PDMI PREPRINT – 4/1996

The q-analogue of Model Space and CGC Generating Matrices

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January 1996

Abstract

The q-analogue $(T^*B)_q$ of the phase space T^*B for the Borel subgroup B of a given simple Lie group G is constructed and studied. It is shown to give a q-analogue of the model representation for the group G. A particular "coordinatization" of $(T^*B)_q$ (in the case of $G = SL(2)$) is given in terms of matrix generating corresponding Clebsch-Gordan coefficients (CGC).

This paper is a short version of the preprint q-alg/9508022

1 Preliminaries

There exist three symplectic manifolds (from physical point of view they are phase spaces) naturally related to a given Lie group G and corresponding Lie algebra $\mathcal{J}$:

1) T^*G – the cotangent bundle for the group G;

2) T^*B – the cotangent bundle for the Borel subgroup B;

3) $\mathcal{O}$ – the orbit of coadjoint action of G on $\mathcal{J}^*$.

In the case of $G = SL(2)$ these spaces are six-, four- and two- dimensional, correspondingly. The method of geometric quantization [2, 3] provides a representation theory for 1), 2), 3).

Turning from classical to quantum groups, one can try to construct a representation theory for the deformed analogues of the manifolds 1), 2), 3). In the present paper we shall deal with deformations of the spaces 1) and 2). First, let us remind some facts concerning $(T^*G)_q$.

Let G_q be a deformation of the group G and $\mathcal{J}_q$ be a deformation of corresponding Lie algebra $\mathcal{J}$. The deformed cotangent bundle $(T^*G)_q$ is a noncommutative manifold, i.e., according to the ideology developed by A.Connes [8], its coordinates are (noncommuting) generators of some associate algebra. A point on this manifold is parametrized by the pair (g, L), where $g \in G_q$ is a coordinate in the base of the bundle, L is a coordinate in a fiber.

The structure of $(T^*G)_q$ is defined by means of commutation relations between coordinates in the base and in a fiber. An appropriate and self-consistent R-matrix form of these relations was given in [5]:

$$R_\pm \overset{1}{g}\overset{2}{g} = \overset{2}{g}\overset{1}{g} R_\pm \,, \tag{1}$$

$$R_- \overset{1}{g}\overset{2}{L} = \overset{2}{L} R_+ \overset{1}{g} \,, \tag{2}$$

$$\overset{1}{L} R_-^{-1} \overset{2}{L} R_- = R_+^{-1} \overset{2}{L} R_+ \overset{1}{L} \,. \tag{3}$$

Here g and L are considered as matrices in the spirit of [9] and we use the standard notations: $\overset{1}{g} = g \otimes 1$, etc. Following [5], we assume that the standard parameter q, which enters explicit formulae in the theory of quantum groups, has the form

$$q = e^{\gamma\hbar}, \tag{4}$$

where $\hbar$ is the Planck constant (the parameter of quantization) and γ is the deformation parameter. In physical applications it is most natural to suppose that γ is either pure real (q belongs to the real axis) or pure imaginary (q belongs to the unit circle at the complex plane). For example, the second form of q is typical for the WZNW theory [10, 11, 14]. We suppose also that q is not a root of unity.

It should be mentioned that for both variants of choice of γ in (4) a q-number defined as

$$[x] \equiv \frac{q^x - q^{-x}}{q - q^{-1}} \tag{5}$$

is invariant with respect to complex conjugation of q. This property becomes important if one discusses involutions of deformed Lie algebras.

If g and L in the relations (1)–(3) are chosen as 2×2 matrices

$$g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}, \quad L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{6}$$

and R-matrices have the form

$$R_+ = q^{-1/2}\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & \omega & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \ \omega \equiv q - q^{-1}; \qquad R_- = PR_+^{-1}P \tag{7}$$

(P denotes a permutation operator: $P\overset{1}{g}P=\overset{2}{g}$, etc.), then (1)–(3) define a cotangent bundle for the quantum group $G_q = GL_q(2)$.

In this case each of R-matrix equations (1) and (3) is equivalent to six independent relations:

$$\begin{gathered} q\,g_1\,g_2 = g_2\,g_1, \quad q\,g_1\,g_3 = g_3\,g_1, \quad q\,g_2\,g_4 = g_4\,g_2, \quad q\,g_3\,g_4 = g_4\,g_3, \\ g_2\,g_3 = g_3\,g_2, \qquad g_1\,g_4 - g_4\,g_1 = -\omega\,g_2\,g_3; \end{gathered} \tag{8}$$

and

$$\begin{gathered} [A,B] = -q^{-1}\omega BD, \quad [A,C] = q^{-1}\omega DC, \quad [A,D] = 0, \\ CD = q^2DC, \quad BD = q^{-2}DB, \quad [B,C] = q^{-1}\omega D(D-A). \end{gathered} \tag{9}$$

It is well known that the algebra generated by the entries of the matrix g obeying (8) possesses the central element:

$$\det{}_q g = g_1\,g_4 - q^{-1}g_2\,g_3. \tag{10}$$

Now consider the algebra with generators A, B, C, D obeying (9). In what follows we shall denote this algebra as $\mathcal{L}$. It is easy to verify that it possesses two central elements:

$$K_1 = qA + q^{-1}D, \quad K_2 = q^{-1}AD - qBC. \tag{11}$$

One can check that the operators $\det_q g$ and K_2 commute with all entries of the matrices g and L. The set of relations (1)–(3) for fixed values of these central elements:

$$\det{}_q g = 1, \quad K_2 = \text{const},$$

defines a cotangent bundle for the quantum group $G_q = SL_q(2)$.

Algebra $\mathcal{L}$, generated by the entries of the matrix L (6) subject to the relations (9), is naturally connected with quantum $6j$-symbols for $sl_q(2)$. To remind this fact we start with the following theorem [5]

Theorem 1 *Let 2×2 matrices U and D satisfy the following conditions*

i) D is an unimodular diagonal matrix

$$D = \begin{pmatrix} q^{p/\hbar} & \\ & q^{-p/\hbar} \end{pmatrix} \tag{12}$$

ii)

$$\overset{1}{D}\overset{2}{U} = \overset{2}{U}\overset{1}{D}\,\sigma, \quad \overset{2}{D}\overset{1}{U} = \overset{1}{U}\overset{2}{D}\,\sigma, \quad \sigma = \mathrm{diag}(q^{-1}, q, q, q^{-1}); \tag{13}$$

iii)

$$R_+\,\overset{1}{U}\overset{2}{U} = \overset{2}{U}\overset{1}{U}\,\mathcal{R}_+(p), \quad R_-\,\overset{1}{U}\overset{2}{U} = \overset{2}{U}\overset{1}{U}\,\mathcal{R}_-(p), \tag{14}$$

where

$$\mathcal{R}_+(p) = q^{-1/2}\begin{pmatrix} q & & & \\ & \frac{\sqrt{[p/\hbar+1][p/\hbar-1]}}{[p/\hbar]} & \frac{q^{p/\hbar}}{[p/\hbar]} & \\ & -\frac{q^{-p/\hbar}}{[p/\hbar]} & \frac{\sqrt{[p/\hbar+1][p/\hbar-1]}}{[p/\hbar]} & \\ & & & q \end{pmatrix}, \quad \mathcal{R}_-(p) = P\mathcal{R}_+^{-1}(p)P \tag{15}$$

(here $[x]$ denotes a "q-number" (5)).

Then matrix L constructed by means of the similarity transformation

$$L = U D U^{-1}, \tag{16}$$

satisfies the relation (3) .

Here U^{-1} denotes an inverse matrix: $UU^{-1} = U^{-1}U = I$. Its explicit form respecting the relations (14) will be given below.

The useful consequence of (13) is a commutativity of L and D

$$\overset{1}{L}\,\overset{2}{D}=\overset{2}{D}\,\overset{1}{L}\,. \tag{17}$$

An important fact is that matrix $\mathcal{R}_+(p)$ appearing in the Theorem 1 can be regarded as $6j$ symbol for algebra $sl_q(2)$ [11, 13, 15, 16, 17, 18]. The corresponding Yang-Baxter equation reads [17, 5]:

$$\overset{1}{Q}\overset{23}{\mathcal{R}_+}(p)(\overset{1}{Q})^{-1}\overset{13}{\mathcal{R}_+}(p)\,\overset{3}{Q}\overset{12}{\mathcal{R}_+}(p)(\overset{3}{Q})^{-1}=\overset{12}{\mathcal{R}_+}(p)\,\overset{2}{Q}\overset{13}{\mathcal{R}_+}(p)(\overset{2}{Q})^{-1}\overset{23}{\mathcal{R}_+}(p), \tag{18}$$

where matrix $Q = \begin{pmatrix} e^{i\xi} & \\ & e^{-i\xi}\end{pmatrix}$ contains an extra variable ξ, conjugated with p:

$$[p, \xi] = -i\hbar, \qquad q^{p/\hbar}\, e^{i\xi} = q\, e^{i\xi}\, q^{p/\hbar}. \tag{19}$$

This variable ξ belongs to the algebra $\mathcal{U}$ but does not enter matrix L. An explicit expression for ξ will be given below.

The algebra $\mathcal{L}$ has several known realizations. We shall essentially exploit the following one [19].

Proposition 1 *Let matrices L_+, L_- obey the exchange relations of type (1):*

$$R_\pm\,\overset{1}{L_+}\overset{2}{L_+}=\overset{2}{L_+}\overset{1}{L_+}\,R_\pm,\quad R_\pm\,\overset{1}{L_-}\overset{2}{L_-}=\overset{2}{L_-}\overset{1}{L_-}\,R_\pm,\quad R_+\,\overset{1}{L_+}\overset{2}{L_-}=\overset{2}{L_-}\overset{1}{L_+}\,R_+\ . \tag{20}$$

Then the matrix L combined of L_+, L_- as follows

$$L = L_+\, L_-^{-1}, \tag{21}$$

satisfies the relation (3)

In the case of quantum algebra $\mathcal{J}_q = sl_q(2)$ (introduced first in [20]) this abstract construction admits the following realization [9]

Proposition 2 *Let R-matrices entering (20) be chosen as in (7). Then one can take $L_\pm$ as the upper and lower triangular matrices:*

$$L_+ = \begin{pmatrix} q^{l_3} & -\omega q^{1/2} l_- \\ 0 & q^{-l_3} \end{pmatrix}, \quad L_- = \begin{pmatrix} q^{-l_3} & 0 \\ \omega q^{-1/2} l_+ & q^{l_3} \end{pmatrix} \tag{22}$$

with entries expressed through the generators of the quantum algebra $sl_q(2)$:

$$[l_+, l_-] = \frac{q^{2l_3} - q^{2l_3}}{q - q^{-1}} \equiv [2l_3], \quad q^{l_3}\, l_\pm = q^{\pm 1} l_\pm\, q^{l_3}. \tag{23}$$

Note that the matrix L in the Proposition 1 is defined only up to a scaling factor. Thus, for L_+, L_- given in (22), we may choose L as follows

$$L = q^2 L_+ L_-^{-1} = \begin{pmatrix} qC - q^{-2l_3} & -q^{5/2}\omega l_- q^{-l_3} \\ -q^{-1/2}\omega l_+ q^{-l_3} & q^2 q^{-2l_3} \end{pmatrix}. \tag{24}$$

Here C stands for the Casimir operator of $sl_q(2)$:

$$C = \omega^2 l_- l_+ + q^{2l_3+1} + q^{-(2l_3+1)} = q^{2\hat{j}+1} + q^{-(2\hat{j}+1)}, \tag{25}$$

where $\hat{j}$ is the spin operator.

According to the Proposition 1 the matrix (24) satisfies (3). Therefore, it provides a representation of the algebra $\mathcal{L}$ introduced above. In this representation the central elements (11) are given by

$$K_1 = q^2 C, \quad K_2 = q^3, \tag{26}$$

Note that the scaling factor q^2 introduced in (24) has changed the values of K_1 and K_2. The choice of such normalization in (24) will be explained later.

2 Generating matrices and $(T^*B)_q$

2.1 The algebra $\mathcal{U}$

Now we shall study the quadratic algebra defined by (14). The R-matrix relations (14) may be written down for 2×2 matrix U

$$U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$$

in following explicit form:

$$U_1 U_3 = q^{-1} U_3 U_1, \quad U_2 U_4 = q^{-1} U_4 U_2, \tag{27}$$

$$U_1 U_2 = U_2 U_1 \sqrt{\frac{[p/\hbar - 1]}{[p/\hbar + 1]}}, \quad U_3 U_4 = U_4 U_3 \sqrt{\frac{[p/\hbar - 1]}{[p/\hbar + 1]}} \tag{28}$$

$$U_1 U_4 = U_4 U_1 \frac{\sqrt{[p/\hbar + 1][p/\hbar - 1]}}{[p/\hbar]} - U_3 U_2 \frac{q^{p/\hbar}}{[p/\hbar]} \tag{29}$$

$$U_3 U_2 = U_2 U_3 \frac{\sqrt{[p/\hbar + 1][p/\hbar - 1]}}{[p/\hbar]} - U_1 U_4 \frac{q^{-p/\hbar}}{[p/\hbar]} \tag{30}$$

The rest of the relations contained in (14) are not independent and can be deduced from (27)–(30).

Since (27)–(30) involve variable $q^{p/\hbar}$ in addition to operators U_i, these relations should be complemented with the formulae derived from (13):

$$q^{p/\hbar}\,U_1 = q^{-1}\,U_1\,q^{p/\hbar}, \quad q^{p/\hbar}\,U_2 = q\,U_2\,q^{p/\hbar},$$
$$q^{p/\hbar}\,U_3 = q^{-1}\,U_3\,q^{p/\hbar}, \quad q^{p/\hbar}\,U_4 = q\,U_4\,q^{p/\hbar}. \tag{31}$$

Let us denote the algebra generated by the operators U_i and p subject to the commutation relations (27)–(31) as $\mathcal{U}$. The following statement may be verified in straightforward way:

Proposition 3 *A central element of $\mathcal{U}$ is given by the "deformed" determinant of the matrix U:*

$$\mathrm{Det}U \equiv U_1U_4\sqrt{\frac{[p/\hbar+1]}{[p/\hbar]}} - U_2U_3\sqrt{\frac{[p/\hbar-1]}{[p/\hbar]}} = qU_4U_1\sqrt{\frac{[p/\hbar-1]}{[p/\hbar]}} - qU_3U_2\sqrt{\frac{[p/\hbar+1]}{[p/\hbar]}}. \tag{32}$$

Hence the algebra $\mathcal{U}$ contains only four independent generators. In classical limit ($\hbar = 0$) they become the coordinates on 4-dimensional phase space.

For further discussion it is convenient to introduce new variables instead of U_i:

$$\widehat{U}_i = U_i\sqrt{[p/\hbar]}. \tag{33}$$

The coordinates $\{p,\ \widehat{U}_i\}$ form a new set of generators of the algebra $\mathcal{U}$. The commutation relations (27)–(31) rewritten in terms of the new generators become slightly simplified:

$$\widehat{U}_1\widehat{U}_3 = q^{-1}\widehat{U}_3\widehat{U}_1, \ \ \widehat{U}_2\widehat{U}_4 = q^{-1}\widehat{U}_4\widehat{U}_2, \ \ \widehat{U}_1\widehat{U}_2 = \widehat{U}_2\widehat{U}_1, \ \ \widehat{U}_3\widehat{U}_4 = \widehat{U}_4\widehat{U}_3 \tag{34}$$

$$\widehat{U}_1\widehat{U}_4 = \widehat{U}_4\widehat{U}_1\frac{[p/\hbar+1]}{[p/\hbar]} - \widehat{U}_3\widehat{U}_2\frac{q^{p/\hbar}}{[p/\hbar]}, \tag{35}$$

$$\widehat{U}_3\widehat{U}_2 = \widehat{U}_2\widehat{U}_3\frac{[p/\hbar+1]}{[p/\hbar]} - \widehat{U}_1\widehat{U}_4\frac{q^{-p/\hbar}}{[p/\hbar]}, \tag{36}$$

$$q^{p/\hbar}\,\widehat{U}_1 = q^{-1}\,\widehat{U}_1\,q^{p/\hbar}, \quad q^{p/\hbar}\,\widehat{U}_2 = q\,\widehat{U}_2\,q^{p/\hbar},$$
$$q^{p/\hbar}\,\widehat{U}_3 = q^{-1}\,\widehat{U}_3\,q^{p/\hbar}, \quad q^{p/\hbar}\,\widehat{U}_4 = q\,\widehat{U}_4\,q^{p/\hbar}. \tag{37}$$

The central element (32) in new variables looks like follows

$$\mathrm{Det}U \equiv (\widehat{U}_1\widehat{U}_4 - \widehat{U}_2\widehat{U}_3)\frac{1}{[p/\hbar]} = (\widehat{U}_4\widehat{U}_1 - \widehat{U}_3\widehat{U}_2)\frac{q}{[p/\hbar]}. \tag{38}$$

The explicit form of the matrix inverse to $\widehat{U}$, which we shall need later, is

$$\widehat{U}^{-1} = \frac{1}{\mathrm{Det}U}\begin{pmatrix} \widehat{U}_4 & -q\widehat{U}_2 \\ -\widehat{U}_3 & q\widehat{U}_1 \end{pmatrix}\frac{1}{[p/\hbar]}. \tag{39}$$

Finally, from (33) we conclude that the expression (16) for the matrix L looks similarly in terms of new matrix $\widehat{U}$:

$$L = UDU^{-1} = \widehat{U}D\widehat{U}^{-1}, \tag{40}$$

where the formula (39) is to be used.

2.2 Description of $(T^*B)_q$

Now we are going to develop an approach to find a realization of matrix U. It has a deep ideology and is based on a connection of the algebra $\mathcal{L}$ introduced above with the deformed cotangent bundle for the Borel subgroup – $(T^*B)_q$. The later can be interpreted as a quantum (deformed) model space and, according to ideology of the Wigner-Eckart theorem, one can try to construct the generating operators for Clebsch-Gordan coefficients using its coordinates (i.e. noncommuting operators).

To clarify the announced connection we start with the following theorem (its analogue for the case of a chain of L-operators with nonultralocal commutation relations appeared first in [10])

Theorem 2 *Let the matrices A and B, obey the relations of type (1):*

$$R_\pm \overset{1}{A}\overset{2}{A}=\overset{2}{A}\overset{1}{A} R_\pm \, , \quad R_\pm \overset{1}{B}\overset{2}{B}=\overset{2}{B}\overset{1}{B} R_\pm \tag{41}$$

and the additional exchange relation

$$\overset{1}{A}\overset{2}{B}=\overset{2}{B}\overset{1}{A} R_+ \, , \quad \overset{2}{A}\overset{1}{B} R_- =\overset{1}{B}\overset{2}{A} \, . \tag{42}$$

Then L-operator constructed by means of similarity transformation [1]

$$L = ABA^{-1} \tag{43}$$

satisfies the relation (3).

The proof is straightforward:

$$\overset{1}{L} R_-^{-1} \overset{2}{L} R_- =\overset{1}{A}\overset{1}{B} (\overset{1}{A})^{-1}R_-^{-1} \overset{2}{A}\overset{2}{B} (\overset{2}{A})^{-1}R_- =\overset{1}{A}\overset{1}{B}\overset{2}{A} R_-^{-1}(\overset{1}{A})^{-1} \overset{2}{B} (\overset{2}{A})^{-1}R_- =$$

$$=\overset{1}{A}\,\overset{2}{A}\,\overset{1}{B}\,\overset{2}{B} R_+^{-1}(\overset{1}{A})^{-1}(\overset{2}{A})^{-1}R_- = R_+^{-1} \overset{2}{A}\,\overset{1}{A}\,\overset{2}{B}\,\overset{1}{B} R_-(\overset{2}{A})^{-1}(\overset{1}{A})^{-1} =$$

$$= R_+^{-1} \overset{2}{A}\overset{2}{B}\overset{1}{A} R_+(\overset{2}{A})^{-1} \overset{1}{B} (\overset{1}{A})^{-1} = R_+^{-1} \overset{2}{A}\overset{2}{B} (\overset{2}{A})^{-1}R_+ \overset{1}{A}\overset{1}{B} (\overset{1}{A})^{-1} = R_+^{-1} \overset{2}{L} R_+ \overset{1}{L} \, .$$

To argue that (41)–(42) give a natural definition of a q-analogue of T^*B, let us turn to the example of $SL(2)$. Note that the nonsymmetric (with respect to R-matrices) form of the relations (42) imposes some restriction on the structure of the matrices A and B. For $R_\pm$ defined as in (7) a possible choice is

$$A = \begin{pmatrix} a & c \\ 0 & a^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} b & 0 \\ d & b^{-1} \end{pmatrix}. \tag{44}$$

One may think of A and B as of coordinates in the deformed Borel subgroup $\mathcal{B}_q$ and in the corresponding deformed Lie subalgebra respectively. In other words, the matrices A and B are coordinate and momentum on the $(T^*B)_q$ respectively. Thus (41)–(42) may be regarded as a definition of $(T^*B)_q$.

We should underline that, although the matrices A and B look similarly on quantum level (see (41),(44)), they transform into different objects when $q \to 1$ (see comments in [5, 10]). Indeed, in the limit $q \to 1$ the matrix L is

$$L \to I + \gamma\hbar L_0,$$

[1] Here A^{-1} is an antipode of matrix A, determined by the structure of matrices $R_\pm$ (see, e.g., [9])

and the corresponding limit forms of A and B are:

$$A \to A_0, \quad B \to I + \gamma\hbar B_0, \tag{45}$$

where A_0 is a group-like element, whereas B_0 is rather an element of algebra.

Commutation relations for the generators of $(T^*B)_q$ following from (41)–(42) are

$$ac = q^{-1}ca, \quad bc = q^{1/2}cb, \quad ab = q^{1/2}ba; \tag{46}$$

$$bd = q^{-1}db, \quad ad = q^{1/2}da, \quad cd = q^{-1/2}dc + q^{-1/2}\omega b^{-1}a. \tag{47}$$

It is useful to perform the following decomposition of d

$$d = d_0 + d_1 = d_0 + q^{1/2}c^{-1}b^{-1}a \tag{48}$$

in order to transform (47) into homogeneous form:

$$bd_0 = q^{-1}d_0 b, \quad ad_0 = q^{1/2}d_0 a, \quad cd_0 = q^{-1/2}d_0 c \tag{49}$$

Thus, (46) and (49) describe 4 variables obeying Weyl-like commutation relations. Using the jargon of conformal field theory, we shall call formulae above "free field representation" and the generators a, b, c, d_0 "free field" variables.

Let us remark here that the last of equations (47) is nothing but a commutation relation from definition of the deformed Heisenberg algebra. Thus, the transformation (48) can be interpreted as "bosonization" of q-oscillators.

Now, substituting (44) in (43), we get expressions for the entries of L in terms of "free fields"

$$\begin{aligned} L &= q^{1/2}\begin{pmatrix} a & c \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} b & 0 \\ d & b^{-1} \end{pmatrix}\begin{pmatrix} a^{-1} & -q\,c \\ 0 & a \end{pmatrix} = \\ &= \begin{pmatrix} q\,(b+b^{-1}) + a^{-1}c\,d_0 & -q^2 a\,c\,(b + q\,a^{-1}c\,d_0) \\ (a\,c)^{-1}(b^{-1} + q^{-1}a^{-1}c\,d_0) & -q^2 a^{-1}c\,d_0 \end{pmatrix}. \end{aligned} \tag{50}$$

This matrix provides a "free field" realization of the algebra $\mathcal{L}$ defined in (9). Note that the additional scaling factor $q^{1/2}$ was introduced in (50) to ensure a coincidence of the Casimir operators calculated by formulae (11) for the matrix (50):

$$K_1 = q^2(b + b^{-1}), \quad K_2 = q^3. \tag{51}$$

with those for the matrix (24). In fact, we redefined the matrix B in (44) as

$$\tilde{B} = q^{1/2}B. \tag{52}$$

Comparing the Casimir operator K_1 given by (51) with one given by (26), we identify the operator b as the power of the spin operator

$$b = q^{2\hat{j}+1}. \tag{53}$$

It follows from (51) that matrix L contains only 3 independent variables (it is easy to see from the explicit form (50) that these are b, ac and $a^{-1}cd_0$). Moreover, direct calculation using (46),(49) shows that all elements of the matrix L commute with operator b. That agrees with the property (17).

Now comparing the statements of the Theorems 1 and 2 and taking into account the equality (40), we can write down the formula

$$L = A\,B\,A^{-1} = \widehat{U}\,D\,\widehat{U}^{-1}, \tag{54}$$

which points out a possibility to construct the matrix $\widehat{U}$ obeying (34)–(37) via the generators of $(T^*B)_q$.

Theorem 3 *The algebra $\mathcal{U} \equiv \{\widehat{U}, p\}$ with defining relations (34)–(37) has the following realization in terms of generators a, b, c, d_0*

$$b = q^{p/\hbar}, \quad \widehat{U} = \left(\begin{array}{cc} \frac{1}{\omega}\, a\,(b + a^{-1}\, c\, d_0)\, e^{-\frac{i\xi}{2}} & c\, e^{\frac{i\xi}{2}} \\ \frac{1}{\omega}\, c^{-1}(b^{-1} + q^{-1} a^{-1}\, c\, d_0)\, e^{-\frac{i\xi}{2}} & a^{-1}\, e^{\frac{i\xi}{2}} \end{array} \right), \tag{55}$$

where $\omega \equiv q - q^{-1}$, d_0 is defined in (48) and

$$e^{i\xi} = a^{-1}\, b^{\gamma}\, c^{-1}\, d_0^{-1} \tag{56}$$

with γ being arbitrary constant.

This theorem gives a "free field" representation of the algebra $\mathcal{U}$. Let us remark that the remaining freedom in (56) corresponds to canonical transformations (since ξ and p are conjugate variables).

The formulated theorem will be proved in several steps. First, we introduce a lower-triangular matrix which diagonalizes the matrix $\tilde{B}$:

$$V = \left(\begin{array}{cc} v_1 & 0 \\ v_3 & v_2 \end{array} \right), \quad \tilde{B} = V \tilde{B}_0 V^{-1}, \quad \tilde{B}_0 = \left(\begin{array}{cc} q^{1/2} b & 0 \\ 0 & q^{1/2} b^{-1} \end{array} \right) \equiv q^{1/2} B_0. \tag{57}$$

Proposition 4 *A possible solution for the entries of the matrix V is*

$$v_1 = v_1(b), \quad v_2 = v_2(b), \quad v_3 = d\, v_1(b)\, f(b), \tag{58}$$

where $v_1(b)$, $v_2(b)$ are arbitrary functions of b and $f(b)$ is given by

$$f(b) = (b - q b^{-1})^{-1}. \tag{59}$$

Thus, matrix L (50) admits decomposition of the form:

$$L = \widehat{U}_0 \tilde{B}_0 \widehat{U}_0^{-1}, \quad \widehat{U}_0 = A\, V. \tag{60}$$

However, this diagonalization is not unique. Using an arbitrary power of the diagonal matrix Q, which depends on the variable conjugate to b,

$$Q = \left(\begin{array}{cc} e^{i\xi} & \\ & e^{-i\xi} \end{array} \right), \quad b\, e^{i\xi} = q\, e^{i\xi}\, b, \tag{61}$$

we obtain a family of diagonalizing matrices:

$$L = \widehat{U}_\delta\, \tilde{B}_\delta\, \widehat{U}_\delta^{-1}, \quad \widehat{U}_\delta = A\, V\, Q^{\delta}, \quad \tilde{B}_\delta = Q^{-\delta} \tilde{B}_0\, Q^{\delta} = q^{\delta}\, \tilde{B}_0 = q^{\delta+1/2}\, B_0. \tag{62}$$

An explicit form of the diagonalizing matrix is

$$\widehat{U}_\delta = A\, V\, Q^{\delta} = \left(\begin{array}{cc} (a\, v_1 + c\, d\, v_1 f)\, e^{i\delta\xi} & c\, v_2\, e^{-i\delta\xi} \\ a^{-1} d\, v_1 f\, e^{i\delta\xi} & a^{-1} v_2\, e^{-i\delta\xi} \end{array} \right). \tag{63}$$

Here we should describe a new object $e^{i\xi}$ which appeared in the matrix $\widehat{U}$. We assume that the following Weyl-like relations hold:

$$a\, e^{i\xi} = q^{\alpha} e^{i\xi}\, a, \quad b\, e^{i\xi} = q\, e^{i\xi}\, b, \quad c\, e^{i\xi} = q^{\beta} e^{i\xi}\, c, \quad d_0\, e^{i\xi} = q^{\gamma} e^{i\xi}\, d_0. \tag{64}$$

Proposition 5 *The set of equations (64) is equivalent to*

$$e^{i\xi} = a^{\beta+(\gamma-1)/2}\, b^{\gamma}\, c^{(\gamma-1)/2-\alpha}\, d_0^{-1}. \tag{65}$$

Now we have to remind that the matrix U (and $\widehat{U}$ as well) described in the Theorem 1 has to satisfy the relation (13) or, equivalently, to the relation

$$\overset{1}{B_0}\,\overset{2}{\widehat{U}_\delta} = \overset{2}{\widehat{U}_\delta}\,\overset{1}{B_0}\;\sigma, \tag{66}$$

where σ and B_0 are determined in (13) and (57), respectively. A straightforward calculation using (46)–(47) leads to

Proposition 6 *The matrix $\widehat{U}_\delta$ given by (63) satisfies the relation (66) only for*

$$\delta = \ -\tfrac{1}{2}\ .$$

It is worth mentioning that such a choice of δ exactly compensates the renormalization of the matrix B in (52), i.e.

$$\tilde{B}_{-1/2} = B_0. \tag{67}$$

Bearing in mind the definitions (48), (59) one can rewrite (63) for $\delta = \ -\frac{1}{2}$ as follows

$$\widehat{U} \equiv \widehat{U}_{-1/2} = \begin{pmatrix} a(b + a^{-1}\,c\,d_0)\,w\,e^{-\frac{i\xi}{2}} & c\,v\,e^{\frac{i\xi}{2}} \\ c^{-1}(b^{-1} + q^{-1}a^{-1}\,c\,d_0)\,w\,e^{-\frac{i\xi}{2}} & a^{-1}\,v\,e^{\frac{i\xi}{2}} \end{pmatrix}, \tag{68}$$

where $w \equiv w(b) = v_1(b)f(b)$, $v \equiv v_2(b)$.

Finally, a direct check shows that the matrix (68) obeys equations (34)–(37) if the functions w, v are constant (we chose them as follows $v(b) = 1$, $w(b) = \frac{1}{\omega}$) and the coefficients in (64)–(65) satisfy the conditions $\beta = -\alpha$, $\gamma = \alpha - \beta - 1 = 2\alpha - 1$. Thus, the Theorem 3 is proven.

2.3 The explicit representation

Now we face the problem of constructing of an explicit representation for the generators a, b, c, d_0. A Weyl-like form of the commutation relations (46),(49) points out the possibility of finding a realization for these generators in terms of two pairs of canonical variables. This also means (due to the interpretation of (48) as "bosonization" of q-oscillators) that the generators a, b, c, d admit a realization via q-oscillators. Evidently, such a representation is not unique.

It is natural to realize a, b, c, d_0 as operators acting on the q-analogue of the space $D(z_1, z_2)$. We shall denote this space as $D_q(z_1, z_2)$. The space $D_q(z_1, z_2)$ is spanned on the basic vectors of form (remember that $[x]$ stands for q-numbers)

$$|j, m\rangle = \frac{z_1^{j+m}\, z_2^{j-m}}{\sqrt{[j+m]![j-m]!}}, \qquad j = \ 0,\ \tfrac{1}{2},\ 1,\ \tfrac{3}{2},\ \ldots \qquad m = -j, .., j. \tag{69}$$

One can define on $D_q(z_1, z_2)$ such a scalar product [2] that the system (69) will be orthonormal

$$< j, m | j', m' >= \delta_{jj'}\delta_{mm'}.$$

[2]This scalar product is a deformation of the standard one. Its explicit form makes use of the q-exponent and the Jackson integral. See [22, 23, 21] for details.

In all formulae concerning the space $D_q(z_1,z_2)$ we suppose that q is chosen as described in the Section 2 (i.e., it belongs either to real axis or to unit circle at the complex plane). In this case the conjugation rule is

$$(z_i)^* = z_i^{-1}\,[z_i\partial_i], \qquad (z_i\partial_i)^* = z_i\partial_i. \tag{70}$$

The formulae (55) and (53) imply that the generator b is a power of the spin operator. Hence, on the space $D_q(z_1,z_2)$ it is given by

$$b = q^{z_1\,\partial_1 + z_2\,\partial_2 + 1} = q\,N_1\,N_2. \tag{71}$$

Next, let us remind that we already know the limit versions of the generators a, b, c, d (see the Proposition 5; one should take into account the rescaling (52)). Their appropriate deformations for generic q are described by

Proposition 7 *The set of operators*

$$\begin{aligned} &a = q^{\lambda_0}\,z_1^{-1/2}\,N_1^{\lambda_1}, \quad c = q^{\nu_0}\,z_1^{-1/2}\,z_2\,N_1^{\lambda_1-2}\,N_2^{\nu_2},\\ &b = q\,N_1\,N_2, \qquad\quad\;\; d = -q^{\lambda_0-\nu_0+\nu_2}\,z_2^{-1}\,(N_2 - N_2^{-1})\,N_1\,N_2^{-\nu_2}. \end{aligned} \tag{72}$$

satisfies the relations (46)–(47).

Although this proposition gives a family of possible representations, we can choose a particular one (which is "natural" in some sense). To this end we impose again an additional condition:

Proposition 8 *The matrix L (50) constructed from the generators of $(T^*B)_q$ coincides with the matrix (24) provided that*

$$b = q\,N_1\,N_2, \quad a\,c = q^{-1/2}\,z_1^{-1}\,z_2\,N_1^{-1/2}\,N_2^{-1/2}, \quad a^{-1}\,c\,d_0 = -N_1^{-1}\,N_2. \tag{73}$$

Comparing the statements of the Propositions 11, 12, we derive:

$$\begin{aligned} &a = q^{\lambda_0}\,z_1^{-1/2}\,N_1^{3/4}, && b = q\,N_1\,N_2, && c = q^{\nu_0}\,z_1^{-1/2}\,z_2\,N_1^{-5/4}\,N_2^{-1/2},\\ &d_0 = -q^{\lambda_0-\nu_0-1/2}\,z_2^{-1}\,N_1\,N_2^{3/2}, && && q^{\lambda_0+\nu_0} = q^{-1/8}. \end{aligned} \tag{74}$$

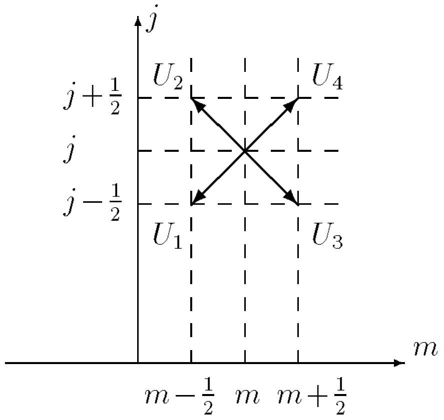
Substituting (74) into (56) (and remember that (56) is defined only up to a coefficient), we get

$$e^{i\xi} = q^{2\epsilon}\,z_1\,N_1^{\gamma-1/2}\,N_2^{\gamma-1}, \tag{75}$$

where γ and ϵ are arbitrary. Finally, substituting (74)–(75) into (68) (one should be accurate when calculating $e^{\pm i\xi/2}$), we obtain (one should remember that the matrix U is defined only up to arbitrary scaling factor)

$$\widehat{U} = \begin{pmatrix} \frac{1}{\omega}\,\alpha_0\,z_1^{-1}\,N_1^{1-\gamma/2}\,N_2^{3/2-\gamma/2}(N_1 - N_1^{-1}) & \beta_0\,z_2\,N_1^{\gamma/2-3/2}N_2^{\gamma/2-1} \\ -\frac{1}{\omega}\,q^{-1}\alpha_0\,z_2^{-1}N_1^{1/2-\gamma/2}\,N_2^{1-\gamma/2}(N_2 - N_2^{-1}) & \beta_0\,z_1\,N_1^{\gamma/2-1}N_2^{\gamma/2-1/2} \end{pmatrix}. \tag{76}$$

Fig. 1: Action of operators U_i



2.4 Quantum Clebsch-Gordan coefficients

Using the connection formula (33) we get from (76) a family of matrices U which provide possible representations of the algebra $\mathcal{U}$. It is natural to study an action on the entries of these matrices on the space $D_q(z_1, z_2)$ described above. On the basic vectors (69) these operators act as follows (see Fig.1)

$$\begin{aligned}
U_1\,|j,m\rangle &= C_1\,q^{\frac{1}{2}(j-m+1)}\sqrt{\tfrac{[j+m]}{[2j+1]}}\,|j-\tfrac{1}{2},m-\tfrac{1}{2}\rangle,\\
U_2\,|j,m\rangle &= C_2\,q^{-\frac{1}{2}(j+m)}\sqrt{\tfrac{[j-m+1]}{[2j+1]}}\,|j+\tfrac{1}{2},m-\tfrac{1}{2}\rangle,\\
U_3\,|j,m\rangle &= -C_3\,q^{-\frac{1}{2}(j+m+1)}\sqrt{\tfrac{[j-m]}{[2j+1]}}\,|j-\tfrac{1}{2},m+\tfrac{1}{2}\rangle,\\
U_4\,|j,m\rangle &= C_4\,q^{\frac{1}{2}(j-m)}\sqrt{\tfrac{[j+m+1]}{[2j+1]}}\,|j+\tfrac{1}{2},m+\tfrac{1}{2}\rangle,
\end{aligned} \tag{77}$$

where the coefficients C_i do not depend on m.

The q-analogues of CGC for the algebra $sl_q(2)$ were studied by many authors (see e.g. [16, 24]). An analogue of the Van-der-Waerden formula reads

$$\begin{aligned}
\left\{\begin{array}{ccc} j & \frac{1}{2} & j'' \\ m & m' & m'' \end{array}\right\}_q &= \delta_{m'',m+m'}\left(\frac{[j+\frac{1}{2}-j'']![j+j''-\frac{1}{2}]![j''+\frac{1}{2}-j]!}{[j+j''+\frac{3}{2}]!}\right)^{1/2}\times\\
&\times q^{\frac{1}{2}(j+\frac{1}{2}-j'')(j+j''+\frac{3}{2})+jm'-\frac{1}{2}m}\times\\
\times\sum_{r\geq 0}&\frac{(-1)^r q^{-r(j+j''+\frac{3}{2})}\left([j+m]![j-m]![j''+m'']![j''-m'']![2j''+1]\right)^{1/2}}{[r]![j+\frac{1}{2}-j''-r]![j-m-r]![\frac{1}{2}+m'-r]![j''-\frac{1}{2}+m+r]![j''-j-m'+r]!}.
\end{aligned} \tag{78}$$

Comparing the matrix elements $< j', m'|U_i|j, m >$ following from (77) with values of CGC given by (78), we establish the following correspondence

$$\begin{aligned}
\langle j'', m'' \mid U_1 \mid j, m\rangle &= \delta_{j'',j-\frac{1}{2}}\, \alpha_0\, q^{(1-\gamma/2)j-1/2} \left\{ \begin{array}{ccc} j & \frac{1}{2} & j'' \\ m & -\frac{1}{2} & m'' \end{array} \right\}_q , \\
\langle j'', m'' \mid U_2 \mid j, m\rangle &= \delta_{j'',j+\frac{1}{2}}\, \beta_0\, q^{(\gamma/2-1)j} \left\{ \begin{array}{ccc} j & \frac{1}{2} & j'' \\ m & -\frac{1}{2} & m'' \end{array} \right\}_q , \\
\langle j'', m'' \mid U_3 \mid j, m\rangle &= \delta_{j'',j-\frac{1}{2}}\, \alpha_0\, q^{(1-\gamma/2)j-1/2} \left\{ \begin{array}{ccc} j & \frac{1}{2} & j'' \\ m & \frac{1}{2} & m'' \end{array} \right\}_q , \\
\langle j'', m'' \mid U_4 \mid j, m\rangle &= \delta_{j'',j+\frac{1}{2}}\, \beta_0\, q^{(\gamma/2-1)j} \left\{ \begin{array}{ccc} j & \frac{1}{2} & j'' \\ m & \frac{1}{2} & m'' \end{array} \right\}_q .
\end{aligned}$$

Thus we derive

Theorem 4 *The family of matrices $U = \hat{U}\frac{1}{\sqrt{[p/\hbar]}}$, where $\widehat{U}$ is given by (76), with entries acting on the space $D_q(z_1, z_2)$ generates (up to coefficients which depend only on j) the Clebsch-Gordan coefficients corresponding to the decomposition of the tensor product $V_j \otimes V_{\frac{1}{2}}$ for $sl_q(2)$.*

Let us remark that among the described family of matrices there exists one which may by called "exact generating matrix". Indeed, the following matrix (which corresponds to the parametrization $\gamma = 2$, $\alpha_0 = q^{1/2}$, $\beta_0 = 1$ in (76))

$$U = \left(\begin{array}{cc} z_1^{-1}\,[z_1\partial_1] q^{\frac{1}{2}(z_2\partial_2+1)} & z_2\, q^{-\frac{1}{2}z_1\partial_1} \\ -z_2^{-1}\,[z_2\partial_2] q^{-\frac{1}{2}(z_1\partial_1+1)} & z_1\, q^{\frac{1}{2}z_2\partial_2} \end{array} \right) \frac{1}{\sqrt{[p/\hbar]}} \tag{79}$$

belongs to the described family and its entries satisfy (77) with $C_i = 1$. For the given matrix the formula (38) reads

$$\mathrm{Det} U = (\hat{U}_1\hat{U}_4 - \hat{U}_2\hat{U}_3)\frac{1}{[p/\hbar]} = q^{1/2}. \tag{80}$$

The question about unitarity of the matrix (79) is considered in the Appendix.

3 Conclusion

In this paper we have constructed the q-analogue of the phase space T^*B and clarified its role in the description of the model representation of the corresponding quantum group G_q. The general formulae are concretized on the example of $G = SL(2)$.

To realize described scheme in case of arbitrary group G one has to use some well known ingredients of the theory of quantum groups [9] as well as others, like the q-analogues of $6j$-symbols $R(p)$, which are discussed in the literature only partially [25, 26].

The results of this paper can be generalized in several directions even for the case of $SL(2)$. The first is consideration of the matrix U with an auxiliary space corresponding to the higher spin representation. It must lead to the expression for generating matrix for all CGC. The work in this direction is now in progress.

The second point to be discussed is the case of q being a root of unity. The structure of $R(p)$ allows to hope that reduction on so-called "good" representations will be quite natural in our formalism. But this case is to be examined much more carefully.

Acknowledgements

We are grateful to A.Yu.Alekseev, P.P.Kulish and V.Schomerus for stimulating discussions and useful comments. We would like to thank Prof. A.Niemi for hospitality at TFT, University of Helsinki, where this work was begun.

This work was partially supported by ISF grant R2H000.

Appendix. Unitarity of U_0 and U.

First we discuss a unitarity of the matrix U_0. Taking into account that $p^* = p$, one can check that the matrix conjugated to U_0 does not coincide with U_0^{-1}. Hence, the matrix U_0 itself is not a unitary one. However, it turns out that the transposed matrix [3]

$$U_0^T = \begin{pmatrix} \partial_1 & -\partial_2 \\ z_2 & z_1 \end{pmatrix} \sqrt{\frac{\hbar}{p}}$$

satisfies unitarity condition:

$$(U_0^T)^* = \sqrt{\frac{\hbar}{p}} \begin{pmatrix} z_1 & \partial_2 \\ -z_2 & \partial_1 \end{pmatrix} = (U_0^T)^{-1}.$$

In the deformed case the question of unitarity of U is more complicated (recall that q can be either real or $|q| = 1$) because now U includes an operator N which cojugates in different ways for the different choices of q.

Let us consider the matrix U given by (79). The conjugated matrix can be constructed according to the rules (70). One can check that the unitarity condition

$$(U^T)^* U^T = U^T (U^T)^* = I$$

(similar to what in nondeformed case) for the transposed matrix hold only for the real q.

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