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NEW METHODS FOR INVESTIGATION OF $\mathbf{m}^2_{\bar{\nu}_e}$ FROM TRITIUM β SPECTRUM EXPEROMENTAL DATA AND SOLUTION OF THE NEGATIVE $\widehat{\mathbf{m}}^2_{\bar{\nu}_e}$ PUZZLE

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ABSTRACT

Solution of the recent negative $m^2_{\bar{\nu}_e}$ puzzle was derived. New methods of the estimation the changing-points were proposed for neutrino mass estimation. The ill-point property of the inverse problem of the reconstruction the theoretical β spectrum from the experimental spectrum was investigated. Bibliography–37.

"If your experiment need
statistics you ought to have
done a better experiment"
E. Rutherford

0. The problem of the finiteness of neutrinos $(\nu_e, \nu_\mu, \nu_\tau)$ masses is very essential in modern physics. This is essential from different points of view – from the cosmological (the density of matter in Universe, dark matter problem), and astrophysical (Solar neutrino puzzle, explosion of SN stars) up to elementary particles physics (neutrinos oscillations, an origin of the finiteness of neutrinos masses in a "New Physics" beyoud Standard Model (SM)). This problem arizen immediately after Pauli prediction of the neutrino and Fermi derivation of the β-decay theory (see, for example, [1]). Fermi pointed that the form of the β-spectrum near the end-point essentially depends from the possible finite neuntrino mass. Fermi and Perrin stated (see in [1]), that neutrino mass is zero or very close to zero, on the base of known at this time experimental data on the β-decay. In 1952 [2] from the more accurate experiments on the β-spectrum in the tritium β-decay ($^3H \rightarrow ^3\mathrm{He} + e^- + \bar{\nu}_e$) it was estimated that $m_{\bar{\nu}_e} < 250\,eV$. After discovery of the P and C violations in weak interactions [3] it was popular assumption [4] that neutrino is two-component Weyl neutrino [5], and so with exactly zero mass. But later, after construction of the SM [6], and ideas of the Grand Unification (GU), it was clear, that donot exist any fundamental limitation for the finiteness of the neutrinos $(\nu_e, \nu_\mu, \nu_\tau)$ masses. And moreover, it was proposed so called "see-saw" mechanism [7] of the generation of finite neutrinos masses. The additional support for the finiteness of the neutrinos masses follows from the Solar neutrino puzzle (see, for example, [8] and references therein), and from the detection of the neutrinos burst from $SN\,1987\,A$ (see on some limitations for m_ν from this detection [9]). All of this, and discussions on the 17 keV neutrino questions (see [10, 11] and references therein) gave the additional attention and support to a new, more accurate experimental investigations of the β-spectrum with higher as before sensitivity and the resolution power of the β-spectrometers. At the same time were done (and doing now) some experiments (see information, for example, in [8, 9]) for observation neutrinos flavor oscillations, which would be possible only if neutrinos have finite (nonzero) masses.

1. After very interesting and delicate improvements of the experimental methods recent experiments gave such estimates $\widehat{m}^2_{\bar{\nu}_e}$ for $m^2_{\bar{\nu}_e}$ from the tritium ^3H β-decay:

$$\begin{aligned}
\widehat{m}^2_{\bar{\nu}_e} &= -147 \pm 41\,eV^2\,(LANL,\ 1990,\ [12]) \\
\widehat{m}^2_{\nu_e} &= -65\ \pm\ 85 \pm\ 65\,eV^2\,(INS\,Tokio,\ 1991,\ [13]) \\
\widehat{m}^2_{\bar{\nu}_e} &= -75 \pm 65 \pm 30\,eV^2\,(LLNL,\ 1991,\ [14]) \\
\widehat{m}^2_{\bar{\nu}_e} &= -24 \pm 48 \pm 61\,eV^2 (Zuerich,\ 1992,\ [16]) \\
\widehat{m}^2_{\bar{\nu}_e} &= -39 \pm 34 \pm 15\,eV^2\,(Mainz,\ 1993,\ [17])
\end{aligned}$$

First error here is statistical, the second is systematical. Let we point out that in the experiments [14, 15] the β-spectrum was measured "first complete from end to end ... ever by β-spectrometer with a well known efficiency and superbresolution in the range 0.0000 keV to about 50 keV" [14]. This gave possibility to test also Simpson assumption [10, 11] on the existence 17 keV neutrino. The general property of all recent results [12–17] (contrary to the old [2] – not so accurate) is that all of they gave the *negative* estimates $\widehat{m}^2_{\bar{\nu}_e}$ for $m^2_{\bar{\nu}_e} \geq 0$ (!?). Contrary to this evident *puzzle* (how can experiment give negative estimates for nonnegative from the definition neutrino mass?) usially give from this *negative* $\widehat{m}^2_{\bar{\nu}_e}$ the estimates for *nonnegative* $m_{\bar{\nu}_e}$ [18]. It is interesting to point out such sentence from Particle Physical Booklet (July, 1994) [18] connected with this puzzle: "The formal upper limit, as obtained from the $m^2_{\bar\nu_e}$ average, is 5.1 eV (for $m_{\bar\nu_e}$ – L. A. Kh.) at the 95%$C.L.$. Caution is urged in interpretive this result, since the $m_{\bar\nu_e}$ average is positive with only 3.5% probability".

Recently [19, 20] V. M. Lobashev team (Troitsk) gave, with some essential improvements of the experimental method and with corresponding here statistics as was done in previous works, such estimates:

in Preprint [19]

$$\widehat{m}^2_{\bar\nu_e} = (-18 \pm 6)eV^2 \tag{1}$$

and in the published work [20]

$$\widehat{m}^2_{\bar\nu_e} = (-22 \pm 4.8)eV^2 \tag{2}$$

"which is essentially lower than those published up to now, and involves a new kind of systematics, which was not seen previously" [19]. Let we point out that the errors ($\pm 6eV^2$[19], $\pm 4.8eV^2$[20]) are much lower than in all previous experimental works (see above). Authors of [19, 20] seriously look on the negative $\widehat{m}^2_{\bar\nu_e} < 0$ puzzle ("The negative values for $\widehat{m}^2_{\bar\nu_e}$ obviously have no physical meaning and indicate that there exist some systematic effects not taking into account in the calculation of the theoretical spectrum"[20]), and assumed that the origin of this (in any case in their experiment) is some unexpected spike-like structure around $6eV^2$ below the end-point in the theoretical β spectrum corresponding integrally to about $(6.3 \pm 1.3)10^{-11}(B.R.)$ of the total decay probability. Taking into account this additional structure authors [19, 20] reinterpretate their experimental data and obtained [20]

$$\widehat{m}^2_{\bar\nu_e} = (-4.1 \pm 10.9)eV^2 \tag{3}$$

and by using the recipe of the Particle Data Group [18] obtained from *negative unphysical* estimate (3) such upper limit on the *non-negative physically meaningful* (anti)neutrino mass

$$m_{\bar\nu_e} < 4.35eV^2/c^2(95\%C.L.) \tag{4}$$

It was also recently published a new work of the LLNL team [21], in which was obtained such result

$$\widehat{m}^2_{barv_e} = (-130 \pm 20)eV^2 \tag{5}$$

and also, as in [19, 20], was obtained the indication om some structure near the end-pint of the theoretical β spectrum. Authors [21] also assumed that "The negative value for the best fit of $\widehat{m}^2_{\bar\nu_e}$ has no physical meaning, it is just an indication from our fit procedure that there are too many counts near the end point". Because the results from [19, 20] are more sensitive and were published with more essential details than in [21] I shall discuss later in main results from [19, 20].

As pointed authors [19]: "The spike-like structure in tritium β-spectrum of $B.R. \simeq 10^{-10}$ could not be seen in the previous experiments [12, 17] (and, of course, in other experiments [2, 13–16] – $L.\ A\ .Kh.$) due to insufficient statistical accuracy". From this statement [19] follows directly that, if, even the phenomena of the negative estimate $m^2_{\bar\nu_e}$ from experiments [19-21] can be explained on the base of the additional (to the usual standard form of the β-spectrum) spike-like structure, the negative $m^2_{\bar{\nu}_e}$ puzzle in all other previous experiments [12–17] cannot be explained by the same manner (by nonseen in [12–17] spike-like structure). Of course and authors of [19, 20] pointed that the existence of the additional spike-like structure must be tested again by future new experiments with much better statistics and with help of some new improvements of the used experimental method. It is essential to point this, because such additional spike-like structure *nonexist* in today standard theory of the β-decay, and it is very unlikely, that such violation of the standard theory of the β-decay, which so good explains so much effects in the β-decay, is true. The solution of the negative $m^2_{\bar{\nu}_e}$ puzzle, which was derived in this work give the additional support for this point of view.

2. From the above consideration naturally to assume, that the solution of the negative $\widehat{m}^2_{\bar{\nu}_e}$ puzzle, which must expain this puzzle for *all* recent [12–17] experiments has the another different from "spike-like structure" solution [19, 20], of general nature. And this solution must be true also for last [19, 21] experiments.

The clue for the solution of the negative $m^2_{\bar{\nu}_e}$ puzzle connected with ill-point using of the well-known $\min \chi^2$-method for the case of $m_{\bar{\nu}_e} > 0$. This $\min \chi^2$-method was used for interpretation of experimental data in *all recent* experiments [12–17, 19-21].

First of all, usual standard theory of the β-decay predicts such expression for theoretical β-spectrum $d\mathcal{N}_{\text{th}}(E)/dE$:

$$\frac{d\mathcal{N}_{\text{th}}(E)}{dE} = A \cdot F(E, E_0) \cdot p \cdot E \cdot (E_0 - E) \cdot \sqrt{(E_0 - E)^2 - m^2_{\bar{\nu}_e}} \cdot \theta(E - E_0 + m^2_{\bar{\nu}_e}) \tag{6}$$

where $\theta(E - E_0 + m^2_{\bar{\nu}_e})$ is the usual step function. In (1) A – is the normalization constant, $F(E, E_0)$ – Fermi function, continious as function of E, uncluding all corrections (relativistic, screening, ...), p –momentum, E – energy of electron, E_0 total decay energy, $m_{\bar{\nu}_e} \geq 0$ – mass of the (anti) neuntrino $\bar{\nu}_e$. If $m_{\bar{\nu}_e} = 0$, than $d\mathcal{N}_{\text{th}}(E)/dE$ is nonnegative continious function, which at $E = E_0$ is zero with derivatives of all orders (excluding of order 2) also equal to 0. But if $m_{\bar{\nu}_e} > 0$, than $d\mathcal{N}_{\text{th}}(E)/dE$ is continious function for $E < E_0 - m_{\bar{\nu}_e}$, with zero value at $E = E_0 - m_{\bar{\nu}_e}$, but with infinite derivatives of all orders at this end-point $E = E_0 - m_{\bar{\nu}_e}$. By definition of the theoretical β spectrum according to (6) we donot taking into account that for real tritium sources there are some effects from the molecular physics (see, for example, [22]), because such effects are not essential for our solution of the negative $\widehat{m}^2_{\bar{\nu}_e} < 0$ puzzle. The effects [22] from the molecular physics possible to include in the consideration by corresponding more complicated (only from the technical point of view) investigations.

Now assume that the β-spectrometer is ideal (below in 3. we investigate the essential influence of the nonideal (finite resolution power) property of the used real β-spectrometers.) With this assumption the experimentally determined β – spectrum $d\mathcal{N}_{\text{exp}}(E)/dE$ is

$$\frac{d\mathcal{N}_{\text{exp}}(E)}{dE} = \frac{d\mathcal{N}_{\text{th}}(E)}{dE} + C \tag{7}$$

Where C is the background (we assume, as usually assumed in experimental works, that C is constant, which is independent from energy E). Let now $d\mathcal{N}_{\text{exp}}(E_i)/dE$, $E_i \in [E_{\text{min}}, E_{\text{max}}]$ are experimental data (a sample in terms of the mathematical statistics). For obtaining the estimates $\widehat{A}, \widehat{C}, \widehat{E}_0, \widehat{m}^2_{\bar{\nu}_e}$ for unknown parameters $A, C, E_0, m^2_{\bar{\nu}_e}$ in all works [12–17, 19-21] was used well-known the $\min \chi^2$-method. If $m_{\bar{\nu}_e} = 0$ then theoretical (and experimental) β-spectrum will be non-negative even if we shall use the usual expression of $d\mathcal{N}_{\text{th}}(E)/dE$ for E_i, which are bigger than E_0, and the using of the $\min \chi^2$-method for estimation A, C, E_0 has not inner contradictions. The situation changes dramatically for the case if $m_{\bar{\nu}_e} > 0$. The estimates $\widehat{A}, \widehat{C}, \widehat{E}_0$ by the min χ^2-method, which were obtained in all experimental works [12–17, 19-21], are *non-negative* (in accordance with non-negativeness of A, C, E_0 from their definitions) and *stable* to changing the $E_{\min}$ of the sample (for $E_{\min} \geqq 18300 eV^2$). But from all experiments [12–17, 19-21] follows *negative* estimates $\widehat{m}^2_{\bar{\nu}_e}$, contrary to the *non-negativeness* of $m^2_{\bar{\nu}_e}$ (in [23] it was pointed: "It is another problem, and not a statistical one, to understand why the weighted mean (for $\widehat{m}^2_{\nu_e}$ – *L. A. Kh.*) comes out so strongly negative"). Our solution of the negative $\widehat{m}^2_{\bar{\nu}_e}$ puzzle, contrary to this statement [23], is *statistical* one. Let we pointed out that the negative estimates $m^2_{\bar{\nu}_e}$ are *nonstable* and change with changing the sample. More precisely, as follows from the most accurate experimental work [19, 20], the estimates $\widehat{m}^2_{\bar{\nu}_e}$ are increasing generally speaking with increasing $E_{\min}$ (we donot include the results for $E_{\min} < 18300 eV^2$ because there are some indications [17, 19, 20] that for such $E_{\min}$ there are some anomalies) [19]:

$$
\begin{aligned}
\widehat{m}^2_{\bar{\nu}_e}(E_{\min} &= 18300\,\mathrm{eV}) = -19.3 \pm 4.8(\mathrm{eV}^2)\\
\widehat{m}^2_{\bar{\nu}_e}(E_{\min} &= 18350\,\mathrm{eV}) = -20.2 \pm 5.3(\mathrm{eV}^2)\\
\widehat{m}^2_{\bar{\nu}_e}(E_{\min} &= 18400\,\mathrm{eV}) = -18.6 \pm 6.1(\mathrm{eV}^2)\\
\widehat{m}^2_{\bar{\nu}_e}(E_{\min} &= 18450\,\mathrm{eV}) = -25.5 \pm 7.3(\mathrm{eV}^2)\\
\widehat{m}^2_{\bar{\nu}_e}(E_{\min} &= 18500\,\mathrm{eV}) = -35.7 \pm 10.2(\mathrm{eV}^2)\\
\widehat{m}^2_{\bar{\nu}_e}(E_{\min} &= 18550\,\mathrm{eV}) = -30.0 \pm 14.0(\mathrm{eV}^2)
\end{aligned}
$$

and [20]

$$
\begin{aligned}
\widehat{m}^2_{\bar{\nu}_e} &= (E_{\min} = 18300\,\mathrm{eV}) = -22.9 \pm 4.4(\mathrm{eV}^2)\\
\widehat{m}^2_{\bar{\nu}_e} &= (E_{\min} = 18350\,\mathrm{eV}) = -22.4 \pm 4.8(\mathrm{eV}^2)\\
\widehat{m}^2_{\bar{\nu}_e} &= (E_{\min} = 18400\,\mathrm{eV}) = -22.9 \pm 5.2(\mathrm{eV}^2)\\
\widehat{m}^2_{\bar{\nu}_e} &= (E_{\min} = 18450\,\mathrm{eV}) = -19.8 \pm 6.0(\mathrm{eV}^2)\\
\widehat{m}^2_{\bar{\nu}_e} &= (E_{\min} = 18500\,\mathrm{eV}) = -30.3 \pm 9.1(\mathrm{eV}^2)
\end{aligned}
$$

The small diferences for $\widehat{m}^2_{\bar{\nu}_e}$ from [19] and [20] connects with corresponding values of the errors.

Why were obtaind the *negative* $\widehat{m}^2_{\bar{\nu}_e} < 0$? The explanation of this evident *puzzle* (puzzle from the physical point of view) has the *statistical* nature and connected with *inner contradiction* of using the min χ^2 method for the case when $m_{\bar{\nu}_e} > 0$. For *all* samples in [19–21] (the same is true for all previous experiments [12–17]) were used the experimental data up $E_{\max}$, which are *bigger* than E_0 and moreover *bigger* than $E_0 - m_{\bar{\nu}_e}$, if $m_{\bar{\nu}_e} > 0$ ($E_{\max} = 18800$ eV in [19], $E_{\max} = 18770$ eV in [20]). Because we *donot know a priori* what is the *true* (positive) $m_{\bar{\nu}_e}$ (and for the estimation by the min χ^2 method we assumed that the parametric form (6),(7) of the β-spectrum is known a priori, and only what we donot know are the *true* values of unknown parameters, including $m_{\bar{\nu}_e}$), then some experimental data are for $E_i > E_0 - m_{\bar{\nu}_e}$. But for such $E_i > E_0 - m_{\bar{\nu}_e}$ the β-spectrum (theoretical (6) and experimental (7)) is *real* and nonnegative *only if $m^2_{\bar{\nu}_e}$ will be negative (!)*. This simple fact gives the solution of the negative $\widehat{m}^2_{\bar{\nu}_e}$ puzzle. From this solution follows also why the obtained estimates $\widehat{m}^2_{\bar{\nu}_e}$ [19, 20] are unstable and change by chanching $E_{\min}$. If $E_{\min}$ increase then weight of the experimental data for $E_i > E_0 - m_{\bar{\nu}_e}$ in χ^2 also increases, and from this follows the explanation of the increasing (more ptecisely- the tendency of increasing taking into account that esimates $\widehat{m}^2_{\bar{\nu}_e}$ are random values) of $\widehat{m}^2_{\bar{\nu}_e}$ with increasing $E_{\min}$ and corresponding increasing of the errors of the estimates.

From the considereation above follows that the negative estimates $\widehat{m}^2_{\bar{\nu}_e} < 0$, obtained by the min χ^2 method are *unadmissible*.Because for the case of zero (anti)neutrino mass $m_{\bar{\nu}_e} = 0$ the negative estimates $\widehat{m}^2_{\bar{\nu}_e}$ canot be originate the fact of the obtained the negative estimates $\widehat{m}^2_{\bar{\nu}_e} < 0$ gives indication on the *nonzero* (anti)neutrino mass $m_{\bar{\nu}_e} > 0$, but such unadmissible negative estimates cannot be used for meaningful estimation physical positive (anti)neutrino mass.

Unfortunately,in all recent experimental works were done "by hand" continuations of the theoretical β spectrum from the physical range $m^2_{\bar{\nu}_e} \geqq 0$ into *unphysical* negative range $m^2_{\bar{\nu}_e} < 0$ (different in different works!). By such continuation was done *ad hoc* change of the initial physical problem of the estimation non-negative physical meaningful (anti)neutrino mass $m_{\bar{\nu}_e} \geqq 0$ to the unphysical meaningless problem of estimation $-\infty < m^2_{\bar{\nu}_e} < \infty$. This continuation was done for more simple calculation by usual programs, like MINUIT [24], the minimum of the χ^2. Of course in this case the negative estimates $\widehat{m}^2_{\bar{\nu}_e} < 0$ are *admissible*, but such estimates are meaningless from the physical point of view and cannot be used for the estimation of *non-negative meaningful (anti)neutrino mass* $m_{\bar{\nu}_e} \geqq 0$.

The derived solution of the negative $\widehat{m}^2_{\bar{\nu}_e}$ puzzle can and must be tested by changing the experimental sample by changing $E_{\min}$ (for more lower than in [19]) and by changing $E_{\max}$. It is essential to point out that in old experiments (like [2]), in which the experimental data were for E_i not close to the end-point, and in any case lower than $E_0 - m_{\bar{\nu}_e}$, if for $m_{\bar{\nu}_e}$ we use the estimates from SN 1987 and from the cosmological limitations, the negative puzzle estimate $\widehat{m}^2_{\bar{\nu}_e}$ was absent. Of course, old results gave less accurate but positive estimates for $m^2_{\bar{\nu}_e}$ and correspondingly for $m_{\bar{\nu}_e}$.The simple way to avoid the origin of the negative estimates $\widehat{m}^2_{\bar{\nu}_e}$ it is not using the experimental data close to the end-point. But of course such recomendation is unnatural, because the main information on $m^2_{\bar{\nu}_e}$ is in the experimental data for E_i, which are close as possible to end-point $E_0 - m_{\bar{\nu}_e}$.

As was pointed by author's [10, 21] in the discussion the 17 keV neutrino problem the min χ^2-method is not effective for estimating "kinks" in the β-spectrum even in the case if kinks are for E_i lower than end-point $E_0 - m_{\bar{\nu}_e}$ and there donot exist the situation, which we discussed above.

And in general the min χ^2 method,derived by K. Pearson at 1900 [25], long ago before Pauli prediction of the neutrino [1], is not effective (from the mathamatical statistics point of view [26]) even if we use the min χ^2 method for the problems, for wich this method is admissible. After 1900 in the mathematical statistics were derived many new effective methods of the estimation, different from the min χ^2 method, especially connected with the problems, when the distributions have changing-points, the points where the distributions have discontinuity or discontinuities of the derivatives of some orders (see [26–29] and the referemces therein). It was proved that the effective estimates in such cases converge, when the volume of the sample goes to infinity, *more rapidly*, than in the usual cases for the smooth distributions. The problem of the estimation the (anti) neutrino mass from the (theoretical) β decay spectrum is the typical problem of the estimation the chanching-points, because the theoretical β spectrum has such discontiniuty property if $m_{\bar{\nu}_e} > 0$:

$$\frac{d}{dm^2}(d\mathcal{N}_{\rm th}(E)/dE)\big|_{E=E_0-m_{\bar{\nu}_e}} = \infty \tag{8}$$

and

$$\Big|\frac{d^n}{dE^n}(d\mathcal{N}_{\rm th}(E)/dE)\big|_{E=E_0-m_{\bar{\nu}_e}}\Big| = \infty \tag{9}$$

I shall not go in detail description of the proposed new interesting effective methods for estimation the change-points, in particular, by so called kernel (or nuclear) estimators (see, for example,

[29] and the references therein), because such methods cannot be directly applied to our more delicate problem. But even for such, more simple cases. for the estimation the changing-points usually used not the min χ^2 method, but asymmetrical kernel estimates [29]. For understanding of such methods I give below such simplest example. Let

$$y_{i,n} = g_(t_{i.n}) + \epsilon_{i,n},\ t_{i,n}\varepsilon[0,1], i = 1, ..., n \tag{10}$$

where $y_{i,n}$ are the results of the measurements (the sample), $\epsilon_{i,n}$ are independent identically distributed (i.d.d.) random variables (noises) with

$$\mathcal{E}\epsilon_{i,n} = 0,\ Var(\epsilon_{i,n}) = \sigma^2 < \infty \tag{11}$$

and $g(t)$ is the regeression function. Let $g(t)$ has an jump at point $t = \tau$

$$g(t) = f(t) + \Delta\mathcal{I}_{[\tau,1]}(t) \tag{12}$$

where $f(t)$ is two times differentiable function in $[0, 1]$, $\Delta > 0$, $t\varepsilon[0, 1]$, and $\mathcal{I}_{[\tau,1]}(t)$ is the characteristic function of the interval $[0, 1]$

$$\mathcal{I}_{[\tau,1]}(t) = \begin{cases} 1, & t \in [\tau, 1] \\ 0, & t \notin [\tau, 1]. \end{cases} \tag{13}$$

Let K_+, K_- are one-side kernel functions with $supportK- = [-1, 0]$ and $supportK_+ = [0, 1]$. Let us define such statistics

$$\widehat{g}_+(t) = \frac{1}{b(n)}\sum_{i=1}^{n} y_{i,n}\int_{s_{i-1,n}}^{s_{i,n}} K_+(\frac{t-u}{b(n)})du \tag{14a}$$

and

$$\widehat{g}_-(t) = \frac{1}{b(n)}\sum_{i=1}^{n} y_{i,n}\int_{s_{i-1,n}}^{s_{i,n}} K_-(\frac{t-u}{b(n)})du \tag{14b}$$

where

$$s_{i,n} = (t_{i,n} + t_{i+1,n})/2, i = 1, ..., n-1; s_{0,n} = 0, s_{n,n} = 1 \tag{15}$$

and $b(n)$ are such that

$$b(n)_{n\to\infty} \to 0, nb(n)_{n\to\infty} \to \infty, lim_{n\to\infty}supnb^3(n) < \infty \tag{16}$$

How to estimate by using the asymmetrical kernel estimators $g_+(t)$, $g_-(t)$ the changing-point τ and the jump Δ? The answer is such [29]. Let $Q \subset (0, 1)$ is such that $\tau\varepsilon Q$. Define

$$\widehat{\Delta}(t) = \widehat{g}_+(t) - \widehat{g}_-(t) \tag{17}$$

Then the estimator $\widehat{\tau}$ of τ is

$$\widehat{\tau} = inf[\rho\varepsilon Q : \widehat{\Delta}(\rho) = sup_{x\varepsilon Q}\widehat{\Delta}(x)] \tag{18}$$

and the estimator of the jump at τ is

$$\widehat{\Delta}(\widehat{\tau}) = \widehat{g}_+(\widehat{\tau}) - \widehat{g}_-(\widehat{\tau}) \tag{19}$$

It was proved [29] that such estimators $\widehat{\tau}$, $\widehat{\Delta}(\widehat{\tau})$ are consistent and have the optimal (are effective) speed of convergence (more rapidly than usual $n^{-1/2}$) when $n \to \infty$.

Our problem of the estimation $m^2_{\bar{\nu}_e}$ is more complicated. In our case within the assumption that the β spectrometer is ideal (we shall later go beyond this assumption and this will be very essential) $d\mathcal{N}_{\exp}(E_i)/dE$ are independent Poisson random variables with the mean values

$$\mathcal{E}(d\mathcal{N}_{\exp}(E_i)/dE) = ApE_iF(E_0, E_i)(E_0 - E_i)[(E_0 - E_i)^2 - m^2_{\bar{\nu}_e}]^{1/2} + C \tag{20}$$

for $E_i \leqq (E_0 - m_{\bar{\nu}_e})$, and

$$\mathcal{E}(d\mathcal{N}_{\exp}(E_i)/dE) = C \tag{21}$$

for $E_i \geqq (E_0 - m_{\bar{\nu}_e})$. We see that our problem of the estimation the changing-point $E = E_0 - m_{\bar{\nu}_e}$ is much more complicated than the problems of type (10). For the solution of our problem I proposed to use (for the first time it was proposed in [30, 31]) the estimators, which I called *asymptotical Fourier estimators* (AFE). The idea of the construction such AFE was based on the such specific properties of the asymptotics of the Fourier transforms of the functions, which have the discontinuity points (changing-points). Let $f(E)$ is such function

$$f(E) = A(E_0 - E)[(E_0 - E)^2) - m^2]^{1/2}\theta(E - E_0 + m) \tag{22}$$

and $F(t)$ is the Fourier transform

$$F(t) = \int_{E_{\min}}^{E_{\max}} f(E)exp(iEt)dE \tag{22a}$$

where $(E_0 - m)\varepsilon(E_{\min}, E_{\max})$. Then the asymtotics of $F(t)$ for $t \to \infty$ is

$$\begin{cases} F(t)_{t\to\infty} \backsimeq A(E_0 - E_{\max}). \\ [(E_0 - E_{\max})^2 - m^2]^{1/2}(it)^{-1}exp(itE_{\max}) \\ -A(E_0 - E_{\min})[(E_0 - E_{\min})^2 - m^2]^{1/2}(it)^{-1}exp(-itE_{\min}) \\ +Am^2it^{-3/2}exp(it(E_0 - m)) \end{cases} \tag{23}$$

From (23) we see that the asymptotics of $F(t)$ give us information on m^2 and E_0. On the base of (23) the AFE $\widehat{m}^2_{\bar{\nu}_e}$ and $\widehat{E}_0$ are defined according to

$$\begin{cases} \int_{E_{\min}}^{E_{\max}}(d\mathcal{N}_{\exp}(E)/dE)exp(iEt)dE_{t\to\infty} \backsimeq . \\ AE_{\max}F(E_{\max}, \widehat{E}_0)(\widehat{E}_0 - E_{\max})[(\widehat{E}_0 - E_{\max})^2 - \widehat{m}^2_{\bar{\nu}_e}]^{1/2}\times \\ (it)^{-1}exp(itE_{\max}) \\ -AE_{\min}F(E_{\min}, \widehat{E}_0)(\widehat{E}_0 - E_{\min})[(\widehat{E}_0 - E_{\min})^2 - \widehat{m}^2_{\bar{\nu}_e}]^{1/2}\times \\ (it)^{-1}exp(-itE_{\min}) \\ +A(\widehat{E}_0 - \widehat{m}_{\bar{\nu}_e})F(\widehat{E}_0 - \widehat{m}_{\bar{\nu}_e}, \widehat{E}_0\widehat{m}^2_{\bar{\nu}_e}it^{-3/2}exp(it(\widehat{E}_0 - \widehat{m}_{\bar{\nu}_e})) \end{cases} \tag{24}$$

It is interesting that proposed method of estimation $m^2_{\bar\nu_e}$ by AFE is nonsensitive in the very general situation to the delicate details of the theoretical β spectrum (in particular, of the Fermi function, and even of some new structures [19–21]) and the response function, if all such functions are sufficiently smooth (see below).

The proposed solution of the negative $\widehat{m}^2_{\bar\nu_e} < 0$ puzzle was verified by computer simulation (cowork with my student A.Liber in progress). I shall give here only one example. For simplcity of simulation we choose $F(E, E_0) = 1$ and for $A = 2, C = 10, m^2 = 12, E_0 = 18573eV$ simulate for any $E_i, i = 1, ..., 256; E_i\varepsilon[18560, 18580]eV$ 100 times independent Poisson random values $\psi(E_i)$ with mean values

$$\mathcal{E}\psi(E_i) = AE_i(E_0 - E_i)[(E_0 - E_i)^2 - m^2]^{1/2}$$

In such way we produced the sample. After this we determine by usual min χ^2 method the estimates for m^2. We obtained such results:

$$\begin{aligned}
\widehat{m}^2(E_{\min} &= 18561eV, E_{\max} = 18579eV) = -9\\
\widehat{m}^2(E_{\min} &= 18567eV, E_{\max} = 18574eV) = -14\\
\widehat{m}^2(E_{\min} &= 18568eV, E_{\max} = 18573eV) = -12\\
\widehat{m}^2(E_{\min} &= 18561eV, E_{\max} = 18566eV) = +3
\end{aligned}$$

Of course,it will be better to test the proposed solution by real experimental data [19–21] and I hope to do this in future.

3. Above we assumed that the β spectrometer was ideal. Even in this case we demonstrated the unadmissibility of the min χ^2 method for the estimation (anti)neutrino mass. But by taking into account the nonideality (the finite resolution power) of the β spectrometer the problem of the estimation of $m^2_{\bar\nu_e}$ (and correspondingly $m_{\bar\nu_e}$) will be much more complicated, because in this case we shall connect with the typical ill-posed inverse problem (from Hadamard's point of view). For real (nonideal) β spectrometer the experimental $d\mathcal{N}_{\rm exp}(E)/dE$ β-spectrum connected with the theoretical $d\mathcal{N}_{\rm th}(E)/dE$ β-spectrum according to

$$\frac{d\mathcal{N}_{\rm exp}(E)}{dE} = \int K(E, E') \cdot \frac{d\mathcal{N}_{\rm th}(E')}{dE'}\, dE' \tag{25}$$

where $K(E, E')$ is the response (resolution) function of the β-spectrometer. For determination of the $d\mathcal{N}_{\rm th}(E')/dE'$ from the experimental data $d\mathcal{N}_{\rm exp}(E)/dE$, and so for estimating parameters of $d\mathcal{N}_{\rm th}(E)/dE$, including $m^2_{\bar\nu_e}$, we must in general to solve the inverse problem, the problem of the solution the integral equation (25). But this inverse problem is ill-posed (from Hadamard point of view) [30, 32]. This means that very small errors in $d\mathcal{N}_{\rm exp}(E)/dE$ can produce very big errors in $d\mathcal{N}_{\rm th}(E)/dE$. Let reproduce this statement here. Let

$$\frac{d\widetilde{\mathcal{N}}_{\rm th}(E')}{dE'} = \frac{d\mathcal{N}_{\rm th}(E')}{dE'} + A \cdot e^{i\Omega E'} \tag{26}$$

where $A \gg 1$. By substituting $d\widetilde{\mathcal{N}}_{\rm th}(E)/dE$ (26) in (25) we obtain:

$$\frac{d\widetilde{\mathcal{N}}_{\rm th}(E)}{dE} = \frac{d\mathcal{N}_{\rm th}(E)}{dE} + A \cdot \int K(E, E') \cdot e^{i\Omega E'} \cdot dE' \tag{27}$$

From (26) follows that

$$\left| \frac{d\widetilde{\mathcal{N}}_{\mathrm{th}}(E')}{dE'} - \frac{d\mathcal{N}_{\mathrm{th}}(E')}{dE'} \right| = A \gg 1 \tag{28}$$

But for any big $A \gg 1$ we can determine such big $\Omega \gg 1$, that on the base of classical Riemann-Lebesque theorem

$$\left| \frac{d\widetilde{\mathcal{N}}_{\mathrm{exp}}(E)}{dE} - \frac{d\mathcal{N}_{\mathrm{exp}}(E)}{dE} \right| \leq \varepsilon \ll 1 \tag{29}$$

where ε can be as small as we wish. So very small errors in the experimental β-spectrum can produce very big errors in the theoretical β-spectrum. Usually, in the experiments [12–17, 19-21] was used not *inverse*, but *direct* problem. Assuming that we know *exactly* the parametric form of the theoretical β-spectrum, and response function $K(E, E')$ we substitute this theoretical spectrum in (25) and evaluate by integration the experimental β-spectrum, parameters of which shall be the same as parameters (like E_0, $m^2_{\nu_e}$) of the unknown theoretical β-spectrum, which must be estimated. Let we point out again that such using of the direct problem possible *only* if we know *exactly* the parametric form of the theoretical β-spectrum and the response function.

It was pointed above that proposed AFE are in general nonsensitive to the delicate details of the theoretical β spectrum and the response function if such functions are sufficiently smooth. More concretly, let kernel of the integral equation (25) has such property $K(E, E^{'}) = K(E - E^{'})$. In this case from (25) we obtain

$$\widetilde{\mathcal{N}}_{\mathrm{exp}}(t) = K(t)\widetilde{\mathcal{N}}_{\mathrm{th}}(t) \tag{30}$$

where

$$\widetilde{\mathcal{N}}_{\mathrm{exp}}(t) = (2\pi)^{-1} \int (d\mathcal{N}_{\mathrm{exp}}(E)/dE) exp(iEt) dE \tag{31a}$$

and

$$\widetilde{\mathcal{N}}_{\mathrm{th}}(t) = (2\pi)^{-1} \int (d\mathcal{N}_{\mathrm{th}}(E)/dE) exp(iEt) dE \tag{31b}$$

and

$$K(t) = (2\pi)^{-1} \int K(E) exp(iEt) dE \tag{31c}$$

Because the asymptotics of $K(t)$ depends only from the smoothness properties of $K(E - E^{'})$, we can obtain the asymptotics of $\widetilde{\mathcal{N}}_{\mathrm{th}}(t)$ from the asymptotics of $\widetilde{\mathcal{N}}_{\mathrm{exp}}(t)$, which can be determined from the experimentally measured β spectrum, and from asymtotics of $K(t)$. After this we can use the AFE $\widehat{m}^2_{\nu_e}$ and $\widehat{E}_0$ according to (24). Of course, taking into account the ill-posed character of the inverse problem such estimators shall converge to real values of $m^2_{\nu_e}$, E_0 very slowly (see [33] for example). In author's works (see [34] and the references therein) was proposed and investigated the statistical approach to the ill-posed inverse problems.

The ill-posed property of the *inverse* problem (25) is extermly essential if somebody wish to investigate some new structures (like in [19–21]) in the theoretical β spectrum. From this point of view the results from recent experimental works [19–21] on the possibility of existness the new unexpected structures in the theoretical β spectrum are very conditional and have not rigorous proof, because in the interpretation of the experimental data were not taking into account the very delicate properties of the inverse ill-posed problems. In any case, the min χ^2 test as the goodness-of-fit test has not mathematical foundation [35] for the case if the distribution has the discontinuity of the type as has the theoretical β spectrum at the end-point.

4. Taking into account all above considerations I shall repeat E. Rutherford sentence: "If your experiment need statistics you ought to have done a better experiment". In author's works [31, 32] was proposed a new *nonstatistical* but new *physical* method of investigation the neutrino physics by investigations the *nonexponential* terms in the nuclear β decay.

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References

1. W. Pauli, "Aufzätze und Vortrüge über Physik und Erkenntnisttheory," Braunnschweig, 1961.
2. L. M. Langer, R. J. D. Moffat, *Phys. Rev.*, **88**, 689 (1952).
3. T. D. Lee, C. N. Yang, *Phys. Rev.*, **104**, 254 (1956).
4. T. D. Lee, C. N. Yang, *Phys. Rev.*, **105**, 1671 (1957); L. D. Landau, *Nucl. Phys.*, **56**, 330 (1957).
5. H. Weyl, *Zs. f. Physik*, **56**, 330 (1929).
6. S. Weinberg, *Rev. Mod. Phys.*, **52**, 451 (1980); A. Salam, *Rev. Mod. Phys.*, **52**, 525 (1980); S. L. Glashow, *Rev. Mod. Phys.*, **52**, 539 (1980).
7. M. Gell-Mann, P. Ramon, R. Slansky, *Rev. Mod. Phys.*, **50**, 721 (1978).
8. P. I. Krastev, S. T. Petkov, *Fermilab preprint*, PUB-99/188-T, July 1994; S. Parke, *Fermilab preprint*, PUB-94/973-T, August 30, 1994; P. I. Krastev, S. T. Petkov, *Phys. Rev*, **D53**, 1665 (1996).
9. D. O. Caldwell, P. Langacker, *Phys. Rev.*, **D44**, 823 (1991).
10. Workshop on the 17 keV Neutrino Question, December 18-20, 1991, Berkeley, USA; Transparences from the Workshop on the 17 keV Neutrino Question, Sponcored by the Center for Particle Astrophysics.
11. D. R. O. Morrison, *Nature*, **366**, 29, 4 November (1993).
12. R. G. H. Robertson et al. *Phys. Rev., Lett.*, **67** 957 (1991).
13. H. Kawakami et al., *Phys. Lett.* **B256**, 105 (1991).
14. W. Stoeffl – report in [10].
15. W. Stoeffl, *Bull. Am. Phys. Soc.*, **37**, 1286 (1992).
16 . E. Holzschuh et al., *Phys. Lett.*, **B287**, 381 (1992).
17. Ch. Weinheimer et al., *Phys. Lett.*, **B300**, 210 (1993).
18. *Phys. Rev.*, **D50**, 1173 (1994); July 1994 Particle Physics Booklet.
19. A. I. Belishev et al., *Preprint INR-862/94*, July 1994; V. M. Lobashev – report on General Session Nuclear Physics Depertment Russian Academy of Scieness, December 21, 1994, Moscow.
20. A. I. Belishev et al., *Phys. Lett.*, **B350**, 263 (1995).
21. W. Stoeffl and D. DEcman, *Phys. Rev. Lett.*, **75**, 3237 (1995).
22. P. Froelich et al., *Phys. Rev. Lett.*, **71**, 2871 (1993).

23. F. James and M. Roos, *Phys. Rev.*, **D44**, 299 (1991).
24. F. James and M. Roos, *Comput. Phys. Comm.*, **10**, 343 (1975).
25. K. Pearson, *Phil. Mag.*, **5**, 157 (1900).
26. I. A. Ibragimov, R. Khasminsky, "Statistical estimation. Asymptotical theory," *Springer Verlag*, 1981.
27. A. N. Kolmogorov, A. N. Shirjaev,- in *Proc. of VI All- Union Conf. on the Probab. Theory and Math. Statistics*, Vilnus, **1960**, (in Russian).
28. A. P. Korostelev, *Theory of Probab. and Applications*, **32**, 796 (1987).
29. H. -G. Muller, *Ann. Stat.*, **20**,737 (1992).
30. L. A. Khalfin – round-table discussion in [10],p.561.
31. L. A. Khalfin, *Soviet Physics Uspekhi*, **35**, 717 (1992).
32. Leonid A. Khalfin, *Research Institute for Theoretical Physics, University of Helsinki preprint*, HU-TFT-92-31, 4 September 1992, Helsinki
33. R. .J. Carroll and P. Hall, *Journ. of the Amer. Stat. Assoc.*, **83**, 1184 (1988).
34. L. A. Khalfin, *Theory of Probability and its Applications*, **29**, 181 (1985).
35. Leonid A. Khalfin, *(in preparation for publication)*, 1996.
36. Leonid A. Khalfin, *PDMI PREPRINT-2/1995*, February 1995, St. Petersburg.
37. Maats Roos and Leonid A. Khalfin, *Research Institute for Theoretical Physics, University of Helsinki preprint*, HU-TFT-96-B, May 1996, Helsinki.