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Vertex operators – from toy model to lattice algebras

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Abstract

For a given quantum Lie algebra $\mathcal{G}$ we construct some associative algebra $\mathcal{V}$ which can be considered as a "toy model" for the algebraof vertex operators associated with the corresponding quantum affine algebra $\widehat{\mathcal{G}}$.

Preliminaries.

1. By definition, a Hopf algebra is a tuple $(\mathcal{G}, \epsilon, \Delta, \mathcal{S})$ of an associative algebra $\mathcal{G}$ (the "*symmetry algebra* ") with a one-dimensional representation $\epsilon : \mathcal{G} \mapsto \mathbb{C}$ (the "*co-unit*"), a homomorphism $\Delta : \mathcal{G} \mapsto \mathcal{G} \otimes \mathcal{G}$ (the "*co-product*") and an anti-automorphism $\mathcal{S} : \mathcal{G} \mapsto \mathcal{G}$ (the "*antipode*"). They obey a set of basic axioms which can be found e.g. in [1]. The Hopf algebra $(\mathcal{G}, \epsilon, \Delta, \mathcal{S})$ is called quasitriangular if there exists an invertible element $R \in \mathcal{G} \otimes \mathcal{G}$ such that

$$R\ \Delta(\xi) = \Delta'(\xi)\ R \quad \text{for all} \quad \xi \in \mathcal{G}, \tag{1}$$

$$(id \otimes \Delta)(R) = R_{13}R_{12}, \qquad (\Delta \otimes id)(R) = R_{13}R_{23}. \tag{2}$$

Here $\Delta' = \omega \circ \Delta$ with $\omega : \mathcal{G} \otimes \mathcal{G} \mapsto \mathcal{G} \otimes \mathcal{G}$ being the permutation map: $\omega(\xi \otimes \eta) = \eta \otimes \xi$ and we used the standard notation for the elements $R_{ij} \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G}$. Later we shall use along with R the following notations

$$R_+ \equiv R, \quad R' \equiv \omega(R), \quad R_- \equiv (R')^{-1}.$$

For a modular Hopf algebra one postulates in addition the existence of a certain invertible central element $v \in \mathcal{G}$ (the "*ribbon element*") which has the properties

$$R'R = (v \otimes v)\Delta(v^{-1}), \quad S(v) = v, \quad \epsilon(v) = 1$$

(see [2, 3] for details). Later we shall also need the element $\kappa \in \mathcal{G}$ which is a certain square root of v.

We suppose that the algebra $\mathcal{G}$ possesses a $*$-operation with following standard extension onto $\mathcal{G}^{\otimes n}$:

$$(\xi_1 \otimes .. \otimes \xi_n)^* = \xi_1^* \otimes .. \otimes \xi_n^* \quad \text{for any } \xi_i \in \mathcal{G} \tag{3}$$

In this paper we consider the case of $\mathcal{G} = U_q(\mathcal{J})$ – the quantized universal enveloping algebra with the complex parameter q taking values on the unit circle. In this case the algebraic structure of the Hopf algebra behaves according to

$$(\Delta(\xi))^* \;=\; \Delta'(\xi^*) \tag{4}$$

$$R^* \;=\; (R')^{-1} \qquad , \qquad v^* = v^{-1}. \tag{5}$$

We restrict our consideration to the case of q being not a root of unity.

2. Let us briefly describe the approach which we will systematically use to define different algebras. Let $\mathcal{G}$ be the quantum Lie algebra. To define some associative algebra $\mathcal{A}$ one may introduce the objects $\{Y_j\} \in \mathcal{G} \otimes \mathcal{A}$ with certain properties and certain maps (homomorphisms) $\{\pi_i\} : \mathcal{G} \mapsto \mathbb{C}$ (the later objects are elements of the dual space $\mathcal{G}'$).

Then the algebra $\mathcal{A}$ can be defined as an algebra generated by the elements $\pi_i(Y_j)$ (we call them components). For example, the algebra $\mathcal{G}$ itself can be defined via properties of $R \in \mathcal{G} \otimes \mathcal{G}$ (see above).

1 Basic axioms.

Let $\mathcal{X}$ be a discrete space of all irreps of the quantum quasitriangular ribbon Hopf algebra $\mathcal{G}$. The coordinate of a point on $\mathcal{X}$ is given by a vector p taking discrete values. For our main example – $\mathcal{G} = U_q(\mathcal{J})$ the space $\mathcal{X}$ coincides with the space of weight vectors.

Definition 1 *Let $\mathcal{C}$ be an $*$-algebra of functions on $\mathcal{X}$, i.e., its elements are the maps $f : \mathcal{X} \mapsto \mathbb{C}$ with the pointwise multiplication $(g \cdot f)(p) = (f \cdot g)(p) = f(p)\, g(p)$ and the involution $*$ induced by the complex conjugation in $\mathbb{C}$ as follows*

$$(f^*)(p) = \overline{f(p)}, \quad (f\, g)^*(p) = f^*(p)\, g^*(p).$$

Remark. The ribbon element v and the quantum dimensions $d_I = \mathrm{tr}_q^I(e^I)$ (e is the unit element of $\mathcal{G}$) are important examples of functions on $\mathcal{X}$.

Let us introduce the following data: the map (homomorphism)

$$\sigma : \mathcal{C} \mapsto \mathcal{G} \otimes \mathcal{C}, \quad \sigma(f\, g) = \sigma(f)\, \sigma(g) \tag{6}$$

and a certain $\mathcal{C}$-valued element of $\mathcal{G} \otimes \mathcal{G}$: $F \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{C}$. In what follows we assume that σ and F satisfy the following basic axioms

$$\textbf{i)} \quad \Big((id \otimes \Delta)\, F\Big)\, (F \otimes e) \;=\; \Big((\Delta \otimes id)\, F\Big)\; \overset{3}{\sigma}\, (e \otimes F); \tag{7}$$

$$\textbf{ii)} \quad \overset{2}{\sigma}\overset{1}{\sigma}\, (f) = \Delta_F(\sigma(f)); \tag{8}$$

$$\textbf{iii)} \quad \overset{1}{D}\, \mathcal{R}_- = \mathcal{R}_+\, \overset{2}{\sigma}\, (\overset{1}{D}), \quad \mathcal{R}_-\, \overset{2}{D} = \overset{1}{\sigma}\, (\overset{2}{D})\, \mathcal{R}_+. \tag{9}$$

Here the notations $\overset{1}{\sigma}$, $\overset{2}{\sigma}$, $\overset{3}{\sigma}$ are used in the standard sense, e.g. $\overset{1}{\sigma}\, (f) = \sigma(f) \otimes e \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{C}$ etc. and we have introduced new objects constructed with the help of σ, F and the ribbon element v as follows

$$\mathcal{R} \equiv (F')^{-1}\, R\, F \quad \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{C}, \tag{10}$$

$$\Delta_F(f) \equiv F^{-1}\, \Delta(f)\, F \quad : \mathcal{G} \mapsto \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{C}. \tag{11}$$

$$D \;=\; \sigma(v)\, (v \otimes v)^{-1} \;\in \mathcal{G} \otimes \mathcal{C}. \tag{12}$$

Notice also that (9) contains only one axiom; the second equation in (9) follows from the first one and from (10).

The extension of $*$-operation on elements of $\mathcal{G}^{\otimes n} \otimes \mathcal{C}$ is defined as follows : for $\theta = \xi_1 \otimes ... \otimes \xi_n \otimes \vartheta \;\in \mathcal{G}^{\otimes n} \otimes \mathcal{C}$ one has

$$\theta^* = \xi_1^* \otimes ... \otimes \xi_n^* \otimes \vartheta^*, \tag{13}$$

where the $*$-operations on elements of $\mathcal{G}$ and $\mathcal{C}$ were defined above.

To complete the description of σ and F we postulate two more relations in addition to axioms (7)–(9)

$$\textbf{iv)} \quad \Big(\sigma(f)\Big)^* = \sigma(f^*), \tag{14}$$

$$\textbf{v)} \quad F^* = F^{-1}\; \Delta(\kappa)\, (\kappa \otimes \kappa)^{-1}\, R^{-1}, \tag{15}$$

where κ-depending factor (recall that κ is defined as a square root of v) was introduced to ensure the self-consistency:

$$(F^*)^* = R'\,(\kappa\otimes\kappa)\,\Delta'(\kappa^{-1})\,R\,(\kappa\otimes\kappa)\,\Delta(\kappa^{-1})\,F = R'R\,(\kappa\otimes\kappa)^2\,\Delta(\kappa^{-2})\,F = F.$$

As a consequence of (14)–(15) and the definitions (10)–(12) we get

$$\mathcal{R}^* \;=\; \mathcal{R}^{-1}, \tag{16}$$

$$(\Delta_F(\xi))^* \;=\; \Delta_F(\xi^*) \tag{17}$$

$$D^* \;=\; D^{-1}. \tag{18}$$

Remark. The formulae (10), (11) show that $\mathcal{R}(p)$ and Δ_F are attributes of quasi-Hopf algebra [4] which is obtained from Hopf algebra (with standard R and Δ) by means of twisting with the element F. Note that the new algebra possesses the same ribbon element, that is

$$\mathcal{R}'\mathcal{R} \;=\; (v\otimes v)\,\Delta_F(v^{-1}). \tag{19}$$

2 Toy model for vertex operators.

Using the approach described above, we introduce the chiral vertex operators algebra $\mathcal{V}$ as follows.

Definition 2 *The algebra $\mathcal{V}$ is an associative algebra generated by the functions $f\in\mathcal{C}\subset\mathcal{V}$ and the components of the elements Φ^{-1}, $\Phi\in\mathcal{G}\otimes\mathcal{V}$ such that*

$$\textbf{i)}\qquad \Phi^{-1}\,\Phi \;=\; \Phi\,\Phi^{-1} \;=\; e\otimes 1; \tag{20}$$

$$\textbf{ii)}\qquad \Phi\, f \;=\; \sigma(f)\,\Phi; \tag{21}$$

$$\textbf{iii)}\qquad \overset{2}{\Phi}\,\overset{1}{\Phi} = \; F^{-1}\,\Delta(\Phi)\,. \tag{22}$$

In the l.h.s. of (21) $f\in\mathcal{C}$ is understood as trivially embedded into $\mathcal{G}\otimes\mathcal{V}$: $f\sim e\otimes f$ and Δ in the r.h.s. of (22) is regarded as action on the $\mathcal{G}$-part of Φ: $\Delta(\Phi)\in\mathcal{G}\otimes\mathcal{G}\otimes\mathcal{C}$.

Let us notice that (21) is consistent with (22) due to the axiom (8). Indeed, since Δ is a homomorphism, we find

$$\Delta\Phi\, f \;=\; \Delta(\Phi\, f) \;=\; \Delta(\sigma(f)\,\Phi) \;=\; \Delta(\sigma(f))\,\Delta(\Phi);$$

but it also follows from (22) that

$$\Delta\Phi\, f \;=\; F\;\overset{2}{\Phi}\,\overset{1}{\Phi}\; f \;=\; F\,(\overset{2}{\sigma}\,\overset{1}{\sigma}\,(f))\,F^{-1}\,\Delta(\Phi),$$

what is consistent with the previous equality if (8) is valid.

Using the definition (10) and the quasitriangularity of Δ (1) one derives from (22) the following quadratic R-matrix relations for Φ

$$\mathcal{R}\;\overset{2}{\Phi}\,\overset{1}{\Phi} = (F')^{-1}\,R\,\Delta(\Phi) \;=\; (F')^{-1}\,\Delta'(\Phi)\,R \;=\; (F^{-1}\,\Delta(\Phi))'\,R = \overset{1}{\Phi}\,\overset{2}{\Phi}\;R. \tag{23}$$

Remark. Taking into account that R-matrix in the r.h.s. of (23) satisfies the standard Yang-Baxter equation and using the property (21), one derives from (23) the following equation for $\mathcal{R}(p)$:

$$\mathcal{R}_{12} \overset{2}{\sigma}(\mathcal{R}_{13})\, \mathcal{R}_{23} = \overset{1}{\sigma}(\mathcal{R}_{23})\, \mathcal{R}_{13} \overset{3}{\sigma}(\mathcal{R}_{12}), \tag{24}$$

which is twisted Yang-Baxter equation for quasi-Hopf algebra [4]. In different physical contexts this equation was discussed in [6, 7, 5, 8, 9, 10]. Its solutions for fundamental representations of $U_q(sl(n))$ and $U_q(sl(n|m))$ were recently found in [11]. Let us give an explicit form of $\mathcal{R}(p)$ in the simplest case of $\mathcal{G} = U_q(sl(2))$:

$$\mathcal{R}_+(p) = q^{-1/2} \begin{pmatrix} q & & & \\ & \frac{\sqrt{[p/\hbar+1][p/\hbar-1]}}{[p/\hbar]} & \frac{q^{p/\hbar}}{[p/\hbar]} & \\ & -\frac{q^{-p/\hbar}}{[p/\hbar]} & \frac{\sqrt{[p/\hbar+1][p/\hbar-1]}}{[p/\hbar]} & \\ & & & q \end{pmatrix}, \tag{25}$$

where $[x] = \dfrac{q^x - q^{-x}}{q - q^{-1}}$ stands for a "q-number" and $p = 2\hat{j} + 1$ with $\hat{j}$ being the operator of spin (details can be found e.g. in [9]).

Now we can extend the conjugation operation already introduced for elements of $\mathcal{C}$ and $\mathcal{G}$ ($f^+ = f^*$, $\xi^+ = \xi^*$) to the elements of $\mathcal{V}$:

$$\Phi^+ = \mathcal{P}\, R_\pm\, \Phi^{-1}\, E^{\pm 1}\, \mathcal{P}^{-1}, \tag{26}$$

where $\mathcal{P} \in \mathcal{C}$ is some selfconjugated element (in (26) it is considered as trivially embedded into $\mathcal{G} \otimes \mathcal{C}$) and E is the element such that

$$E = \sigma(\kappa)\,(\kappa \otimes \kappa)^{-1} \in \mathcal{G} \otimes \mathcal{C}, \quad E^2 = D,$$

where κ is a square root of the ribbon element v.

Remark. The definition (26) essentially makes use of the equality $\Phi^{-1} D \Phi = N = R_-^{-1} R_+$ which will be discussed below.

Using (18) and (5), one can check the self-consistency of (26) :

$$(\Phi^+)^+ = \mathcal{P}^{-1}\, E^*\, \mathcal{P}\, E^{-1}\, \Phi\, R_-\, \mathcal{P}^{-1}\, (R_-)^*\, \mathcal{P}^* = D^{-1}\, \Phi\, R_+ R_-^{-1} = \Phi.$$

Remark. One should remember that (26) takes place for $|q| = 1$. For real q the conjugation of Φ looks simpler: $\Phi^+ = \mathcal{P}\Phi^{-1}\mathcal{P}^{-1}$.

Remark. It is easy to see that the product $\Phi^+\,\Phi$ acts trivially on $\mathcal{C}$:

$$\Phi^+\, \Phi\, f = f\, \Phi^+\, \Phi.$$

Definition 3 *$\mathcal{L}$ is an algebra generated by components of elements $L \in \mathcal{G} \otimes \mathcal{L}$ and $L^{-1} \in \mathcal{G} \otimes \mathcal{L}$ such that*

$$\Delta(L) = R_+^{-1} \overset{2}{L} R_+ \overset{1}{L} \tag{27}$$

and L^{-1} is an inverse to L in the sense that $L^{-1} L = e \otimes e = L L^{-1}$.

The relation (27) leads to the R-matrix equation

$$R_+ \overset{1}{L} R_-^{-1} \overset{2}{L} = \overset{2}{L} R_+ \overset{1}{L} R_-^{-1}. \tag{28}$$

Relations of this form were found to describe the quantum enveloping algebras of simple Lie algebras [12]. Under certain technical assumptions it was shown in [13] that the algebras $\mathcal{L}$ and $\mathcal{G}$ are isomorphic.

Proposition 1 (Embedding of $\mathcal{L}$ into $\mathcal{V}$) *There exists an embedding (or injective *-homomorphism) $\rho : \mathcal{L} \mapsto \mathcal{V}$ of algebra $\mathcal{L}$ into the algebra $\mathcal{V}$. Explicitly it is given by the formula*

$$\rho(L) = \Phi^{-1} D \Phi \quad .$$

Remark. The same construction was considered in [5] but our statement is stronger (because (28) follows from (27) but not vice versa).

Proof.

$$R_+ \Delta(\rho(L)) \;=\; R_+ \overset{1}{\Phi}{}^{-1} \overset{2}{\Phi}{}^{-1} F^{-1} \Delta(D) F \overset{2}{\Phi} \overset{1}{\Phi} =$$

$$= \overset{2}{\Phi}{}^{-1} \overset{1}{\Phi}{}^{-1} \mathcal{R}_+ \Delta_F(D) \overset{2}{\Phi} \overset{1}{\Phi} = \overset{2}{\Phi}{}^{-1} \overset{1}{\Phi}{}^{-1} \overset{1}{\sigma} (\overset{2}{D}) \mathcal{R}_+ \overset{2}{\sigma} (\overset{1}{D}) \overset{2}{\Phi} \overset{1}{\Phi} =$$

$$= \overset{2}{\Phi} \overset{2}{D} \overset{1}{\Phi}{}^{-1} \mathcal{R}_+ \overset{2}{\Phi} \overset{1}{D} \overset{1}{\Phi} = \overset{2}{\Phi}{}^{-1} \overset{2}{D} \overset{2}{\Phi} R_+ \overset{1}{\Phi}{}^{-1} \overset{1}{D} \overset{1}{\Phi} = \rho(\overset{2}{L}) R_+ \rho(\overset{1}{L}).$$

Here we used the equality

$$\mathcal{R}_+ \Delta_F(D) = \overset{1}{\sigma} (\overset{2}{D}) \mathcal{R}_+ \overset{2}{\sigma} (\overset{1}{D}),$$

which is a consequence of (8)–(9) and (12). □

To conclude this section we make some abstract objects introduced above more concrete for the case $\mathcal{G} = U_q(sl(n))$. Let us denote the highest weight vector which labels the irreps as $\vec{J}$ and let $\{\lambda\}$ is the set of vectors corresponding to the "magnet numbers". Introduce the matrix-valued map $\sigma^J : \mathcal{C} \mapsto V^J \otimes \mathcal{C}$ (V^J is a carrier space of the irreps)

$$\sigma^J(p)_{\lambda',\lambda} \;=\; (p+\lambda)\, \delta_{\lambda',\lambda}, \tag{29}$$

where the sum is understood as an addition on the weight lattice. The relation (29) gives an exact expression for the map σ (6) taken in fixed representation.

Now let H_i, $i = 1, .., n$, $\sum_i H_i = 0$ are the generators of the Cartan subalgebra of $U_q(sl(n))$ (in standard basis they have the form $\rho^K(H_i) = \frac{1}{2}(E^K_{i,i} - E^K_{i+1,i+1})$); one can consider them as components of vector $\vec{H}$. Denote by $\vec{p}$, $p_i \in \mathcal{C}$ the corresponding weight vector (it is convenient to take it as n-component vector that is as vector on $(n-1)$-dimensional simplex $\sum_i p_i = 0$ in n-dimensional space.) In these notations the map (29) takes the form

$$\sigma(\vec{p}) \;=\; e \otimes \vec{p} + \vec{H} \otimes e. \tag{30}$$

For the case of $\mathcal{G} = U_q(\mathcal{J})$ there exists the following exact expression for the ribbon element [3] : $v_{\vec{J}} = q^{2(\vec{J}, \vec{J}+2\vec{\rho})}$, where $\vec{J}$ stands for the highest weight and $\vec{\rho}$ is the sum of simple roots. Introducing the variable $\vec{p} = 2\vec{J} + \vec{\rho}$, we get

$$v_{\vec{p}} \;=\; q^{(\vec{p}^2 - \vec{\rho}^2)/2}. \tag{31}$$

Substituting (30)–(31) into (12) and using the property $\vec{H}^2 = \frac{1}{2}e$, we found the exact form of D (in the fixed representation ρ^K):

$$D^K(\vec{p}) \;=\; q^{1/4}\, v_K^{-1}\, q^{\vec{H}_K \otimes \vec{p}}. \tag{32}$$

Remark. This form of D enables us to introduce the matrix Q :

$$Q^K(\vec{\xi}) \;=\; e^{i\,\vec{H}_K \otimes \vec{\xi}},$$

where ξ_i are the variables conjugated to p_i: $[p_j, \xi_k] = -i\hbar\, \delta_{jk}$. It is easy to see that D^K and Q^K form a matrix Weyl pair that is $D^K\, Q^K = const\, Q^K\, D^K$. In [5, 9] it was drawn attention to the important role of the matrix Q.

Remark. An explicit prescription for the constructing of the another basic object $F(p)$ is less simple (in particular, it uses the notion of fusion matrix or, equivalently, quantum $6j$-symbol) and we will describe it somewhere else.

3 Theory on the lattice.

In the previous paragraph we have described a toy model for the theory of q-vertex operators. Of course, the real theory of q-vertex operators deals not with a quantum Lie algebra $\mathcal{G}$ but with a quantum affine algebra $\widehat{\mathcal{G}}$. Now we are going to extend our model theory to the case of lattice current algebra. Here we give only the first step of this procedure; more details will be given somewhere else.

Let us consider one-dimensional closed chain which consists of N sites. Let $\mathcal{G}$ denotes a ribbon quasitriangular Hopf algebra and let R denotes its R-matrix. Following [14, 15], one defines the lattice Kac-Moody algebra $\mathcal{K}_N$:

Definition 4 *The algebra $\mathcal{K}_N$ is an associative algebra generated by components of the elements J_n, J_n^{-1}, N_n, $N_n^{-1} \in \mathcal{G} \otimes \mathcal{K}_N$ such that*

$$J_n\, J_n^{-1} = J_n^{-1}\, J_n = e \otimes e, \qquad N_n\, N_n^{-1} = N_n^{-1}\, N_n = e \otimes e,$$

$$\Delta\, J_n \;=\; R'\; \overset{2}{J}_n\, \overset{1}{J}_n, \tag{33}$$

$$\Delta\, N_n \;=\; R^{-1}\; \overset{1}{N}_n\;\, R\;\, \overset{2}{N}_n, \tag{34}$$

$$\overset{1}{J}_n\;\, R'\;\, \overset{2}{J}_{n+1} = \overset{2}{J}_{n+1}\, \overset{1}{J}_n, \tag{35}$$

$$\overset{1}{J}_n\, \overset{2}{N}_n = \overset{2}{N}_n\;\, R\;\, \overset{1}{J}_n\;\, R', \qquad \overset{1}{J}_n\, \overset{2}{N}_{n-1} = R\;\, \overset{2}{N}_{n-1}\;\, R'\;\, \overset{1}{J}_n, \tag{36}$$

$$|n-m| \neq 0 \;\;\Longrightarrow\;\; \overset{1}{N}_n\, \overset{2}{N}_m = \overset{2}{N}_m\, \overset{1}{N}_n,$$

$$|n-m| \geq 2 \;\;\Longrightarrow\;\; \overset{1}{J}_n\, \overset{2}{J}_m, = \overset{2}{J}_m\, \overset{1}{J}_n,$$

where $n \in \mathbf{Z}$ mod N.

As explained above , the relation (34) leads to R-matrix relations of type (28) which define an algebra isomorphic to $\mathcal{G}$. Thus, each site of the lattice is equipped with a copy of the algebra $\mathcal{G}$ and those copies for different sites commute.

The relation (33) leads to the following R-matrix relations

$$R' \overset{2}{J}_n \overset{1}{J}_n = \overset{1}{J}_n \overset{2}{J}_n \; R, \tag{37}$$

$$R \; \overset{1}{J}_n \overset{2}{J}_n = \overset{2}{J}_n \overset{1}{J}_n \; R'. \tag{38}$$

These equations together with (35) define the lattice deformation of current algebra [14, 15, 16]. In the continuous limit $q \to 1$, $N \to \infty$, $x = n/N \in [0, 2\pi]$ they transfer into the usual relations for Kac-Moody algebra. It is natural to think that the current J_n is associated with the edge between n-th and $(n+1)$-th sites.

It is convenient to introduce "partial monodromies" U_n :

$$U_n \; = \; J_1 \ldots J_n, \quad n = 1, .., N-1. \tag{39}$$

Lemma 1 *The following relations hold for the "partial monodromies"*

$$\Delta U_n \; = \; R' \; \overset{2}{U}_n \overset{1}{U}_n, \tag{40}$$

$$R' \; \overset{2}{U}_n \overset{1}{U}_n = \overset{1}{U}_n \overset{2}{U}_n \; R, \qquad R \; \overset{1}{U}_n \overset{2}{U}_n = \overset{2}{U}_n \overset{1}{U}_n \; R', \tag{41}$$

$$n > m \quad \Longrightarrow \; \overset{1}{U}_n \overset{2}{U}_m = \; R' \; \overset{2}{U}_m \overset{1}{U}_n, \tag{42}$$

$$n < m \quad \Longrightarrow \; \overset{1}{U}_n \overset{2}{U}_m = \; R^{-1} \; \overset{2}{U}_m \overset{1}{U}_n \; . \tag{43}$$

Proof. (40) follows from (34) and (35); (41) follow from (40); to check (42) one can represent U_n as $U_n = U_m \, J_{m+1} \ldots J_n$ and make use of the relations (41) and (35); (43) can be checked similarly. $\square$

Let $\mathcal{V}$ stands for the algebra of q-vertex operators introduced above. Following the ideology of lattice models, we can extend $\mathcal{V}$ up to the algebra of vertex operators on a lattice $\mathcal{V}_{lat}$ which contains $\mathcal{V}$ as a subalgebra.

Definition 5 *The algebra $\mathcal{V}_{lat}$ is generated by components of the elements Φ_n, $\Phi_n^{-1} \in \mathcal{G} \otimes \mathcal{V}_{lat}$, where $\Phi_1 \in \mathcal{G} \otimes \mathcal{V}$ and*

$$\Phi_n \; = \; \Phi_1 \, U_n, \tag{44}$$

$$\overset{1}{\Phi}_1 \; R' \; \overset{2}{J}_1 = \overset{2}{J}_1 \overset{1}{\Phi}_1, \qquad \overset{1}{\Phi}_1 \; R \; \overset{2}{J}_N = \overset{2}{J}_N \overset{1}{\Phi}_1, \tag{45}$$

$$n \neq 1, N \quad \Longrightarrow \; \overset{1}{\Phi}_1 \overset{2}{J}_n = \overset{2}{J}_n \overset{1}{\Phi}_1 \; . \tag{46}$$

Let us describe some properties of the introduced objects.

Proposition 2 *Φ_n has the co-product of type (22) :*

$$\Delta(\Phi_n) \; = \; F \; \overset{2}{\Phi}_n \overset{1}{\Phi}_n \tag{47}$$

and the R-matrix relations of type (23) :

$$\mathcal{R} \; \overset{2}{\Phi}_n \overset{1}{\Phi}_n = \overset{1}{\Phi}_n \overset{2}{\Phi}_n \; R. \tag{48}$$

Proof. (47) can be verified directly with the help of (22), (40) and (45):

$$\Delta(\Phi_n) \;=\; \Delta(\Phi_1\, U_n) \;=\; \Delta(\Phi_1)\,\Delta(U_n) \;=\; F\; \overset{2}{\Phi}_1 \overset{1}{\Phi}_1\; R'\; \overset{2}{U}_n \overset{1}{U}_n =$$
$$= \; F\; \overset{2}{\Phi}_1 \overset{2}{U}_n \overset{1}{\Phi}_1 \overset{1}{U}_n = \; F\; \overset{2}{\Phi}_n \overset{1}{\Phi}_n,$$

and (48) is its consequence. □

At the same time, the introduced vertex operators are nonlocal objects as shown in the next statement.

Proposition 3

$$n > m \quad \Longrightarrow \quad \mathcal{R}'\; \overset{1}{\Phi}_n \overset{2}{\Phi}_m = \overset{2}{\Phi}_m \overset{1}{\Phi}_n, \tag{49}$$

$$n < m \quad \Longrightarrow \quad \mathcal{R}\; \overset{1}{\Phi}_n \overset{2}{\Phi}_m = \overset{2}{\Phi}_m \overset{1}{\Phi}_n\; . \tag{50}$$

Proof. Using (48), (45) and (42), we find for $n > m$

$$\mathcal{R}'\; \overset{1}{\Phi}_n \overset{2}{\Phi}_m = \; \mathcal{R}\; \overset{1}{\Phi}_1 \overset{1}{U}_n \overset{2}{\Phi}_1 \overset{1}{U}_m = \; \mathcal{R}\; \overset{1}{\Phi}_1 \overset{2}{\Phi}_1\; R\; \overset{1}{U}_n \overset{2}{U}_m =$$
$$= \overset{2}{\Phi}_1 \overset{1}{\Phi}_1\; R'\; \overset{2}{U}_m \overset{1}{U}_n = \overset{2}{\Phi}_1 \overset{2}{U}_m \overset{1}{\Phi}_1 \overset{1}{U}_n = \overset{2}{\Phi}_m \overset{1}{\Phi}_n\; .$$

The relation (50) is a consequence of (49). □

Remark. The relations (49)–(50) point out that $\mathcal{R}(p)$ is an object similar to the braiding matrix in conformal field theory [17].

References

[1] Abe E. Hopf algebras. – Cambridge tracts in mathematics., No 74, Cambridge Univ. Press, 1980.

[2] Reshetikhin N.Yu., Turaev V.G. Ribbon graphs and their invariants derived from quantum groups. – Comm. Math. Phys., 1990, v.127, p. 1–26.

[3] Drinfeld V.G. On almost cocommutative Hopf algebras. – Algebra & analysis, 1989, v.1, No.2, p. 30–46.

[4] Drinfeld V.G. Quasi-Hopf algebras. – Algebra & analysis, 1989, v.1, No.6, p. 114–148.

[5] Alekseev A.Yu., Faddeev L.D. $(T^*G)_t$: a toy model for conformal field theory. – Comm. Math. Phys., 1991, v.141, No 2, p. 413–422.

[6] Gervais J-L., Neveu A. Novel triangle relation and absence of tachions in Liouville string field theory. – Nucl. Phys., 1984, v. B 238, p. 125–141.

[7] Faddeev L.D. On the exchange matrix for WZNW model. – Comm. Math. Phys., 1990, v.132, No 1, p. 131–138.

[8] Cremmer E., Gervais J-L. The quantum group structure associated with non-linearly extended Virasoro algebras. – Comm. Math. Phys., 1990, v. 134, No 3, p. 619–632.

[9] Bytsko A.G., Faddeev L.D. The q-analogue of model space and CGC generating matrices, – preprint q-alg/9508022, 1995, p. 1–28.

[10] Avan J., Babelon O., Billey E. The Gervais-Neveu-Felder equation and the quantum Calogero-Moser systems. – preprint hep-th/9505091, 1995, p. 1–13.

[11] Isaev A.P. Twisted Yang-Baxter equations for linear quantum (super) groups. – preprint IFUP-TH 62/95, 1995, p. 1–11.

[12] Reshetikhin N.Yu., Semenov-Tian-Shansky M.A. Central extensions of the quantum current groups. – Lett. Math. Phys., 1990, v.19, p.133–142.

[13] A. Yu. Alekseev, V. Schomerus, *Representation theory of Chern-Simons observables*, preprint (1995), q-alg 9503016.

[14] Alekseev A.Yu., Faddeev L.D., Semenov-Tian-Shansky M.A. Hidden quantum groups inside Kac-Moody algebras. – Comm. Math. Phys., 1992, v.149, No 2, p. 335–347.

[15] Alekseev A.Yu., Faddeev L.D., Schomerus V. Representation theory of lattice Kac-Moody algebras. – to appear.

[16] Falceto F., Gawedzki K. Lattice Wess-Zumino-Witten model and quantum groups. – J. Geom. Phys., 1993, v.11, p. 251–279.

[17] Moore G., Seiberg N. Classical and quantum conformal field theory. – Comm. Math. Phys., 1989, v.123, p. 177–254.