k,\, \Omega_k^{\perp}\right) = O(t^{\varepsilon}) \tag{2.14}$$

In § 5 we will give corresponding sufficient conditions of the lattice.

In is worth noting that there are reasons to expect that the bounds (2.13), and (2.14) (as well as (1.16)) can be replaced by logarithmically small ones. However, this questin is closely related with some unsolved problems in diophautine approximations and we will not discuss it here.

Presumably, anomalously small, particularly logarithmically small, errors may appear only polyhedra. We conjucture that for all regions $\mathcal{O} \subset \mathbb{R}^n$ and lattices $\Gamma \subset \mathbb{R}^n$ one has the following omega-bound similar to (1.11):

$$R(t\mathcal{O}, \Gamma) = \Omega\left(t^{\frac{q}{2}}\right) \tag{2.15}$$

where q $(0 \leq 1 \leq n-1)$ is a maximal number of nonvanishing principal curvatures on the boundary $\partial\mathcal{O}$. By this means (cf. (1.13)) even a minor distortion of the boundary prevents the appearence of anomalously small errors. Unfortunately, in dimensions $n > 2$ the conjucture (2.15) seems to be rather difficult. The lattice point problem for regions (like cylinders) where only some of principal curvatures of the boundaty are equal to zero identically was studied in [NSkr 1–2].

Theorems 2.1, 2.2, 2.3 will be proved in § 6 on the base of a new general bound for the lattice point counting in a polyhedron (Theorem 6.1).

Our approach to the study of the behavior of the error $R(tP, \Gamma)$ for a given polyhedron P and a lattice Γ can be outlined as follows. With the set of faces of P and the lattice Γ we associate a finite set of lattices $\Gamma_{\mathfrak{f}}^{\perp} \in \mathcal{L}_n$. For example, in the simplest case where the polyhedron is a parallelepiped Π, this set coinsists of a single lattice $\Gamma^{\perp}$ dual to Γ. Next we consider the orbits $\{\delta\Gamma_{\mathfrak{f}}^{\perp} : \delta \in \Delta\}$ of the lattices $\Gamma_{\mathfrak{f}}^{\perp}$ under the zction of a discrete subgroup $\Delta \subset D$ of the group of unimodular diagonal matrices $D \subset \mathrm{SL}(n, \mathbb{R})$. It turns out that the rate of approzch of these orbits to "infinity" (the *cusp* of the homogeneous space $\mathcal{L}_n = \mathrm{SL}(n,\mathbb{R}) \setminus \mathrm{SL}(n,\mathbb{Z})$) determines the behavior of the corresponding error $R(tP, \Gamma)$ as $t \to \infty$.

For example, if all of the lattices $\Gamma_{\mathfrak{f}}^{\perp}$ are addmissible, then their orbits $\{\delta\Gamma_{\mathfrak{f}}^{\perp} : \delta \in \Delta\}$ are relatively compact (cf. Mahler's criterion § 3), and Theorem 6.1 leads to a minimal bound for the corresponding error given in Theorem 1.2.

At the same time, if at least one of the orbits tends to infinity exponentially rapidly, we get the worst bound for the error (cf. (1.10)–(1.11)).

We show (cf. 3.2, 4.3, 4.4, 4.5 below) that from the 'metric' point of view for almost all lattices $\Lambda \in \mathcal{L}_n$ the orbits $\{\delta\Lambda : \delta \in \Delta\}$ tend to infinity very slowly. This circumstance leads to Theorems 2.1 and 2.2. Similarly, Theorem 2.3 follows from the fact (cf. Lemmas 5.2, 5.4 below) that the orbits $\{\delta\Omega_k^{\perp} : \delta \in \Delta\}$ of special algebraic lattices $\Omega_k^{\perp}$ (cf. (2.11), (2.12)) also tend to infinity very slowly.

It should be noted that there are reasons to believe that the methods of the present paper may enjoy wider application, say, for regions $\mathcal{O} \subset \mathbb{R}^n$ other than polyhedra. It should be also emphasized that there are certain parallels between our approach to the lattice point problem and known applications of ergodic theory on $\mathrm{SL}(n)$ to number-theoretic problems given in the papers [M] by Margulis and [DM] by Dani and Margulis.

A peculiarity of the present paper is the use of the methods of geometry of numbers (like dyadic minima of a lattice, integration over the space of latties, etc.) in combination with rather refined Fourier analysis (like the Fourier transform of characteristic functions of polyhedra and bounds for certain Fourier integrals). Conceivably this is a most challenging part of the work. Because of this, a development of a straighrforward approach would have a particular importance. One can look forward to a successful consideration of this matter.

§ 3. Ergodic properties of the flow of diagonal matrices on the space of lattices

Let $D \subset \mathrm{SL}(m,\mathbb{R})$ be the subgroup of diagonal matrices,

$$D = \left\{\delta_Y = \mathrm{diag}\,(2^{y_1},\dots,2^{y_n}) \in \mathrm{SL}(n,\mathbb{R}) \,:\, Y = {}^{\mathrm{T}}(y_1,\dots,y_n) \in \mathcal{L}\right\}, \tag{3.1}$$

where

$$\mathcal{L} = \left\{Y = {}^{\mathrm{T}}(y_1,\dots,y_n) \in \mathbb{R}^n \,:\, y_1 + \ldots + y_n = 0\right\} \tag{3.2}$$

is the $n-1$ dimensional subspace in $\mathbb{R}^n$. Let $\Delta \subset D$ be a discrete subgroup of the shape:

$$\Delta = \left\{\delta_m = \mathrm{diag}\,(2^{m_1},\dots,2^{m_n}) \in \mathrm{SL}(n,\mathbb{R}) \,:\, m = {}^{\mathrm{T}}(m_1,\dots,m_n) \in M = \mathbb{Z}^n \cap \mathcal{L}\right\}. \tag{3.3}$$

Here M is the $n-1$ dimensional lattice in the subspace $\mathcal{L}$, so that, Δ is a free Abelian group with $n-1$ generators. Let

$$\Delta_r = \{\delta_m \in \Delta : \|m\| < r\}, \quad r > 0, \tag{3.4}$$

be a system of finite subsets in Δ. Moreover $\Delta_r \to \Delta$ as $r \to \infty$, and

$$n_r = \mathrm{card}\,\Delta_r = c_n r^{n-1} + 0(r^{n-1}) \tag{3.5}$$

with a constant $c_n > 0$.

Let

$$\|\Lambda\| = \min\{\|\gamma\| : \gamma \in \Lambda \setminus \{0\}\} \tag{3.6}$$

be the homogeneous minimum of a lattice $\Lambda \subset \mathbb{R}^n$ with respect to the Euclidean metric $\|X\|$, $X \in \mathbb{R}^n$. One has the inequality

$$h_n = \sup\{\|\Lambda\| : \Lambda \in \mathcal{L}_n\} \le \left(\frac{4}{3}\right)^{\frac{n-1}{4}}, \tag{3.7}$$

where h_n is the Hermite constant, cf. [Cas 1], [GL].

The quantities $\|\Lambda\|$ and $Nm\Lambda$ (cf. (1.17)) satisfy the relation (cf. [Skr 3, Eq. (3.6)])

$$Nm\Lambda = n^{-\frac{n}{2}} \int \{\|\delta\Lambda\|^n \,:\, \delta \in D\}. \tag{3.8}$$

Moreover, the quantities are closely related with topology of the homogeneous $\mathcal{L}_n$. In particular, one has the following *Mahler's compactness criterion* (cf. [Cas 1], [GL]): *A subset* $\mathcal{E} \subset \mathcal{L}_n$ *relatively compact if and only if*

$$int\,\{\|\Lambda\| \,:\, \Lambda \in \mathcal{E}\} > 0. \tag{3.9}$$

The comparison of (3.8) and (3.9) shows that the orbit $\{\delta\Lambda : \delta \in D\} \subset \mathcal{L}_n$ of a lattice $\Lambda \in \mathcal{L}_n$ under the action of the group of diagonal matrices is relatively compact if and only if $Nm\Lambda > 0$, i.e., the lattice Λ is admissible. Thereby the set of admissible lattice in $\mathcal{L}_n$ can be identified with the set of relatively compact orbits of the group D. This implies, by the way, that the invariant measure $\mu_{\mathcal{L}}$ of the set of admissible lattices is equal to zero, cf. Dani [D; Proposition 2.4 and Corollary 2.5].

It is known (cf. [Skr 3; Assertion 2 of Lemma 3.1]) that the conditions

$$Nm\Lambda > 0 \quad \text{and} \quad Nm\Lambda^{\perp} > 0 \tag{3.10}$$

are equivalent. Therefore, both orbits

$$\{\delta\Lambda \, : \, \delta \in D\} \quad \text{and} \quad \{\delta\Lambda^{\perp} \, : \, \delta \in D\} \tag{3.11}$$

are relatively compact or noncompact simultaneously.

We recall that a subgroup $G \subset \mathrm{SL}(n, \mathbb{R})$ is called to be *ergodic* on the homogeneous space $\mathcal{L}_n$ if for every G-invariant measurable subset $\mathcal{E} \subset \mathcal{L}_n$ (i.e., $G\mathcal{E} = \mathcal{E}$) the measure $\mu_{\mathcal{L}}(\mathcal{E}) = 0$ or 1. One has the following Moore's ergodic theorem (cf. Moore [Mo] and Zimmer [Z]): *A subgroup $G \subset \mathrm{SL}(n, \mathbb{R})$ is ergodic on $\mathcal{L}_n$ if and only if G* is not contained in any compact subgroup of $\mathrm{SL}(n, \mathbb{R})$.

Note that Moore's theorem is given in [Mo] and [Z] in much more general form.

The subgroup $\Delta \subset \mathrm{SL}(n, \mathbb{R})$ generates a discrete flow of lattice on $\mathcal{L}_n$:

$$\delta \, : \, \mathcal{L}_n \ni \Lambda \to \delta\Lambda, \quad \delta \in \Delta. \tag{3.12}$$

From Moor's theorem it follows that the flow (3.12) is ergodic. This leads to the following *Individual ergodic theorem: Let the function $\psi(\Lambda) \in L_1(\mathcal{L}_n, \mu_{\mathcal{L}})$,*[*)] *then for almost all $\Lambda \in \mathcal{L}_n$ (in the sense of the invariant measure $\mu_{\mathcal{L}}$) one has the relation*

$$\lim_{r\to\infty} \frac{1}{n_r} \sum_{\delta\in\Delta_r} \psi(\delta\Lambda) = \int_{\mathcal{L}_n} \psi(\Lambda) d\mu_{\mathcal{L}}(\Lambda). \tag{3.13}$$

Indeed, since Δ is Abelian group the existence of the limit in the left-hand side of (3.13) for almost all $\Lambda \in \mathcal{L}_n$ follows from the well-known results of ergodic theory, cf. the papers [W] by Wiener and [Cal] by Calderon or more modern book [Kre] by Krengel. Now the reference to the ergodic of the flow (3.12) completes the proof of the relation (3.13).

In order to use Individual ergodic theorem we need the following assertion, which will be proved in Appendix 1 (see Lemma 13.1 below).

Lemma 3.1. *Let the function $\psi(\Lambda)$, $\Lambda \in \mathcal{L}_n$, be defined by*

$$\psi(\Lambda) = \left(\|\Lambda\|^n \varphi\left(\frac{1}{\|\Lambda\|}\right)\right)^{-1}, \tag{4.13}$$

where $\varphi(\cdot)$ is arbitrary majorant, then $\psi(\Lambda) \in L_1(\mathcal{L}_n, \mu_{\mathcal{L}})$.

We introduce the following averaged characteristic of the flow (3.12):

$$S(\Lambda, r) = \sum_{\delta\in\Delta_r} \|\delta\Lambda\|^{-n}, \qquad r \geq 0, \tag{3.15}$$

and $S(\Lambda, r) = 0$, $r < 0$.

We are interested in the behavior of the characteristic (3.15) as $r \to \infty$. If the lattice Λ is admissible, then one has the bound

$$S(\Lambda, r) = O\left(r^{n-1}\right), \tag{3.16}$$

which follows from (3.5), (3.8), and (3.15). However, for special Λ the sum (3.15) can increase exponentially. For example, it is easy to check that

$$S(\mathbb{Z}^n, r) \geq 2^r, \quad r > 0. \tag{3.17}$$

[*)] $L_1(\mathcal{L}_n, \mu_{\mathcal{L}})$ denotes the class of functions absolutely integrable over $\mathcal{L}_n$ with respect to the invariant measure $\mu_n\mathcal{L}$.

Lemma 3.2. *For almost all lattices $\Lambda \in \mathcal{L}_n$ (in the sense of the invariant measure $\mu_{\mathcal{L}}$) one has the bound*

$$S(\Lambda, r) = O\left(r^{n-1+\varepsilon}\right), \tag{3.18}$$

with arbitrary majorant $\varphi(\cdot)$. In particular, for almost all $\Lambda \in \mathcal{L}_n$ one has the bound

$$S(\Lambda, r) = O\left(r^{n-1+\varepsilon}\right) \tag{3.19}$$

with arbitrarily small $\varepsilon > 0$.

Proof. We wish to estimate the sum (3.15) with the help of Individual ergodic theorem (3.13). However, this can not be done immediately because the function $\|\Lambda\|^{-n}$ is not integrable over $\mathcal{L}_n$.

From (3.13) and Lemma 3.1 (with $\varphi(t) = \exp \varepsilon t$, $\varepsilon > 0$) we obtain for almost all $\Lambda \in \mathcal{L}_n$ the bound

$$\sum_{\delta \in \Delta_r} \|\delta\Lambda\|^{\varepsilon - n} = O\left(r^{n-1}\right). \tag{3.20}$$

Therefore, we have for almost all $\Lambda \in \mathcal{L}_n$ the bound

$$\min\left\{\|\delta\Lambda\| \,:\, \delta \in \Delta_r\right\} > c_\varepsilon(\Lambda) r^{-1+\frac{1}{n}-\varepsilon} \tag{3.21}$$

with arbitrary small $\varepsilon > 0$ and some constant $c_\varepsilon(\Lambda) > 0$.

Now let $\varphi(\cdot)$ be arbitrary majorant, we set

$$S_\varphi(\Lambda, r) = \sum_{\delta \in \Delta_r} \left(\|\delta\Lambda\|^n \varphi\left(\log \frac{1}{\|\delta\Lambda\|}\right)\right)^{-1}. \tag{3.22}$$

From (3.13) and Lemma 3.1 we obtain for almost all $\Lambda \in \mathcal{L}_n$ the bound

$$S_\varphi(\Lambda, r) = O\left(r^{n-1}\right). \tag{3.23}$$

The comparison of the definitions (3.15) and (3.22) gives the inequality

$$S_\varphi(\Lambda, r) \geq \max\left\{\left(\varphi\left(\log \frac{1}{\|\delta\Lambda\|}\right)\right)^{-1} \,:\, \delta \in \Delta_r\right\} S(\Lambda, r). \tag{3.24}$$

Since the majorant $\varphi(\cdot)$ is a monotone increasing function, we get from (3.24) the inequality

$$S(\Lambda, r) \leq \varphi\left(\log\left(\min\left\{\|\delta\Lambda\| \,:\, \delta \in \Delta_r\right\}\right)^{-1}\right) S_\varphi(\Lambda, r). \tag{3.25}$$

Substituting the bounds (3.21) and (3.24) in (3.25) we obtain the bound (3.19).

The proof of Lemma 3.2 is complete.

§ 4. Metrical properties of diophantine approximations on the space of lattices

We introduce the following diophantine characteristic of a lattice $\Lambda \subset \mathbb{R}^n$:

$$\nu(\Lambda, \rho) = \min \{ |Nm\,\gamma| \,:\, \gamma \in \Lambda,\ 0 < \|\gamma\| < \rho \}, \quad \rho > 0, \tag{4.1}$$

which is a generalization of the definition (1.17).

Note that $\nu(\Lambda, \rho) - 0$ for $0 < \rho \le \|\Lambda\|$. The comparision of (1.17), (3.6) and (4.1) gives

$$\nu(\Lambda, \rho) \ge Nm\Lambda \quad \text{for} \quad \rho > \|\Lambda\|. \tag{4.2}$$

The relation (4.2) implies that for admissible lattices $\nu(\Lambda, \rho) > 0$ as $\rho \to \infty$. However, it is easy to check that $\nu(\mathbb{Z}^n, \rho) = 0$ for all $\rho \ge 0$. We give the following Definition: A lattice $\Lambda \subset \mathbb{R}^n$ is called to be *weakly admissible* if $\nu(\Lambda, \rho) > 0$ as $\rho \to \infty$. It is obvious that a lattice Λ is weakly admissible if and only if there are no points $\gamma \in \Lambda \setminus \{0\}$ on the coordinate planes.

The characteristics (3.15) and (4.1) are related as follows.

Lemma 4.1. *For every weakly admissible lattice $\Lambda \in \mathcal{L}_n$ one has the relation*

$$S(\Lambda, r) \le n^{-\frac{n}{2}} \left(\nu(\Lambda, \rho)\right)^{-1} = O\left(\frac{r^{n-1}}{\nu(\Lambda, \rho)}\right), \quad r \to \infty \tag{4.3}$$

with

$$\rho = h_n 2^r \le \left(\frac{4}{3}\right)^{\frac{n-1}{4}} 2^r, \tag{4.4}$$

cf. (3.7).

Proof. Let $\delta_m = \operatorname{diag}(2^{m_1}, \dots, 2^{m_n}) \in \Delta_r$ (cf. (3.3, (3.4)), and let the homogeneous minimum $\|\delta_m \Lambda\|$ of the lattice $\delta_m \Lambda \in \mathcal{L}_n$ be attained on a vector $\delta_m \gamma_m$, $\gamma_m \in \Lambda \setminus \{0\}$, i.e., $\|\delta_m \Lambda\| = \|\delta_m \gamma_m\|$ (cf. (3.7). Using (3.4) and (2.7) we obtain

$$h_n \ge \|\delta_m \Lambda\| = \|\delta_m \gamma_m\| \ge \min\{2^{m_1}, \dots, 2^{m_n}\} \|\gamma_m\| \ge 2^{-\|m\|} \|\gamma_m\| \ge 2^{-r} \|\gamma_m\|.$$

Therefore (cf.(4.3)),

$$\|\gamma - M\| \le h_n\, 2^r = \rho \tag{4.5}$$

for all $\delta_m \in \Delta_r$.

From the inequality between the arithmetic and geometric means we derive the inequality (cf. (1.17), (1.18))

$$\|X\|^n \ge n^{\frac{n}{2}} |Nm\,X|, \quad X \in \mathbb{R}^n. \tag{4.6}$$

Moreover, we have the relation

$$Nm\delta X = Nm\,X, \quad X \in \mathbb{R}^n, \tag{4.7}$$

for every diagonal matrix $\delta \in D$.

From (4.1), (4.5), (4.6), (4.7) we obtain

$$\|\delta_m \Lambda\|^n = \|\delta_m \gamma_m\|^n \ge n^{\frac{n}{2}} |Nm\,\gamma_m| \ge n^{\frac{n}{2}} \nu(\Lambda, \rho). \tag{4.8}$$

Now the relation (4.3) follows from the definition (3.15) with the help of (3.5) and (4.8):

$$S(\Lambda, r) = \sum_{\delta_m \in \Delta_r} \|\delta_m \Lambda\|^{-n} \le n^{-\frac{n}{2}} \left(\nu(\Lambda, \rho)\right)^{-1} \sum_{\delta_m \in \Delta_m} 1 = n^{-\frac{n}{2}} \left(\nu(\Lambda, \rho)\right)^{-1} \operatorname{card} \Delta_r = O\left(\frac{r^{n-1}}{\nu(\Lambda, \rho}\right).$$

The proof of Lemma 4.1 is complete.

We are interest in the behavior of the characteristic (4.1) as $\rho \to \infty$.

Lemma 4.2. *Let $\Lambda \in \mathcal{L}_n$ be arbitrary lattice, then for almost all orthogonal matrices $U \in \mathrm{SO}(n)$ (in the sense of the Haar measure on $\mathrm{SO}(n)$) the lattive $U\Lambda$ is weakly admissible and one has the bound*

$$\nu(U\Lambda, \rho) > c_\omega(U, \Lambda)\omega(\rho), \quad \rho \to \infty, \tag{4.9}$$

where $\omega(\rho)$, $\rho \geq \|\Lambda\|$, is arbitrary positive monotone nonincreasing function satisfy the following condition

$$\sum_{\gamma \in \Lambda \setminus \{0\}} \|\gamma\|^{-n} \left(\log \frac{\|\gamma\|^n}{\omega(\|\gamma\|)}\right)^{n-2} \omega(\|\gamma\|) < \infty, \tag{4.10}$$

and $c_\omega(U, \Gamma) >$ is a constant.

Proof. Let π_i, $1 \leq i \leq n$, be the i-th coordinate plane. For a fiven point $\gamma \in \Lambda \setminus \{0\}$, let Q_i denote the set of orthogonal matrices such that $U\gamma \in \pi_i$ for $U \in Q_i$. We have $Q_i = G_i V_i$, where $V_i \in Q_i$ is one of such matrices, and $G_i \backsimeq \mathrm{SO}(n-1)$ is the stabolozer of π_i in the group of rotations. Thus, $\mu_{\mathrm{SO}}(Q_i) = o$, and therefore, for almost all $U \in \mathrm{SO}(n)$ and all $\gamma \in \Lambda \setminus \{0\}$ we have

$$|Nm\, U\gamma| > 0. \tag{4.11}$$

This proves first assertion of Lemma 4.2.

The prrof of the bound (4.9) is based on the Borel–Cantelli lemma (cf. Sprindžuk [Sp]). We start with evaluation of measures of some sets. Let

$$\mathbb{K}^l = \left\{X = {}^{\mathrm{T}}(x_1, \ldots, x_l) \in \mathbb{R}^l \,:\, 0 < x_i < 1, \quad 1 \leq i \leq l\right\} \tag{4.12}$$

be the unit cube in $\mathbb{R}^l$, and let

$$v_l(\theta) \left\{X \in \mathbb{K}^l \,:\, Nm\, X < \theta\right\}, \quad 0 < \theta < 1. \tag{4.13}$$

We obtain immediately from (4.12), (4.13)

$$v_2(\theta) = \theta + \int_\theta^1 \frac{\theta}{x} dx = \theta + \theta \ln \frac{1}{\theta}, \tag{4.14}$$

and further for all integer $l \geq 2$

$$v_{l+1}(\theta) = \theta + \int_\theta^1 v_l\left(\frac{\theta}{x}\right) dx = \theta \left(1 + \theta \ln \frac{1}{\theta}\right)^l. \tag{4.15}$$

Let

$$\Sigma^n = \{X \in \mathbb{R}^n \,:\, \|X\| = 1\} \tag{4.16}$$

be the unit sphere in $\mathbb{R}^n$, and let σ be the unique $\mathrm{SO}(n)$-invariant measure on Σ^n normalized by the condition $\sigma(\Sigma^n) = 1$. Let

$$s_n(\theta) = \sigma\left(\{X \in \Sigma^n \,:\, |Nm\, X| < \theta\}\right). \tag{4.17}$$

Since the set in (4.17) is invariant under reflections with respect to coordinate axes, we have the bound

$$s_n(\theta) \leq 2^{2n-1} \sigma\left(\{X \in \Omega^n \,:\, |Nm\, X| < \theta\}\right). \tag{4.18}$$

where the set $\Omega^n \subset \Sigma^n$ is defined by

$$\Omega^n = \left\{ X = {}^{\mathrm{T}}\!(x_1, \dots, x_l) \in \Sigma^n \,:\, 0 < x_i < 1, \quad 1 \le i \le n-1, \quad x_n > \frac{1}{\sqrt{n}} \right\}. \tag{4.19}$$

We note that $x_1, \dots, x_{n-1}$ form a smooth coordinate system on Ω^n, moreover the element $d\sigma$ of the invariant measure σ has in these coordinates the shape

$$d\sigma = a_n \frac{dx_1, \dots, dx_{n-1}}{x_n}, \quad a_n = \frac{\Gamma(\frac{n}{2})}{2\pi^{n/2}}. \tag{4.20}$$

The comparison of (4.16)–(4.20) with (4.12)–(4.13) gives

$$s_n(\theta) \le 2^{2n-1} \sqrt{n} a_n v)n - 1(\sqrt{n}\,\theta) \tag{4.21}$$

for $0 < \sqrt{n}\theta < 1$.

Let $\gamma \in \Lambda \setminus \{0\}$, and let

$$m_\gamma(\theta) = \mu_{\mathrm{SO}}\left(\{U \in \mathrm{SO}(n) \,:\, |Nm\, U\gamma| < \theta\}\right). \tag{4.22}$$

We regard the unit spare as homogeneous space, $\Sigma^n \backsimeq \mathrm{SO}(n)/G_\gamma$, where $G_\gamma \backsimeq \mathrm{SO}(n-1)$ is the stabilizer of the poinr $\widehat{\gamma} = \|\gamma\|^{-1}\gamma \in \Sigma^n$. This leads to the relation

$$m_\gamma(\theta) = s_n\left(\|\gamma\|^{-n}\theta\right). \tag{4.23}$$

From the relations (4.14), (4.15), (4.21), (4.22), (4.23) we obtain the bound

$$m_\gamma(\theta) < c_n \|\gamma\|^{-n} \theta \left(1 + \ln \frac{\|\gamma\|^n}{\theta}\right)^{n-2} \tag{4.24}$$

with a positive constant $c(n)$ depebding only on n.

Using the Borel–Cantelli lemma, we derive from the condition (4.10) and the bound (4.24) that for almost all orthogonal matrices $U \in \mathrm{SO}(n)$ the inequality

$$|Nm\, U\gamma| \ge \omega(\|\gamma\|) \tag{4.25}$$

is valid for all $\gamma \in \Lambda \setminus \{0, \gamma_1, \dots, \gamma_q\}$, where the exceptional set of points $\gamma_1, \dots, \gamma_q \in \Lambda$ is finite, but may depend on U. Substituting the bounds (4.11) and (4.25) in the definition (4.1), and taking monotonicity of the function $\omega(\cdot)$ into account, we obtain for $\rho \ge \|\Lambda\|$ the following inequalities:

$$\begin{aligned} \nu(U\Lambda, \rho) &\ge \min\left\{\min\left\{\omega(\|\gamma\|) : \gamma \in \Lambda,\ 0 < \|\gamma\| < \rho\right\},\ |Nm\,\gamma_1|, \dots, |Nm\,\gamma_q|\right\} = \\ &= \min\left\{\omega(\rho),\ |Nm\,\gamma_1|, \dots, |Nm\,\gamma_q|\right\} \ge C_\omega(U, \Lambda)\omega(\rho) \end{aligned} \tag{4.26}$$

with

$$C_\omega(U, \Lambda) = \min\left\{1, \frac{|Nm\,\gamma_1|}{\omega(\|\Lambda\|)}, \dots, \frac{|Nm\,\gamma_q|}{\omega(\|\Lambda\|)}\right\}. \tag{4.27}$$

Certaon, the constant (4.27) in the bound (4.26) is ineffective, that is typical for applications of the Borel–Cantelli lemma.

The proof of Lemma 4.2 is complete.

From Lemma 4.2 we derive the following corollary.

Lemma 4.3. *Let $\Lambda \in \mathcal{L}_n$ be arbitrary lattice, then for almost all orthogonal matrices $U \in \mathrm{SO}(n)$ (in the sense of the Haar measure on $\mathrm{SO}(n)$) one has the bound*

$$\nu(U\Lambda, \rho) > c_\varphi(U, \Lambda)\left((\log \rho)^{n-1}\varphi(\log\log \rho)\right)^{-1}, \quad \rho \to \infty, \tag{4.28}$$

where $\varphi(\cdot)$ is arbitrary majorant and $c_\varphi(U, \Lambda) > 0$ is a constant. In particulat, for almost all $U \in \mathrm{SO}(n)$ one has the bound

$$\nu(U\Lambda, \rho) > c_\varepsilon(U, \Lambda)(\log \rho)^{n-1-\varepsilon}, \quad \rho \to \infty, \tag{4.29}$$

with arbitrary small $\varepsilon > 0$ and some constant $c_\varepsilon(U, \Lambda) > 0$.

Proof. We put in Lemma 4.2

$$\omega(t) = \left((\log t)^{n-1}\varphi(\log\log t)\right)^{-1} \tag{4.30}$$

where $\varphi(\cdot)$ is arbitrary majorant. From (2.1) and (4.30) we obtain the bound

$$\omega(t) > c_b(\log t)^{-b}, \quad t \to \infty, \tag{4.31}$$

with some $b > n - 1$.

Using the bound (4.31) and monotonicity of $\varphi(\cdot)$ we conclude that the condition (4.10) is valid because of the condition (2.2). Now the bound (4.28) follows from the bound (4.9).

The proof of Lemma 4.3 is complete.

Now we wish to consider the behavior of the characteristic (3.15) as $r \to \infty$ in the case of inapplicability of ergodic theorems.

Lemma 4.4. *Let $\Lambda \in \mathcal{L}_n$ be arbitrary lattice, then for almost all orthogonal matrices $U \in \mathrm{SO}(n)$ (in the sense of the Haar measure on $\mathrm{SO}(n)$) one has the bound*

$$S(U\Lambda, r) = O\left(r^{2(n-1)}\varphi(\log r)\right) \tag{4.32}$$

with arbitrary majorant $\varphi(\cdot)$. In particular, for almost all $U \in \mathrm{SO}(n)$ one has the bound

$$S(U\Lambda, r) = O\left(r^{2(n-1)+\varepsilon}\right) \tag{4.33}$$

with arbitrary small $\varepsilon > 0$.

Proof. Substituting the bound (4.28) in the relation (4.3), and taking (4.4) into account, we obtain the bounds (4.32) and (4.33).

The proof of Lemma 4.4 is complete.

From 'metrical' results on the behavior of $\nu(\Lambda, \rho)$ with respect to the Haar measure μ_{SO} on the group $\mathrm{SO}(n)$ we can derive similar results concerning the behavior of $\nu(\Lambda, \rho)$ with respect to the measure $\mu_{\mathcal{L}}$ on the space of lattice $\mathcal{L}_n$.

Lemma 4.5. *For almost all lattices $\Lambda \in \mathcal{L}_n$ (in the sense of the invariant measure $\mu_{\mathcal{L}}$) one has the bound*

$$\nu(U\Lambda, \rho) > c_\varphi(\Lambda)\left((\log\rho)^{n-1}\varphi(\log\log\rho)\right)^{-1}, \quad \rho \to \infty, \tag{4.34}$$

where $\varphi(\cdot)$ is arbitrary majorant and $c_\varphi(\Lambda) > 0$ is a constant. In particulat, for almost all $\Lambda \in \mathcal{L}_n$ one has the bound

$$\nu(\Lambda, \rho) > c_\varepsilon(\Lambda)(\log\rho)^{n-1-\varepsilon}, \quad \rho \to \infty, \tag{4.35}$$

where $\varepsilon > 0$ is arbitrary small and $c_\varepsilon(\Lambda) > 0$ is a constant.

Proof. We use here notations and facts given in Appendix 1 (§ 13). Using the polar decomposition of matrices (cf. (13.13)), we derive from Lemma 4.2 that the bound

$$\nu\left(UA^{\frac{1}{2}\mathbb{Z}^n}, \rho\right) > c_\varphi(U, A)\left((\log\rho)^{n-1}\varphi(\log\log\rho)\right)^{-1}, \quad \rho \to \infty, \tag{4.36}$$

is valid with arbitrary majorant $\varphi(\cdot)$ and some constant $c_\varphi(U, A) > 0$ for all symmetric positive definite matrices $A \in \mathcal{F}_n$, and all orthogonal matrices $U \in \mathrm{SO}(n) \setminus \mathcal{E}_A$, where the exceptional sunset $\mathcal{E}_A \subset \mathrm{SO}(n)$ and $\mu_{\mathrm{SO}}(\mathcal{E}_A) = 0$. From (4.36) we conclude, that the bounds (4.34), (4.35) are valid for all lattices $\Lambda \in \mathcal{L}_n \setminus \mathcal{E}$, where the exceptional subset $\mathcal{E} \subset \mathcal{L}_n$ is given by

$$\mathcal{E} = \left\{\Lambda = UA^{\frac{1}{2}}\mathbb{Z}^n \in \mathcal{L}_n \,:\, A \in \mathcal{F}_n,\, U \in \mathcal{E}_A\right\}. \tag{4.37}$$

Using the relations (13.12)–(13.14), we obtain from (4.37)

$$\mu_{\mathcal{L}}(\mathcal{E}) = \int_{\mathcal{F}_n} d\mu_{\mathcal{F}}(A)\, \mu_{\mathrm{SO}}(\mathcal{E}_A) = 0.$$

This proves Lemma 4.5.

In conclusion of this section we wish to state the following question, which naturally arises from the above ewsults. Let

$$Nm_q\Lambda = \inf\left\{(\log(2 + \|\gamma\|))^q \mid Nm\gamma \mid :\, \gamma \in \Lambda \setminus \{0\}\right\}. \tag{4.38}$$

The quantity (4.38) can also be regarded as a generalization of the definition (1.17).

Lemma 4.5 implies that for almost all $\Lambda \in \mathcal{L}_n$ the quantity $Nm_q > 0$ when $q > n-1$, while $Nm_0\Lambda = 0$ (as it was mentioned above in § 3). In the spirit of Khinchin's type theorems (cf. [Sp]) we can conjecture the existence of a critical exponent $\varkappa(n)$, such that for almost all lattices $\Lambda \in \mathcal{L}$ the quantity $Nm_q\Lambda > 0$ when $q > \varkappa(n)$ while $Nm_1\Lambda = 0$ when $q \le \varkappa(n)$; moreover $0 < \varkappa(n) \le n - 1$. With the help of continned fraction this conjecture can be proved in two dimensions, moreover, it turns out that $\varkappa(2) = 1$. Thus our bounds (4.34), (4.35) are best possible for $n = 2$. In dimensions $n > 0$ the question concerning the existence of the critical exponent $\varkappa(n)$ remains open.

§ 5. Simultaneous diophantine approximations and algebraic lattices

We recall definitions and facts from theory of simultaneous diophantine approximations of algebraic number. For details we refer to the book [Sch] by Schmidt.

A subspace $E \subset \mathbb{R}^n$ is called to be *rational* is E is defined by linear equations

$$r_{iq}x_1 + \ldots + r_{in}x_n = 0, \quad i = 1, \ldots, n-d, \quad 0 \le d \le n, \tag{5.1}$$

with a rational $(n-d)\times n$ matrix of coefficients $R=(r_{ij})$, $1\le i\le n-d$, $1\le j\le n$. We assume that the equation (5.1) are linearly independent. Therefore,

$$\dim E=\operatorname{rank} R=d. \tag{5.2}$$

A lattice $\Lambda\subset\mathbb{R}^n$ is called to be *rational* if $\Lambda=\alpha\mathbb{Z}^n$ (cf. (1.1)) with $\alpha=cA$, where the matrix $A\in\mathrm{GL}(n,\mathbb{Q})$ and $c\neq 0$ is a constant. For rational subspace E and lattice Λ the intersection $\Lambda\cap E$ is a rational lattice in E.

We consider a system of n linear forms

$$L_i(X)-a_{i1}x_1+\ldots+a_{in}x_n,\quad 1\le i\le n, \tag{5.3}$$

with real matrix of coefficients $A=(a_{ij})$. One has the following Schmidt's subspace theorem (cf. [Sch]):???

Let the forms (5.3) *be linerly independent with algebraic coefficients, and let* $\Lambda\subset\mathbb{R}^n$ *be a rational lattice. Then, for every* $\varepsilon>0$*, there are finitely many proper rational subspaces* $E_1,\ldots,E_w\subset\mathbb{R}^n$*,* $w=w(A,\Lambda,\varepsilon)$*, such that every point* $\zeta\in\Lambda\setminus\{0\}$ *with*

$$\mid L_1(\zeta)\ldots L_n(\zeta)\mid<\|\zeta\|^{-\varepsilon} \tag{5.4}$$

lies in one these subspaces.

We are interested in the question when the inequality (5.4) has no more than finitely many solutions $\zeta\in\Lambda\setminus\{0\}$. We give the following Definition: A system of forms (5.3) and the corresponding matrix of coefficients A are called to be *completely irrational* if every d forms $L_{i_1}(X),\ldots,L_{i_d}(X)$, $1\le d\le n$, from (5.3) are linearly independent on every D-dimensional rational subspace $E\subset\mathbb{R}^n$.

With the help of elementary linear algebra it is easy to check that the forms $L_{i_1}(X),\ldots,L_{i_d}(X)$ restricted on the subspace (5.1) are linearly independent if and only if

$$\det A_E(i_1,\ldots,i_j)\neq 0, \tag{5.5}$$

where $A_E(i_1,\ldots,i_j)$ is a $n\times n$ matrix whose first d rows coincide with the rows $i_1,\ldots,i_j$ of the matrix $A=(a_{ij})$ from (5.3), and last $n-d$ rows coincide with rows $1,\ldots,d$ of the matrix $R=(r_{ij})$ from (5.1). The set of conditions (5.5) written down for all indeces $i_1,\ldots,i_j$, $1\le d\le n$, and all rational subsapces $E\subset\mathbb{R}^n$ is a criterion for the system of forms (5.3) no be completely irrational.

Lemma 5.1. *Let* (5.3) *be a system of completely irrational forms with slgebraic coefficients, and let* $\Lambda\subset\mathbb{R}^n$ *be a rational lattice. Then for every* $\varepsilon>0$ *the inequality* (5.4) *has finitely many solutions* $\zeta\in\Lambda\setminus\{0\}$*. Moreover, for all* $\zeta\in\Lambda\setminus\{0\}$ *one has the inequality*

$$\mid L_1(\zeta)\ldots L_n(\zeta)\mid<C_\varepsilon\|\zeta\|^{-\varepsilon} \tag{5.6}$$

with a constant $C_\varepsilon>0$

Proof. We use induction on dimensions n. Obviously, Lemma 5.1 is valid for $n=1$. Let Lemma 5.1 be valid for all dimensions $d\le n-1$. Consider dimension n. By Schmidt's subspace theorem, all solutions of the inequality (5.4) lie in a finite number of proper rational subspaces $E_1,\ldots,E_w$. Let $E\subset\mathbb{R}^n$, $1\le d=\dim E\le n-1$, be one of these subspaces. Since the system of forms (5.3) is completely irrational, every d forms $L_{i_1}(X),\ldots,L_{i_d}(X)$ are linearly independent on E. Moreover, the system of d forms $L_{i_1}(X), \dots, L_{i_d}(X)$ is completely irrational on E. Using Lemma 5.1 for dimensions $d \leq n-1$, we obtain that for every $\theta > 0$ one has the inequality

$$| L_{i_1}(\zeta) \dots L_{i_d}(\zeta) |> c_\delta \|\zeta\|^{-\theta} \tag{5.7}$$

for all $\zeta \in \Lambda_1 \setminus \{0\}$, where $\Lambda_1 La \cap E$ is a rational lattices in the subspace E.

Writting down (5.7) for all $\binom{d}{n}$ choice of d forms from the system of n forms (5.3) and multiplying together all these inequalities we find that

$$| L_1(\zeta) \dots L_n(\zeta) |^{\binom{d-1}{n-1}}> \left(c_\delta \|\zeta\|^{-\theta}\right)^{\binom{d}{n}} \tag{5.8}$$

for every $\zeta \in \Lambda_1 \setminus \{0\}$. The inequality (5.8) is valid for all subsapces $E_1, \dots, E_w$. Therefore,

$$| L_1(\zeta) \dots L_n(\zeta) |> c_\delta^{\frac{n}{d}} \|\zeta\|^{-\frac{n}{d}\theta} \tag{5.9}$$

for every $\zeta \in \Lambda \setminus \{0\}$.

From (5.9) we conclude that for every $\varepsilon > \frac{n}{d}\theta$ the inequality (5.4) has finitely many solutions $\zeta \in \Lambda \setminus \{0\}$, and that the inequality (5.6) holds for all $\zeta \in \Lambda \setminus \{0\}$. Since $\theta > 0$ is arbitrarily small, Lemma 5.1 is valid for dimension n.

The proof of Lemma 5.1 is complete.

Let $\Lambda = A\mathbb{Z}^n \subset \mathbb{R}^n$ be a lattice (cf. (1.1)), and let (5.3) be a system of forms with the same matrix of coefficients $A = (a_{ij})$. Then points $\gamma \in \Lambda$ have the shape

$$\gamma = A\zeta \left(L_1(\zeta), \dots, L_n(\zeta)\right), \qquad \zeta \in \mathbb{Z}^n. \tag{5.10}$$

Moreover (cf. (1.18)),

$$Nm\, \gamma = L_1(\zeta) \dots L_n(\zeta), \tag{5.11}$$

and

$$c_1\|\zeta\| > \|\gamma\| < c_2\|\zeta\| \tag{5.12}$$

with some sonstants $c_i = c_i(A) > 0$, $i = 1, 2$.

We give the following Definition: A lattice $\Lambda = A\mathbb{Z}^n$ is called to be *completely irrational* if the matrix A and the corresponding system of forms (5.3) are completely irrational. A lattice $\Lambda = \alpha\mathbb{Z}^n$ is called to be *algebraic* if the matrix $\alpha = cA$, where $A \in \mathrm{GL}(n, k)$ over a real algebraic number field k, and $c \neq 0$ is a constant.

Lemma 5.2. *Let $\Lambda \in \mathcal{L}_n$ be a completely irrational algebraic lattice, then the characteristivs $\nu(\Lambda, \rho)$ and $S(\Lambda, r)$ (cf. (4.1) and (3.15)) satisfy the bounds*

$$\nu(\Lambda, \rho) > c_\varepsilon \rho^{-\varepsilon}, \quad \rho \to \infty, \tag{5.13}$$

and

$$S(\Lambda, r) \leq c_\varepsilon\, 2^{\varepsilon r}, \tag{5.14}$$

with arbitrary small $\varepsilon > 0$ and some constants $c_\varepsilon = c_\varepsilon(\Lambda) > 0$.

Proof. For a completely irrational algebraic lattice Λ the corresponding forms (5.3) satisfy the assumption of Lemma 5.1. Thus, from the inequality (5.6) with the help of (5.10)–(5.12) we derive for all $\gamma \in \Lambda \setminus \{0\}$ the inequality

$$| Nm\, \gamma |> c_\varepsilon \|\gamma\|^{-\varepsilon} \tag{5.15}$$

with arbitrary snall $\varepsilon > 0$ and a constant $c_\varepsilon > 0$. The comparison of (5/15) with the definition (4.1) gives the bound (5.13). The bound (5.14) follows from (5.13) and Lemma 4.1.

The proof of Lemma 5.2 is complete.

Now we wish to give examples of completely irrational algebraic lattices. Let

$$\widetilde{k} : k = k_0 \subset k_1 \subset \ldots \subset k_n \tag{5.16}$$

be a tower of real algebraic number fields of finite degrees, satisfying the conditions

$$[k_{j+1} \, : \, k_j] \geq n + 1, \qquad j = 0, \ldots , n - 1. \tag{5.17}$$

Let $\mathrm{Cl}(\widetilde{k})$ denote a class of $n \times n$ matrices $A = (\zeta_{ij})$, where the elements $\xi_{qi}, \ldots , \xi_{ni}$ of the i-th column belong to the field k_j, $1 \leq i \leq n$, from the tower (5.16), and algebraic numbers $1, \xi_{i1}, \ldots , \xi_{in}$ are linerly independent over the field $k_{j-1}ck_j$. Because of the condition (5.17) the class $\mathrm{Cl}(\widetilde{k})$ is not empty.

It is not difficult to check that

$$\mathrm{GL}(n, k)\mathrm{Cl}(\widetilde{k}) = \mathrm{Cl}(\widetilde{k}), \tag{5.18}$$

i.e., for every matrices $\alpha \in \mathrm{GL}(n, k)$ and $A \in \mathrm{Cl}(\widetilde{k})$ the matrix $\alpha A \in \mathrm{Cl}(\widetilde{k})$. Firthermore, elements of the trnsor product of the following sets of algebraic numbers

$$\begin{aligned} &\{1;\, \xi_{11}; \ldots ; \xi_{n1}\} \times \ldots \times \{1;\, \xi_{n1}; \ldots ; \xi_{nn}\} = \\ &= \{1;\, \xi_{11}; \ldots ; \xi_{nn};\, \xi_{11}\xi_{21}; \ldots ; \xi_{n-1,n}\xi_{nn};\, \xi_{11} \ldots \xi_{n1}; \ldots ; \xi_{n1} \ldots \xi_{nn}\} \end{aligned} \tag{5.19}$$

are linearly independent over $\mathbb{Q}$. In particular, for every, d, $1 \leq d \leq n$, elements of the following set of algebraic numbers

$$\left\{ \prod_{l=1}^{d} \zeta_{li_l}, \; 1 \leq i_1 < \ldots < i_d \leq n \right\} \tag{5.20}$$

are linearly independent over $\mathbb{Q}$. This implies by the way, that all matrices $A \in \mathrm{Cl}(\widetilde{k})$ are nondegenerate. Indeed,

$$\det A = \sum_{1 \leq i_1 < \ldots < i_n \leq n} \pm \prod_{l=1}^{d} \zeta_{li_l} \neq 0. \tag{5.21}$$

In fact, a more general assertion can be proved.

Lemma 5.3. *Every matrix $A \in \mathcal{C}_k$ is completely irrational.*

Proof. It is sufficient to check that the conditions (5.5) are valid. Let E be a rational subspace of dimension d given by (5.1). Choose any d forms from the system (5.3) defined by a given matrix $A \in \mathrm{Cl}(\widetilde{k})$. Taking the relation (5.18) into account, we nay assume that these d forms are $L_1(X), \ldots , L_d(X)$. Therefore, the matrix $A_E(\cdot)$ in (5.5) takes the shape

$$A_E(1, \ldots , d) = \begin{pmatrix} \xi_{11} & \ldots & \xi_{1n} \\ \vdots & & \vdots \\ \xi_{j1} & \ldots & \xi_{dn} \\ r_{11} & \ldots & r_{1n} \\ \vdots & & \vdots \\ r_{n-d,d} & \ldots & r_{n-d,n} \end{pmatrix} \tag{5.22}$$

The Laplace expansion with respect to $n-d$ rational rows of the matrix (5.22) gives

$$\det A_E(1,\dots d) = \sum_{1\le i_1<\dots<i_j\le d} \pm R(t_1,\dots,i_{i_d}) \prod_{l=1}^{d} \zeta_{li_l}, \tag{5.23}$$

where $R(i_1,\dots,i_j)$ is the minor of order $n-d$ obtained from the rational $(n-d)\times n$ matrix $R=(r_{ij})$, $1\le i\le n-d$, $1\le j\le n$, by cancellation of columns $i_1,\dots,i_j$. Not all these minors are iqual to zero in view of (5.2).

Hence the determinant (5.23) is not equal to zero, since algebraic numbers (5.20) are linearly independent over $\mathbb{Q}$.

The prrof of Lemma 5.3 is complete.

By definition, a lattice $\Lambda\subset\mathbb{R}^n$ is said to belong to the class $\mathrm{Cl}(\widetilde{k})$ if $\Lambda=\alpha\mathbb{Z}^n$, and $\alpha=cA$ where the matrix $A\in\mathrm{Cl}(\widetilde{k})$ and $c\neq 0$ is a constant.

Lemma 5.4. *Let a lattice $\Lambda\in\mathcal{L}$ belong to the class $\mathrm{Cl}(\widetilde{k})$, and $\delta\in D$ be arbitrary diagonal unimodular matrix, $\beta\in\mathrm{GL}(n,k)$, and $\widehat{\beta}=(\det\beta)^{-\frac{1}{4}}\beta\in\mathrm{SL}(n,\mathbb{R})$, then the characteristics $\nu(\Lambda,\rho)$ and $A(\Lambda,r)$ (cf. (4.1) and (3.15)) satisfy the bounds*

$$\nu\left(\delta\widehat{\beta}\Lambda,\rho\right) > c_\varepsilon\rho^{-\varepsilon}, \qquad \rho\to\infty, \tag{5.24}$$

and

$$S\left(\delta\widehat{\beta}\Lambda,r\right) < c_\varepsilon 2^{\varepsilon r}, \tag{5.25}$$

with arbitrary small $\varepsilon>0$ and some constants $c_\varepsilon=c_\varepsilon(\delta,\beta,\Lambda)>0$.

Proof. Let $\delta=\mathrm{diag}(d_1,\dots,d_n)\in D$, $d_j>0$, $1\le j\le n$, (cf. (3.1)). Then for every lattice $\Gamma\in\mathcal{L}_n$ one has the inequalities

$$\nu\left(\Gamma,\frac{\rho}{d_+}\right)\ge\nu(\delta\Gamma,\rho)\ge\nu\left(\Gamma,\frac{\rho}{d_-}\right) \tag{5.26}$$

where

$$d_-=\min\{d_1,\dots,d_n\},\quad d_+=\min\{d_1,\dots,d_n\}.$$

Indeed, we obtain from the definition (4.1)

$$\nu(\delta\Gamma,\rho)=\min\left\{|\,Nm\,\delta\gamma\,|:\gamma\in\Gamma,\ 0<\|\delta\gamma\|<\rho\right\}.$$

At the same time we have $Nm\,\delta\gamma=Nm\,\gamma$ and

$$\|d_-\|\gamma\|\le\|\delta\gamma\|\le d_=\|\gamma\|.$$

This proves the inequalities (5.26).

Now the bound (5.24) follows from (5.13) with the help of (5.18) and (5.26). The bound (5.25) follows from the bound (5.24) and Lemma 4.1.

The proof of Lemma 5.4 is complete.

§ 6. Main bound for the lattice point counting for a polyhedron. Proof of Theorems 2.1, 2.2, 2.3

Now we are in position to state our main boind for the lattice point counting for a polyhedron. Let $P \subset \mathbb{R}^n$ be a compact convex polyhedron, and let

$$\mathfrak{f} = \left\{ P = P^n \supset P^{n-1} \supset \ldots \supset P^0 ,\ \dim P^j = j,\ n \geq j \geq o \right\} \tag{6.1}$$

be a flag of faces $P^j = \mathrm{Face} P$, $1 \leq j \leq n$, of P (cf. Berger [Ber, Chapter 12]). Let $\nu_{\mathfrak{f}} \in \mathrm{SO}(n)$ be an orthogonal matrix associated with the flag (6.1) by the relation

$$^{T}V_{\mathfrak{f}} = \left[l^n, l^{n-1}, \ldots, l^1 \right] . \tag{6.2}$$

The columns of the matrix $^{T}v_g f$ in (6.2) are orthonormal vectors l^j, $n \geq j \geq 1$, given by the following rule: the unit vector l^j is the external normal to the face P^{j-1} parallel to the face P^j. Thus coordinates of the vector l^j forms the j-th row of the matrix $V_{\mathfrak{f}}$.

For given lattice $\Gamma \in \mathcal{L}_n$ and flag of faces $\mathfrak{f}$ of P we define a lattice $\Gamma_{\mathfrak{f}} \in \mathcal{L}_n$ and the dual lattice $\Gamma_{\mathfrak{f}}^{\perp}$ by

$$\Gamma_{\mathfrak{f}} = V_{\mathfrak{f}} \Gamma, \quad \Gamma_{\mathfrak{f}}^{\perp} = V_{\mathfrak{f}} \Gamma^{\perp}. \tag{6.3}$$

Theorem 6.1. *Let arbitrary compact convex polyhedron $P \subset \mathbb{R}^n$ and a lattice $\Gamma \in \mathcal{L}_n$ be given, then the error $R(tP, \Gamma)$ (cf. (1.7)) satisfies the bound*

$$R(tP, \Gamma) < c(P, \Gamma, \theta) \left(t^{n-1} \rho^{-\theta} + \sum_{\mathfrak{f}} S\left(\Gamma_{\mathfrak{f}}^{\perp}, t_{\mathfrak{f}}\right) \right) , \tag{6.4}$$

with

$$t_{\mathfrak{f}} = \log \left(\frac{\rho^n}{\nu(\Gamma_{\mathfrak{f}}^{\perp}, \rho)} \right) , \tag{6.5}$$

here $\rho > 1$ is arbitrarily large parameter, $0 < \theta < 1$ is arbitrarily fixed, the lattaces $\Gamma_{\mathfrak{f}}^{\perp}$ are defined in (6.3), the characteristics $S(\cdot,\cdot)$ and $\nu(\cdot,\cdot)$ are given in (3.15) and (4.1), respectively, the summation in (6.4) is taken over all flags of faces $\mathfrak{f}$ of the polyhedron P; finally, the constant $c(P, \Gamma, \theta)$ in (6.4) is independent of the parameters $t \to \infty$ and $\rho \to \infty$.

The proof of Theorem 6.1 will be given in § 11.

It should be noted that the bound (6.4) has meaning only if all lattices $\Gamma_{\mathfrak{f}}^{\perp}$ are weakly admissible. Otherwise the right-hand side of (6.4) is equal to infinity.

At first, we wish to show how to use Theorem 6.1 in proving of the bound (1.19) of Theorem 1.2. Let $\Pi \subset \mathbb{R}^n$ be a parallelepiped with edgse parallel to the coordinate exes. In this case the matrices $V_{\mathfrak{f}}$ in (6.2) coincide with all permutations and reorientations of the coordinate axes, which form a finite group of order $2^n\, n!$. Therefore, we have the relations

$$S\left(\Gamma_{\mathfrak{f}}^{\perp}, r\right) = S\left(\Gamma^{\perp}, r\right) , \quad \nu\left(\Gamma_{\mathfrak{f}}^{\perp}, \rho\right) = \nu\left(\Gamma^{\perp}, \rho\right) \tag{6.6}$$

for every flag of faces $\mathfrak{f}$ of the parallelepiped Π. With the help of (6.6) the bound (6.4) takes the shape

$$R(t\Pi, \Gamma) < C(\Pi, \Gamma, \theta) \left(t^{n-1} \rho^{-\theta} + S(\Gamma^{\perp}, r) \right) \tag{6.7}$$

with

$$r = \log\left(\frac{\rho^n}{\nu(\Gamma^\perp, \rho)}\right), \tag{6.8}$$

moreover, the constant $C(\Pi, \Gamma, \theta)$ in (6.7) is independent of the parameters $t \to \infty$ and $\rho \to \infty$.

Now let $\Gamma \in \mathcal{L}_n$ be an admissible lattice, then the lattice $\Gamma^\perp$ is also admissible (cf. (3.10)). Thus, $\nu(\Gamma^\perp, \rho) \geq Nm\, \Gamma^\perp > 0$ as $\rho \to \infty$ (cf. (4.2)), and we obtain from (6.8)

$$r = O(\log \rho). \tag{6.9}$$

Using the bounds (3.16) and (6.9) we conclude that

$$S(\Gamma^\perp, r) = O(r^{n-1}) = O\left((\log \rho)^{n-1}\right). \tag{6.10}$$

Substituting (6.10) in (6.7), we obtain the bound

$$R(t\Pi, \Gamma) = O\left(t^{n-1}\rho^{-\theta} + (\log \rho)^{n-1}\right). \tag{6.11}$$

Choosing $\rho^\theta = t^{n-1}$ in (6.11) we obtain the bound (1.19). This is a new proof of the bound (1.19) different from the proof given in [Skr 3].

Turning to the proof of Theorems 2.1, 2.2, 2.3 we wish to show that without loss of generality the polyhedron P may be assumed to be convex.

Let $P \subset= BR^n$ be arbitrary compact polyhedron. The hyperplanes of the $n-1$ dimensional faces of P devide P into a finite number of disjoint convex polyhedra $\{\mathcal{P}_j\}_1^h$. This leads to the inequality (cf. (1.5))

$$\left| N(tP + X, \Gamma) - \sum_{j=1}^{h} N(t\mathcal{P}_j + X, \Gamma) \right| \leq \sum_{j=1}^{h} N(\partial t\mathcal{P}_j + X, \Gamma), \tag{6.12}$$

where $N(\partial t\mathcal{P}_j + X, \Gamma)$ denotes the number of lattice points $\gamma \in \Gamma$ lying on the boundary $\partial t\mathcal{P}_j + X$ of the polyhedron $t\mathcal{P}_j + X$. Moreover,

$$\operatorname{vol} P = \sum_{j=1}^{h} \operatorname{vol} \mathcal{P}_j. \tag{6.13}$$

We have the relation

$$t\mathcal{P}_j + X \subset (t+\varepsilon)\mathcal{P}_j + Y, \quad 0 < \varepsilon < 1,$$

with a vector $Y = Y_j(X, \varepsilon)$. Thus,

$$N(\partial t\mathcal{P}_j + X, \Gamma) \leq N\left((t+\varepsilon)\mathcal{P}_j + Y, \Gamma\right) - N(t\mathcal{P}_j + X, \Gamma). \tag{6.14}$$

Substituting (1.6) in (6.14) and using (1.7) we obtain the inequality

$$N(\partial t\mathcal{P}_j + X, \Gamma) \leq ((t+\varepsilon)^n) \operatorname{vol} \mathcal{P}_j + R\left((t+\varepsilon)\mathcal{P}_j, \Gamma\right) + R)t\mathcal{P}_j, \Gamma),$$

which leads to the following bound as $\varepsilon \to 0$

$$N(\partial t\mathcal{P}_j + X, \Gamma) \leq R\left((t+o)\mathcal{P}_j, \Gamma\right) + R(t\mathcal{P}_j, \Gamma). \tag{6.15}$$

Substituting the bound (6.15) in (6.12) and using (1.6), (1.7), we find that

$$R(tP,\Gamma) \leq 2\sum_{j=1}^{h}\left(R(t\mathcal{P}_j,\Gamma) + R((t+o)\mathcal{P}_j,\gamma)\right). \tag{6.16}$$

From the relation (6.16) we immediately conclude that if the assertion of Theorems 2.1, 2.2, 2.3 are valid for the convex polyhedra $\{\mathcal{P}_j\}_1^h$, then these assertions are valid for the (nonconvex) polyhedron P.

Thus it is sufficient to prove Theorems 2.1, 2.2, 2.3 only for convex polyhedra.

Proof of Theorem 2.1. Let $P \subset \mathbb{R}^n$ be arbitrary compact convex polyhedron. Since the sets of flags of faceses $\mathfrak{f}$ of P and associated orthogonal matrices $V_{\mathfrak{f}}$ are finite, we derive from Lemmas 3.2 and 4.5 that for every flag $\mathfrak{f}$ and for almost all lattices $\Gamma \in \mathcal{L}_n$ (in the sense of the invariant measure $\mu_{\mathcal{L}}$) one has the bounds

$$S\left(\Gamma_{\mathfrak{f}}^{\perp}, r\right) = O\left(r^{n-1}\varphi(\log r)\right), \quad r \to \infty, \tag{6.17}$$

with arbitrary majorant $\varphi(\cdot)$, and the bounds

$$\nu\left(\Gamma_{\mathfrak{f}}^{\perp}, r\right) > c_{\varepsilon}(\Gamma_{\mathfrak{f}}^{\perp})\,(\log r)^{1-n-\varepsilon}, \quad \rho \to \infty, \tag{6.18}$$

with arbitrarily small $\varepsilon > 0$, and constants $c_{\varepsilon}(\Gamma_{\mathfrak{f}}^{\perp}) > 0$. Recall that the lattice $\Gamma_{\mathfrak{f}}^{\perp}$ in (6.17) and (6.18) are defined in (6.3).

From the bounds (6.18) and the relation (6.5) we find that for every flag $\mathfrak{f}$ and for almost all lattices $\Gamma \in \mathcal{L}_n$ one has the bound

$$r_{\mathfrak{f}} < \log\left(c_{\varepsilon}^{-1}(\Gamma_{\mathfrak{f}}^{\perp})\rho^n(\log\rho)^{-1+n+\varepsilon}\right) = O(\log\rho). \tag{6.19}$$

Substituting (6.17) and (6.19) in the main bound (6.4) we find that for almost all lattices $\Gamma \in \mathcal{L}_n$ one has the bound

$$R(tP,\Gamma) = O\left(t^{n-1}\rho^{-\theta} + (\log\rho)^{n-1}\varphi(\log\log\rho)\right). \tag{6.20}$$

Choosing $\rho^{\theta} - t^{n-1}$ in (6.20) we obtain the bound (2.5).

The prrof of Theorem 2.1 is complete.

Proof of Theorem 2.2. Let $P \subset \mathbb{R}^n$ be arbitrary compact convex polyhedron, and let $\Gamma \in \mathcal{L}_n$ be arbitrary lattice. Let $\Gamma_U = U\Gamma$, $U \in \mathrm{SO}(n)$, then we obtain from (6.3) the following relations

$$(\Gamma_U)_{\mathfrak{f}} = V_{\mathfrak{f}}U\Gamma, \quad (\Gamma_U)_{\mathfrak{f}}^{\perp} = V_{\mathfrak{f}}Ga^{\perp}, \tag{6.21}$$

which are valid for all flags $\mathfrak{f}$ of faces of the polyhedron P.

We derive from Lemmas 4.3 and 4.4 and from the relations (6.21) that for every flag $\mathfrak{f}$ and for almost all orthogonal matrices $U \in \mathrm{SO}(n)$ (in the sense of the Haar measure on the group $\mathrm{SO}(n)$) one has the bounds

$$\nu\left((\Gamma_U)_{\mathfrak{f}}^{\perp}, \rho\right) > c_{\varepsilon}\left(V_{\mathfrak{f}}U, \Gamma^{\perp}\right)\,(\log\rho)^{1-n-\varepsilon}, \quad \varepsilon \to \infty, \tag{6.22}$$

with arbitrary small $\varepsilon > 0$ and a constant $c_\varepsilon(V_\mathfrak{f} U, \Gamma^\perp) > 0$, and the bounds

$$S\left((\Gamma_U)_\mathfrak{f}^\perp, r\right) = O\left(r^{2(n-1)}\varphi(\log r)\right), \quad r \to \infty, \tag{6.23}$$

with arbitrary majorant $\varphi(\cdot)$.

From the bounds (6.22) and the relation (6.5) we find that for every flag $\mathfrak{f}$ and for almost all orthogonal matrices $U \in \mathrm{SO}(n)$ one has the bound

$$r_\mathfrak{f} = O(\log \rho). \tag{6.24}$$

Substituting (6.23) and (6.24) in the mean bound (6.4) we find that for almost all $U \in \mathrm{SO}(n)$ one has the bound

$$R(tP, U\Gamma) = O\left(t^{n-1}\rho^{-\theta} + (\log\rho)^{2(n-1)}\varphi(\log\log\rho)\right). \tag{6.25}$$

Choosing $\rho^\theta = t^{n-1}$ in (6.25) we obtain the bound (2.7).

The proof of Theorem 2.2 is complete.

Proof of Theorem 2.3. Let $P_k \subset \mathbb{R}^n$ be arbitrary compact convex polyhedron, such that all of its vertices have coordinates belonging to a real algebraic number field k of finite degree, and let $\Omega_k \in \mathcal{L}_n$ be a special algebraic lattice constructed in terms of the tower of real algebraic number fields (2.9) by the relations (2.11) and (2.12).

Consider the orthogonal matrices $V_\mathfrak{f}$ associated with the flags of faces $\mathfrak{f}$ of the polyhedron P_k (cf. (6.2)). By the construction of the orthonormal vectors $l_1, \dots, l_n$ in (6.2) can be written dowm as follows

$$l_j = \frac{q_j}{\|q_j\|}, \quad 1 \le j \le n, \tag{6.26}$$

where $q_1, \dots, q_n \neq 0$ are pairwise orthogonal vectors with coordinates in the field k. Substituting (6.26) in (6.2) we obtain the following representation of the matrices $V_\mathfrak{f}$:

$$V_\mathfrak{f} = \delta_\mathfrak{f}\, \widehat{\beta}_\mathfrak{f}, \tag{6.27}$$

where

$$\delta_g f = c_\mathfrak{f} \mathrm{diag}\left(\|q_1\|^{-1}, \dots, \|q_n\|^{-1}\right) \in D \subset \mathrm{SL}(n, \mathbb{R}),$$
$$\beta_\mathfrak{f} =^{\mathrm{T}} (q_1, \dots, q_n) \in \mathrm{GL}(n, k), \quad \widehat{\beta}_g f = c_\mathfrak{f}^{-1}\beta_\mathfrak{f} \in \mathrm{SL}(n, \mathbb{R}),$$
$$c_\mathfrak{f} = (\det\beta_\mathfrak{f})^{\frac{1}{n}} = \left(\prod_{j=1}^n \|q_j\|\right)^{\frac{1}{n}}.$$

By definition (cf. § 5), the dual lattice $\Omega_k^\perp$ (cf. (2.12)) belongs to the class $\mathrm{Cl}(\widetilde{k})$. From Lemma 5.4 and (6.27) we derive that for every flag $\mathfrak{f}$ of the polyhedron P_k one has the bounds

$$\nu\left(V_\mathfrak{f}\,\Omega_k^\perp, \rho\right) > c_\varepsilon \rho^{-\varepsilon}, \quad \rho \to \infty \tag{6.28}$$

and

$$S\left(V_\mathfrak{f}\,\Omega_k^\perp, r\right) < c_\varepsilon\, 2^{epr}, \quad r \to \infty \tag{6.29}$$

with arbitrarily small $\varepsilon > 0$ and some constants $c_\varepsilon = c_\varepsilon(f, \Omega_k^\perp) > 0$.

From the bounds (6.28) and the relation (6.5) we find that for every flag $\mathfrak{f}$ one has the bound

$$r_\mathfrak{f} < \log\left(c_\varepsilon^{-1}\rho^{n+\varepsilon}\right) = O(\log\rho). \tag{6.30}$$

Substituting (6.29) and (6.30) in the main bound (6.4) we find that one has the bound

$$R\left(tP_k, \Omega_k^\perp\right) = O\left(t^{n-1}\rho^{-\theta} + \rho^\varepsilon\right) \tag{6.31}$$

with arbitrarily small $\varepsilon > 0$. Choosing $\rho^\theta = t^{n-1}$ in (6.31) we obtain the bound (2.13).

The proof of Theorem 2.3 is complete.

CHAPTER II. PROOF OF THE MAIN BOUND FOR THE LATTICE POINT COUNTING IN A POLYHEDRON

§ 7. Dyadic minima of a lattice

Consider the action of the discrete subgroup Δ (cf. (3.3)) in the positive orthant $\mathbb{R}_+^n$. We introduce the following *hyperbolic coordinates* in $\mathbb{R}_+^n$: for $X = {}^{\mathrm{T}}(x_1, \dots, x_n) \in \mathbb{R}_+^n$ we put

$$x_j = (Nm\, X)^{\frac{1}{n}} 2^{y_j}, \quad 1 \leq j \leq n, \tag{7.1}$$

where $Y = {}^{\mathrm{T}}(y_1, \dots, y_n) \in \mathcal{L}$ (cf. (3.2)). In the coordinates (7.1) the action of the group Δ takes the form (cf. (3.3))

$$\Delta \ni \delta_m : X \to \delta_m X = (Nm\, X)^{\frac{1}{n}}\ {}^{\mathrm{T}}\!\left(2^{m_1+y_1}, \dots, 2^{m_n+y_n}\right). \tag{7.2}$$

From (7.2) we conclude that the following cone in $\mathbb{R}_+^n$ is a fundamental set $\mathcal{F}_\Delta = \mathbb{R}^n/\Delta$ of the discrete subgroup Δ:

$$\mathcal{F}_\Delta = \left\{X \in \mathbb{R}_+^n : X = t\,(2^{y_1}, \dots, 2^{y_n}), \quad t > 0, \quad Y = {}^{\mathrm{T}}(y_1, \dots, y_n) \in F_M\right\}, \tag{7.3}$$

where $F_M = \mathcal{L}/M$ is a fundamentas set of the lattice $M = \mathbb{Z}^n \cup \mathcal{L}$ in the subspace $\mathcal{L}$ (cf. (3.3)).

Further we assume that the fundamental set F_M in (7.3) is taken to be the Dirichlet–Voronoǒ region of the lattice $M \subset \mathcal{L}$ centered at the origin (cf. [GL]), and let

$$2\tau_n = \text{diameter} \quad F_M. \tag{7.4}$$

Obviously, the quantity (7.4) depends only the dimension n.

From the definitions (7.3) and (7.4) we directly derive the following.

Lemma 7.1. *Let $X = {}^{\mathrm{T}}(x_1, \dots, x_n) \in \mathcal{F}_\Delta$, then one has the following inequalities:*

$$2^{-\tau_n}(Nm\, X)^{\frac{1}{n}} \leq x_j \leq 2^{\tau_n}(Nm\, X)^{\frac{1}{n}}, \tag{7.5}$$

$$\frac{1}{\sqrt{n}} 2^{-2\tau_n}\|X\| \leq x_j \leq \frac{1}{\sqrt{n}} 2^{2\tau_n}\|X\|, \tag{7.6}$$

$$2^{-2\tau_n} \leq \frac{x_i}{x_j} \leq 2^{2\tau_n} \tag{7.7}$$

for any $i, j = 1, \dots, n$.

We wish to evaluate the size of a matrix $\delta \in \Delta$ which is needed to transfer a given point $X \in \mathbb{R}_+^n$ into the fundamental set (7.3).

Theorem 7.2. *Let $X \in \mathbb{R}_+^n$, and let a diagonal matrix $d\langle X\rangle \in \Delta$ be uniquely defined by*

$$d\langle X\rangle X \in \mathcal{F}_\Delta, \tag{7.8}$$

then both of the matrices $d\langle X\rangle$ and $\{d\langle X\rangle\}^{-1}$ belong to the finite set (cf. (3.4)) with

$$r \leq 2\tau_n + \left|\log \frac{\|X\|^n}{Nm\, X}\right|. \tag{7.9}$$

(Notice that in Lemma 7.2 as well as Lemmas 7.6 and 7.7 below, we use logarithms with the base 2).

Proof. Let (cf. (3.3))

$$d\langle X\rangle = \mathrm{diag}\,(2^{m_1},\dots,2^{m_n}),\ \{d\langle X\rangle\} = \mathrm{diag}\,\left(2^{-m_1},\dots,2^{-m_n}\right).$$

Using the inequality (7.5) for the point $d\langle X\rangle X\in\mathcal{F}_\Delta$ we obtain

$$2^{-\tau_n}(Nm\,X)^{\frac{1}{n}}\le 2^{m_j}x_j\le 2^{\tau_n}(Nm\,X)^{\frac{1}{n}}, \tag{7.10}$$

since $Nm\,d\langle X\rangle X = Nm\,X$. Moreover,

$$Nm\,X = x_1\dots x_n\le \|X\|^{n-1}\min\{x_1\dots x_n\}.$$

Therefore, the ineqiality (7.10) gives

$$2^{m_j}\le 2^{\tau_n}\frac{(Nm\,X)^{\frac{1}{n}}}{x_j}\le 2^{\tau_n}\frac{(Nm\,X)^{\frac{1}{n}}}{\min\{x_1,\dots,x_n\}}\le 2^{\tau_n}\left(\frac{\|X\|^n}{Nm\,X}\right)^{\frac{n-1}{n}}, \tag{7.11}$$

and similarly,

$$2^{m_j}\ge 2^{-\tau_n}\frac{(Nm\,X)^{\frac{1}{n}}}{x_j}\ge 2^{-\tau_n}\frac{(Nm\,X)^{\frac{1}{n}}}{\|X\|}= 2^{-\tau_n}\left(\frac{\|X\|^n}{Nm\,X}\right)^{-\frac{1}{n}} \tag{7.12}$$

Taking logs in (7.11) and (7.12) we obtain the inequality

$$-\tau_n-\frac{1}{n}\log\frac{\|X\|^n}{Nm\,X}\le m_j\le r_n+\left(1-\frac{1}{n}\right)\log\frac{\|X\|^n}{Nm\,X},$$

which implies the following

$$|m_j|\le 2\tau_n+\left|\log\frac{\|X\|^n}{Nm\,X}\right|.$$

The prrof of Lemma 7.2 is complete.

We consider the following family of parallelepipeds in $\mathbb{R}^n$ with edges parallel to the coordinate axes depending on a vector $X\in\mathbb{R}^n_+$:

$$\Pi\langle X\rangle = \left\{Y = {}^{\mathrm{T}}(y_1,\dots,y_n)\in\mathbb{R}^n : |y_j|<x_j,\quad 1\le j\le n\right\}. \tag{7.13}$$

For a given lattice $\Lambda\subset\mathbb{R}^n$ a parallelepiped $\Pi\langle X\rangle$ is called to be *empty* if $\Pi\langle X\rangle$ does not contain lattice points $\gamma\in\Lambda$ other than the origin.

In two dimensions empty parallelepipeds (parallelograms) $\Pi\langle X\rangle$ can be explicitly given in terms of continned fractions. In dimensions $n>2$ an explicit description of empty parallelepipeds (7.13) is a very difficult problem in geometry of numbers. This problem can be treated as a geometric background of the multi-dimensional continned fraction algorithms. For details we refer to the books [Br] by Brentjes and [DF] by Delone and Faddeev. Fortunately, for the purposes of the present work we need to study empty parallelepipeds $\Pi\langle X\rangle$ only for very special vectors $X\in\mathbb{R}^n_+$.

We give the following **Definition**: Points of the shape

$$Q_m = {}^{\mathrm{T}}(2^{m_1}, \dots, 2^{m_{n-1}}, q_n(m)) \in \mathbb{R}^n_+, \quad m = {}^{\mathrm{T}}(m_1, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}, \tag{7.14}$$

are called to *dyadic minima* of a weakly admissible lattice $\Lambda \subset \mathbb{R}^n$ if the n-th coordinate $q_n(m)$ is defined by

$$q_n(m) = \sup\{q: \text{ the parallelepiped } \Pi\langle {}^{\mathrm{T}}(2^{m_1}, \dots, 2^{m_{n-1}}, q)\rangle \text{ is empty }\}, \tag{7.15}$$

or, that is the same,

$$q_n(m) = \min\left\{\lambda_n : \lambda = {}^{\mathrm{T}}(\lambda_1, \dots, \lambda_n) \in \Lambda, \quad |\lambda_j| < 2^{m_j}, \quad 1 \le j \le n-1\right\}. \tag{7.16}$$

Notice that for every dyadic minimum Q_m the parallelepiped $\Pi\langle Q_m\rangle$ is empty but $\Pi\langle Q_m\rangle$ has lattice points $\pm\gamma \in \Lambda \setminus \{0\}$ on the boundary.

We denote ny $\mathcal{M}(\Lambda 0$ the set of all dyadic minima of a given weakly admissible lattice Λ. Obviously, for every diagonal matrix $\delta \in \Delta$ one has the relation

$$\delta\mathcal{M}(\Lambda) = \mathcal{M}(\delta\Lambda). \tag{7.17}$$

Dyadic minima are related with homogeneous minima (3.6) and the characteristic (4.1) by the following.

Lemma 7.3. *For every dyadic minimum $Q_m \in \mathcal{M}(\Lambda 0$ one has the relations*

$$2^{-\tau_n}(Nm\, Q_m)^{\frac{1}{n}} \le \|d\langle Q_m\rangle\Lambda\| \le \sqrt{n}2^{\tau_n}(Nm\, Q_m)^{\frac{1}{n}} \tag{7.18}$$

and

$$Nm\, Q_m \ge \nu\left(\Lambda, \|Q_m\|\right). \tag{7.19}$$

Proof. We put $Y = d\langle Q_m\rangle Q_m$ and $\Lambda_1 = d\langle Q_m\rangle\Lambda$. By the relation (7.17) the point $Y = {}^{\mathrm{T}}(y_1, \dots, y_n) \in \mathbb{R}^n_+$ is a dyadic minimum of the lattice Λ_1. Hence the parallelepiped $\Pi\langle Y\rangle$ is empty but $\Pi\langle Y\rangle$ has points $\pm\gamma \in \Lambda_1 \setminus \{0\}$ on the biundary. Now using this gact together with the definition (3.6) of the homogeneous minimum we derive the inequalities

$$\min\{y_1, \dots, y_n\} \le \|\Lambda_1\| \le \|Y\| \le \sqrt{n}\max\{y_1, \dots, y_n\}. \tag{7.20}$$

By the definition (7.8) the point $Y \in \mathcal{F}_\Delta$. Using the inequalities (7.5) and (7.20) we obtain the relation (7.18).

Let $\pm\gamma \in \Lambda_1 \setminus \{0\}$ be lattice points on the boundary of an empty parallelepiped $\Pi\langle Q_m\rangle$, $Q_m \in \mathcal{M}(\Lambda)$. Obviously,

$$Nm\, Q_m \ge \mid Nm\,\gamma_1 \mid \quad \text{and} \quad \|Q_m\| \ge \|\gamma_1\|. \tag{7.21}$$

Using (7.21) and the definition of the characteristic $\nu(\Lambda, \rho)$ (4.1) we obtain

$$Nm\, Q_m \ge \mid Nm\,\gamma_1 \mid \nu(\Lambda, \|\gamma_1\|) \ge \nu(\Lambda, \|Q_m\|),$$

since $\nu(\Lambda, \rho)$ is a nonincreasing function of ρ. This proves the relation (7.19).

The proof of Lemma 7.3 is complete.

The positive orthant $\mathbb{R}^n_+$ is a *partially ordered* set. Namely, we write $X \lhd Y$ for points $X = {}^{\mathrm{T}}(x_1, \dots, x_n)$ and $Y = {}^{\mathrm{T}}(y_1, \dots, y_n)\mathbb{R}^n_+$ if $x_j < y_j$, $1 \le j \le n$. A set $\mathcal{E} \subset \mathbb{R}^n_+$ is called to be *completely disordered* if the relation $X \lhd Y$ is not valid for any pair of points X and $Y \in \mathcal{E}$.

Lemma 7.4. *For every weakly admissible lattice Λ the set of dyadic minima $\mathcal{M}(\Lambda)$ is completely disordered.*

Proof. Let Q_m and $Q_{m'} \in \mathcal{M}(\Lambda)$. Let us assume that $Q_m \lhd Q_{m'}$, then (cf. (7.14))

$$m_j < m'_j \quad \text{for} \quad 1 \le j \le n-1, \tag{7.22}$$

and $q_n(m) < q_n(m')$. But the conditions (7.22) together with the definitions (7.15), (7.16) imply that $q_n(m) \ge q_n(m')$. This contradiction proves Lemma 7.4.

We consider the following family of regions in $\mathbb{R}^n_+$ depebdinding on a vector $Y \in \mathbb{R}^n_+$:

$$V(Y) = \left\{ X \in \mathbb{R}^n_+ : \frac{1}{\sqrt{n}} 2^{-4\tau_n - 1} \|Y\| \le x_j \le 2^{2\tau_n - 1} \|Y\|, \ \ 1 \le j \le n \right\}. \tag{7.23}$$

Lemma 7.5. *Let $Y \in \mathcal{F}_\Delta$, and let $X \in \mathcal{F}_\Delta$ but $X \notin V(Y)$, then either $Y \lhd X$ or $X \lhd Y$.*

Proof. Since $X \notin V(Y)$, we have (cf. (7.23))

$$\text{either} \quad x_J > 2^{2\tau_n - 1} \|Y\| \quad \text{or} \quad x_J < 2^{-4\tau_n - 1} \|Y\| \tag{7.24}$$

for an index J, $1 \le J \le n$.

Now we use the inequalities (7.6) and (7.7) for $X \in \mathcal{F}_\Delta$ and $Y \in \mathcal{F}_\Delta$. In the former case in (7.24) we obtain

$$x_j - y_j = x_j \left(1 - \frac{y_j}{x_j}\right) \ge x_j \left(1 - \frac{\|Y\|}{x_j}\right) \ge x_j \left(1 - 2^{2\tau_n} \frac{\|Y\|}{x_J}\right) > \frac{1}{2} x_j > 0$$

for all $j = 1, \dots, n$. Thus $Y \lhd X$. Similarly, in the latter case in (7.24) we obtain

$$y_j - x_j = y_j \left(1 - \frac{x_j}{y_j}\right) \ge y_j \left(1 - 2^{2\tau_n} \frac{x_J}{y_j}\right) \ge y_j \left(1 - \sqrt{n}\, 2^{4\tau_n} \frac{x_J}{\|Y\|}\right) > \frac{1}{2} y_j > 0$$

for all $j = 1, \dots, n$. Thus $X \lhd Y$.

The proof of Lemma 7.5 is complete.

Lemma 7.6. *For every weakly admissible lattice $\Lambda \subset \mathbb{R}^n$ the number of dyadic minima $Q_m \in \mathcal{M}(\Lambda)$ lying in the fundamental set $\mathcal{F}_\Delta$ (cf. (7.3)) is bounded by a constant depending only on the dimension n:*

$$\operatorname{card} \{\mathcal{M}(\Lambda) \cap \mathcal{F}_\Delta\} \le \left(6\tau_n + \frac{1}{2} \log n + 1\right)^{n-1}. \tag{7.25}$$

Proof. If the set $\mathcal{M}(\Lambda) \cap \mathcal{F}_\Delta$ is empty, then the bound (7.25) is obviously valid. Thus we assume that the set $\mathcal{M}(\Lambda) \cap \mathcal{F}_\Delta$ contains at least one dyadic minimum Q_{m_0}. By Lemma 7.4 the set $\mathcal{M}(\Lambda)$ is com[letely disordered. With the help of this fact we derive from Lemma 7.5 that every dyadic minimum $Q_m \in \mathcal{M}(\Lambda) \cap \mathcal{F}_\Delta$ belongs also to the region $V(Q_{m_0})$ (cf. (7.23)). Therefore, for every $Q_m \in \mathcal{M}(\Lambda) \cap \mathcal{F}_\Delta$ we have the inequalities (cf. (7.14) and (7.23))

$$\frac{1}{\sqrt{n}} 2^{-4\tau_n - 1} \|Q_{m_0}\| \le 2^{m_j} \le 2^{2\tau_n - 1} \|Q_{m_0}\|, \ \ 1 \le j \le n-1. \tag{7.26}$$

Taking logs in (7.26) we obtain

$$-4\tau_n - 1 - \frac{1}{2} \log n + \log \|Q_{m_0}\| \le m_j \le 2\tau_n - 1 + \log \|Q_{m_0}\|, \ \ 1 \le j \le n-1. \tag{7.27}$$

The inequalities (7.27) have at most $(6\tau_n + \frac{1}{2} \log n + 1)^{n-1}$ solutions $M = ^{\mathrm{T}}(m_1, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}$. This proves the bound (7.25).

The proof of Lemma 7.6 is complete.

Lemma 7.7. *Let $\Lambda \subset \mathbb{R}^n$ be a weakly admissible lattice, then for every diagonal matrix $\delta \in \Delta$ one has the bound*

$$\operatorname{card}\{Q_m \in \mathcal{M}(\Lambda) : d\langle Q_m\rangle = \delta\} \leq \left(6\tau_n + \frac{1}{2}\log n + 1\right)^{n-1}. \tag{7.28}$$

Proof. Using the relation (7.17) and the definition (7.8) we obtain

$$\begin{aligned}&\operatorname{card}\{Q_m \in \mathcal{M}(\Lambda) : d\langle Q_m\rangle = \delta\} = \\ &= \operatorname{card}\left\{Q_m \in \mathcal{M}(\delta^{-1}\Lambda) : d\langle Q_m\rangle = e\right\} = \\ &= \operatorname{card}\left\{Q_m \in \mathcal{M}(\delta^{-1}\Lambda) \cap \mathcal{F}_\Delta\right\},\end{aligned} \tag{7.29}$$

where e denotes the identity matrix. Now the bound (7.28) follows from (7.29) and the bound (7.25).

The proof of Lemma 7.7 is complete.

§ 8. Bounds for sums over lattices and sums over dyadic minima

First we recall some facts from geometry of numbers (cf. [Cas 1] and [Gl]). Let

$$\|\Lambda\| = \|\Lambda\|_1 \leq \|\Lambda\|_2 \leq \ldots \leq \|\Lambda\|_n \tag{8.1}$$

be the successive minima of a lattice $\Lambda \subset \mathbb{R}^n$ with respect to the Euclidean metric $\|X\|$, $X \in \mathbb{R}^n$. Recall that the j-th successive minimum $\|\Lambda\|_j$ is defined to be the lower bound of the numbers $t > 0$ such that the ball

$$tB^n = \{X \in \mathbb{R}^n : \|X\| \leq t\} \tag{8.2}$$

contains j linearly independent lattice points $\gamma \in \Lambda$.

The successive minima of an unimodular lattice $\Lambda \in \mathcal{L}_n$ satisfy the relation of Minkowski

$$1 \leq \prod_{j=1}^{n} \|\Lambda\|_j \leq h_n^n, \tag{8.3}$$

where relations of Mahler

$$1 \leq \|\Lambda\|_j\, \|\Lambda^\perp\|_{n+1-j} \leq n!, \quad 1 \leq j \leq n, \tag{8.4}$$

where $\Lambda^\perp$ is the lattice dual to Λ. In particular

$$\|\Lambda\|_n \leq n! \|\Lambda^\perp\|^{-1}. \tag{8.5}$$

Lemma 8.1. *Let $\Lambda \in \mathcal{L}_n$ be arbitrary lattice, then the ball $t_\Lambda B^n$ (cf. (8.2)) with a radius t_Λ given by*

$$t_\Lambda = \frac{n}{2}\|\Lambda\|_n \leq \frac{n\, n!}{2}\, \|\Lambda\|^{-1} \tag{8.6}$$

contains a fundamental set $F_\Lambda = \mathbb{R}^n/\Lambda$ of the lattice Λ.

Proof. By the definition of the n-th successive minimum the ball (8.2) with the radius $t = \|\Lambda\|_n$ contains n linearly independent lattive points $\gamma_1, \ldots, \gamma_n \in \Lambda$. It is obvious that the parallelepiped spanned by the vectors $\gamma_1, \ldots, \gamma_n$ contains a functional set of the lattice Λ. The diameter of this parallelepiped does not exceed the quantity $|\gamma_1| + \ldots + |\gamma_n| \leq n\|\Lambda\|_n = 2t_\Lambda$.

It remains to observe that every translation of a fundamental set $F_\Lambda = \mathbb{R}^n?\Lambda$ by a vector $X \in \mathbb{R}^n$ is also a fundamental set of the lattice Λ. This follows from the commutativity of the group of translation of the space $\mathbb{R}^n$.

The proof of Lemma 8.1 is complete.

Lemma 8.2. *Let $\Lambda \in \mathcal{L}_n$ be arbitrary lattice, and let $N(tB^n + X, \Lambda)$ (cf. (1.5)) be the number of lattice points $\gamma \in \Lambda$ lying in the ball $tB^n + X$, then one has the bound*

$$N\left(tB^n + X, \Lambda\right) \leq 2^n(t^n + \|\Lambda\|_n^n) \leq 2^n(n\,n!)^n(t^n + \|\Lambda^{\perp}\|^{-n}). \tag{8.7}$$

Proof. Let F_Λ denote the fundamental set of the lattice Λ contained in the ball $t_\Lambda\, B^n$ (cf. Lemma 8.1). Consider the following coverning of the space $\mathbb{R}^n$ by disjoint translations of F_Λ:

$$\mathbb{R}^n = \bigcup_{\gamma\in\Lambda}(F_\Lambda + \gamma). \tag{8.8}$$

Let for a given $\gamma \in \Lambda$ the intersection $(\mathcal{F}_\Lambda + \gamma) \cap (tB^n + X)$ be not empty, then the fundamental set $\mathcal{F}_\Lambda + \gamma$ belongs to the ball $(t + 2t_\Lambda)B^n + X$. Moreover, $\mathrm{vol}\,(\mathcal{F}_\Lambda + \gamma) = \det\Lambda = 1$, and every set $\mathcal{F}_\Lambda + \gamma$ contains exactly one lattice point. Therefore,

$$\begin{aligned} N(tB^n + X, \Lambda) &\leq \mathrm{card}\,\{\gamma \in \Lambda : (\mathcal{F}_\Lambda + \gamma) \cap (tB^n + X) \neq \varnothing\} \leq \\ &\leq \mathrm{vol}\,((t + 2t_\Lambda)B_n) = (t + 2t_\Lambda)^n \mathrm{vol}\, B^n < 2^n(t + 2t_\Lambda)^n. \end{aligned} \tag{8.9}$$

Now the bound (8.7) follows from (8.6) and (8.9).
The proof of Lemma 8.2 is complete.

We wish to give a generalization of Lemma 8.2. We consider the following sum over a lattice $\Lambda \in \mathcal{L}_n$:

$$f(\mu, \Lambda) = \max_{X\in\mathbb{R}^n} \sum_{\gamma\in\Lambda} f\left(\frac{\gamma - X}{\mu}\right), \quad \mu > 0, \tag{8.10}$$

where

$$f(X) = (1 + \|X\|)^{-A}, \quad X \in \mathbb{R}^n, \tag{8.11}$$

with an exponent $A > n$.

Lemma 8.3. *Let $\Lambda \in \mathcal{L}_n$ be arbitrary lattice, then one has the bound*

$$f(\mu, \Lambda) < c_n\left(\mu^n + \|\Lambda^{\perp}\|^{-n}\right) \tag{8.12}$$

with a constant c_n depending only on the dimension n.

Proof. Let F_Λ denote the fundamental set of the lattice Λ contained in the ball $t_\Lambda\, B^n$ (cf. Lemma 8.1). We set

$$f(\mu, \Lambda) = \max_X f^{(1)}(X, \mu, \Lambda) + \max_X f^{(2)}(X, \mu, \Lambda), \tag{8.13}$$

where

$$f^{(1)}(X, \mu, \Lambda) = \sum_{\|\gamma - X\| \leq \mu + 3t_\Lambda} f\left(\frac{\gamma - X}{\mu}\right), \quad f^{(2)}(X, \mu, \Lambda) = \sum_{\|\gamma - X\| > \mu + 3t_\Lambda} f\left(\frac{\gamma - X}{\mu}\right), \tag{8.14}$$

Notice that both of the functions (8.14) are periodic in X with the period lattice Λ. Thus, without loss of generality, we may assume that $X \in \mathcal{F}_\Lambda$, and therefore, $X \in t_\Lambda\, B^n$.

Using Lemma 8.2 and the relations (1.5), (8.11), (8.14) we obtain the bound

$$f^{(1)}(X,\mu,\Lambda) \leq \sum_{\|\gamma - X\| \leq \mu + 3t_\Lambda} f\left((\mu + 3t_\Lambda)B^n + X, \Lambda\right) < c_1(n)\left(\mu^n + \|\Lambda^\perp\|^{-n}\right). \tag{8.15}$$

Now we estimate the second sum in (8.14). We have

$$\begin{aligned}
&\int_{F_\Lambda + \gamma} f\left(\frac{Y - X}{\mu}\right) dY \\
&\geq \min\left\{ f\left(\frac{Y - X}{\mu}\right) : Y \in F_\Lambda + \gamma \right\} \\
&\geq \min\left\{ f\left(\frac{Y - X}{\mu}\right) : Y \in t_\Lambda B^n + \gamma \right\} \\
&= \min\left\{ \left(1 + \frac{\|Z + \gamma - X\|}{\mu}\right)^{-A} : Z \in t_\Lambda B^n \right\} \\
&\geq \min\left\{ \left(1 + \frac{\|Z\| + \|\gamma - X\|}{\mu}\right)^{-A} : Z \in t_\Lambda B^n \right\} \\
&\geq \left(1 + \frac{\|\gamma - X\|}{\mu}\right)^{-A} \min\left\{ \left(1 + \frac{\|Z\|}{\mu}\right)^{-A} : Z \in t_\Lambda B^n \right\} \\
&\geq \left(1 + \frac{t_\Lambda}{\mu}\right)^{-A} \left(1 + \frac{\|\gamma - X\|}{\mu}\right)^{-A}.
\end{aligned} \tag{8.16}$$

Summing the inequalities (8.16) over $\gamma \in \Lambda$ with $\|\gamma - X\| > \mu + 3t_\Lambda$, and taking (8.6) and (8.8) into account we obtain

$$\begin{aligned}
f^{(2)}(X,\mu,\Lambda) &\leq \left(1 + \frac{t_\Lambda}{\mu}\right)^{A} \sum_{\|\gamma - X\| > \mu + 3t_\Lambda} \int_{F_\Lambda + \gamma} \left(\frac{Y - X}{\mu}\right) dY \\
\left(1 + \frac{t_\Lambda}{\mu}\right)^{A} \int_{\|Y\| > \mu + t_\Lambda} f(Y\mu)\, dY &\leq c_2(n) \left(1 + \frac{t_\Lambda}{\mu}\right)^{A} \mu^A \int_{\mu + t_\Lambda}^{\infty} t^{n-1-A} dt \\
&\leq \frac{c_2(n)}{A - n} (\mu + t_\Lambda)^n \leq c_3(n)\left(\mu^n + \|\Lambda^\perp\|^{-n}\right).
\end{aligned} \tag{8.17}$$

It is worth to be noted that on second step in the calculation (8.17) we use the fact that the diameter of the fundamental set F_Λ does not exceed $2t_\Lambda$.

Substituting the bounds (8.15) and (8.17) in (8.13) we obtain the bound (8.12).

The proof of Lemma 8.3 is complete.

Let $X =^{\mathrm{T}} (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $Y =^{\mathrm{T}} (y_1, \ldots, y_n) \in \mathbb{R}^n_+$, we set

$$g(Y, X) = \prod_{j=1}^{n} (1 + y_j |x_j|)^{-A} \tag{8.18}$$

with an exponent $A > n$. Obviously, that for every diagonal matrix $\delta \in \Delta$ one has the relation

$$g(\delta Y, \delta^{-1} X) = g(Y, X). \tag{8.19}$$

Moreover, one has the bound

$$g(Y,X) \leq (1+\min\{y_1,\dots,y_n\}\|X\|)^{-A}. \tag{8.20}$$

We consider the following sum over a lattice $\Lambda \in \mathcal{L}_n$:

$$G(Y,\Lambda) = \max_{X\in\mathbb{R}^n} \sum_{\gamma\in\Lambda} g(Y,\gamma - X). \tag{8.21}$$

Notice that (8.19) imply the relation

$$G(\delta Y, \delta^{-1}\Lambda) = G(Y,\Lambda). \tag{8.22}$$

Lemma 8.4. *Let $\Lambda \in \mathcal{L}_n$ be arbitrary lattice, then one has the bound*

$$G(Y,\Lambda) < c_n \left((Nm\,Y)^{-1} + \|d\langle Y\rangle\Lambda^{\perp}\|^{-n}\right) \tag{8.23}$$

with a constant c_n depending only on the dimension n.

Proof. Substituting the bound (8.20) in (8.21), and using the definitions (8.10) and (8.11) we obtain

$$G(Y,\Lambda) \leq f(\mu_0,\Lambda) \tag{8.24}$$

with

$$\mu_0 = (\min\{y_1,\dots,y_n\})^{-1}. \tag{8.25}$$

In (8.22) we put $\delta = d\langle Y\rangle$. Recall that $d\langle Y\rangle Y \in \mathcal{F}_\Delta$ (cf. (7.8)). Substituting the inequality (8.24) in (8.22) and the inequality (7.5) in (8.25) we obtain

$$G(Y,\Lambda) = G\left(d\langle Y\rangle Y,\, \{d\langle Y\rangle\}^{-1}\Lambda\right) \leq f\left(\mu_1, \{d\langle Y\rangle\}^{-1}\Lambda\right) \tag{8.26}$$

with

$$\mu_1 \leq 2^{\tau_n}(Nm\,Y)^{-\frac{1}{n}}. \tag{8.27}$$

Using the bound (8.12) and inequalities (8.26) and (8.27) we obtain the bound (8.23).
The proof of Lemma 8.4 is complete.

Lemma 8.5. *Let a lattice $\Lambda \in \mathcal{L}_n$ and the dual lattice $\Lambda^{\perp} \in \mathcal{L}_n$ be weakly admissible (cf. § 4), and let $Q_m \in \mathcal{M}(\Lambda^{\perp})$ be a dyadic minimum of the lattice $\Lambda^{\perp}$ (cf. § 7), then the sum (8.21) satisfies the bound*

$$G(Q_m,\Lambda) < c_n \|d\langle Q_m\rangle\Lambda^{\perp}\|^{-n} \tag{8.28}$$

with a constant c_n depending only on the dimension n.

Proof. It is sufficient to use the bound (8.23) and take the inequality (7.18) into account.
The proof of Lemma 8.5 is complete.

Now we are in a position to give a main result of this section concerning sums over dyadic minima of a lattice.

Lemma 8.6. *Let $\Lambda \in \mathcal{L}_n$ be a weakly admissible lattice then one has the bound*

$$\sum_{Q_m \in \mathcal{M}(\Lambda) \cap \rho B^n} \|d\langle Q_m \rangle \Lambda\|^{-n} < c_n S(\Lambda, r) \tag{8.29}$$

with

$$r = 2\tau_n + \left| \log \frac{\rho^n}{\nu(\Lambda, \rho)} \right| . \tag{8.30}$$

The summation in (8.29) *is taken over all dyadic minima Q_m of the lattice Λ lying in the ball ρB^n (cf.* (8.2)*), the characteristics $S(\cdot,\cdot)$ and $\nu(\cdot,\cdot)$ are given in* (3.15) *and* (4.1) *respectively, the constant c_n in* (8.29) *depends only on the dimension n.*

Proof. Using Lemma 7.2 and the inequality (7.19) we conclude that for every dyadic minima $Q_m \in \mathcal{M}(\Lambda)$ with $\|Q_m\| \le \rho$ the corresponding diagonal matrix $d\langle Q_m \rangle$ btlongs to the finite set Δ_r (cf. (3.4)) where r is given by (8.30).

Now, from Lemma 7.7 and the definition (3.15) we obtain

$$\begin{aligned} \sum_{Q_m \in \mathcal{M}(\Lambda) \cap \rho B^n} \|d\langle Q_m \rangle \Lambda\|^{-n} &= \sum_{\delta \in \Delta_r} \|\delta \Lambda\|^{-n} \operatorname{card}\{Q_m \in \mathcal{M}(\Lambda) : d\langle Q_m \rangle = \delta\} \\ &\le \left(6\tau_n = \frac{1}{2}\log n + 1\right)^{n-1} \sum_{\delta \in \Delta_r} \|\delta \Lambda\|^{-n} = c_n \, SLa, r). \end{aligned}$$

The proof of Lemma 8.6 is complete.

§ 9. Dyadic decomposition of Fourier series

We consider an absolutely convergent Fourier series

$$W(X) = \sum_{\gamma \in \Gamma^\perp \setminus \{0\}} \widetilde{w}(\gamma) e^{2\pi i \langle \gamma, X \rangle}. \tag{9.1}$$

The series (9.1) is a periodic functions of X with the period lattice: $W(X+\gamma) = W(X)$, $\gamma \in \Gamma$. The Fourier coefficients are given by

$$\widetilde{w}(\gamma) - (\det \Gamma)^{-1} \int\limits_{\mathbb{R}^n/\Gamma} W(X) e^{-2\pi i \langle \gamma, X \rangle} dX, \quad \gamma \in \Gamma^\perp, \tag{9.2}$$

moreover,

$$\widetilde{w}(0) = (\det \Gamma)^{-1} \int\limits_{\mathbb{R}^n/\Gamma} W(X) dX = 0. \tag{9.3}$$

For details concerning the Fourier series with arbitrary period lattices we refer to [Skr 7].

Let the lattice $\Gamma^\perp \in \mathcal{L}_n$ be weakly admissible (cf. § 4). We wish to introduce a special decomposition of the Fourier series (9.1) associated with the set of dyadic minima $Q_m \in \mathcal{M}(\Gamma^\perp)$. For admissible lattices this decomposition was used in [Skr 3]. It should be noted that decompositions of such kind are known in Fourier analysis in the Littlwood–Paley theory, cf. the book [St] by Stain, and also [Skr 8].

We introduce the following auxiliary functions $\xi_j(t)$, $t \in \mathbb{R}$, $1 \le j \le n$. Let

$$\xi_1(t) = \ldots = \xi_{n-1}(t) = \xi(t), \tag{9.4}$$

where $\xi(t)$, $t \in \mathbb{R}$, is an even function of class C^∞ with a compact support,

$$\text{support } \ \xi(\cdot) \subseteq \left[-1; -\tfrac{1}{4}\right] \cup \left[\tfrac{1}{4}; 1\right], \tag{9.5}$$

satisfying the relation

$$\sum_{m=-\infty}^{\infty} \xi\left(2^m\, t\right) = 1 \tag{9.6}$$

for every $t \neq 0$. A number of such functions can easily be given in an explicit form (cf. [Skr 3], Eq. (5.16)).

Let $\xi_n(t)$, $t \in \mathbb{R}$, be an even function of the class C^∞ satisfying the conditions

$$\xi_n(t) = \begin{cases} 0 & \text{if} \quad 0 \le |t| \le \tfrac{1}{4}, \\ 1 & \text{if} \quad |t| \ge 1, \end{cases} \tag{9.7}$$

moreover

$$\text{support } \ \xi_n(\cdot) \subseteq \left(-\infty; -\tfrac{1}{4}\right] \cup \left[\tfrac{1}{4}; \infty\right), \tag{9.8}$$

We define a function $\Xi(Y)$, $Y =^{\mathrm{T}}(y_1, \ldots, y_n) \in \mathbb{R}^n$, by setting

$$Xi(Y) = \prod_{j=1}^{n} \xi_j(y_j). \tag{9.9}$$

From the definitions (9.4)–(9.8) of the functions $\xi(\cdot)$, $1 \le j \le n$, we conclude that the function (9.9) belongs to class C^∞, and the support of $\Xi(\cdot)$ is concentrated in the following temzor product of segments:

$$\text{support } \ \Xi(\cdot) \subseteq \prod_{j=1}^{n-1} \otimes \left\{\left[-1; -\tfrac{1}{4}\right] \cup \left[\tfrac{1}{4}; 1\right]\right\} \otimes \left\{\left(-\infty; -\tfrac{1}{4}\right] \cup \left[\tfrac{1}{4}; \infty\right)\right\}. \tag{9.10}$$

Let (cf. (7.14))

$$\begin{gathered} Q_m =^{\mathrm{T}}(q_1(m), \ldots, q_n(m)) ==^{\mathrm{T}}(2^{m_1}, \ldots, 2^{m_{n-1}}, q_n(m)), \\ m =^{\mathrm{T}}(m_1, \ldots, m_{n-1}) \in \mathbb{Z}^{n-1}, \end{gathered} \tag{9.11}$$

be a dyadic minimum of a weakly admissible lattice $\Lambda \in \mathcal{L}_n$. We introduce a positive diagonal matrix $\mathcal{Q}_m \in \mathrm{GL}(n, \mathbb{R})$ by setting

$$\mathcal{Q}_m = \operatorname{diag}\left(q_1(m), \ldots, q_n(m)\right). \tag{3.12}$$

The determinant of the matrix (9.12) satisfies the bound

$$\det \mathcal{Q}_m \le 1. \tag{9.13}$$

To prove (9.13) we notice that

$$\det \mathcal{Q}_m = 2^{-n} \mathrm{vol}\,\Pi\langle Q_m\rangle, \tag{9.14}$$

where $\Pi\langle\,.\,\rangle$ is the parallelepiped (7.13).

By the definition of dyadic minima (cf. (7.15)) the parallelepiped $\Pi\langle Q_m\rangle$ is empty, i.e., $\Pi\langle Q_\rangle$ does not contain lattice points $\gamma \in \Lambda \setminus \{0\}$. Therfore, $\mathrm{vol}\,\Pi\langle Q_m\rangle \leq 2^n$ by Minkowski's convex body theorem. Now the bound (9.13) follows from (9.14).

Consider the following function

$$\Xi(\mathcal{Q}_m^{-1}Y) = \prod_{j=1}^{n} \xi_j\left(\frac{y_j}{q_j(m)}\right) = \left\{\prod_{j=1}^{n} \xi\left(2^{-m_j}y_j\right)\right\} \xi_n\left(\frac{y_n}{q_n(m)}\right). \tag{9.15}$$

The inclusion (9.10) implies that

$$\begin{aligned}\text{supported}\ \ \Xi(\mathcal{Q}_m^{-1}\,.\,) \subseteq \prod_{j=1}^{n} &\left\{\left[-2^{m_j}; -\frac{1}{4}2^{m_j}\right] \bigcup \left[\frac{1}{4}2^{m_j}; 2^{m_j}\right]\right\} \\ &\otimes \left\{\left(-\infty; -\frac{q_n(m)}{4}\right] \bigcup \left[\frac{q_n(m)}{4}; \infty\right)\right\}.\end{aligned} \tag{9.16}$$

Lemma 9.1. *Let $\Lambda \in \mathcal{L}_n$ be an arbitrary weakly admissible lattice, then one has the relation*

$$\Xi(\mathcal{Q}_m^{-1}Y) = 0 \tag{9.17}$$

provided that $\|Q_m\| > 4\|Y\|$, and the relation

$$\sum_{m\in\mathbb{Z}^{n-1}} \Xi(\mathcal{Q}_m^{-1}\gamma) = 1 \tag{9.18}$$

for every lattice point $\gamma \in \Lambda \setminus \{0\}$.

Proof. The inclusion (9.16) implies that the support of the function $\Xi(\mathcal{Q}_m^{-1}Y)$ is separated from the origin by a distance no less than $\frac{1}{4}\|Q_m\|$. This proves the relation (9.17).

Let $\gamma = {}^{\mathrm{T}}(\lambda_1, \ldots, \lambda_n) \in \Lambda \setminus \{0\}$, then $\lambda_j \neq 0$ for each $j = 1, \ldots, n$, since the lattice Λ is weakly admissible. We have the following relation

$$\Xi(\mathcal{Q}_m^{-1}Y) = \prod_{j=1}^{n-1} \sigma\left(2^{-m_j}\lambda_j\right). \tag{9.19}$$

Really, if $\sigma\left(2^{-m_j}\lambda_j\right) = 0$ at least for one of the indeces j, $1 \leq j \leq n-1$, then the relation (9.19) is obviously valid (cf. (9.15)). Now, let $\sigma\left(2^{-m_j}\lambda_j\right) \neq 0$ for each j, $1 \leq j \leq n-1$, the $|\lambda_j| \leq 2^{-m_j}$ for $1 \leq j \leq n-1$ by the relations (9.4)–(9.5), and $|\lambda_n| \geq q_n(m)$ by the definition (7.14)–(7.16) of the dyadic minima. Hence $\xi_n(\frac{\lambda_n}{q_n(m)} = 1$ (cf. (9.7)), and the relation (9.19) is also valid.

Using the relations (9.5) and (9.19) we obtain

$$\sum_{m\in\mathbb{Z}^{n-1}} \Xi(\mathcal{Q}_m^{-1}\gamma) = \sum_{m_1,\ldots,m_{n-1}=-\infty}^{\infty} \prod_{j=1}^{n-1} \sigma\left(2^{-m_j}\lambda_j\right) = \sum_{m_j=-\infty}^{\infty} \prod_{j=1}^{n-1} \sigma\left(2^{-m_j}\lambda_j\right) = 1.$$

This proves the relation (9.18).

The proof of Lemma 9.1 is complete.

By applying the relation (9.18) with $\Lambda = \Gamma^{\perp}$, the Fourier series (9.1) can be transformed imto the following sum over the dyadic minima $Q_m \in \mathcal{M}(\Gamma^{\perp})$ of the dual lattice $\Gamma^{\perp}$:

$$W(X) = \sum_{Q_m \in \mathcal{M}(\Gamma^{\perp})} W(Q_m, X), \tag{9.20}$$

where

$$W(Q_m, X) = \sum_{\gamma \in \Gamma^{\perp}} \Xi(\mathcal{Q}^{-1}\gamma)\widetilde{w}(\gamma)e^{2\pi i\langle\gamma,\lambda\rangle}. \tag{9.21}$$

Actually, the sum (9.21) is taken over the lattice points $\gamma \in \Gamma^{\perp}$ lying in the region given on the right-hand side of (9.16).

The transformation of the series (9.1) into (9.20)–(9.21) will be called the *dyadic decomposition* of the Fourier series.

§ 10. Bounds for spectral Fourier series

We consider an application of the dyadic decomposition to special Fourier series which arise naturally from the study of errors in the lattice point problem for polyhefra.

Let $\Gamma \in \mathcal{L}_n$ be an arbitrary lattice. Assume that the dual lattice $\Gamma^{\perp}$ is weakly admissible. We introduce the following Fourier series

$$W_{+}\tau, \rho(X) = \sum_{\gamma \in \Gamma^{\perp}\setminus\{0\}} \widetilde{w}_{\tau,\rho}(\gamma)e^{2\pi i\langle\gamma,X\rangle}, \quad X \in \mathbb{R}^n, \tag{10.1}$$

depending on two parameters $0 < \tau < 1$ and $\rho > 0$. The Fourier coefficients in (10.1) are defined by the formula

$$\widetilde{w}_{\tau,\rho}(Y) = \frac{1}{Nm\,Y}\Phi(\widehat{Y})\omega_1(\rho^{-1}Y)\omega_2(\tau Y), \quad Y \in \mathbb{R}^n, \tag{10.2}$$

for $Y = \gamma \in \Gamma^{\perp} \setminus \{0\}$. The factors in (10.2) are given as follows:

– The function $\Psi(\widehat{Y})$, $\widehat{Y} \in \Sigma^n$, depends only on the angular variables $\widehat{Y} = ^{\mathrm{T}}(\|Y\|^{-1}\,y_1, \dots, \|Y\|^{-1}y_n)$ on the unite spere Σ^n (cf. (4.16)), the function $\Phi(\widehat{Y})$ belongs to class C^{∞}, and is even in each variable $\|Y\|^{-1}y_j$, $1 \le j \le n$.

– The functions $\omega_1(Y)$ and $\omega_2(Y)$, $Y \in \mathbb{R}^n$, belong to class C^{∞}, and are even in each coordinate y_j, $1 \le j \le n$. Moreover, $\omega_1(\cdot)$ is compactly supported in the ball (8,2) with radius $\frac{1}{4}$:

$$\text{supported} \quad \omega_1(\cdot) \subseteq \tfrac{1}{4}B^n, \tag{10.3}$$

and $\omega_2(\cdot)$ satisfies the bounds

$$\left| \frac{\partial^{k_1+\ldots+k_n}}{\partial y_1^{k_1}\ldots\partial y_n^{k_n}}\omega_2(Y) \right| < c_{A,k}(1 + \|Y\|)^{-A}, \tag{10.4}$$

The inclusion (10.3) implies that

$$\text{supported} \quad \omega_1(\rho^{-1}\cdot) \subseteq \tfrac{\rho}{4}B^n. \tag{10.5}$$

Hence the sum (10.1) contains only finitely many terms. Since the lattice $\Gamma^\perp$ us weakly admissible, the denominators in (10.1)–(10.2) are not equal to zero: $Nm\,\gamma \neq 0$, $\gamma \in \Gamma^\perp \setminus \{0\}$. Thus the Fourier series (10.1) is well defined.

The dyadic decomposition (9.20)–(9.21) for the Fourier series (10.1) takes the form

$$W_{\tau,\rho}(X) = \sum_{Q_m \in \mathcal{M}(\Gamma^\perp)} W_{\tau,\rho}(Q_m, X), \tag{10.6}$$

where

$$W_{\tau,\rho}(Q_m, X) = \sum_{\gamma \in \Gamma^\perp} \Xi(\mathcal{Q}_m^{-1}\,\gamma)\widetilde{w}_{\tau,\rho}(\gamma)e^{2\pi i\langle\gamma,X\rangle}. \tag{10.7}$$

Using the relations (9.17) and (10.5) we obtain that

$$W_{\tau,\rho}(Q_m, X) = 0 \quad \text{if} \quad \|Q_m\| > \rho. \tag{10.8}$$

With the help of (10.8) we can write (10.6) in the following form

$$W_{\tau,\rho}(X) = \sum_{Q_m \in \mathcal{M}(\Gamma^\perp)\cap\rho B^n} W_{\tau,\rho}(Q_m, X). \tag{10.9}$$

Lemma 10.1. *Let $\Gamma \in \mathcal{L}_n$ be an arbitrary lattice. Assume that the dual lattice $\Gamma^\perp$ is weakly admissible. Then the special Fourier series (10.1)–(10.2) satisfies the bound*

$$\max_{X \in \mathbb{R}^n} |W_{\tau,\rho}(X)| < c\, S(\Gamma^\perp\, r) \tag{10.10}$$

with

$$r = 2\tau_n + \left|\log \frac{\rho^n}{\nu(\Gamma^\perp, \rho)}\right|. \tag{10.11}$$

In the bound (10.10) the constant $c = c(\Phi, \omega_1, \omega_2)$ is independent of the parameters $0 < \tau < 1$ and $\rho > 0$. The characteristics $S(\cdot,\cdot)$ and $\nu(\cdot,\cdot)$ are defined in (3.15) and (4.1), respectively.

Proof. We need the Poisson summation formula for an arbitrary unimodular lattice $\Gamma \in \mathcal{L}_n$ (see e.g., [Skr 6]):

$$\sum_{\gamma \in \Gamma} f(\gamma - X) = \sum_{\gamma \in \Gamma^\perp} \widetilde{f}(\gamma)e^{2\pi i\langle\gamma,X\rangle}, \tag{10.12}$$

where the functions f and $\widetilde{f}$ are related by the Fourier transform

$$\widetilde{f}(Y) = \int_{\mathbb{R}^n} e^{2\pi i\langle Y,X\rangle} f(X)dX, \quad f(Y) = \int_{\mathbb{R}^n} e^{-2\pi i\langle Y,X\rangle}\widetilde{f}(Y)dY. \tag{10.13}$$

Notice that the sum over $\gamma \in \Gamma$ in (10.12) represent a periodic function, and the sum over $\gamma \in \Gamma^\perp$ is the Fourier series of this function. Clearly, the formula (10.12) is valid identically in $X \in \mathbb{R}^n$ if one of the functions f of $\widetilde{f}$ is compactly supported and belongs to class ∞.

Applying the Poisson summation formula (10.12) with

$$\widetilde{f}(Y) = \Xi(\mathcal{Q}_m^{-1}\,Y)\widetilde{w}_{\tau,\rho}(Y), \quad Y \in \mathbb{R}^n, \tag{10.14}$$

we obtain the following representation for the seroes (10.7):

$$W_{\tau,\rho}(Q_m, X) = \sum_{\gamma\in\Gamma} u_m(\tau,\rho;\gamma - X), \quad m \in \mathbb{Z}^{n-1}, \tag{10.15}$$

where

$$u_m(\tau,\rho;X) = \int\limits_{\mathbb{R}^n} e^{2\pi i\langle Y,X\rangle}\Xi(\mathcal{Q}_m^{-1}Y)\widetilde{w}_{\tau,\rho}(Y)dY. \tag{10.16}$$

Substituting (10.2) in (10.16) and replacing Y by $\mathcal{Q}_m^{-1}Y$, we obtain

$$u_m(\tau,\rho;X) = \det \mathcal{Q}_m \int\limits_{\mathbb{R}^n} e^{2\pi i\langle \mathcal{Q}_m Y,X\rangle} v_m(\tau,\rho,Y)dY \tag{10.17}$$

with

$$v_m(\tau,\rho,Y) = \frac{1}{Nm\,Y}\Xi\Phi(\widehat{\mathcal{Q}_m\,Y})\omega_1)\rho^{-1}\mathcal{Q}_m\,Y)\omega_2(\tau\mathcal{Q}_m\,Y). \tag{10.18}$$

The indergand in (10.17) is an odd function in each coordinate y_j, $1 \le j \le n$. Hence we may represent the integral (10.17) in the form

$$u_m(\tau,\rho;X) = (-2i)^n \det \mathcal{Q}_m I_m(\tau,\rho;2\pi\mathcal{Q}_m X), \tag{10.19}$$

where we introduced the following Fourie integral

$$I_m(\tau,\rho;X) = \int\limits_{\mathbb{R}^n} dY\, J_m(\tau,\rho,Y)\prod_{j=1}^n \sin(x_j y_j) \tag{10.20}$$

with

$$J_m(\tau,\rho,Y) = \frac{1}{Nm\,Y}\Xi\Phi(\widehat{\mathcal{Q}_m\,Y})\omega_1)\rho^{-1}\mathcal{Q}_m\,Y)\omega_2(\tau\mathcal{Q}_m\,Y). \tag{10.21}$$

The above expressions (10.14)–(10.21) are rather combersome but, fortunately, the Fourier integral (10.20) is very well behaved. The following assertion will be proved in Appendix 2 (§ 14).

Lemma 10.2. *The fourier integral* (10.19) $- -$(10.20) *satisfies the bound*

$$|\, I_m(\tau,\rho;X)\,| < c_A \prod_{j=1}^n (1+|x_j|)^{-A} \tag{10.22}$$

with arbitrarily large $A > 0$. In the bound (10.22) *the constant $c_A = c_A(\Phi,\omega_1,\omega_2)$ is independent of the parameters $0 < \tau < 1$, $\rho > 0$ and the diagonal matrix $\mathcal{Q}_m$, $m \in \mathbb{Z}^{n-1}$, (cf.* (9.12)).

We use Lemma 10.2 to continue the proof of Lemma 10.1. Subsituting the bound (9.13) and the bound (10.22) with $A > n$ in (10.19) and (10.15), we obtain

$$\begin{aligned}\max_{X\in\mathbb{R}^n} |W(Q_m,X)| <& c_A \max_{X\in\mathbb{R}^n} \sum_{\gamma={}^t(\lambda_1,\dots,\lambda_n)\in\Gamma}\ \prod_{j=1}^n (1+q_j(m)|\lambda_j - x_j|)^{-A} \\ =& c_A \max_{X\in\mathbb{R}^n} \sum_{\gamma\in\Gamma} g(Q_m,\gamma - X) = c_A G(Q_m,\Gamma)\end{aligned} \tag{10.23}$$

with the same constant c_A as in (10.22). In the calculation (10.23) we used the definitions (8.18) and (8.21).

Subsituting the bound (10.23) in (10.9), we obtain

$$\max_{X\in\mathbb{R}^n} |W_{\tau,\rho}(X)| < c_A \sum_{Q_m\in\mathcal{M}(\Gamma^\perp\cap\rho B^n} G(Q_m,\Gamma). \tag{10.24}$$

Using Lemmas 8.5 and 8.6 we derive from (10.24) that

$$\max_{X\in\mathbb{R}^n} |W_{\tau,\rho}(X)| < c_1 \sum_{Q_m\in\mathcal{M}(\Gamma^\perp\cap\rho B^n} \|d\langle Q_m\rangle\Gamma^\perp\|^{-n} < c_2 S(\Gamma^\perp, r), \tag{10.25}$$

where the parameter r is given in (10.11). The constants c_1 and c_2 are independent of the lattice $\Gamma\in\mathcal{L}_n$ and parameters $0<\tau<1$ and $\rho>0$.

The prrof of Lemma 10.1 is complete.

§ 11. The Voronoĭ–Hardy formula. The Fourier transform of the caracteristic function of a lolyhedron

The error $\mathcal{R}(\mathcal{O}{+}X,\Gamma)$ (cf. (1.9)) is a periodic function of $X\in\mathbb{R}^n$ with the period lattice $\Gamma\in\mathcal{L}_n$. The Voronoĭ–Hardy formula (cf. [Kr]) links the Fourier espansion of the error $\mathcal{R}(\mathcal{O}{+}X,\Gamma)$ and the Fourier transform of the caracteristic function $\chi(\mathcal{O},X)$, $X\in\mathbb{R}^n$, of the region $\mathcal{O}\in\mathbb{R}^n$:

$$\mathcal{R}(\mathcal{O}+X,\Gamma) = \sum_{\gamma\in\Gamma^\perp\setminus\{0\}} \widetilde{\chi}(\mathcal{O};\gamma)e^{-2\pi i\langle\gamma,X\rangle}, \tag{11.1}$$

and

$$\widetilde{\chi}(\mathcal{O},Y) = \int_{\mathbb{R}^n} \chi(\mathcal{O},X)e^{2\pi i\langle\gamma,X\rangle}\,dX = \int_{\mathcal{O}} \widetilde{\chi}(\mathcal{O};\gamma)e^{2\pi i\langle\gamma,X\rangle}dX. \tag{11.2}$$

However, the Fourier series (11.1) is not abso;utely convergent. Thereby, a suitable 'smoothing' method is required to handle the series (11.1) in the pointwise sense (cf. [H], Kr], [Ra]). Here we use a method given in [Skr 3].

We recall the following **Definition** (cf. [Skr 3], § 7): given a compact region $\mathcal{O}\subset\mathbb{R}^n$ and a number $r>0$, a couple of compact regions $\mathcal{O}_\tau^-$ and $\mathcal{O}_\tau^+$, and the points of the boundaries $\partial\mathcal{O}_\tau^\pm$ are at a distance at least τ from the boundary $\partial\mathcal{O}$.

We fix a nonnegative function $\omega(X)$, $X\in\mathbb{R}^n$, of class C^∞, supperted inside the unit ball B^n (cf. (8.2)), and such that

$$\int_{\mathbb{R}^n} \omega(X)\,dX = 1. \tag{11.3}$$

Notice that the Fourier transform $\widetilde{\omega}(\cdot)$ (cf. (10.13)) of the function $\omega(\cdot)$ satisfies the bound

$$|\,\widetilde{\omega}(Y)\,| < c_A(1+\|Y\|)^{-A} \tag{11.4}$$

with arbitrary large $A>0$.

We introduce the following absolutely convergent Fourier series

$$\mathcal{R}_\tau^\pm(\mathcal{O},X) = \sum_{\gamma\in\Gamma^\perp\setminus\{0\}} \widetilde{\chi}(\mathcal{O}_\tau^\pm,\gamma)\widetilde{\omega}(\tau\,\gamma)e^{2\pi i\langle\gamma,X\rangle}, \tag{11.5}$$

where $\widetilde{\chi}(\mathcal{O}_\tau^\pm,Y)$, $Y\in\mathbb{R}^n$, are the Fourier transform (11.2) of the characteristic functions $\chi(\mathcal{O}_\tau^\pm,Y)$, $X\in\mathbb{R}^n$, of the regions $\mathcal{O}_\tau^\pm$.

The following assertion was given in [Skr 3, Lemma 7.1]. For completeness we give here a short proof of this assertion.

Lemma 11.1. *Let a compact region $\mathcal{O} \subset \mathbb{R}^n$ and a lattice $\Gamma \in \mathcal{L}_n$ be given. Then, for any τ-coapproximation $\mathcal{O}^{\pm}_{\tau}$ of $\mathcal{O}$, one has the following bound for the error $R(\mathcal{O},\Gamma)$ (cf. (1.7)):*

$$R(\mathcal{O},\Gamma) \leq \operatorname{vol} \mathcal{O}^{\pm}_{\tau} - \operatorname{vol} \mathcal{O}^{-}_{\tau} + \sup_{X \in \mathbb{R}^n} \left(\left| \mathcal{R}^{+}_{\tau}(\mathcal{O},X) \right| + \left| \mathcal{R}^{-}_{\tau}(\mathcal{O},X) \right| \right), \tag{11.6}$$

where $\mathcal{R}^{\pm}_{\tau}(\mathcal{O},X)$ are the Fourier series (11.5).

Proof. We set $\omega_\tau(X) = \tau^{-n}\omega(\tau^{-n}X)$, $X \in \mathbb{R}^n$, $0 < \tau < 1$, moreover, $\widetilde{\omega}_\tau(Y) = \widetilde{\omega}(\tau Y)$, $Y \in \mathbb{R}^n$. We consider the convolutions of the characteristic functions $\chi(\mathcal{O}^{\pm}_{\tau}) = \chi(\mathcal{O}^{\pm}_{\tau},\cdot)$ with $\omega_\tau(\cdot)$:

$$\left(\omega_\tau * \chi(\mathcal{O}^{\pm}_{\tau})\right)(X) = \int_{\mathbb{R}^n} dY \omega_\tau(X-Y)\chi(\mathcal{O}^{\pm}_{\tau},Y). \tag{11.7}$$

It is obvious that nonnegative functions (11.7) are of ckass C^∞ and are compactly supported in τ-neighborhoods of the regions $\mathcal{O}^{\pm}_{\tau}$, respectively. From the definition of τ-coapproximation we derive the inequalities

$$\left(\omega_\tau * \chi(\mathcal{O}^{-}_{\tau})\right)(X) \leq \chi(\mathcal{O},X) \leq \left(\omega_\tau * \chi(\mathcal{O}^{+}_{\tau})\right)(X). \tag{11.8}$$

Replacing X by $\gamma - X$ in (11.8) and summing these inequalities over $\gamma \in \Gamma$, we find that

$$V^{-}_{\tau}(\mathcal{O},X) \leq N(\mathcal{O},X) \leq N^{+}_{\tau}(\mathcal{O},X), \tag{11.9}$$

where

$$N(\mathcal{O},X) = \sum_{\gamma\in\Gamma} \chi(\mathcal{O},\gamma - X) = \sum_{\gamma\in\Gamma} \chi(\mathcal{O}+X,\gamma) \tag{11.10}$$

is the number of lattice points $\gamma \in \Gamma$ lying inside the region $\mathcal{O}+X$, and

$$N^{\pm}(\mathcal{O},X) = \sum_{\gamma\in\Gamma} (\omega_\tau * \chi(\mathcal{O}^{\pm}_{\tau}))(\gamma - X) \tag{11.11}$$

are sums with finitely many terms.

Applying the Poisson summation formula (10.12) to the sums (11.11), we obtain

$$N^{\pm}(\mathcal{O},X) == \operatorname{vol} \mathcal{O}^{\pm}_{\tau} + \mathcal{R}^{\pm}_{\tau}(\mathcal{O},X), \tag{11.12}$$

where $\mathcal{R}^{\pm}_{\tau}(\mathcal{O},X)$ are given in (11.5).

Substituting the relations (11.10)–(11.12) in the inequality (11.9), and using the definitions (1.7), (1.9) of the error $R(\mathcal{O},\Gamma)$, we obtain the inequality (11.6).

The proof of Lemma 11.1 is complete.

Now we wish to estimate the Fourier transform of the characteristic function of a polyhedron $P \subset \mathbb{R}^n$:

$$\widetilde{\chi}(P,Y) = \int_{\mathbb{R}^n} e^{2\pi i \langle Y,X\rangle}\, dX. \tag{11.13}$$

Much research was devoted to this rather nontrivial matter (see e.g. a survey paper [B] by Barvinok and references therein). Nowever, we need only relatively simple information concerning the Fourier integral (11.13).

Let $P = P^k \subset \mathbb{R}^k$ be an arbitrary compact polehedron. Assume that the boundary ∂P^k consists of the $(k-1)$-dimensional faces P_j^{k-1}, $1 \le j \le k$,

$$\partial P^k = \bigcup_{j=1}^{h} P_j^{k-1}. \tag{11.14}$$

We denote by $l_j = l_j^k$, $1 \le j \le h$, the unit vectors of the externel normals to the faces P_j^{k-1} (cf. (6.2)). One has the following formula

$$\widetilde{\chi}(P^k, Y) = \frac{1}{2\pi i \|Y\|^2} \sum_{j=1}^{h} \langle Y, l_j \rangle e^{2\pi i \langle Y, \rangle l_j} \widetilde{\chi}\left(p_j^{k-1},\, Y_j\right), \tag{11.15}$$

where $Y_j = Y - \langle Y, l_j \rangle l_j \in \mathbb{R}^{k-1}$ is a component of the vector Y parallel to the face P_j^{k-1}. For simolicity we use in (11.15) the same notation P_j^{k-1} for the orthogonal projection of the face P_j^{k-1} onto the $(k-1)$-dimensional subspace parallel to P_j^{k-1}. Moreover, we put $\widetilde{\chi}(P_j^0, Y) = 1$ for zero-dimensional faces (verteces) P_j^0.

To prove (11.15) we use the following identity (cf. (11.13))

$$-4\pi^2 \|Y\|^2 \widetilde{\chi}(P^k, Y) = \int\limits_{P^k} \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j}\, e^{2\pi i \langle Y, X \rangle} \right) d^k X, \tag{11.16}$$

where $d^k X$ denotes the Euclidean volume element in $\mathbb{R}^k$. Integrating in (11.16) by parts and using (11.14), we obtain

$$-4\pi^2 \|Y\|^2 \widetilde{\chi}(P^k, Y) = \int\limits_{\partial P^{k-1}} \left(\frac{\partial}{\partial \nu_x}\, e^{2\pi i \langle Y, X \rangle} \right) d^{k-1} X =$$

$$= \sum_{j=1}^{h} \int\limits_{P_j^{k-1}} \left(\sum_{j=1}^{n} \frac{\partial}{\partial \nu_x}\, e^{2\pi i \langle Y, X \rangle} \right) d^{k-1} X =$$

$$= \sum_{j=1}^{h} 2\pi i \langle Y, l_j^k \rangle e^{2\pi i \langle Y, l_j^k \rangle} \int\limits_{P_j^{k-1}} e^{2\pi i \langle Y_j, X \rangle} d^{k-1} X, \tag{11.17}$$

where $\frac{\partial}{\partial \nu_X}$ denotes the normal derivative at a point $X \in \partial P^k$. The relation (11.17) proves the formula (11.15).

Obviously, we have the relation

$$\widetilde{\chi}(tP, Y) = t^n \widetilde{\chi}(P, Y), \tag{11.18}$$

where tP is a dilatation of P by a factor $t > 0$.

The following assertion follows immediately from formulas (11.15) and (11.18).

Lemma 11.2. *Let $P \subset \mathbb{R}^n$ be a arbitrary compact polyhedron, then one has the bound*

$$| \widetilde{\chi}(tP, Y) | \leq \frac{t^{n-1}}{2\pi \|Y\|} \quad \text{area}\, \partial P, \tag{11.19}$$

where area ∂P *is the* $(n-1)$*-dimensional Euclidean volume of the boundary* ∂P.

Using the definition (6.1)–(6.2) of the flags of faces $\mathfrak{f}$ of a given polyhedron P and the orthogonal matrices $V_{\mathfrak{f}}$ associated with flags $\mathfrak{f}$, and applying the formula (11.15) consecutively in dimensions $k = n, n-1, \dots, 1$, we immediately obtain the following assertion.

Lemma 11.3. *Let $P \subset \mathbb{R}^n$ be an arbitrary compact polyhedron, then one has the following explicit formula*

$$\widetilde{\chi}(tP, Y) = \left(\frac{1}{2\pi i} \right)^n \sum_{\mathfrak{f}} \lambda(V_{\mathfrak{f}}\, Y) e^{2\pi i t \langle P^0, Y \rangle}. \tag{11.20}$$

The summation in (11.20) *is taken over all flags of faces of the polyhedron* P *(cf.* (6.1)*),*

$$\mathfrak{f} = \left\{ P = P^n \supset P^{n-1} \supset \ldots \supset P^0, \; \dim P^j = j, \; n \geq j \geq 0 \right\}, \tag{11.21}$$

and $V_{\mathfrak{f}} \in \mathrm{SO}(n)$ are the orthogonal matrices associated with the flags (cf. (6.2)*). The function $\lambda(\cdot)$ is given by*

$$\lambda(Y) = \frac{1}{Nm\, Y} \Phi(\widehat{Y}), \quad Y \in \mathbb{R}^n, \tag{11.22}$$

where $\Phi(\cdot)$ if the following function of class C^∞ depending only on angular variables $\widehat{Y} = \|Y\|^{-1} Y \in \Sigma^n$ (cf. (4.16)*):*

$$\Phi(\widehat{Y}) = \Phi\left(\frac{y_1}{\|Y\|}, \dots, \frac{y_n}{\|Y\|} \right) = \prod_{j=1}^{n-1} \frac{y_j^2}{y_j^2 + \ldots + y_n^2}. \tag{11.23}$$

§ 12. Proof of Theorem 6.1

Now we are in position to prove the main bound (6.4) for the lattice point counting for a polyhedron. Let $P \subset \mathbb{R}^n$ be an arbitrary compact convex polyhedron. By the definition (1.7) of the error $R(tP, \Gamma)$ we may consider the polyhedron P up to the translations by vectors $X \in \mathbb{R}^n$. Thus, without loss of generality, we can assume that the origin $O \in P$. Then the polyhedron P can be given by the inequalities

$$P = \{ X \in \mathbb{R}^n : \langle l_j, X \rangle < \beta_j, \;\; 1 \leq j \leq h \}, \tag{12.1}$$

where l_j, $1 \leq j \leq h$, are the unit vectors of the extremal normals to the $(n-1)$-dimensional faces of P, and $\beta_j > 0$ are some constants. In these notations the dilatation tP of P by a factor $t > 0$ can be given by

$$tP = \{ X \in \mathbb{R}^n : \langle l_j, X \rangle < \beta_j\, t, \;\; 1 \leq j \leq h \}, \tag{12.2}$$

Let $\beta = \{\beta_1, \dots, \beta_n\}$ and $t_\tau^{\pm} = t \pm \beta^{-1}\tau$, $0 < \tau < 1$. We consider the following two polyhedra

$$t_\tau^{\pm} P = \left\{ X \in \mathbb{R}^n : \langle l_j, X \rangle < \beta_j\, t_\tau^{\pm}, \;\; 1 \leq j \leq h \right\}, \tag{12.3}$$

Comparing the definition (12.2) and (12.3) we conclude that the polyhedrons $t_\tau^- P$ and $t_\tau^+ P$ form a τ-coapproximation of the polyhedron tP (cf. § 11). Since the polyhedra P. tP, and $t_\tau^\pm P$ are homotetic their flags of faces are also homothetic. Therefore, all of the polyhedra P. tP, and $t_\tau^\pm P$ have the same set of orthogonal matrices $V_{\mathfrak{f}} \in \mathrm{SO}(n)$ associated with their flags.

We wish to apply Lemma 11.1 wuth $\mathcal{O} - tP$ and $\mathcal{O}_\tau^\pm = t_\tau^\pm P$ to estimate the error $R(tP, \Gamma)$.

First, we have the bound

$$0 < \operatorname{vol} t_\tau^+ P - \operatorname{vol} t_\tau^- P = \left((t + \beta^{-1}\tau)^n - (t - \beta^{-1}\tau)^n\right) \operatorname{vol} P \sim \left(2n\beta^{-1} \operatorname{vol} P\right) r t^{n-1} \tag{12.4}$$

as $t \to \infty$.

Now we consider the Fourier series (11.5) with $\mathcal{O} = tP$ and $\mathcal{O}_\tau^\pm = t^\pm P$. Let $\omega_1(X)$, $X \in \mathbb{R}^n$, be a function of class C^∞ supported inside the ball $\frac{1}{4}B^n$ (cf. (8.2)) and satisfying the following conditions

$$\left.\begin{aligned} 0 \le \omega_1(X) \le 1 \quad &\text{if} \quad X \in \mathbb{R}^n, \\ \omega_1(X) = 1 \quad &\text{if} \quad \|X\| \le \frac{1}{8}. \end{aligned}\right\} \tag{12.5}$$

Let $\omega_2(X) = \widetilde{\omega}_1(X)$, $X \in \mathbb{R}^n$, be the Fourier transform of the function $\omega_1(\cdot)$. Obviously, the function $\omega_2(\cdot)$ satisfies the bound (11.4). Moreover, we assume that both of the functions $\omega_1(\cdot)$ and $\omega_2(\cdot)$ are spherically symmetric:

$$\omega_j(VX) = \omega_j(X), \quad V \in \mathrm{SO}(n), \quad j = 1, 2. \tag{12.6}$$

The Fourier series (11.5) can be written

$$\mathcal{R}_\tau^\pm(tP, X) = \mathcal{A}_{\tau,\rho}^\pm(tP, X) + \mathcal{B}_{\tau,\rho}^\pm(tP, X), \tag{12.7}$$

where $\rho \ge 1$ is an arbitrary large parameter,

$$\mathcal{A}_{\tau,\rho}^\pm(tP, X) = \sum_{\gamma \in \Gamma^\perp \setminus \{0\}} \widetilde{\chi}(t_\tau^\pm P, \gamma)\omega_1(\tau\gamma)\omega_2(\rho^{-1}\gamma)e^{-2\pi i\langle\gamma, X\rangle} \tag{12.8}$$

are sums with finitelu many terms, and

$$\mathcal{B}_{\tau,\rho}^\pm(tP, X) = \sum_{\gamma \in \Gamma^\perp} \widetilde{\chi}(t_\tau^\pm P, \gamma)\omega_1(\tau\gamma)\left(1 - \omega_2(\rho^{-1}\gamma)\right) e^{-2\pi i\langle\gamma, X\rangle} \tag{12.9}$$

are absolutely convergent series over the lattice points $\gamma \in \Gamma^\perp$ with $\|\gamma\| > \frac{1}{8}\rho$ (cf. (12.5)).

The sums (12.8) can be explicitly expressed in terms of the special Fourier series (10.1). Substituting the formula (11.20) for the Fourier transform of the characteristic functions of the polyhedra $t_\tau^\pm P$ in (12.8), we obtain

$$\mathcal{A}_{\tau,\rho}^\pm(tP, X) = \left(\frac{1}{2\pi i}\right)^n \sum_{\gamma \in \Gamma^\perp \setminus \{0\}} \sum_{\mathfrak{f}} \lambda(V_{\mathfrak{f}}\,\gamma)\omega_1(\tau\gamma)\omega_2(\rho^{-1}\gamma)e^{-2\pi i\langle\gamma, t_\tau^\pm P^0 - X\rangle} \tag{12.10}$$

In the second sum in (12.10) the summation is taken over all flags of faces $\mathfrak{f}$ of the polyhedron P. The function $\lambda(\cdot)$ is given by (11.22). The points P^0 in (12.10) are zero-dimensional faces (verteces) of the polyhedron P belonging to the corresponding flags $\mathfrak{f}$ (cf. (11.20)–(11.21)).

Using the definition (6.3) of lattices $\Gamma_{\mathfrak{f}} = V_{\mathfrak{f}}\Gamma$ and $\Gamma_{\mathfrak{f}}^{\perp} = V_{\mathfrak{f}}\Gamma^{\perp}$, the relation (12.6), and the formula (12.10), we find that

$$\mathcal{A}^{\pm}_{\tau,\rho}(tP,X) = \left(\frac{1}{2\pi i}\right)^n \sum_{\mathfrak{f}} \sum_{\gamma\in\Gamma_{\mathfrak{f}}^{\perp}\setminus\{0\}} \lambda(\gamma)\omega_1(\tau\gamma)\omega_2(\rho^{-1}\gamma)e^{-2\pi i\langle\gamma,X^{\pm}_{\tau,\mathfrak{f}}\rangle} =$$

$$= \left(\frac{1}{2\pi i}\right)^n \sum_{\mathfrak{f}} W_{\tau,\rho}\left(\Gamma_{\mathfrak{f}}, X^{\pm}_{\tau,f}\right), \tag{12.11}$$

where we used the notation $X^{\pm}_{\tau,\mathfrak{f}} = V_{\mathfrak{f}}(t^{\pm}P - X)$.

From Lemma 10.1 and the relation (12.11) we obtain the bound

$$\max_{X\in\mathbb{R}^n} \left|\mathcal{A}^{\pm}_{\tau,\rho}(tP,X)\right| < c\sum_{\mathfrak{f}} S(\Gamma_{\mathfrak{f}}^{\perp}, r_{\mathfrak{f}}) \tag{12.12}$$

with

$$r_{\mathfrak{f}} = 2\varkappa_n + \left|\log\frac{\rho^n}{\nu(\Gamma_{\mathfrak{f}}^{\perp},\rho)}\right|. \tag{12.13}$$

In the biund (12.12) the constant $c = c(\omega_1,\omega_2)$ is independent of the lattice Γ and the parameters $t>0$, $\rho>0$, and $- < \tau < 1$.

Now we consider the series (12.9). Substituting the bound (11.19) for the Fourier transform of the caracteristic functions of the polyhedra $t^{\pm}_{\tau}P$, and the bound (11.4) for the function $\omega_1(\cdot) = \widetilde{\omega}(\cdot)$ with $A>n$, we obtain as $t\to\infty$

$$\max_{X\in\mathbb{R}^n} \left|\mathcal{B}^{\pm}_{\tau,\rho}(tP,X)\right| \le \left(\frac{c_A}{2\pi}\text{ area }\partial P\right) t^{n-1}\tau^{-A} \sum_{\gamma\in\Gamma^{\perp}:\|\gamma\|>\frac{1}{8}\rho} \|\gamma\|^{-A-1}. \tag{12.14}$$

Notice that for every lattice $\Lambda\in\mathbb{R}^n$ one has the bound (see e.g. [Kr])

$$\sum_{\gamma\in\Gamma^{\perp}:\|\gamma\|>\frac{1}{8}\rho} \|\gamma\|^{-A-1} < c_A(\Lambda)\rho^{n-A-1} \quad (A>n) \tag{12.15}$$

as $\rho\to\infty$. The bounds (12.14)–(12.15) imply

$$\max_{X\in\mathbb{R}^n} \left|\mathcal{B}^{\pm}_{\tau,\rho}(tP,X)\right| < c_A(P,\Gamma)t^{n-1}\tau^{-A}\rho^{n-A-1}. \tag{12.16}$$

Substituting the bounds (12.4), (12.12)–(12.13), and (12.16) in (12.7) and (11.6), we obtain

$$R(tP,\Gamma) < C_A(P,\Gamma)\left(t^{n-1}\tau + t^{n-1}\tau^{-A}\rho^{n-A-1} + \sum_{\mathfrak{f}} S(\Gamma_{\mathfrak{f}}^{\perp}, r_g f)\right). \tag{12.17}$$

Further we assume that

$$\tau = \rho^{\theta} \quad\text{with}\quad \theta = \frac{A+1-n}{A+1} = 1 - \frac{n}{A+1}, \tag{12.18}$$

moreover, the exponent $\theta\in(0,1)$ may be chosen arbitrary close to 1, since $A>n$ is arbitrary large. From (12.18) we obtain

$$t^{n-1}\tau = t^{n-1}\tau^{-A}\rho^{n-A-1} = t^{n-1}\rho^{-\theta}. \tag{12.19}$$

Now the bound (6.4)–(6.5) follows from (12.17) and (12.19).

The proof of Theorem 6.1 is complete.

APPENDICES

§ 13. Appendix 1. Invariant integration over the space of lattices. Proof of Lemma 3.1

Recall basic facts concerning invariant integration on $\mathrm{SL}(n)$. We closely follow the paper [Sig] by Siegel. Additional details can be found in [GL] and [R].

Let $\mathbb{R}^N$, $N = \frac{n^2+n}{2}$, be the space of all real symmetric $n \times n$ matrices $A = (a_{ij})$, $a_{ij} = a_{ji}$, $1 \leq i,\ j \leq n$, and let $\mathcal{K} \subset \mathbb{R}^N$ be an open cone of all positive definite matrices $A \in \mathbb{R}^N$. The action of the group $\mathrm{SL}(n, \mathbb{R})$ in the cone $\mathcal{K}$ is defined by

$$\mathcal{K} \ni A \to g[A] =^{\mathrm{T}}g\ Ag \in \mathcal{K}, \quad g \in \mathrm{SL}(n, \mathbb{R}), \tag{13.1}$$

From reduction theory we know that with respect to the action (13.1) the discrete subgroup $\mathrm{SL}(n, \mathbb{Z}) \subset \mathrm{SL}(n, \mathbb{R})$ has a well-defined foundamental set

$$\mathcal{R} = \mathcal{K}/\mathrm{SL}(n, \mathbb{R}), \tag{13.2}$$

which can be realized as a cone in $\mathcal{K}$ given by finitely many kinear inequality on the coefficients a_{ij} of matrices $A \in \mathbb{R}^N$. In particular, one has the inclusion

$$\mathcal{R} \subset \mathcal{R}_0, \tag{13.3}$$

where $\mathcal{R}_0 \subset \mathcal{K}$ is the following cone

$$\mathcal{R}_0 = \left\{ A \in \mathcal{K} : 0 < a_{11} \leq a_{22} < \ldots \leq a_{nn},\ |a_{ij} \leq \frac{1}{2} a_{ii},\ 1 \leq i < j \leq n \right\}. \tag{13.4}$$

We consider the following homogeneous dpaces

$$\begin{aligned} \mathcal{H}_n &= \mathrm{SO}(n) \setminus \mathrm{SL}(n, \mathbb{R}), \\ \mathcal{F}_n &= \mathcal{H}_n/\mathrm{SL}(n, \mathbb{Z}) = \mathrm{SO}(n) \setminus \mathcal{L}_n. \end{aligned} \tag{13.5}$$

Using the group action (13.1), we can identify the homogeneous spaces (13.5) with the following submanifolds in the cone $\mathcal{K}$:

$$\begin{aligned} \mathcal{H}_n &= \{A \in \mathcal{K}\ :\ \det A = 1\}, \\ \mathcal{F}_n &= \{A \in \mathcal{R}\ :\ \det A = 1\}. \end{aligned} \tag{13.6}$$

The space $\mathcal{F}_n$ admits the unique $\mathrm{SL}(n, \mathbb{R})$ – invariant (with respect to the froup action (13.1)) measure $\mu_{\mathcal{F}}$ normalized by the condition

$$\mu_{\mathcal{F}}(\mathcal{F}_n) = 1. \tag{13.7}$$

Using (13.6), we obtain the following formula for every measurable function $\psi(A)$, $A \in \mathcal{F}_n$:

$$\int_{\mathcal{F}_n} \psi(A) d\mu_{\mathcal{F}}(A) = v_n^{-1} \int_{\mathcal{R}(1)} \psi\left(\{\det A\}^{-\frac{1}{n}} A \right) dA, \tag{13.8}$$

where

$$dA = \prod_{1 \leq i \leq j \leq n} da_{ij} \tag{13.9}$$

in the Euclidean volume element in $\mathbb{R}^N$ (invariant with respect to the group action (13.1)),

$$\mathcal{R}(1) = \{A \in \mathcal{R} \,:\, 0 < \det A \leq 1\}\,, \tag{13.10}$$

and v_n is the well known Minkowski's constant:

$$v_n = \operatorname{vol}\mathcal{R}(1) = \frac{2}{n+1}\pi^{-\frac{n(n+1)}{4}}\Gamma\left(\frac{1}{2}\right)\ldots\Gamma\left(\frac{n}{2}\right)\zeta(2)\ldots\zeta(n).$$

Let μ_{SO} denote the Haar measure on the orthogonal group $\mathrm{SO}(n)$ normalized by the condition

$$\mu_{\mathrm{SO}}(\mathrm{SO}(n)) = 1. \tag{13.11}$$

The elements of the measures $\mu_{\mathcal{L}}$ (cf. (1,3)), $\mu_{\mathcal{F}}$, and μ_{SO} are related by

$$d\mu_{\mathcal{L}} = d\mu_{\mathcal{F}} \times d\mu_{\mathrm{SO}}. \tag{13.12}$$

Precisely, let $\Lambda = \alpha\mathbb{Z}^n \in \mathcal{L}_n$ (cf. (1.1)), and let

$$\alpha = VA^{\frac{1}{2}}, \quad V \in \mathrm{SO}(n), \quad A =^{\mathrm{T}}\alpha\alpha \in \mathcal{F}_n \tag{13.13}$$

be the polar decomposition of the matrix $al \in \mathrm{SL}(n,\mathbb{R})$, then one has the following formula for every integrable function $\psi(\Lambda)$, $\Lambda \in \mathcal{L}_n$:

$$\int\limits_{\mathcal{L}_n} \psi(\Lambda)d\mu_{\mathcal{L}}(A) = \int_{\mathcal{F}_n} d\mu_{\mathcal{F}}(A)\int_{\mathrm{SO}(n)} d\mu_{\mathrm{SO}}(V)\psi\left(VA^{\frac{1}{2}}\mathbb{Z}^n\right), \tag{13.14}$$

where the integration over the measure $\mu_{\mathcal{F}}$ is given (13.8).

The expression $\langle AX, X\rangle$, $A \in \mathcal{K}$, is a positive definite quadratic form in n variables $X =^{\mathrm{T}} (x_1, \ldots, x_n) \in \mathbb{R}^n$. Let

$$M(A) = \min\{\langle A\gamma, \gamma\rangle \,:\, \gamma \in \mathbb{Z}^n \setminus \{0\}\} \tag{13.15}$$

be the minimum of this form. Comparing the definition (3.6) of the homogeneous minimum of a lattice $\Lambda = \alpha_\Lambda\mathbb{Z}^n$ (cf. (1.1)) with the definition (13.15), we obtain the relation

$$\|\Lambda\| = \left(M\left({}^{\mathrm{T}}\alpha_\Lambda\,\alpha_\Lambda\right)\right)^{\frac{1}{2}}. \tag{13.16}$$

Thus, the function $\|\Lambda\|$, $\Lambda \in \mathcal{L}_n$, is well-defined on the homogeneous space $\mathcal{F}_n$ by the relation (13.16).

Since the cone (13.4) is a *Siegel domain*, one has the following inequalities for every matrix $A = (a_{ij}) \in \mathcal{R}_0$ (cf. [Cas 2], Chapter 12, Lemma 4.1)

$$\eta^{-1}a_{11} \leq M(A) \leq a_{11}, \tag{13.17}$$

$$\eta^{-1}\prod_{j=1}^{n} a_{jj} \leq \det A \leq \prod_{j=1}^{n} a_{jj} \tag{13.18}$$

with a constant $\eta > 1$ depending inly on dimension n. It is worth noting that the inequalities on the right-hand sides of (13.17) and (13.18) are valid for every matrix $A \in \mathcal{K}$.

Proof of Lemma 3.1. The following assertion implies Lemma 3.1.

Lemma 13.1. *Let a function $\Omega(\Lambda)$, $\Lambda \in \mathcal{L}_n$, be given by the formula*

$$\Omega(\Lambda) = \|\Lambda\|^{-n} \omega \left(\log \frac{1}{\|\Lambda\|} \right), \tag{13.19}$$

where $\omega(t)$, $t \in \mathbb{R}$, is a positive monotone nonincreasing function, then one has the inequality

$$\int\limits_{\mathcal{L}_n} \Omega(\Lambda)\, d\mu_{\mathcal{L}}(\Lambda) < c(n) \int\limits_{\varkappa(n)}^{\infty} \omega(t)\, dt \tag{13.20}$$

with constants $c(n)$ and $\varkappa(n) > 0$ depending only on dimension n.

In particular, let $\omega(.) = \frac{1}{\varphi(.)}$ where $\varphi(.)$ is an arbitrary majorant (cf. (2.1)−−(2.2)), then the corresponding function $\Omega(\Lambda) \in L_1(\mathcal{L}_n, \mu_{\mathcal{L}} :)$.

Proof. Consider the integral

$$J = \int\limits_{\mathcal{L}_n} \Omega(\Lambda)\, d\mu_{\mathcal{L}}(\Lambda). \tag{13.21}$$

Using the relations (13.15), (13.16) and (13.8), we find that

$$\begin{aligned} J &= \int\limits_{\mathcal{F}_n} \{M(A)\}^{-\frac{n}{2}}\, \omega \left(\log \{M(A)\}^{-\frac{1}{2}} \right) d\mu_{\mathcal{F}}(A) \\ &= v_n^{-1} \int\limits_{\mathcal{R}(1)} \left\{ \frac{(\det A)^{\frac{1}{n}}}{M(A)} \right\}^{\frac{n}{2}} \omega \left(\left\{ \frac{(\det A)^{\frac{1}{n}}}{M(A)} \right\}^{\frac{1}{2}} \right) dA. \end{aligned} \tag{13.22}$$

From the definition (13.2)–(13.4), (13.10), and the ineqialities (13.18) we derive the following inclusions

$$\mathcal{R}(1) \subset \mathcal{R}_0(1) \subset \mathcal{O}, \tag{13.23}$$

where

$$\mathcal{R}_0(1) = \{A \in \mathcal{R}_0 \,:\, 0 < \det A \leq 1\} \tag{13.24}$$

and

$$\mathcal{O} = \left\{ A \in \mathcal{R}_0 \,:\, 0 < \prod_{j=1}^{n} a_{jj} \leq \eta \right\}, \tag{13.25}$$

moreover, η is the constant from (13.18).

From (13.14), (13.17), and (13.18) we obtain the following relations for every matrix $A \in \mathcal{R}_0$

$$\frac{det A)^{\frac{1}{n}}}{M(A)} \leq \eta \frac{1}{a_{11}} \left(\prod_{j=1}^{n} a_{jj} \right)^{\frac{1}{n}} \tag{13.26}$$

and

$$\begin{aligned} \frac{det A)^{\frac{1}{n}}}{M(A)} &\geq \eta \frac{1}{a_{11}} \left(\prod_{j=1}^{n} a_{jj} \right)^{\frac{1}{n}} \geq \\ &\geq \eta^{-1} \frac{(a_{11} a_{22}^{n-1})^{\frac{1}{n}}}{a_{11}} = \eta^{-1} \left(\frac{a_{22}}{a_{11}} \right)^{\frac{n-1}{n}}. \end{aligned} \tag{13.27}$$

Since $\omega(\cdot)$ is a monotone nonincreasing function the inequality (13.27) implies that

$$\omega\left(\left\{\frac{(\det A)^{\frac{1}{n}}}{M(A)}\right\}^{\frac{1}{2}}\right) \le \omega\left(\log\left\{\eta^{-1}\left(\frac{a_{22}}{a_{11}}\right)^{\frac{n-1}{n}}\right\}^{\frac{1}{2}}\right). \tag{13.28}$$

Now we wish to estimate the integral (13.22) as folloes: replace the region $\mathcal{R}(1)$ by $\mathcal{O}$ (cf. (13.23)–(13.25)), and estimate the integrand by bounds (13.26) and (13.28). This gives

$$J \le v_n^{-1}\, \eta^{\frac{n}{2}}\, I, \tag{13.29}$$

where

$$I = \int\limits_{\mathcal{O}} \frac{(a_{11}\dots a_{nn})^{\frac{1}{n}}}{a_{11}^{\frac{n}{2}}}\omega\left(\log\left\{\eta^{-1}\left(\frac{a_{22}}{a_{11}}\right)^{\frac{n-1}{n}}\right\}^{\frac{1}{2}}\right)\left\{\prod_{1\le i<j\le n} da_{ij}\right\} da_{11}\dots da_{nn} \tag{13.30}$$

$$= \overbrace{\int\dots\int}^{n \text{ times}}_{\substack{0<a_{11}\le a_{22}\le\dots\le a_{nn},\\ 0<a_{11}\dots a_{nn}\le\eta}} \frac{(a_{11}\dots a_{nn})^{\frac{1}{n}}}{a_{11}^{\frac{n}{2}}}\omega\left(\log\left\{\eta^{-1}\left(\frac{a_{22}}{a_{11}}\right)^{\frac{n-1}{n}}\right\}^{\frac{1}{2}}\right) a_{11}^{n-1}\dots a_{n-1,n-1}da_{11}\dots da_{nn}.$$

To calculate this integral, we make consecutively changes of variables in(13.30). By setting $a_{jj} = \eta^{\frac{1}{n}}\exp x_j$, $1\le j\le n$, we obtain

$$I = \eta^{\frac{n+1}{2}} \overbrace{\int\dots\int}^{n \text{ times}}_{\substack{x_1\le x_2\le\dots\le x_n,\\ x_1+\dots x_n\le 0}} dx_1\dots dx_n\omega\left(\frac{n-1}{2n}\{x_2 - x_1 + \varkappa(n)\}\right)\times\exp(\nu_1 x_1+\dots+\nu_n x_n), \tag{13.31}$$

where

$$\varkappa(n) = \frac{2n}{n-1}\log\eta,$$

and $\nu_1 = 0$, and $\nu_j = \frac{3}{2} + n - j$ if $2\le j\le n$. Notice that $\varkappa(n) > 0$ since $\eta > 1$.

Now, let $y_j = x_{j+1} - x_j$ if $1\le j\le n-1$, and $y_n = -x_1 - \dots - x_n$, then the integral (13.31) takes the form

$$I = \eta^{\frac{n+1}{2}} n^{-1}\int\limits_0^\infty dy_1\omega\left(\frac{n-1}{2n}\{y_1+\varkappa(n)\right)\prod_{j=2}^n\int\limits_0^\infty \exp(-\lambda_j\, y_j), \tag{13.32}$$

where $\lambda_j = \frac{1}{2}(n-j)(j-1)$ if $2\le j\le n-1$, and $\lambda_n = \frac{n+1}{2}$.

From (13.32) we derive that

$$I = c(n)\int\limits_{\varkappa(n)}^\infty \omega(t)\,dt, \tag{13.33}$$

where

$$c(n) = \eta^{\frac{n+1}{2}}\, 2^n\frac{1}{n^2-1}\prod_{j=2}^n\frac{1}{(n-j)(j-1)}.$$

The bound (13.20) follows from (13.29) and (13.33).

The proof of Lemma 13.1 is complete.

§ 14. Appendix 2. Biunds for Fourier integrals. Proof of Lemma 10.2

We consider the following Fourier integral

$$I(X) = \int\limits_{\mathbb{R}^n_+} dY\, J(Y) \prod_{j=1}^{n} \sin(x_j\, y_j), \quad X = {}^{\mathrm{T}}(x_1, \dots, x_n) \in \mathbb{R}^n, \tag{14.1}$$

where $J(Y) = J(y_1, \dots, y_n)$, $Y = {}^{\mathrm{T}}(y_1, \dots, y_n) \in \mathbb{R}^n_+$ is a function of class C^∞ supported in the following region

$$\text{support } J(\cdot) \subseteq \left\{ Y \in \mathbb{R}^n_+ : \tfrac{1}{4} \leq j \leq 1,\, 1 \leq j \leq n-1,\, y_n \geq \tfrac{1}{4} \right\}, \tag{14.2}$$

moreover, the function $J(\cdot)$ satisfies the bounds

$$\left| \frac{\partial^{k_1 + \dots + k_n}}{\partial y_1^{k_1} \dots \partial y_n^{k_n}} J(Y) \right| < b\,(k_1 + \dots + k_n) y_n^{-1-k_n}, \tag{14.3}$$

for all integers $k_1 \geq 0, \dots, k_n \geq 0$.

Obviously, the function $J9\cdot) \in L_2(\mathbb{R}^n_+)$. Therefore, the Fourier integral (14.1) converges at least in the L_2-norm, and $I(\cdot) \in L_2(\mathbb{R}^n)$. However, a much more precise assertion may be stated. For functions of the form $J(Y) = j_1(y_1) \dots j_n(y_n)$ this assertion was given previously in [Skr 3, Lemma 4.2].

Lemma 14.1. *The Fourier integral* (14.1) *satisfies the bound*

$$|I(X)| < c_A \prod_{j=1}^{n} (1 + |x_j|)^{-A} \tag{14.4}$$

with arbitrary large $A > 0$.

Proof. The integral (14.1) is an odd function of each variable x_j, $1 \leq j \leq n$. Thus, without loss of generality, we may assume that $X \in \mathbb{R}^n_+$. We prove that for every integer $l > 0$ the Fourier integral (14.1) satisfies the bound

$$|I(X)| < c_l \prod_{j=1}^{n} (1 + x_j^{4l})^{-\frac{1}{n}}, \quad X \in \mathbb{R}^n_+. \tag{14.5}$$

Obviously, the bounds (14.5) imply the bounds (14.4).

We use the following coordinates in $\mathbb{R}^n_+$:

$$Y = {}^{\mathrm{T}}({}^{\mathrm{T}}y, y_n), \quad y = {}^{\mathrm{T}}(y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1}_+, \quad y_n \in \mathbb{R}^1_+. \tag{14.6}$$

In the coordinates (14.6) the integral (14.1) takes the form

$$I(X) = \int\limits_{\mathbb{R}^{n-1}_+} dy\, A(x_n, y) \prod_{j=1}^{n-1} \sin(x_j\, y_j), \tag{14.7}$$

where

$$A(x_n, y) = \int\limits_{1/4}^{\infty} dy_n\, J(y_n, y) \sim (x_n\, y_n). \tag{14.8}$$

Integrating in (14.8) $4l$ times by parts, we obtain

$$x_n^{4l}\, A(x_n, y) = \int\limits_{1/4}^{\infty} dy_n\, \left\{ \left(\frac{\partial}{\partial y_n} \right)^{4l} j(y_n, y) \right\} \sin(x_n\, y_n). \tag{14.9}$$

From (14.3) and (14.9) we find that

$$x_n^{4l}\, A(x_n, y) < b(4l) \int\limits_{1/4}^{\infty} dy_n\, y_n^{-1-4l} \ll 1. \tag{14.10}$$

For simplicity we use here Vinogradov's symbolism. Moreover the constants implicit in $\ll$ are independent of vectors X and Y.

We wish to prove the following bound

$$|A(x_n, y)| \ll 1. \tag{14.11}$$

If $x_n \geq 4$, then the bound (14.11) follows from the bound (14.10) with $l = 1$.

If $0 < x_n < 4$, then we put

$$A(x_n, y) = A_1(x_n, y) + A_2(x_n, y), \tag{14.12}$$

where

$$A_1(x_n, y) = \int\limits_{1/4}^{1/x_n} dy_n\, J(y_n, y) \sin(x_n\, y_n). \tag{14.13}$$

and

$$A_2(x_n, y) = \int\limits_{1/x_n}^{\infty} dy_n\, J(y_n, y) \sin(x_n\, y_n). \tag{14.14}$$

The integral (14.13) can be estimated as follows (cf. (14.3))

$$|A_1(x_n, y)| < b(0) \int\limits_{1/4}^{1/x_n} dy_n \frac{1}{y_n} |\sin(x_n, y_n)| \ll 1. \tag{14.15}$$

Integrating in (14.14) by parts, we obtain

$$x_n\, A_2(x_n, y) = J\left(\frac{1}{x_n}, y \right) \cos 1 + \int\limits_{1/x_n}^{\infty} dy_n \left\{ \frac{\partial}{\partial y_n} J(y_n, y) \right\} \cos(x_n\, y_n).$$

Hence (cf. (14.3))

$$x_n |A_2(x_n, y)| < b(0) x_n + b(1) \int\limits_{1/x_n}^{\infty} dy_n \, y_n^{-2} \ll x_n.$$

Therefore,

$$|A_2(x_n, y)| \ll 1. \tag{14.16}$$

From (14.12), (14.15) and (14.16) we obtain the bound (14.11) for $0 < x_n < 4$. Thus the bound (14.11) is valid for all $x_n \in \mathbb{R}^1_+$.

Substituting the bound (14.11) in (14.7) and taking (14.2) into account, we find that

$$|I(X)| \ll \int\limits_{1/4}^{1} dy_1 \ldots \int\limits_{1/4}^{1} dy_{n-1} \ll 1. \tag{14.17}$$

Similarly, substituting the bound (14.10) in (14.7) and taking (14.2) into account, we find that

$$x_n^{4l} |I(X)| \ll 1. \tag{14.18}$$

Adding the bounds (14.17) and (14.18) together, we obtain the bound

$$\left(1 + x_n^{4l}\right) \, |I(X)| \ll 1. \tag{14.19}$$

Integrating in (14.7) $4l$ times by parts in variable y_j, $1 \leq j \leq n-1$, we obtain

$$x_j^{4l} I(X) = \int\limits_{\mathbb{R}^{n-1}_+} dy \left\{ \left(\frac{\partial}{\partial y_j} \right)^{4l} A(x_n, y) \right\} \prod_{j=1}^{n-1} \sin\left(x_j \, y_j\right), \tag{14.20}$$

moreover

$$\left(\frac{\partial}{\partial y_j} \right)^{4l} A(x_n, y) = \int\limits_{1/4}^{\infty} dy_n \, J_{j,l}(y_n, y) \sin\left(x_n \, y_n\right), \tag{14.21}$$

where

$$J_{j,l}\left(y_n, y\right) = \left(\frac{\partial}{\partial y_j} \right)^{4l} J(y_n, y). \tag{14.22}$$

It is obvious that the functions (14.22) are of class C^∞ and are supported in the region (14.2). Moreover, the bounds (14.3) imply the following bounds for the functions (14.22):

$$\left| \frac{\partial^{k_1 + \cdots + k_n}}{\partial y_1^{k_1} \ldots \partial y_n^{k_n}} J_{j,l}(Y) \right| < b\left(k_1 + \cdots + k_n + 4l\right) y_n^{-1-k_n}, \tag{14.23}$$

The foregoing shows that the bound (14.11) with $J = J_{j,l}$ can be applied to estimate the integral (14.21). The gives

$$\left| \left(\frac{\partial}{\partial y_j} \right)^{4l} A(x_n, y) \right| \ll 1. \tag{14.24}$$

Substituting the bound (14.24) in (14.20) and taking (14.2) into account, we find that

$$x_j^{4l}\,|I(X)| \ll 1. \tag{14.25}$$

Adding the bounds (14.17) and (14.25) together, we obtain the bound

$$\left(1+x_j^{4l}\right)\,|I(X)| \ll 1, \quad q \le j \le n-1. \tag{14.26}$$

Finally, multiplyimg together the bound (14.19) and the bounds (14.26) with $j = 1, \ldots, n-1$, we obtain the bound (14.5).

The proof of Lemma 14.1 is complete.

Proof of Lemma 10.2. We consider the following function

$$J_{\mathcal{Q}}(\tau,\rho,Y) = \frac{1}{Nm\,Y}\,V_{\mathcal{Q}}\,(\tau,\rho,Y), \quad Y \in \mathbb{R}_+^n \tag{14.27}$$

where

$$V_{\mathcal{Q}}\,(\tau,\rho,Y) = \Xi(Y)\,\Phi(\widehat{\mathcal{Q}\,Y})\omega_1(\rho^{-1}\,\mathcal{Q}Y)\omega_2(\tau\,\mathcal{Q}Y). \tag{14.28}$$

The functions $\Xi(\cdot)$ and $\Phi(tc)$, $\omega_1(\cdot)$, $\omega_2(\cdot)$ involved in (14.28) are defined in § 9 (cf. (9.9)) and § 10 (cf. (10.2)–(10.4)), respectively. In (14.28) τ and ρ are positive parameters, and $\mathcal{Q} \in \mathrm{GL}(n,\mathbb{R})$ is an arbitrary diagonal matrix,

$$\mathcal{Q} = \mathrm{diag}\,(q_1,\ldots,q_n). \tag{14.29}$$

with $q_j > 0$, $1 \le j \le n$.

If $\mathcal{Q} = \mathcal{Q}_m$ (cf. (9.12)), then (cf. (10.21))

$$J_{\mathcal{Q}}\,(\tau,\rho,Y) = J_m\,(\tau,\rho,Y). \tag{14.30}$$

Lemma 14.2. *The function $J_{\mathcal{Q}}\,(\tau,\rho,Y)$, $Y \in \mathbb{R}_+^n$, given in* (14.27)*, satisfies the bounds* (14.3) *with constants $b(k)$, $k = 0,1,\ldots$, independent of the parameters $\tau > 0$ and $\rho > 0$, and the diagonal matrix $\mathcal{Q} \in \mathrm{GL}(n,\mathbb{R})$.*

Proof. We introduce a differential operator $\mathcal{L}_Y^k$ by setting

$$\mathcal{L}_Y^k = \frac{\partial^{k_1+\cdots+k_n}}{\partial y_1^{k_1}\ldots\partial y_n^{k_n}}, \tag{14.31}$$

where $k = {}^{\mathrm{T}}(k_1,\ldots,k_n)$, and $k_1 \ge 0,\ldots,k_n \ge 0$ are integers.

We wish to prove the following bounds

$$\left|\mathcal{L}_Y^l\,V_{\mathcal{Q}}(\tau,\rho,Y)\right| < c(k)\,y_n^{-k_n} \tag{14.32}$$

with constants $c(.)$ independent of the parameters $\tau > 0$ and $\rho > 0$, and the diagonal matrix $Q \in \mathrm{GL}(n,\mathbb{R})$.

Obviously, the bounds (14.32) imply Lemma 14.2.

To prove the bounds (14.32) we consecutively apply the differential operator $\mathcal{L}_Y^k$ to each factor in the formula (14.28). First we hace (cf. (9.9))

$$\mathcal{L}_Y^k\,\Xi(Y) = \prod_{j=1}^{n}\left(\frac{d}{dy_j}\right)^{k_j}\,\xi_j\,(Y_j). \tag{14.32}$$

From (14.32) we derive the bound

$$\left|\mathcal{L}_Y^k\, \Xi(Y)\right| < c_1\,(k)\, y_n^{-k_n}, \tag{14.33}$$

since the derivative $(\frac{d}{dy_n})^{k_n}\xi_n(y_n) = 0$ for $k_n > 0$ and $y_n \geq 1$ (cf. (9.7)).

Notice that $\Phi(\widehat{Y}) = \Phi(\|Y\|^{-1}Y)$, $Y \in \mathbb{R}^n$, is a homogeneous function of zero degree. Hence

$$\mathcal{L}_Y^k\, \Phi(\widehat{Y}) = \|Y\|^{-k_1-\cdots-k_n}\, \Phi_k(\widehat{Y}) \tag{14.34}$$

where $\Phi_k(\widehat{Y})$ is a new function of class C^∞ in angular variables $\widehat{Y} \in \Sigma^n$. From (14.29) and (14.34) we derive the following relation

$$\mathcal{L}_Y^k\, \Phi(\widehat{\mathcal{Q}\,Y}) = q_1^{k_1}\ldots q_n^{k_n}\mathcal{L}_{\mathcal{Q}Y}^k\, \Phi(\widehat{\mathcal{Q}\,Y}) = q_1^{k_1}\ldots q_n^{k_n}\|\mathcal{Q}Y\|^{-k_1-\cdots-k_n}\, \Phi(\widehat{\mathcal{Q}\,Y}). \tag{14.35}$$

From (14.35) we obtain the bound

$$\mid \mathcal{L}_Y^k\, \Phi(\widehat{\mathcal{Q}\,Y}) \mid < c_2(k) q_1^{k_1}\ldots q_n^{k_n}\|\mathcal{Q}Y\|^{-k_1-\cdots-k_n}, \tag{14.36}$$

where the constant $c_2(k)$ is given by

$$c_2(k) = \sup\left\{\mid \Phi_k(\widehat{Y}) \mid :\ \widehat{Y} \in \Sigma^n\right\}. \tag{14.37}$$

If $Y \in$ support Ξ (cf. (9.10) and (14.2)), then we have

$$q_1^{k_1}\ldots q_n^{k_n}\|\mathcal{Q}Y\|^{-k_1-\cdots-k_n} = y_1^{-k_1}\ldots y_n^{-k_n}\,\frac{(q_1y_1)^{k_1}\ldots(q_ny_n)^{k_n}}{(q_1^2y_1^2+\cdots+q_n^2y_n^2)^{\frac{k_1+\cdots+k_n}{2}}}$$
$$= y_1^{-k_1}\ldots y_n^{-k_n}\prod_{j=1}^n\left(\frac{q_j^2y_j^2}{(q_1^2y_1^2+\cdots+q_n^2y_n^2)}\right)^{\frac{k_j}{2}} \leq y_1^{-k_1}\ldots y_n^{-k_n} \leq 4^{k_1+\cdots+k_n}\, y_n^{-k_n}. \tag{14.38}$$

From (14.36) and (14.38) we obtain the bound

$$|\mathcal{L}_Y^k\, \Phi(\widehat{\mathcal{Q}Y}) < c_2\,(k)\, y_n^{-k_n}. \tag{14.39}$$

Obviously, the constant $c_2(k)$ in (14.39) (cf. (14.37)) is ondependent of the diagonal matrix $\mathcal{Q} \in \mathrm{GL}(n,\mathbb{R})$.

Recall that $\omega_1(Y)$, $Y \in \mathbb{R}^n$, is a function of class C^∞ with a compact support (cf. (10.3)). We have

$$\mathcal{L}_Y^k\, \omega_1\,(\rho^{-1}\,\mathcal{Q}Y) = \rho^{-k_1-\cdots-k_n}\, q_1^{k_1}\cdots q_n^{k_n}\, \mathcal{L}_{\rho^{-1}\,\mathcal{Q}Y}^k\, \omega_1(\rho^{-1}\,\mathcal{Q}Y)$$
$$= y_1^{-k_1}\ldots y_n^{-k_n}\left\{(\rho^{-1}\,q_1y_1)^{k_1}\ldots(\rho^{-1}\,q_ny_n)^{k_n}\, \mathcal{L}_{\rho^{-1}\,\mathcal{Q}Y}^k\, \omega_1(\rho^{-1}\,\mathcal{Q}Y)\right\}. \tag{14.40}$$

From (14.40) we obtain the bound

$$\left|\mathcal{L}_Y^k\, \omega_1\,(\rho^{-1}\,\mathcal{Q}Y)\right| < c_3\,(k) y_1^{-k_1}\cdots y_n^{-k_n} \leq 4^{k_1+\cdots+k_{n-1}}\, c_3\,(k)\, y_n^{-k_n}, \tag{14.41}$$

where the constant $c_3(k)$ is given by

$$c_3(k) = \sup\left\{\left|y_1^{k_1}\cdots y_n^{k_n}\,\mathcal{L}_Y^k\,\omega_1(Y)\right| \;:\; Y \in \mathbb{R}^n\right\}. \tag{14.42}$$

Obviously, the constant (14.42) is independent of the parameter $\rho > 0$.

Similarly, we have the following relation with the function $\omega_2(\cdot)$:

$$\mathcal{L}_Y^k\,\omega_2(\tau\,QY) = y_1^{-k_1}\cdots y_n^{-k_n}\left\{(\tau q_1 y_1)^{k_1}\ldots(\tau q_n y_n)^{k_n}\,\mathcal{L}_{\tau\,QY}^k\,\omega_2(\tau\,QY)\right\}. \tag{14.43}$$

From (14.43) we obtain the bound

$$|\,\mathcal{L}_Y^k\,\omega_2(\tau\,QY)\,| \le c_4(k)y_1^{-k_1}\cdots y_n^{-k_n} \le 4^{k_1+\cdots+k_{n-1}}\,c_4(k)\,y_n^{-k_n} \tag{14.44}$$

where the constant $c_4(k)$ is given by

$$c_4(k) = \sup\left\{|y_1^{k_1}\cdots y_n^{k_n}\,\mathcal{L}_Y^k\,\omega_2(Y)| \;:\; Y \in \mathbb{R}^n\right\}. \tag{14.45}$$

Using the bound (10.4) with $A > k_1 + \cdots + k_n$, we conclude that

$$c_4(k) < c_{A,k}\sup\left\{|y_1^{k_1}\cdots y_n^{k_n}|\,(1+\|Y\|)^{-A} \;:\; Y \in \mathbb{R}^n\right\} < \infty. \tag{14.46}$$

Obviously, the constants (14.45)–(14.46) are independent of the parameter $\tau > 0$.

Now the bounds (14.32) follow from the bounds (14.33), (14.39), (14.41), and (14.44).

The proof of Lemma 14.2 is complete.

Comparing Lemma 14.1 with Lemma 14.2 and the relation (14.30), we obtain the bounds (10.21) with constants independent of the parameters $0 < \tau < 1$ and $\rho > 0$, and the diagonal matrix $Q_m \in \mathrm{GL}(n,\mathbb{R})$.

The proof of Lemma 10.2 is complete.

References

[B] A. U. Barvinok, *Statistical sums in optimization and computational probles*, Algebra Analiz **4**, No. 1, (1992) 3–53; English transl.: St. Petersburg Math. J., **4**, No. 1, (1993), p. 1–42.

[Be] J. Beck, *Probabilistic diophantine approximation. Part* I. *Kronecker sequences*, Ann. of Math. vol 140, No. 2 (1994), 109–160.

[Ber] M. Berger, *Géométrie*, CEDIC et Nathan, Paris (1978).

[BLSWZ] A. Bjorner, M. Las Vergans, B. Sturmfels, N. White, G. Ziegler, *Oriented matroids*, Cambridge Univ. Press, (1993).

[Br] A. J. Brentjes, *Multi-dimensional continned fraction algorithms*, Math. Centree Tracts, **145**, Amsterdam (1981).

[By] V. A. Bykovskii, *On the bounds of quadratic discrepancies of certain lattice grids, in Numerical methods in the problems of mathematical physics and cybernetics*, DVO AN SSSR, Vladivostok, (1987), p. 3–14 (in Russian).

[Cal] A. P. Calderon, *A general ergodic theorem*, Ann. of Math. vol 58, No. 1 (1953), 182–191.

[Cas1] J. W. A. Cassels, *An introduction to the geometry of numbers*, Springer-Verlag, Berlin etc., (1959).

[Cas2] J. W. A. Cassels, *Rational quadratic forms*, Academic Press, London etc., (1978).

[D] S. G. Dani, *Divergent trajectories of flows on homogeneous spaces and diophantine approximation*, J. reine und angew. Math., Bd. 359 (1985), 5.55–89.

[DM] S. G. Dani, G. A. Margulis, *Values of quadratic forms at primitive integral point*, Invent. math. vol 98, No. 2 (1989), 405–424.

[DF] B. N. Delone, D. K. Faddeev, *The theory of irrationalities of the 3-rd degree*, Proc. Steklov Math. Inst. **11** (1940); English translation, Translation of Mathematical Monographs, **10**, Amer. Math. Soc. (1964).

[E] G. R. Everest, *Mean values of algebraic linear forms*, Proc. London Math. Soc. vol 70, No. 3 (1995), 529–555.

[GL] P. M. Gruber, C. F Lekkerkerker, *Geometry of numbers*, North-Holland, Amsterdam (1987).

[H] E. Hlawka, *Integrale auf konvexen Körpen* I, Monatsh. Math., Bd. 54 (1950), 5.1–36.

[Kr] E. Kratzel, *Lattice points*, VEB, Berlin (1988).

[Kre] U. Krengel, *Ergodic theorems*, Walter de Gruyter, Berlin-etc. (1985).

[M] G. A. Margulis, *Formes quadratiques indefinies et flots unipotents sur les espaces homogeneous*, C. R. Acad. Sci. Paris, t.304. Série I vol No. 10 (1989), 249–253.

[Mo] C. C. Moore, *Ergodic of flows on homogeneous spaces*, Amer. J. Math. vol 88 (1966), 154–178.

[NSkr1] N. A. Nikichine, M. M. Skriganov, *La formule asymptotique de Lang pour les parallelotopes dans les corps algebriques*, P'epublication PAR-LPTHE 93/41, Université Paris 6 (1993).

[NSkr2] N. A. Nikichine, M. M. Skriganov, *Nombre de points d'un réseau dans on produit cartésien de domaines convexes*, C. R. Acad. Sci. Paris, t.321, SérieI (1995), 671–675.

[R] M. S. Raghunathan, *Discrete subgroups of Lie groups*, Springer-Verlag, Berlin etc. (1972).

[Ra] B. Randol, *A lattice point problem: Parts* I *and* II, Trans. Amer. Math. Soc. **121**, (1866), 257–268; **125**, (1966), 101–113.

[Sch] W. M. Schmidt, *Diophantine approximation*, Lecture Notes in Math., **785**, Springer-Verlag, Berlin etc. (1980).

[Sig] C. L. Siegel, *A mean value theorem in geometry of numbers*, Ann. of Math, vol 46, No. 2 (1945), 340–347.

[Skr1] M. M. Skriganov, *Lattices in algebraic number fields and uniform distributions modulo 1*, Algebra Analiz, **1**, No. 2, (1989), 207–228; English transl., Leningrad Math. J., **1**, No. 2, (1990), 535–558.

[Skr2] M. M. Skriganov, *On integer points in polygons*, Ann. Inst. Fourier, t. 43 vol No. 2 (1993), 313–323.

[Skr3] M. M. Skriganov, *Constructions of uniform distributions in terms of geometry of numbers*, Prépublication Inst. Fourier (1992), No. 200, Algebra Analiz. **6**, No. 3 (1994), 200–230; Reprinted in St. Petersburg Math. J., **6**, No. 3 (1995), 635–664.

[Skr4] M. M. Skriganov, *Ergodic theory on homogeneous spaces and the lattice point counting for polyhedra*, Doklady RAN, (1996) (to appear).

[Skr5] M. M. Skriganov, *Anomalies in spectral asymptotics*, Doklady RAN, **340**, No. 5 (1995), 597–599; English transl.: Doklady Mathematics, **51**, No. 1, (1995), 104–106.

[Skr6] M. M. Skriganov, *On the Littlwood–Paley theory for multidimensional Fourier series*, Zap. Nauchn. Semin. POMI, **226**, (1996), 155–169; English transl.: Journal of Math. Sciences (Plenum Publish. Corporaton), to appear.

[Sp] V. G. Sprindžuk, *Metric theory of diophantine approximations*, Nauka, Moscow, (1977); English transl.: V. H. Winston and Sons, Washington, D. C. (1979).

[St] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton, Univ. Press (1970).

[T-W] M. Tarnopolska-Weiss, *On the number of lattice points in a compact n-dimensional polyhedron*, Proc. Amer. Math. Soc. vol 74 (1979), 124–127.

[W] N. Wiener, *The ergodic theorem*, Duke Math. J. vol 5, No. 1 (1939), 1–18.

[Z] R. J. Zimmer, *Ergodic theory and semisimple groups*, Birkhauser, Basel (1984).