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EXTREMAL PROBLEMS ON CONFORMAL MODULI AND ESTIMATES FOR HARMONIC MEASURES

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ABSTRACT

We obtain numerous new estimates for the harmonic measures of continua, concerning several problems which have been considered for a long time by T. Hall, D. Gaier, J. A. Jenkins, W. K. Hayman and W. Wu, and others. We usually give a complete solution of a problem, including a description of the extremal configurations. Our approach is based on some new results in the Jenkins Module theory and in the theory of symmetrization.

Introduction

The present paper is devoted to applications of the method of extremal metrics in certain problems concerning harmonic measures. This approach is due to J. A. Jenkins who used it for solving several problems [23]–[29].

Let us consider the general statement of such a problem. Let D be a domain in $\overline{\mathbb{C}}$ and let $z \in D$. If E is a Borel subset in $\overline{D}$, the harmonic measure

$$\omega_E(z) = \omega(z, E, D)$$

will mean the harmonic measure of $E \cap \partial(D \setminus E)$ with respect to the component of $D \setminus E$ containing z. Let $\mathcal{E}$ be a family of closed sets E in $\overline{D}$ determined by some conditions.

Let D, $z \in D$, and $\mathcal{E}$ be given. The problem is to compute

$$\inf \omega_E(z), \tag{1}$$

or

$$\sup \omega_E(z), \tag{2}$$

under the condition

$$E \in \mathcal{E} \tag{3}$$

and to find all extremizers of (1) (or (2)) in $\mathcal{E}$.

The condition (3) is usually given in geometrical terms. Specifying D, z, and $\mathcal{E}$, one comes to several well-known problems. Due to conformal invariance of the harmonic measure we may identify D with a canonical domain. Thus in the simply connected case we usually deal with the unit disk $U = \{z : |z| < 1\}$ or with the upper half-plane $H = \{z : \operatorname{Im} z > 0\}$, we also write $U_\pm$ and $H_\pm$ for $\{z \in U : \pm\operatorname{Im} z > 0\}$ and $\{z \in H : \pm\operatorname{Re} z > 0\}$.

Perhaps the first such problem was the Carleman–Milloux problem (Problem CM). In our notation it is the problem (1), (3) where $D = U$, $z = 0$, and $\mathcal{E}$ is the family of compacts $E \subset \overline{U}$ with given circular projection E_{rad} onto $\mathbb{R}_+$ (here $\mathbb{R}_\pm = \{x \in \mathbb{R} : \pm x > 0\}$). By the well-known Beurling's result [4] we have

$$\omega(0, E, U) \geq \omega(0, E_{\mathrm{rad}}, U).$$

The similar problem for compacts in $\overline{H}$ is due to T. Hall. We call it Problem H. For continua touching the real axis $\mathbb{R}$, Problem H was considered by D. Gaier [13] and by Jenkins [25]. In accordance with their results the inequality

$$\omega(z, E, H) \geq \omega(z, E_{\mathrm{rad}}, H) \tag{4}$$

holds true for every continuum E in $\overline{H}$ such that $0 \in E$ and for each z in $\overline{H}_+$.

Theorem 7 below shows that (4) still holds even without the additional restrain $0 \in E$.

W. K. Hayman and W. Wu used in [16] an estimate for $\omega(z, E, H)$ given in terms of the orthogonal projection E_x of E onto $\mathbb{R}$. That is why by Problem HW we mean the problem (1), (3) where $D = H$, $z \in H$, $a \leq \operatorname{Re} z \leq b$, and $\mathcal{E}$ is a family of continua which separate z from the segment $[a, b]$ inside the semistrip $\Pi(a, b) = \{z \in H : a < \operatorname{Re} z < b\}$, $-\infty \leq a, b \leq \infty$. It seems a natural conjecture that the segment $[a, b]$ minimizes the harmonic measure in Problem HW. We shall prove this in Theorem 6.

The problem dual to CM deals with the radial projection E_{circ} of a compact $E \subset \overline{U}$ onto the circle $\mathbb{T}$ where $\mathbb{T} = \mathbb{T}_1$ and $\mathbb{T}_r = \{z : |z| = r\}$. In the case $E_{circ} = \mathbb{T}$ this is an open problem raised by W. H. J. Fuchs [3, Problem 3.22]. A special case of Fuchs' problem when E is a continuum in $\overline{U}$ was considered by Jenkins in [27], where a class of extremal continua was characterized. For continua γ a natural condition in this problem is a restriction of the index $\operatorname{ind}_0 \gamma$ of γ about the origin (the precise definition will be given below). We call this case "Problem FJ".

In the case $\operatorname{ind}_0 \gamma = 1$, D. E. Marshall and C. Sundberg [33] determined the essentially unique extremal continuum and computed the minimal value of the harmonic measure. Jenkins [29] gave a much simplified derivation of this solution. In our Theorem 8 we give the solution of Problem FJ for continua of an arbitrary half-integer index $n/2,\ n \in \mathbb{Z}$. These results were presented at the conference "Boundary Problems and Entire Functions " (Krasnodar, 1988), dedicated to the memory of Professor N. V. Govorov in his sixtieth birthday but never have been published. Of course, in the case $n = 2$ we have the Marshall–Sundberg theorem.

If the index of γ is no greater than π, then by the theorem of C. FitzGerald, B. Rodin, and S. Warschawski [11]

$$\omega(0, \gamma, U) \geq \omega(0, \gamma_{\mathrm{circ}}, U). \tag{5}$$

In [11] the authors pose the question on the maximal number φ_0 such that (5) remains true for all continua γ satisfying the condition

$$|\operatorname{ind}_0 \gamma| \leq \varphi_0.$$

Here we will give only an estimate for φ_0 and solve the local version of this problem called "Problem FRW" below. But our method gives a complete qualitative solution of the problem and allows one to reduce it to numerical computations. We are going to publish these results in the second paper on this subject.

The estimate (5) is closely connected with the Rodin problem on the harmonic measure of continua $E \subset \overline{U}$ having a fixed diameter. This problem was solved by FitzGerald, Rodin, and Warschawski [11], by Jenkins [28], and by the author [37]. Unfortunately the original text was distorted under the publication in [37].

An other type of problems on the harmonic measure was first considered by Gaier [12]. By Problems G1 and G2 we mean the problem (1), (3) where $D = U,\ z = 0$, and $\mathcal{E}$ consists of all continua E in $\overline{U}$ containing a given system of points $A = \{a_1, \dots, a_n\}$ and such that $E \cap \mathbb{T} \neq \emptyset$ in the case of Problem G1. Here $A \subset \overline{U}$ in Problem G1 and $A \subset U$ in Problem G2. Gaier himself considered only the cases $A = \{1, a\}$ and $A = \{a, b\},\ |a| < 1,\ |b| < 1$. In the first of them the complete solution was found by Jenkins [24]. The second case was studied by F. Huckemann [18], L. Liao [32], S. Huckemann [19], J. A. Hummel and B. Pinchuk [20], but the question of the uniqueness of the extremal continuum (this is one of the main questions in such problems) remains open.

F. Huckemann and Liao considered Problem G2 under the following additional assumption: continua $E \in \mathcal{E}$ have fixed hyperbolic capacity. The analogous condition for Problem G1 is as follows: the connected component $\Delta(E)$ of $U \setminus E$ such that $0 \in \Delta(E)$ has fixed conformal radius at the origin: $R(\Delta(E), 0) = r,\ 0 < r \leq 1$. Here we will give the complete solution of the latter problem, called "G3" below. This solution involves proving the monotone property of the harmonic measure as a function of the parameter r.

A well-known problem on the harmonic measure of a system of radial slits due to A. A. Gonchar gives an example of a problem (2), (3) with $D = U,\ z_0 = 0$, and $\mathcal{E} = \{\cup_{k=1}^{n} E(\theta_k)\}$, where $E(\theta_k) = e^{i\theta_k} E,\ \theta_k \in \mathbb{R}$, and E is a given compact in $(0, 1]$. In the simplest but the most interesting case when $E = [r, 1],\ 0 < r < 1$, this problem was solved by V. N. Dubinin [5]. For the general case see [1,8].

Our proofs are based on the application of the extremal metric method combined with certain kinds of symmetrizations. The necessary results and some auxiliary assertions concerning these methods are given in Section 1. Problems G1 and G3 are considered in Section 2. In Sections 3 and 4 we give solutions of Problems HW and H and Problem FJ respectively. In Section 5 we get some results concerning Problem FRW, some open questions are also considered.

1. Preliminaries

1.1. Module problems. We restrict our attention to the plane case of the Jenkins Module problem, see [21]. This case was considered in detail by G. V. Kuz'mina [30]. We closely follow the definitions and notation of [21,22,30].

Let $a_1, \dots, a_n$ and $b_1, \dots, m$ be distinct points on $\overline{\mathbb{C}}$, $n+2m \geq 4$, $n+m \geq 3$. By an admissible family of domains associated with a free family of homotopy classes $\Gamma_1, \dots, \Gamma_p$ of closed curves on $\mathbb{C}' = \overline{\mathbb{C}} \setminus \{a_1, \dots, a_n, b_1, \dots, b_m\}$ we mean an ordered $(p+m)$-tuple of nonoverlapping domains $D_1, \dots, D_{p+m}$ such that:

(i) D_j is a doubly connected domain associated with the nontrivial class Γ_j on $\mathbb{C}'$ if $1 \leq j \leq p$;

(ii) D_{p+k} is a simply connected domain such that $b_k \in D_{p+k} \subset \mathbb{C}' \cup \{b_k\}$ if $1 \leq k \leq m$.

For a given $\alpha_j \geq 0$, $j = 1, \dots, p+m$, we look for the maximum $\mathcal{M} = \mathcal{M}(\alpha_1, \dots, \alpha_{p+m}, a_1, \dots, a_n, b_1, \dots, b_m)$ of the sum

$$\sum_{j=1}^{p} \alpha_j^2 \operatorname{mod}(D_j) + \sum_{j=1}^{m} \alpha_{p+j}^2 \operatorname{mod}(D_{p+j}, b_j) \tag{1.1}$$

among all admissible families $\{D_1, \dots, D_{p+m}\}$ associated with a given free family of homotopy classes. Here and below $\operatorname{mod}(D)$ denotes the module of a doubly connected domain (or a quadrangle) with respect to the family of curves separating its boundary components (or joining its distinguished sides), $\operatorname{mod}(D, z)$ denotes the reduced module of a simply connected domain D at the point $z \in D$; $\operatorname{mod}(D, z) = (1/2\pi) \log R(D, z)$.

By the general result of Jenkins [21] (see also [30, Theorem 0.1]) there exists a unique extremal configuration corresponding to the quadratic differential

$$Q(z)\, dz^2 = P(z) \prod_{j=1}^{n} (z - a_j)^{-1} \prod_{l=1}^{m} (z - b_l)^{-2}\, dz^2, \tag{1.2}$$

whose trajectory structure consists entirely of ring domains D_j^*, $j = 1, \dots, p$ (some of them possibly being degenerate) and of circle domains D_{p+l}^* centered at b_l, $l = 1, \dots, m$. Here $P(z)$ is a polynomial of degree $\leq n + 2m - 4$.

The dependence of the maximum $\mathcal{M}$ on the parameters is described by gradient theorems [36,9]. In this paper we use only the following expression for the gradient of $\mathcal{M} = \mathcal{M}(a_j)$ as a function of a simple pole a_j:

$$\operatorname{grad} \mathcal{M}(a_j) = c \lim_{z \to a_j} \overline{(z - A_j) Q(z)}, \tag{1.3}$$

c is a positive constant.

The relation (1.3) defines the direction of the maximal growth of $\mathcal{M}$ and is very important for applications. According to (1.3) the direction of the gradient coincides with the tangent to the corresponding critical trajectory of $Q(z)\, dz^2$. Initially this simple principle appeared without proof in [35], after that its proof was given independently by the author [36] and by E. G. Emel'janov [9]. We have $c = 1$ in (1.3) if $Q(z)\,dz^2$ is normalized by the following condition:

$$|l|_Q \overset{def}{=} \int_l |Q|^{1/2}\,|dz| = \alpha_j$$

for any regular trajectory $l \in \Gamma_j,\ j = 1, \dots, p+m$. We will demonstrate the situation concerning the gradient formula in Figures 1a and 1b corresponding to Lemma 2 and Lemma 3 respectively.

Now we consider the well-known Teichmüller problem on the maximum of the module of doubly connected domains which reduces to one of the simplest special cases of the Jenkins Module problem (see, for example, [30, Ch.5]).

Let $\mathcal{M}(a_1, a_2; b_1, b_2)$ denote the maximal module in the family of doubly connected domains separating pairs of points a_1, a_2 and b_1, b_2 and let $\mathcal{M}(a_1, a_2) = \mathcal{M}(a_1, a_2; -1, 1)$. The function $\mathcal{M}(a_1, a_2)$ (especially in the case $a_1 = \infty,\ a_2 = a$) is widely used in the theory of conformal and quasiconformal mappings. Now many properties of $\mathcal{M}(a_1, a_2)$ are known, for instance, monotonicity along certain curves (see [30,44]). The next three lemmas connected with research in [44] give new information of this kind. These results will be used in Section 3.

Lemma 1. *Let $a_1 \in H_- \cup \mathbb{R}_- \setminus \{-1\}$. Then*

$$\mathcal{M}(a_1, -\overline{a}_2) < \mathcal{M}(a_1, a_2)$$

for all $a_2 \in H_+ \cup \mathbb{R}_+ \setminus \{1\}$.

Proof. Let $w = \varphi(z)$ be a conformal or anticonformal automorphism of $\overline{\mathbb{C}}$ such that

$$\varphi(1) = -\varphi(-1) = 1,\ \varphi(a_2) = -\varphi(-\overline{a}_2) = k > 1,\ \varphi(a_1) = \zeta,\ \operatorname{Im}\zeta \geq 0.$$

Let us consider the function

$$\psi(\zeta) = \mathcal{M}(k, \zeta) - \mathcal{M}(-k, \zeta).$$

As follows from Theorem 5.8 in [30] $\psi(\zeta)$ is a harmonic function bounded below in H_+. Clearly, $\psi(\zeta) > 0$ if $\zeta > 1$ and $\psi(\zeta) = 0$ if $\operatorname{Re}\zeta = 0$.

Let $0 < \zeta < 1$. Using the mapping

$$\tau(w) = (1 - \zeta w)/(w - \zeta)$$

we obtain

$$\psi(\zeta) = \mathcal{M}((1 - k\zeta)/(k - \zeta),\ \infty) - \mathcal{M}((1 + \zeta k)/(k + \zeta),\ \infty).$$

It follows from the relation (3.21) in [44] that the inequality $|(1-k\zeta)/(k-\zeta)| < (1+\zeta k)/(k+\zeta)$, being true for $k > 1$ and $0 < \zeta < 1$, yields $\psi(\zeta) > 0$. Therefore, by the maximum principle for harmonic functions

$$\psi(\zeta) > 0 \quad \text{if} \quad \zeta \in H_+,$$

and the proof is complete. $\square$

Lemma 2. *Let a_1 be fixed in $\overline{H}_- \setminus \{-1\}$ and let $a_2 = re^{i\theta}$. For fixed $r > 0$ the function $\mathcal{M}(a_1, re^{i\theta})$ increases in $0 \le \theta \le \pi/2$.*

Proof. Let the quadratic differential

$$Q(z)\,dz^2 = A(a_2)[(z^2-1)(z-a_1)(z-a_2)]^{-1}\,dz^2 \tag{1.4}$$

be associated with the problem on the quantity $\mathcal{M}(a_1, a_2)$ considered as a function of a_2. If $\varphi(a_2) = \arg \operatorname{grad} \mathcal{M}(a_1, a_2)$ (see Figure 1a) then by (1.3), (1.4)

$$\varphi(a_2) = \arg[(a_2^2 - 1)(a_2 - a_1)/A(a_2)].$$

Using the relation 5.8 of Theorem 5.8 in [30] one can see that $\arg[A(a_2)]$ is a bounded function harmonic in H_+, as well as the function

$$\psi(a_2) = \varphi(a_2) - \arg a_2.$$

Let γ_1 denote the trajectory of $Q(z)\,dz^2$ joining the poles a_1 and a_2. Using the polarization of a doubly connected domain with respect to $\mathbb{R}$ (see [42] or Section 1.2 for such a transformation of quadrangles and triangles) we see at once that $\gamma_1 \subset \overline{H}$. Therefore, for $a_2 = x \ge 0,\ x \ne 1,\ \gamma_1$ starting from a_2 goes into the upper half-plane. Hence for such x

$$0 \le \psi(x) \le \pi.$$

Let $a_2 = ih,\ h \ge 0$. By Lemma 1 the inequality

$$\mathcal{M}(a_1, ih + x) \le \mathcal{M}(a_1, ih - x). \tag{1.5}$$

holds for all $x > 0$.

Hence

$$\frac{\partial \mathcal{M}(a_1, ih + x)}{\partial x}\Big|_{x=0} \le 0\,.$$

This means that γ_1 starting from a_2 goes into the half-plane $\{z : \operatorname{Re} \le 0\}$. Therefore,

$$0 \le \psi(ih) \le \pi \quad \text{if} \quad h > 0.$$

Now the maximum principle shows that

$$0 < \psi(a_2) = \varphi(a_2) - \arg a_2 < \pi$$

for all $a_2 \in H_+$ and the lemma follows. $\square$

Lemma 3. *Let $h > 0$ and $r > 0$ be fixed. Then $\mathcal{M}(ih, ih + re^{i\theta})$ is an increasing function of θ for $-\arctan h \le \theta \le \pi/2$.*

The proof follows by the same method as in Lemma 2 but without references to the inequality (1.5).

1.2. Connected polarization of certain configurations. Here we extend the usual symmetrization technique to some cases used below. We refer for the definitions and general

FIGURE 1: TEICHMÜLLER'S PROBLEM. MONOTONICITY OF SOLUTIONS

properties of the symmetrization and polarization to Dubinin's papers [7,8], some concretizations of the general symmetrization and polarization results are given in [40,41,42].

(a) Quadrangles. Let G be a quadrangle in an annulus $K(r_1, r_2) = \{z : r_1 < |z| < r_2\}$, with distinguished sides τ_1 and τ_2 on the circles $\mathbb{T}_{r_1}$ and $\mathbb{T}_{r_2}$. Let G^*, τ_1^*, and τ_2^* be the results of polarization of G, τ_1, and τ_2 with respect to the line $p(\theta) = \{z = te^{i\theta}, \ -\infty < t < \infty\}$. The set G^* contains the unique connected component, say G', which intersects the half-plane $H(\theta) = e^{i\theta} H$. Let G^θ be the smallest simply connected domain in $K(r_1, r_2)$ containing G'.

By the connected polarization of G with respect to $p(\theta)$ we mean the domain G^θ considered as a quadrangle with distinguished sides τ_1^* and τ_2^*.

(b) Continua. Let E be a continuum in $\overline{K(r_1, r_2)}$ such that $K(r_1, r_2)\backslash E$ is simply connected. If E^* is the polarization of E with respect to $p(\theta)$ and E' is a connected component of E^* intersecting $\overline{H}(\theta)$, then by the connected polarization of E with respect to $p(\theta)$ we mean the smallest continuum E^θ in $\overline{K(r_1, r_2)}$ containing E' and such that $K(r_1, r_2)\backslash E^\theta$ is simply connected.

(c) Pairs of quadragles. Let G_1 and G_2 be nonoverlapping quadrangles in $K(r_1, r_2)$ and let G_k have distinguished sides $s_{1,k}$ and $s_{2,k}$ on the circles $\mathbb{T}_{r_1}$ and $\mathbb{T}_{r_2}$ respectively, $G_k' = G_k \cup s_{1,k} \cup s_{2,k}$. Let E be one of the two connected components of $\overline{K(r_1, r_2)} \setminus G_1' \setminus G_2'$; let $s_{k,3} = E \cap \mathbb{T}_{r_k}$, $k = 1, 2$. Consider the quadrangle $G' = G_1 \cup G_2 \cup (E \setminus s_{1,3} \setminus s_{2,3})$ with distinguished sides $\tau_k = s_{1,k} \cup s_{2,k} \cup s_{3,k}$, $k = 1, 2$.

Let G^θ and E^θ be the connected polarization of G' and E with respect to $p(\theta)$. The quadrangle G^θ has its sides, say τ_k^θ, on the circles $\mathbb{T}_{r_k}$, $k = 1, 2$. The set $G^\theta \setminus E^\theta$ consists of two components G_1^θ and G_2^θ considered below as quadrangles having the corresponding components of the sets $\tau_1^\theta \setminus E^\theta$ and $\tau_2^\theta \setminus E^\theta$ as their distinguished sides.

The pair of quadrangles G_1^θ and G_2^θ will be called the connected polarization of G_1 and G_2 with respect to $p(\theta)$.

(d) Triangles. By a triangle G we mean a simply connected domain on $\overline{\mathbb{C}}$ with three distinguished boundary points e_0, e_1, and e_2, called vertices. Let $m(G; e_0|e_1, e_2)$ denote the reduced module of G at the vertex e_0 (for the definition see [39]).

Let G have angle $\varphi > 0$ at e_0 and let $\zeta = f_G(z)$ be a conformal mapping of G onto the sector $\{\zeta : |\zeta| < 1, 0 < \arg z < \varphi\}$ such that $f(e_0) = 0$. In extremal problems considered below we deal with triangles bounded by arcs of trajectories of quadratic differentials. So throughout this paper we restrict ourselves by the consideration of triangles G such that the mappings $f_G(z)$ are conformal at the vertex e_0. This implies the existence of all reduced moduli considered below.

Let a triangle G in U have a vertex e_0 at the origin and a side e_1e_2 on $\mathbb{T}$. We shall assume that $\gamma_\varepsilon = \mathbb{T}_\varepsilon \cap G$ is connected for all $\varepsilon > 0$ small enough. Let $G(\varepsilon)$ be the quadrangle $G \setminus \overline{U}_\varepsilon$ having distinguished sides e_1e_2 and γ_ε and let $G^\theta(\varepsilon)$ and $e_1^*e_2^*$ be the connected polarization of the quadrangle $G(\varepsilon)$ and the side e_1e_2 with respect to $p(\theta)$.

By the connected polarization of the triangle G with respect to $p(\theta)$ we shall mean the triangle $G^\theta = \cup_{\varepsilon>0} G^\theta(\varepsilon)$ having the vertex $e_0 = 0$ and the opposite side $e_1^*e_2^*$.

(e) Pairs of triangles. Using the definition of the connected polarization of a pair of quadrangles one can define in the manner of the part (d) the connected polarization of a pair of nonoverlapping triangles G_1, G_2 which are similar to the triangle G considered in (d). We need this kind of polarization only to prove the assertion on the case of equality in the relation (1.8) below.

Remark 1. In the same manner as in (a – e) one can define the connected polarization of a bilateral G with vertices $e_1 = 0$ and $e_2 = \infty$ and all kinds of the connected polarizations for the configurations laying in a strip or in a half-plane instead of an annulus or disk.

Lemma 4. *Let G, $\{G_1, G_2\}$, and Ω be a quadrangle, a pair of quadrangles, and a triangle described in (a), (c), and (d) respectively and let G^θ, $\{G_1^\theta, G_2^\theta\}$, and Ω^θ be the corresponding connected polarization. Then*

$$\operatorname{mod}(G) \le \operatorname{mod}(G^\theta), \tag{1.6}$$

$$\operatorname{mod}^{-1}(G_1) + \operatorname{mod}^{-1}(G_2) \ge \operatorname{mod}^{-1}(G_1^\theta) + \operatorname{mod}^{-1}(G_2^\theta), \tag{1.7}$$

$$m(\Omega; 0|e_1, e_2) \ge m(\Omega^\theta; 0|e_1^*, e_2^*). \tag{1.8}$$

Equality in (1.6), (1.7), or in (1.8) occurs iff G^θ, $\{G_1^\theta, G_2^\theta\}$, or Ω^θ coincides (up to the symmetry with respect to $p(\theta)$) with G, $\{G_1, G_2\}$, or Ω respectively.

Proof. The inequality (1.7) immediately follows from Dubinin's Theorem 1.1 [8] and from the definition of the connected polarization given in part (a).

To prove (1.6) consider the level curve γ which joins distinguished sides of G and divides G into two quadrangles G_1 and G_2 such that

$$\operatorname{mod}(G_1) = \operatorname{mod}(G_2) = (1/2)\operatorname{mod}(G).$$

Let $\{G_1^\theta, G_2^\theta\}$ be the connected polarization of $\{G_1, G_2\}$ with respect to $p(\theta)$. The quadrangles G_1^θ and G_2^θ form a partition of G^θ. Using Theorem 3 of [39] and (1.7) we obtain the desired inequality,

$$\begin{aligned}\operatorname{mod}^{-1}(G^\theta) &\le (1/4)(\operatorname{mod}^{-1}(G_1^\theta) + \operatorname{mod}^{-1}(G_2^\theta)) \\ &\le (1/4)(\operatorname{mod}^{-1}(G_1) + \operatorname{mod}^{-1}(G_2)) = \operatorname{mod}^{-1}(G).\end{aligned}$$

The equality statement of Theorem 3 of [39] and that of the relation (1.7) imply the assertion on the case of equality in (1.6).

Inequality (1.8) immediately follows from (1.6). To prove the equality statement we consider the conformal mapping $\zeta = f(z)$ from G onto the semistrip $\Pi(-1,1)$ such that $f(0) = +i\infty,\ f(\widehat{e_1 e_2}) = [-1,1]$. For $0 < t < 1$ we consider the triangles $G_{0,t},\ G_{1,t}$, and $G_{2,t}$ which are preimages of the semistrips $\Pi(-t,t)$, $\Pi(t,1)$, and $(-1,-t)$ respectively (see Figures 2a, 2b). Let $m(\cdot, z)$ denote the reduced module of the corresponding triangle at the point z. By Theorem 3 of [39]

$$m(G,0) = t^2\, m(G_{0,t},0) + (1/2)(1-t)^2\, m(G_{1,t},0). \tag{1.9}$$

Let $G^{\theta}_{0,t}$ and $\{G^{\theta}_{1,t},\ G^{\theta}_{2,t}\}$ denote the connected polarization of the triangle G and the pair of the triangles $G_{1,t},\ G_{2,t}$ with respect to $p(\theta)$ (see Figure 2c). Each of the triangles $G_{1,t},\ G_{2,t},\ G^{\theta}_{1,t}$, and $G^{\theta}_{2,t}$ has the same opening of the angle at the origin. So using (1.7) one obtains the inequality

$$m(G^{\theta}_{1,t},0) + m(G^{\theta}_{2,t},0) \le m(G_{1,t},0) + m(G_{2,t},0) \tag{1.10}$$

which holds for all $t,\ 0 \le t < 1$.

Theorem 3 [39], (1.9), (1.10) and the inequality

$$m(G_{0,t},0) \ge m(G^{\theta}_{0,t},0)$$

imply

$$\begin{aligned} m(G^{\theta},0) &\le t^2 m(G^{\theta}_{0,t},0) + (1/4)(1-t)^2\left(m(G^{\theta}_{1,t},0) + m(G_{2,t}\theta,0)\right) \\ &\le t^2 m(G_{0,t},0) + (1/2)(1-t^2)\, m(G_{1,t},0) = m(G,0). \end{aligned} \tag{1.11}$$

Let $\zeta = f_\theta(z)$ be a conformal mapping from G^θ onto $\Pi(-1,1)$ satisfying the conditions $f_\theta(0) = +i\infty,\ f_\theta(\widehat{e_1^* e_2^*}) = [-1,1]$.

Equality in (1.8) implies those in both inequalities (1.11). Therefore by the equality statement of Theorem 3 of [39] we have

$$f_\theta(G^{\theta}_{0,t}) = \Pi(-t,t).$$

Hence the sides of $G^{\theta}_{0,t}$ (also of $G_{0,t}$) which have the endpoint $z = 0$ are analytic curves. This fact and the definition of the polarization of a triangle imply that for all $t,\ 0 < t < 1,\ G^{\theta}_{0,t}$ coincides (up to the symmetry with respect to $p(\theta)$) with $G_{0,t}$, and the desired assertion follows. □

1.3. Harmonic measure and the module of a triad. Two results concerning harmonic measure are extensively used in the paper. The first one is the Carleman principle: if $z_0 \in D_1 \subset D_2$ and $E_1 \supset E_2$, then

$$\omega(z_0, E_1, D_1) \ge \omega(z_0, E_2, D_2). \tag{1.12}$$

The equality holds iff the domains $D_1,\ D_2$ and the sets $E_1,\ E_2$ define "essentially" the same configurations $D_k \setminus E_k,\ k = 1,2$.

To state the second result we consider (following Jenkins [22,27]) a triad (Δ, α, z_0), i.e., a configuration consisting of a simply connected domain Δ, an open border arc α of Δ, and a point $z_0 \in \Delta$. Let $\Delta(\alpha, z_0)$ be a quadrangle of the maximal module among all quadrangles G in $\Delta \setminus z_0$ having distinguished sides on $\partial\Delta \setminus \alpha$ and separating α from z_0 inside Δ. This maximal

FIGURE 2: CONNECTED POLARIZATION OF A TRIANGLE

module is connected with the harmonic measure $\omega = \omega(z_0, \alpha, \Delta)$ via the following formula due to J. Hersch [17]:

$$\operatorname{mod}(\Delta(\alpha, z_0)) = 2\mathbf{K}'(\cos(\pi\omega/2))\,/\,\mathbf{K}(\cos(\pi\omega/2)), \tag{1.13}$$

where $\mathbf{K}(\cdot)$ and $\mathbf{K}'(\cdot)$ are the complete elliptic integrals.

The relation (1.13) leads to the following lemma due to Jenkins.

Lemma 5 (on a triad). *Let D be a simply connected domain, let $z_0 \in D$, and let $\mathcal{E}$ be a family of continua E in $\overline{D} \setminus z_0$ such that $\Delta = D \setminus E$ is simply connected. Let $\Delta(E, z_0)$ be a quadrangle of the maximal module defined for the triad $(\Delta, E \cap \partial\Delta, z_0)$.*

A continuum $E_0 \in \mathcal{E}$ is extremal for Problem (1), (3) iff the quadrangle $\Delta(E_0, z_0)$ has the maximal module among all quadrangles $\Delta(E, z_0)$ such that $E \in \mathcal{E}$.

2. CONTINUA WITH A GIVEN SYSTEM OF POINTS

2.1. Consider Gaier's problem G1 in a canonical domain S which is either the disk U or the half-plane H. By S_+ we denote the semidisk U_+ if $S = U$ or the quadrant H_+ if $S = H$, $S_- = S \setminus \overline{S}_+$.

Let $A = \{a_1, \dots, a_n\}$, $n \geq 2$. We shall give a solution of Problem G1 under assumptions guaranteeing the uniqueness of the extremal continuum. Without the uniqueness statement, Theorem 1 below is still true for an arbitrary set $A \subset \overline{S}$.

Theorem 1. *Under the conditions*

$$A \subset \overline{S}_+, \qquad A \not\subset \overline{S}_- \tag{2.1}$$

there exists a unique continuum $E(A)$ realizing the infimum in (1) for Problem G1. It is the closure of a finite number of arcs on critical trajectories of a quadratic differential $Q(z)\,dz^2$ which is real on ∂S (perhaps except two points from A). In the case $S = U$

$$Q(z)\,dz^2 = \frac{P(z)}{z\prod_{k=1}^{n_1}(z-a_k)(1-\overline{a}_k z)\prod_{k=n_1+1}^{n}(z-a_k)}\,dz^2, \tag{2.2}$$

and in the case $S = H$

$$Q(z)\,dz^2 = \frac{P(z)}{(z^2-1)\prod_{k=1}^{n_1}(z-a_k)(z-\overline{a}_k)\prod_{k=n_1+1}^{n}(z-a_k)}\,dz^2, \tag{2.3}$$

$P(z)$ is a polynomial of degree $n-2$ in both cases.

The closure of the critical trajectories of $Q(z)\,dz^2$ laying in $\overline{S}$ consists of a continuum E_A and an arc γ_0 joining z_0 with ∂S. Moreover

$$E_A \subset \overline{S}_+ \tag{2.4}$$

and the set $\overline{S} \setminus E_A \setminus \gamma_0$ is a quadrangle Q_A added with a pair of its distinguished sides on ∂S.

If a quadrangle $Q \subset S$ has distinguished sides on $\partial S \setminus A$ and separates z_0 from A inside S, then

$$\operatorname{mod}(Q) \leq \operatorname{mod}(Q_A). \tag{2.5}$$

Equality in (2.5) holds iff $Q = Q_A$.

If

$$A \subset \overline{S}_+ \cap \overline{S}_-, \tag{2.6}$$

then there exist two continua E_A and $\widetilde{E}_A$ symmetric to each other with respect to the line passing throw the segment $\overline{S}_+ \cap \overline{S}_-$ which are extremal for Problem G1. These continua have the same properties as E_A in the case (2.1).

Proof. The assertion on the connection between the harmonic measure problem and the module problem follows from Lemma on a triad. Therefore via the Jenkins Module theory we have the desired description of extremal configurations. The relation (2.4) and therefore the uniqueness of the extremal continuum can easily be proved by applying the polarization transformation to corresponding quadrangles (see [42]). $\square$

2.2. The complete solution of Problem G1 for the disk U and any two points set $A = \{1, a\}$, $a \in \overline{U}_+$ was proved by Jenkins [24]. We consider a special case of this problem (called Problem G3) under the additional assumption

$$R(\Delta(E), 0) = r, \quad \text{if} \quad 0 < r \leq r(|a|), \tag{2.7}$$

where

$$r(t) = 4t/(1+t)^{-2}$$

is the conformal radius of $U \setminus [t,1]$, $0 < t \leq 1$, at the origin.

Problem G3 is connected with two problems on an extremal decomposition of U into a pair of nonoverlapping domains. The first of them is the problem on the maximum of the sum

$$t^2\, m(D,0) + m(Q), \quad 0 \leq t < \infty, \tag{2.8}$$

in the family of pairs $\{D, Q\}$, where D is a simply connected domain, $0 \in D \subset D \setminus \{a\}$, and Q is a quadrangle in $U \setminus \{a\}$ having distinguished sides on $\mathbb{T}$ and separating 0 from the points 1 and a inside U.

The second one is the problem on the maximum of the sum

$$-t^2\, m(T; 0|e_1, e_2) + m(Q), \quad 0 \leq t < \infty, \tag{2.9}$$

in the family of pairs $\{T, Q\}$ where T is a triangle in $U \setminus \{a\}$ having the angle 2π at the vertex $e_0 = 0$ and a side $\widehat{e_1 e_2}$ on $\mathbb{T}$ and Q is a quadrangle defined in problem (2.8).

For each $a = \rho e^{i\theta}$, $0 < \rho < 1$, $0 \leq \theta \leq \pi$, let us define $t_1(a) > 1$ by

$$1/t_1(a) = 1 - \frac{1}{2\pi} \int_0^{2\pi-\theta} \frac{(e^{i\varphi} - 1)\, d\varphi}{\sqrt{(e^{i\varphi} - \rho)(e^{i\varphi} - 1/\rho)}}. \tag{2.10}$$

A solution of Problem (2.8) is given in

Theorem 2. *Let $a \in U_+$ or $0 < a < 1$. In the family $\{D, Q\}$ there exists a unique pair $\{D^*, Q^*\}$ realizing the maximum of the sum (2.8). If $t \geq t_1(a)$, then the quadrangle Q^* is degenerate ($Q^* = \emptyset$) and $D^* = U \setminus [a, a/|a|]$; if $t = 0$, then $D^* = \emptyset$. For $0 < t < t_1(a)$ both D^* and Q^* are respectively a circular domain and a quadrangle nondegenerate with respect to the quadratic differential*

$$Q(z)\, dz^2 = \frac{P(z)}{z^2(z - 1(z-a)(1 - \overline{a}z)}\, dz^2. \tag{2.11}$$

The polynomial $P(z)$ has one of the following forms:

(1) $P(z) = e^{-i(\theta_1+\theta_2+\theta_3)/2}(z - e^{i\theta_1})(z - e^{i\theta_2})(z - e^{i\theta_3})$, if $1 < t < t_0$, where

$$0 < \theta_1 < \arg a < \theta_2 < \theta_3, \ \theta_2 < \pi + \arg a, \ \pi < \theta_3 < 2\pi, \quad \textit{if } a \in U_+$$

and

$$\theta_1 = 0, \ 0 < \theta_2 < \pi, \ \theta_3 = -\theta_2 \qquad \textit{if } 0 < a < 1;$$

(2) $P(z) = e^{-i(\theta_1+2\theta_2)/2}(z - e^{i\theta_1})(z - e^{i\theta_2})^2$, if $t = 1$ where

$$0 < \theta_1 < \arg a, \ \pi < \theta_2 < \pi + \arg a, \quad \textit{if } a \in U_+,$$

and

$$\theta_1 = 0, \ \theta_2 = \pi \qquad \textit{if } 0 < a < 1;$$

(3) $P(z) = e^{-i\theta/2}(z - e^{i\theta})(z - p)(1 - \overline{p}z)$, if $0 < t < 1$ where

$$0 < \theta < \arg a, \ |p| < 1, \ \pi < \arg p < \pi + \arg a, \qquad \textit{if } a \in U_+;$$

and

$$\theta_1 = 0, \ -1 < p < 0 \quad \textit{if} \quad 0 < a < 1.$$

In the case $-1 < a < 0$ there exist two extremal configurations $\{D^*, Q^*\}$ and $\{\widetilde{D}^*, \widetilde{Q}^*\}$ which are symmetric to each other with respect to the real axis. One of them, say $\{D^*, Q^*\}$, has the same properties as in the case $a \in U_+$.

The assertion of Theorem 2 follows from the well-known results of the Jenkins Module theory (cf. [30, Ch. 0], [36]) and from a result on changing in the corresponding moduli under polarization [42, Lemma 1.2].

Now we describe a set of critical trajectories of $Q(z)\,dz^2$ laying in $\overline{U}$. All possible configurations are depicted in Figures 3a – 3e. The pictures also illustrate some notation used in this Section. Figures 3f – 3h correspond with the extremal configurations of the problem (2.9) (see Theorem 3). For $1 < t < t_0$ the set is shown in Figure 3b. It consists of arcs γ_k, $k = 1, \dots, 4$, such that $\gamma_1 \subset U_+$ joins the points a and $e^{i\theta_1}$, $\gamma_2 = \{e^{i\varphi} : 0 < \varphi < \theta_1\}$, $\gamma_3 \subset U$ joins the points $e^{i\theta_2}$ and $e^{i\theta_3}$, $\gamma_4 = \{e^{i\varphi} : \theta_2 < \varphi < \theta_3\}$. In the case $t = 1$ (Figure 3c) there are only the arcs γ_1, γ_2 and γ_3.

For $0 < t < 1$ (Figure 3d) the differential $Q(z)\,dz^2$ has critical trajectories γ_1, γ_2 mentioned above. Moreover, there exist critical trajectories γ_3 and γ_4 such that $\gamma_3 \subset U$ has both end points at the point $z = p$, and $\gamma_4 \subset U_-$ joins $z = p$ with a certain point $e^{i\theta_0}$, $\pi < \theta_0 < 2\pi$.

The notation $D^*(t)$, $\gamma_k(t), \dots$ will mean dependence of the corresponding objects on the parameter t. Let

$$E(t) = \overline{\gamma}_1 \cup \overline{\gamma}_2 \quad \text{ and } D(t) = U \setminus E(t).$$

Lemma 6. *The reduced module $m(D(t), 0)$ decreases for $0 \le t \le t_1(a)$.*

Proof. Assume that there exist t_1 and t_2, $0 \le t_1 < t_2 \le t_1(a)$, such that

$$m(D(t_1), 0) = m(D(t_2, 0). \tag{2.12}$$

Let us consider the case

$$\omega_{E(t_1)}(0) \ge \omega_{E(t_2)}(0). \tag{2.13}$$

The opposite case can be considered by the same way.

Let $f(z)$ be a conformal mapping of $D(t_1)$ onto $D(t_2)$ with

$$f(0) = 0, \quad f(1) = 1. \tag{2.14}$$

By (2.13), (2.14) the pair $\{f(D^*(t_1)), f(Q^*(t_1))\}$ is admissible for the problem on the maximum of the sum (2.8). This fact, together with equality (2.7), contradicts the uniqueness of the extremal configuration in the problem considered.

Hence (2.12) is impossible and the statement of the lemma follows from the well-known inequality $m(D(t_0), 0) > m(D(t), 0)$, $0 \le t < t_1(a)$. □

Corollary 1. *Let $r_1(a) = R(D(0), 0)$. For each r, $r_1(a) \le r \le r(|a|)$, there exists a unique number $t = t(r)$, $0 \le t(r) \le t_1(a)$, such that $R(D(t(r)), 0) = r$.*

The continuum $E(t(r))$ corresponding to $t(r)$ will be denoted by E_r (see Figures 1).

Let us consider the problem on the maximum of (2.9). Its solution is given by the following theorem.

FIGURE 3: PROBLEM G3

Theorem 3. *Let $a \in U_+$ or $0 < a < 1$. In the family of pairs $\{T, Q\}$ there exists a unique pair $\{T^*, Q^*\}$ realizing the maximum of the sum (2.9). Besides, there exists a number $t_2(a)$, $0 < t_2(a) < \infty$, such that the quadrangle Q^* is degenerate for $t \geq t_2(a)$; for $-\infty < t < t_2(a)$ the extremal domains T^* and Q^* are respectively nondegenerated triangle and quadrangle associated* *with the quadratic differential $Q(z)\,dz^2$ of the form (2.11) having the numerator*

$$P(z) = e^{-i\theta/2}(z - e^{i\theta})(z - p)(z - \overline{p}z), \tag{2.15}$$

where

$$0 < \theta < \arg a,\ \ 0 < \arg p < \arg a,\ \ |p| < 1 \ \ \text{if}\ a \in U_+, \tag{2.16}$$

and

$$\theta = 0,\ \ 0 < p < a \qquad \text{if} \quad 0 < a < 1.$$

In the case $-1 < a < 0$ there exist (Theorem 2) two extremal configurations $\{T^*, Q^*\}$ and $\{\widetilde{T}^*, \widetilde{Q}^*\}$ which are symmetric to each other with respect to the real axis. One of them, say $\{T^*, Q^*\}$, has the same properties as in the case $a \in U_+$.

The set of pairs $\{T, Q\}$ is divided in a natural way into a countable number of classes associated with the homotopy classes of curves on $U\setminus\{a\}$ which join the point $z = 0$ with the punctured circle $\mathbb{T}\setminus\{1\}$. The results of [10,31] imply that in each such a class there exists a unique configuration $\{T_j^*, Q_j^*\}$, $j = 1, 2 \dots$, which maximizes the sum (2.9) in a given homotopy class. Applying the polarization transformation to the quadrangle Q_j^* and to the triangle T_j^* (cf. Section 1.2) we come to a configuration $\{T', Q'\}$ of the simplest topological structure (the curves $\gamma \in T'$ joining the vertex $z = 0$ with a side of T' laying on $\mathbb{T}\setminus\{0\}$ are homotopic to the segment $[0, -i]$ on $U\setminus\{0\}$). Using the statements on the change in the corresponding moduli under polarization (see [42, Lemma 1.2] and Lemma 4) we get the assertions on the uniqueness of the extremal configuration and on the localization of the parameters θ and p.

The existence of the "transition" value for t can be proved as in the case of the classical Module problem (see [36, Theorem 2]). To find this transition value let us consider the problem on the minimum of the reduced module $m(T; 0|e_1, e_2)$ in the family $\{T\}$ of triangles $T \subset U\setminus\{a\}$ having a vertex at $z = 0$ and a side $\widehat{e_1e_2}$ on the punctured circle $\mathbb{T}\setminus\{1\}$. This problem is just a spesial case of Problem (2.9), nevertheless it is convenient to give its solution separately.

In the following lemma and below we use the notation $U(\alpha, \beta) = \{z \in U :\ \alpha < \arg z < \beta\}$, $\mathbb{T}(\alpha, \beta) = \{z = e^{i\theta} :\ \alpha < \theta < \beta\}$.

Lemma 7. *For $a \in U_+$ there exists a unique triangle T^* realizing the minimum of the reduced module $m(T; 0|e_1, e_2)$ in the family $\{T\}$, and the following relations hold: $\overline{T}^* = \overline{U}$ and T^* is a triangle with respect to the quadratic differential $Q(z)\,dz^2$ of the form (2.11) with polynomial $P(z)$ having the form (2.15), (2.16) or the form*

$$P(z) = -e^{-i(\theta/2+\theta_1)}(z - e^{i\theta})(z - e^{i\theta_1})^2 \tag{2.17}$$

where $0 < \theta_1 \le \theta < \arg a$. In any case $Q(z)\,dz^2$ has four critical trajectories γ_k, $k = 1, \dots , 4$ in $\overline{U}$, satisfying the condition $\gamma_k \subset \overline{U(0, \theta)}$. In the cases corresponding to the relations (2.15) and (2.16), γ_1, γ_3, and γ_4 join the point p with the points $z = a$, $z = e^{i\theta}$, and $z = 0$, respectively; $\gamma_2 = \mathbb{T}(0, \theta)$. In the remaining case, γ_1 joins a and $e^{i\theta}$, γ_4 joins 0 and $e^{i\theta_1}$, $\gamma_2 = \mathbb{T}(0, \theta_1)$, $\gamma_3 = \mathbb{T}(\theta_1, \theta)$.

For $0 < a < 1$ the extremal triangle of the problem under consideration is $T^* = U\setminus[0, 1]$ with vertices at the points 0, $1 + 0i$ and $1 - 0i$.

In the case $-1 < a < 0$ there exist two extremal triangles T^* and $\widetilde{T}^*$ which are symmetric to each other with respect to $\mathbb{R}$. One of them, say T^*, has the same properties as in the case $a \in U_+$.

It is not difficult to see that all possibilities for the polynomial $P(z)$ actually realize for certain values of a.

The transition value of t is defined as:

$$t_2(a) = |\mathbb{T}(\theta, 2\pi)|_{Q_1} / (|\mathbb{T}(0, \theta)|_{Q_1} + 2|\gamma_1|_{Q_1} + 2|\gamma_3|_{Q_1})$$

if the parameters of $P(z)$ satisfy (2.16) and as

$$t_2(a) = |\mathbb{T}(\theta, 2\pi)|_{Q_1} / (|\mathbb{T}(0, \theta)|_{Q_1} + 2|\gamma_1|_{Q_1})$$

otherwise.

In particular, for $0 < a < 1$ we have

$$t_2(a) = 2\pi / \log(1/a).$$

In the general case of Theorem 3 the trajectory structure of $Q(z)\,dz^2$ is shown in Figures 1f – 1h. The quadratic differential $Q(z)\,dz^2$ has five critical trajectories γ_k, $k = 1, \dots, 5$, in $\overline{U}$, $\gamma_1 \subset U(0, \arg a)$ and joins a and $e^{i\theta}$, $\gamma_2 = \mathbb{T}(0, \theta)$, $\gamma_3 \subset U(0, \arg a)$ and joins the points 0 and p, γ_4 and γ_5 join p with the circle $\mathbb{T}$ such that the curve $\overline{\gamma}_4 \cup \overline{\gamma}_5$ separates γ_3 from γ_1 and γ_2 inside U.

Let $E_2(t) = \overline{\gamma}_1 \cup \overline{\gamma}_2$ and $D_2(t) = U \setminus E(t)$.

Lemma 8. *The reduced module $m(D_2(t), 0)$ decreases for $-\infty < t \le t_2(a)$.*

Proof. To prove the monotonicity we just repeat arguments of the proof of Lemma 6. Thus to get the desired assertion we only need to prove the inequality

$$R(D_2(t_2(a)), 0) < R(D_1(0), 0). \tag{2.18}$$

Let $g(z)$ map $D_2(t_2(a))$ onto $D_1(0)$ such that $g(0) = 0$, $g(1) = 1$. If (2.18) is false, then

$$|g'(0)| \le 1.$$

By a theorem of Jenkins [24]

$$\omega(0,\ E_2(t_2(a),\ U) > \omega(0,\ E_1(0),\ U).$$

Therefore, the triangle $g(T^*)$ is admissible for the problem on the minimal reduced module for the family $\{T\}$. Clearly, $g(T^*)$ does not coincide with the extremal triangle T^*.

Applying the lemma on the change in the reduced module of a triangle under a conformal mapping [39, Lemma 1], we obtain

$$m(g(T^*); 0|1, g(e^{i(\theta+0)}) = m(T^*; 0|1, e^{i(\theta+0)}) + (1/2\pi)\log|g'(0)|,$$

which contradicts the extremality of T^*. $\square$

Corollary 2. *Let $r_2(a) = R(D_2(t_2(a)), 0)$. For each r, $r_2(a) \le r < r_1(a)$, there exist a unique value $t = t(r)$, $-\infty < t(r) < t_1(a)$, and a continuum $E_r \overset{\text{def}}{=} E_2(t(r))$ corresponding to $t(r)$ such that $1 \in E_r \subset \overline{U}$ and*

$$R(\Delta(E_r), 0) = r.$$

For $r \in (0, r_2(a))$ we define a continuum E_r by

$$E_r = E_2(t_2(a)) \cup \gamma'(r)$$

where $\gamma'(r)$ is an arc of the trajectory γ_4 of $Q(z)\,dz^2$ corresponding to the value $t = t_2(a)$ such that

$$\overline{\gamma}'(r) \cap E_2(t_2(a)) \ne \emptyset \quad \text{and} \quad R(\Delta(E_r), 0) = r.$$

Note that for $0 < a < 1$ the continuum E_r, $0 < r \le r(a)$, is the segment $[\tau_0, 1]$ where

$$\tau_0 = (2 - r - 2\sqrt{1 - r})/r.$$

Now we can give the solution of Problem G3.

Theorem 4. *Let $a \in U_+ \cup [-1,1] \setminus \{0\}$ and let $0 < r \le r(|a|)$. If the points a and 1 belong to a continuum $E \subset \overline{U} \setminus \{0\}$ satisfying the condition*

$$R(\Delta(E), 0) = r,$$

then

$$\omega_E(0) \ge \omega_{E_r}(0). \tag{2.19}$$

In the case $a \notin (-1,0)$ equality in (2.19) is attained only for the continuum E_r, in the case $-1 < a < 0$, — only for the continua E_r and $\widetilde{E}_r$, where $\widetilde{E}_r$ is symmetric to E_r with respect to $\mathbb{R}$.

The harmonic measure $\omega_{E_r}(0)$ decreases in $0 < r \le r_1(a)$ and increases in $r_1(a) \le r \le r(|a|)$.

Thus, in view of Theorem 4, Figures 1a – 1h show the dynamics of varying in the continua E_r extremal for Problem G3.

Proof. Suppose that E, $E \ne E_r$ and $E \ne \widetilde{E}_r$ in the case $-1 < a < 0$, satisfies the conditions of the theorem but

$$\omega_E(0) \le \omega_{E_r}(0). \tag{2.20}$$

Let $f(z)$ be a conformal mapping of $\Delta(E_r)$ onto the domain $\Delta(E)$, $f(0) = 0$. Then $|f'(0)| = 1$. Moreover, (2.20) implies that $\arg f'(0)$ can be normalized in such a way that the image $f(E_r)$ of the boundary set $E_r \subset \partial\Delta(E_r)$ includes the boundary set $E' = E \cap \partial\Delta(E)$.

Let us consider the case $r_2(a) \le r \le r_1(a)$ (other cases can be considered analogously). Let $\{T^*,\ D^*\}$ be an extremal pair of the problem on the maximum of the sum (2.9) for $t = t(r)$. (By Lemma 8 such a number $t(r)$ exists and is unique.)

By our hypothesis and due to the normalization of $f(z)$, the triangle $f(T^*)$ and the quadrangle $f(D^*)$ form an admissible pair for the problem considered. The conformal invariance of the module of a quadrangle, the lemma on the changing in the reduced module of a triangle under a conformal mapping ([39]), and the equality $|f'(0)| = 1$ yield the extremality of the pair $\{f(T^*),\ f(D^*)\}$ for the problem, which contradicts the uniqueness statement of Theorem 3. Hence the inequality (2.19) has been proved together with the equality cases.

To prove the monotonicity of $\omega_{E_r}(0)$ we suppose that there exist r_1 and r_2, $0 < r_1 < r_2 \le r_1(a)$ such that

$$\omega_{E_{r_1}}(0) = \omega_{E_{r_2}}(0).$$

Considering a conformal mapping of $\Delta(E_{r_1})$ onto $\Delta(E_{r_2})$ we obtain , as above, a contradiction with the extremality of the correspond pair of domains for the problem on the maximum of the sum (2.9).

The case $r_1(a) \le r_1 < r_2 \le r(|a|)$ is considered by the same way. $\square$

Let $S_1(\tau)$, $0 < \tau < 1$, be a class of functions $f(z)$ univalent in U and satisfying the conditions

$$f(0) = 0, \quad |f'(0)| = \tau, \quad |f(z)| < 1.$$

As a corollary of Theorem 4, we easily get the following theorem.

Theorem 5. *Let $0 < |a| < 1$, $\operatorname{Im} a \ge 0$, $0 < \tau \le r(|a|)$, let E be an arc of $\mathbb{T}$, and let $f(z) \in S_1(\tau)$. If $f(z) \ne a$ for all $z \in U$ and $1 \notin f(E) \subset \mathbb{T}$, then*

$$\operatorname{mes} E \le 2\pi(1 - \omega_{E_\tau}(0)). \tag{2.21}$$

For $a \notin (-1,0)$ the equality in (2.21) is attained only for a function mapping U onto $\Delta(E_\tau)$, and for $-1 < a < 0$ only for functions mapping U onto $\Delta(E_\tau)$ or onto $\Delta(\widetilde{E}_\tau)$.

Remark 2. In the case $0 < a < 1$, Theorems 4 and 5 are special cases of Theorem 3.4 of [41] where general inequalities for convex integral means of the harmonic measures are proved.

3. Harmonic measures of continua and their projections

3.1. Let A_x denote the projection of an arbitrary set A onto $\mathbb{R}$. We say that a continuum $\gamma \subset \overline{H}$ separates a point $z \in H$ from the projection $\gamma_x = [a, b]$ if $z \in \overline{\Pi(a, b)}$ and every curve $\alpha \subset \overline{\Pi(a, b)}$ joining z and γ_x intersects γ, see Figure 4a which also shows some notation used in the proof of Theorem 6; Figure 4b is related to Problem H (cf. Theorem 7).

Theorem 6. *If a continuum $\gamma \subset \overline{H}$ separates a point $z \in H$ from the projection γ_x, then*

$$\omega_\gamma(z) \geq \omega_{\gamma_x}(z). \tag{3.1}$$

If $\gamma_x \neq \mathbb{R}$, then the equality occurs only in the case $\gamma = \gamma_x$.

Proof. Let $\gamma_x = [a, b]$. For simplicity we assume that a and b are finite and $a = 0,\ b = 1$. Besides, we may assume that $\gamma \subset \Pi(0, 1) \cup (0, 1)$ (except the endpoints of γ). Let $\Delta(\gamma)$ be the connected component of $\Pi(0, 1) \setminus \gamma$ such that $z \in \Delta(\gamma)$. The boundary of $\Delta(\gamma)$ consists of two rays $l_1 = \{z = it,\ t \geq t_1\}$ and $l_2 = \{z = 1 + it,\ t \geq t_2\}$; $t_1, t_2 \geq 0$, and the continuum γ^+ (γ^+ is the upper shore of the slit along γ, see notation in Figure 4).

Let us consider the function

$$u(z) = \omega_\gamma(z) - \omega_{\gamma_x}(z)$$

harmonic in $\Delta(\gamma)$. If $\gamma^+ \neq [0, 1]$ and $z \in \gamma^+ \setminus [0, 1]$, then $u(z) > 0$.

Let $z \in l_1$. By a well-known result of Gaier [13] and Jenkins [25], (3.1) holds when $t_1 = 0$ (in this case, $0 \in \gamma$).

Let $t_1 > 0$ and $D = H \setminus [0, it_1]$. For $z \in D$ we have

$$\omega_\gamma(z) \geq \omega(z,\, \gamma,\, D).$$

Let $z' = it_0,\ t_0 > t_1$, and let

$$f(z) = t_0 \sqrt{z^2 + t_1^2} \Big/ \sqrt{t_0^2 - t_1^2}, \qquad f(z') = z'.$$

By the maximum principle for harmonic functions we have

$$\operatorname{Re} f(z) > \operatorname{Re} z \qquad \text{if} \qquad \operatorname{Re} z > 0.$$

Therefore, the map $\zeta = f(z)$ takes the continuum γ into a continuum γ_1 in $\overline{H}$ joining 0 with some point ζ_1 such that Re $\zeta_1 > 1$. Again using the results of [13,25] we conclude that (3.1) holds in the case considered. Certainly (3.1) also holds for $z \in l_2$.

Now by the maximum principle for harmonic functions

$$u(z) > 0$$

for all $z \in \Delta(\gamma) \cup l_1 \cup l_2$ if $\gamma \neq [0, 1]$. $\square$

Clearly the harmonic measure does not admit nontrivial lower bound if γ does not separate z from γ_x.

FIGURE 4: PROBLEMS HW AND H

3.2. The solution of the Hall problem is given by the following theorem.

Theorem 7.
Let γ be a continuum in $\overline{H}$ and $z \in H$, $\operatorname{Re} z \le 0$. Then

$$\omega_\gamma(z) \ge \omega_{\gamma_{\rm rad}}(z). \tag{3.2}$$

In the case $\operatorname{Re} z < 0$ *the equality here occurs only for continua* $\gamma \subset \mathbb{R}_+ \cup \{0\}$, *and in the case* $\operatorname{Re} z = 0$ *only for continua* $\gamma \subset \mathbb{R}_+ \cup \{0\}$ *or* $\gamma \subset \mathbb{R}_- \cup \{0\}$.

Under the additional condition $0 \in \gamma$ (or $\infty \in \gamma$) Theorem 7 has been proved by D. Gaier [13] and J. A. Jenkins [25].

Proof. Let $\gamma_{rad} = [r_1, r_2]$, $0 < r_1 < r_2 < \infty$ and $\gamma \ne \pm\gamma_{rad}$.
It is enough to consider the case $|z| \ge r_2 = 1$. For $|z| = r$, $r_1 \le r \le r_2$, one can reduce Problem H to Problem HW using the conformal map $\zeta = (z - r)/(z + r)$.
The function

$$u(z) = \omega_\gamma(z) - \omega_{\gamma_{\rm rad}}(z)$$

is harmonic in the domain $D_0 = \{z : |z| > 1, \ \pi/2 < \arg z < \pi\}$. Clearly

$$u(z) = 0 \qquad \text{if} \qquad \operatorname{Im} z = 0, \quad \operatorname{Re} z < -1. \tag{3.3}$$

As mentioned above

$$u(z) \ge 0 \qquad \text{if} \qquad z = e^{i\theta}, \quad \pi/2 < \theta \le \pi. \tag{3.4}$$

In view of the maximum principle it remains only to prove the inequality $u(it) \ge 0$ for $t \ge 1$. Let $h = \min\{\operatorname{Im} z : \ z \in \gamma\}$, $0 \le h \le r_1$, and let $\Delta = \{z : \ \operatorname{Im} z > h\}$. By Carleman's principle

$$\omega_\gamma(it) \ge \omega(it, \gamma, \Delta). \tag{3.5}$$

To estimate $\omega(it, \gamma, \Delta)$ we will consider a problem on the minimum of the harmonic measure $\omega_\gamma(it)$ among all continua $\gamma \in \overline{H}$ joining the arcs $C^1(h) = \{z \in \overline{H} : |z + ih| = r_1\}$ and

$C^2(h) = \{z \in \overline{H} : |z + ih| = r_2\}$ such that $\gamma \cap \mathbb{R}_+ \neq \emptyset$. Let $z_k = \gamma \cap C^k(h)$, $k = 1, 2$. In accordance with Lemma on a triad this problem is equivalent to the problem on the maximal module in a family $\mathcal{D}(z, z_0)$ of quadrangles Q in $H' = H \setminus \{z_1, z_2, it\}$ having distinguished sides s_1 and s_2 on $\mathbb{R}$ and separating the point it from z_1 and z_2 inside H'. Let $\widetilde{Q}$ be a quadrangle symmetric to Q with respect to $\mathbb{R}$; $D = Q \cup \widetilde{Q} \cup s_1 \cup s_2$. The set D is a doubly connected domain with

$$\operatorname{mod}(D) = (1/2)\operatorname{mod}(Q). \tag{3.6}$$

Having in mind to apply the estimates for the module of a doubly connected domain given in Section 1.1 let us consider the problem on the quantity $\mathcal{M}(\overline{z}_1, z_2; it, -it)$ (see Section 1.1) under the conditions

$$z_k \in C^k(h),\ k = 1, 2 \quad \text{and} \quad \operatorname{Re} z_k \geq 0.$$

Using the connected polarization of a doubly connected domain (see [42]) one can show that $\mathcal{M}(\overline{z}_1, z_2; it, -it)$ takes the maximal value only if

$$\operatorname{Re} z_2 \geq \operatorname{Re} z_1.$$

Consequently applying Lemmas 2 and 3 we get

$$\mathcal{M}(\overline{z}_1.z_2; it, -it) \leq \mathcal{M}(|z_1|, z_2; it, -it) \leq \mathcal{M}(|z_1|, \rho; it, -it), \tag{3.7}$$

where $\rho = |z_1| + r_2 - r_1$. Equality in the first case is attained only if $z_1 = |z_1|$ and in the second case only under the conditions $h = 0,\ z_2 = r_2$.

Let Q_0 be a quadrangle having the maximal module in the family $\mathcal{D}(|z_1|, \rho)$. Then (3.6) and (3.7) imply

$$\operatorname{mod}(Q_0) > \operatorname{mod}(Q).$$

In accordance with Lemma on a triad this is equivalent to the following inequality for harmonic measures

$$\omega(it, \gamma, \Delta) > \omega_{\gamma_0}(it_0), \tag{3.8}$$

where

$$\gamma_0 = [\rho_1, \rho_2],\ 0 \leq \rho_1 \leq \sqrt{r_1^2 - h^2},\ \rho_2 = \rho_1 + r_2 - r_1,\ t_0 = t - h.$$

After simple computations one finds

$$\begin{aligned} \omega_{\gamma_0}(it_0) =& (1/\pi)[\arctan(\rho_2/t_0) - \arctan(\rho_1/t_0)] \geq \\ & (1/\pi)[\arctan(\rho_2/t) - \arctan(\rho_1/t)] = \omega_{\gamma_{\mathrm{rad}}}(it). \end{aligned} \tag{3.9}$$

Now (3.5), (3.8), and (3.9) yield the inequality

$$u(it) > 0 \qquad \text{for} \qquad t \geq 1. \tag{3.10}$$

By the maximum principle and (3.3), (3.4), (3.10) we have $u(z) > 0$ for all $z \in \overline{D}_0 \setminus [-\infty, -1]$ and the proof is completed. $\square$

4. The Fuchs-Jenkins problem

Let $f:[0,1]\mapsto \mathbb{C}\setminus\{0\}$ define a continuous curve γ. By definition

$$\operatorname{ind}_0\gamma=(1/2\pi)(\arg f(1)-\arg f(0))$$

is the index of γ about the point $z=0$. The index $\operatorname{ind}_0 E$ of a continuum $E,\ 0\notin E$, about the origin is defined by

$$\operatorname{ind}_0 E=\max_{\gamma\subset E}\operatorname{ind}_0\gamma.$$

We will consider Problem FJ for an arbitrary half-integer index $n/2,\ n\in\mathbb{Z}\setminus\{0\}$. The main result of this section is

Theorem 8. *Let $\mathcal{E}_n$ be a family of continua $E\subset\overline{U}$ having index $n/2$ about the origin, $n\in\mathbb{Z}\setminus\{0\}$.*

There exists a unique (up to rotation about the origin and reflection in the real axis) continuum E_n which realizes the minimum of the harmonic measure $\omega_E(0)$ among all continua $E\in\mathcal{E}_n$. The minimizer E_n consists of two (closed) critical trajectories γ_0 and $\gamma_{1,n}$ of the quadratic differential $Q(z,(-1)^n r_n)\,dz^2$ where

$$Q(z,r)\,dz^2\;=-i(z+1)\,[z(z-1)(z-r)(1-rz)]^{-1}\;dz^2. \tag{4.1}$$

The trajectory $\gamma_{1,n}$ joins the point $z=-1$ with $(-1)^n r_n$ and $\gamma_0=\mathbb{T}^+=\mathbb{T}(0,\pi)$ is the upper semicircle. Here $r_n=(1-\nu_n)/(1+\nu_n)$ where the parameter $\nu_n,\ 0<\nu_n<1$ is defined from the equation

$$\int_0^1\frac{\sqrt{t}\,dt}{\sqrt{(1-t^2)(1-k^2t^2)}}=(n-1)\int_1^{1/k}\frac{\sqrt{t}\,dt}{\sqrt{(t^2-1)(1-k^2t^2)}}, \tag{4.2}$$

which has a unique solution in the interval $(0,1)$.

The numbers $r_n,\ n=1,2,\dots$ satisfy the relations

$$r_{n+1}<r_n\quad\text{and}\quad r_n\to 0\quad\text{as}\quad n\to\infty.$$

The minimal value of the harmonic measure is

$$\omega_{E_n}(0)=(4/\pi)\operatorname{arccot}\sqrt{k_n}, \tag{4.3}$$

where k_n is a solution of the well-known modular equation

$$\mathbf{K}(k)=(n-1/2)\mathbf{K}'(k). \tag{4.4}$$

Proof. The assumption $E\in\mathcal{E}_n$ means that there exists a Jordan curve $\gamma\subset E$ with endpoints a and b satisfying the condition $\operatorname{ind}_0\gamma=n/2$. Without loss of generality we may assume that $n\geq 1,\ E=\gamma$, and $0<(-1)^n b\leq a\leq 1$.

Let $d_k,\ k=1,2$, be the distance from the point $z=(-1)^k$ to γ along the segment $[-1,1]$. In the case where $d_1+d_2>0$ we consider the domain $U'=U\setminus[1-d_1,1]\setminus[-1,-1+d_2]$. Let $g(z)$ conformally map U' onto U so that $g(0)=0,\ g(1-d_1)=1$. Obviously the continuum $\gamma'=g(\gamma)$ lies in $\mathcal{E}_n$ and $\pm1\in\gamma'$. By Carleman's principle

$$\omega_\gamma(0)>\omega(0,\gamma,U')=\omega_{\gamma'}(0).$$

Hence to solve Problem FJ we may restrict ourselves by the continua γ including the points ± 1.

As can be easily verified, in the case $a < 1$ there exists a subarc $\gamma'' \subset \gamma$, $\gamma'' \neq \gamma$, having an endpoint at $z = 1$ if n is odd or having an endpoint at one of the points $z = \pm 1$ if n is even and such that

$$\operatorname{ind}_0 \gamma'' = n/2.$$

Again, Carleman's principle shows that we need only consider the curves starting from $z = 1$ and ending at some point $(-1)^n r$, where $0 < r \leq 1$ and $r = 1$ is possible only if $n = \pm 1$.

By Lemma on a triad, Problem FJ is equivalent to the problem on the maximal module of a quadrangle among all quadrangles having a topological structure determined by the family $\mathcal{E}_n$. By symmetry with respect to $\mathbb{T}$ the latter problem is equivalent to the problem on the maximal module among the corresponding doubly connected domains. Now applying the Jenkins Module theory (see Section 1.1 and [24,26] for examples of such applications) we conclude that each continuum which is extremal for Problem FJ consists of two critical trajectories of the quadratic differential

$$Q(z, r, \varphi)\, dz^2 = \frac{e^{-i\varphi/2}(z - e^{i\varphi})}{z(z-1)(z-(-1)^n r)(1-(-1)^n r z)}\, dz^2, \tag{4.5}$$

where $0 < r < 1$ and φ is a certain function of r and n. One of these trajectories is an arc $\gamma_0 = \mathbb{T}(0, \varphi)$ and the second trajectory, say $\gamma_{1,n} = \gamma_{1,n}(r, \varphi)$, joins the point $(-1)^n r$ and $e^{i\varphi}$.

Below we will consider the trajectory structure of (4.5) only on $\overline{U}$. In the case $n = 3$ and $\varphi = \pi$ this structure is depicted in Figure 5 which also illustrates some notation used below. The differential (4.5) has, besides, a critical trajectory $\gamma_{2,n}$ joining $z = 0$ to some point of $\mathbb{T}$. The domain $Q_{n,r} = U \setminus \gamma_{1,n} \setminus \gamma_{2,n}$ will be considered below as a quadrangle with distinguished sides on $\mathbb{T}$ additional to sides $\gamma_0 \cup \gamma_{1,n}$ and $\gamma_{2,n}$.

FIGURE 5: PROBLEM FJ

To find the parameter φ we will apply the gradient formula (1.3). For fixed r, $0 < r < 1$, let $\psi(\varphi)$, $0 \leq \psi(\varphi) < 2\pi$, denote the argument of the tangent to $\gamma_{1,n}$ at the point $z = (-1)^n r$. One can find from (4.5) that

$$\begin{aligned}\psi(\varphi) =& \lim_{z\to(-1)^n r}\{\arg{(z-(-1)^n r)Q(z,r,\varphi)}\}\\ =&\arg\left\{-e^{-i\varphi/2}(r-(-1)^n e^{i\varphi})\right\}.\end{aligned}$$

By the gradient formula (1.3) $\psi(\varphi)$ is the argument of the gradient of the module of the extremal doubly connected domain mentioned above. Here the module is considered as a function of a simple pole of $Q(z,r,\varphi)\,dz^2$ at the point $z=(-1)^n r$.

Using the connection between Problem FJ and the problem on the maximum of module of doubly connected domains we come to a necessary condition for a continuum to be extremal for Problem FJ,

$$\psi(\varphi)=(-1)^n\pi/2$$

or equivalently

$$\arg\left\{-e^{-i\varphi/2}(r-(-1)^n e^{i\varphi})\right\}=(-1)^n\pi/2 \qquad (\mathrm{mod}\ 2\pi). \tag{4.6}$$

Since

$$\begin{aligned}\psi'(\varphi)=&-\frac{1}{2}+(\arg{(r-(-1)^n e^{i\varphi})})'_\varphi\\ =&\frac{1-r^2}{2(1-2(-1)^n\cos\varphi+r^2)}>0\,,\end{aligned}$$

the function $\psi(\varphi)$ is monotone. Since in addition $\psi(\pi)=\arg{(-1)^n}-\pi/2$ and the variation of $\psi(\varphi)$ on the segment $[\pi,2\pi]$ is at most 2π, the equation (4.6) has a unique solution $\varphi=\pi$ common for all n. Therefore the quadratic differential (4.5) has the form (4.1) in the extremal case.

Here we should emphasize the remarkable fact, that $\varphi=\pi$, which allows us to give a complete solution of Problem FJ. Besides, it allows to compute the extremal value of the harmonic measure without proving the uniqueness of the extremal continuum and even without calculating the parameter r_n.

Let us consider the case of odd n, the case of even n requires a similar argument.

By Hersch's formula we may compute the module M_n of the quadrangle $Q_{n,r}$ corresponding to the extremal configuration. The function

$$f_{n,r}(z)=\int_0^z [Q(z,r)]^{1/2}\,dz$$

maps $Q_{n,r}$ conformally onto the rectangle $R(l,h)=\{z:\,|\mathrm{Re}\,z|<l,\ 0<\mathrm{Im}\,z<h\}$ where $2h$ is the $Q(\cdot,r)$-length of the lower semicircle $\mathbb{T}^-=\mathbb{T}\setminus\overline{H}$ and l is the $Q(\cdot,r)$-length of $\gamma_{2,n}$. In these notation

$$M_n=h/2l. \tag{4.7}$$

To find the $Q(\cdot,r)$-lengths h and $2l$ we need study the trajectory structure of $Q(z,r)\,dz^2$ more detailed. Since

$$\mathrm{Re}\,Q(x,r)\neq 0 \quad \text{if} \quad x\in\mathbb{R}\setminus\{0,r,1,1/r,-1\} \tag{4.8}$$

the trajectories and orthogonal trajectories of $Q(z,r)$ have no tangency with $\mathbb{R}$.

The diameter $[-1,1]$ subdivides $\gamma_{1,n}$ and $\gamma_{2,n}$ into subarcs, say $\gamma_{1,n}^{(k)}$ and $\gamma_{2,n}^{(j)}$, with endpoints a'_k, a''_k and b'_j, b''_j. We started the numeration from the points $z=0$ $(z=r)$ for $\gamma_{1,n}$ $(\gamma_{2,n})$.

I claim that the number of such arcs $\gamma_{1,n}^{(k)}$ is $(n-1)$ and the number of arcs $\gamma_{2,n}^{(j)}$ is n. Moreover, every such an arc laying in U_+ starts from $[-1,0]$ and ends on $[r,1)$. Analogously every arc lying in U_- starts from $[r,1]$ and ends on $[-1,0]$ except the arc $\gamma_{2,n}^{(n)}$ which ends at the point $z=e^{i\varphi}$.

Indeed, if $\gamma_{1,n}^{(1)}$ (the arc starting from the pole $z=r$) ends at $a''_1 \in (0,1)\setminus\{r\}$, then this arc with the segment $[r,a''_1]$ added bounds a simply connected domain Δ filled with trajectories of $Q(z,r)\,dz^2$. Hence by Pirl's lemma [34] there is a point of tangency of (r,a''_1) with a trajectory of $Q(z,r)\,dz^2$, contrary to (4.8). The same argument shows that $\gamma_{2,n}^{(1)}$ ends at a point $b''_1 \in (-1,0)$.

If an arc $\gamma_{j,n}^{(s)}$ does not satisfy the codition considered, then it has both ends in the interval $(-1,0)$ or in $(r,1)$. Again Pirl's lemma yields the existence of a tangency point of a trajectory of $Q(z,r)\,dz^2$ with $\mathbb{R}$, which contradicts (4.8).

Now the established properties of the arcs $\gamma_{j,n}^{(s)}$ and the relation

$$\operatorname{ind}_0(\gamma_{1,n}\cup\gamma_0)=n/2$$

imply the assertion on the number of these arcs.

Let Δ_j^+, $j=1,\dots,n$, and Δ_k^-, $k=1,\dots,n+1$, denote simply connected domains obtained respectively from U_+ and U_- by deleting the trajectories $\gamma_{1,n}$ and $\gamma_{2,n}$. We numerate Δ_j^+ and Δ_k^- in a natural order so that $r\in\partial\Delta_1^+$ and $0\in\partial\Delta_1^-$.

Returning to calculating the $Q(\cdot,r)$-lengths h and $2l$ choose one of the domains, say Δ_j^+. The boundary $\partial\Delta_j^+$ consists of two segments of $\mathbb{R}$, say $[a,b]$ and $[c,d]$, and of two arcs γ' and γ'' where $\gamma''\subset\gamma_{2,n}$ and $\gamma'\subset\gamma_{1,n}$ if $j=1,\dots,n-1$ and $\gamma'=\mathbb{T}^+$ if $j=n$ (see Figure 5b).

By Cauchy's theorem we have

$$\int_{\partial\Delta_j^+}[Q(z,r)]^{1/2}\,dz = I_1+I_2+I_3+I_4=0, \tag{4.9}$$

here $[Q(z,r)]^{1/2}>0$ if $z\in\gamma'\cup\gamma''$ and I_1, I_2, I_3, and I_4 denote the integrals along $[a,b]$, γ', $[c,d]$, and γ'' respectively.

Since the segments $[a,b]$ and $[c,d]$ connect opposite sides of the quadrangle $Q_{n,r}$, we find that

$$\operatorname{Im} I_1 = -\operatorname{Im} I_3 = h. \tag{4.10}$$

Now using the explicit expression of $Q(z,r)\,dz^2$ $(\varphi=\pi\,!)$ one finds

$$\operatorname{Re} I_1 = -\operatorname{Re} I_3. \tag{4.11}$$

(4.9)–(4.11) imply

$$I_2=-I_4.$$

Therefore the arcs γ' and γ'' have the equal $Q(\cdot,r)$-lengths. The latter, being hold true for all domains Δ_j^+, implies

$$|\gamma_{1,n}^{(k)}|_{Q(\cdot,r)} = |\gamma_{2,n}^{(j)}|_{Q(\cdot,r)} = |\mathbb{T}^+|_{Q(\cdot,r)} \tag{4.12}$$

for all $\gamma_{1,n}^{(k)}$ and $\gamma_{2,n}^{(j)}$ in the upper semidisk.

Analogously the equalities

$$|\gamma_{1,n}^{(k)}|_{Q(\cdot,r)} = |\gamma_{2,n}^{(j)}|_{Q(\cdot,r)} = |\gamma_{1,n}^{(n)}|_{Q(\cdot,r)} \tag{4.13}$$

hold for all arcs in U_- except $\gamma_{2,n}^{(n+1)}$.

Applying Cauchy's theorem to the domain Δ bounded by $\gamma_{1,n}^{(n)}$, $\mathbb{T}^-$ and the segment $[d,1]$ which connects the point $z=1$ with $\gamma_{1,n}^{(n)}$ we get as before

$$|\gamma_{1,n}^{(n)}|_{Q(\cdot,r)} = |\mathbb{T}^-|_{Q(\cdot,r)}.$$

By symmetry of $Q(z,r)\,dz^2$ with respect to $\mathbb{R}$

$$|\mathbb{T}^+|_Q(\cdot,r) = |\mathbb{T}^-|_Q(\cdot,r) = 2h. \tag{4.14}$$

Now (4.12)–(4.14) imply

$$|\gamma_{j,n}^{(s)}|_{Q(\cdot,r)} = 2h \tag{4.15}$$

for all $\gamma_{j,n}^{(s)}$ except $\gamma_{2,n}^{(n+1)}$ which satisfies the relation

$$|\gamma_{2,n}^{(n+1)}|_{Q(\cdot,r)} = h. \tag{4.16}$$

It follows from (4.15), (4.16)

$$2l = |\gamma_{2,n}|_{Q(\cdot,r)} = 2\sum_{k=1}^{n} |\gamma_{2,n}^{(k)}|_{Q(\cdot,r)} = 2(2n-1)h \tag{4.17}$$

for all odd n. The similar consideration shows that (4.17) still holds for all even n too.

Now (4.7) and (4.17) imply

$$M_n = 1/2(2n-1)$$

that via Hersch's formula (1.13) leads to (4.3) and (4.4).

We will give two proofs of the uniqueness of the extremal configuration.

(i) Suppose there are two continua γ_1 and γ_2 in $\mathcal{E}_n$ minimizing the harmonic measure in Problem FJ. Then γ_1 and γ_2 have the form described in Theorem 8.

Let r_1 and r_2, $r_1 \neq r_2$, be poles of the quadratic differentials associated with γ_1 and γ_2 respectively. For some positive q_j, $j=1,2$, the function $q_j f_{n,r_j}(z)$ maps Q_{n,r_j} conformally onto the rectangle $R(2n-1,1)$. The relations (4.12)–(4.15) imply the equalities

$$q_j f_{n,r_j}(-1+0) = 2n-3+i, \qquad q_j f_{n,r_j}(r_j) = -1+i,$$

which show that the composition

$$\tau(z) = f_{n,r_1}^{-1}(q_1 q_2^{-1} f_{n,r_2}(z))$$

is continued to a conformal automorphism of U such that

$$\tau(0)=0, \quad \tau(1)=1$$

and therefore $\tau(z) = z$. In particular, $r_1 = r_2$ and the uniqueness of the extremal continuum is proved.

The presented method of proving of the uniqueness due to Jenkins is applied to many other problems. However it seems to be not convenient for computations. To compute r_n and to prove the inequality $r_n > r_{n+1}$ we apply another method more adapted to the problem under consideration.

(ii) First we use Möbius mapping

$$z \mapsto \frac{z+r}{1+rz}$$

to reduce the quadratic differential $Q(z, -r_n)\, dz^2$ in the case of even n to the form

$$\widetilde{Q}(z)\, dz^2 = \frac{-i(z+1)}{z(z-1)(z-r_n)(1-r_n z)}\, dz^2$$

(now the differentials have the unified form for all odd and even n). Using established earlier properties of the geometrical structure of the trajectories and relations (4.10) (or similar relations in the case of even n) we get the equation

$$\operatorname{Im} \int_{-1}^{0} [Q(x,r)]^{1/2}\, dx = (n-1) \operatorname{Im} \int_{0}^{r} [Q(x,r)]^{1/2}\, dx \tag{4.18}$$

for determining the parameter r_n. Here the branches of the square roots are choosen such a way that $\operatorname{Im}[Q(x,r)]^{1/2} > 0$ for $x \in (-1,0)$ and for $x \in (0,r)$.

Substituting (4.1) into (4.18) and changing the variable of integration we transform (4.18) into

$$\begin{aligned}(n-1) \int_0^1 \left[\frac{1+rx}{x(1-x)(1-rx)(1-r^2x)}\right]^{1/2} dx = \\ \int_0^1 \left[\frac{1-x}{x(x+1)(x+r)(1+rx)}\right]^{1/2} dx.\end{aligned} \tag{4.19}$$

Clearly, the right side of (4.19) decreases from $+\infty$ to a finite number and the left side increases from some number to $+\infty$ when r runs from 0 to 1. Therefore for all $n > 1$ the equation (4.19) has a unique solution $r_n \in (0,1)$. Moreover, (4.19) implies immediately that $r_{n+1} < r_n$ and $r_n \to 0$ when $n \to \infty$.

After changing a variable in (4.19) we come to the desired equation (4.2). The proof is complete. $\square$

Remark 3. As it was found in [33] and in [29] the explicit value of r_2 is $7-4\sqrt{3}$. Moreover, for $n = 2$ the extremal quadratic differential can be reduced to another one having a radial structure of critical trajectories. This implies that equation (4.2) (with $n = 2$) has a unique solution

$$\nu_2 = \sqrt{3}/2$$

in the interval $(0,1)$.

It seems there is no such a simple expression for ν_n and r_n when $n > 2$.

5. Further results and questions

5.1. In this Section we establish some results concerning Problem FRW on the maximal index φ_0 (see Introduction). Of course a solution of Problem FJ, being obtained for an arbitrary index

θ, $0 < \theta < 2\pi$, gives a solution of Problem FRW too. Now we describe some properties of extremizes of these problems which follow from the Jenkins Module theory.

First, we note the existence of extremal continua, each of those is a Jordan arc in a sector of opening θ, say in the sector $U(0,\theta)$, having ends at some points $a_1 = r_1$ and $a_2 = r_2 e^{i\theta}$, where $0 < r_1,\ r_2 \le 1$.

Next, a well-known argument shows to solve problems considered it is enough to deal with continua which intersect $\mathbb{T}$. Indeed, if $E \in \mathcal{E}(\theta)$, $E \subset U$ and $r = \max\{|z| :\ z \in E\}$, $0 < r < 1$, then the continuum $E_r = \{z :\ rz \in E\}$ belongs to $\mathcal{E}(\theta)$ and, moreover,

$$\omega_E(0) > \omega(0, E, U_r) = \omega_0(E_r). \tag{5.1}$$

Further, in accordance with Lemma on a triad, the problem on the minimum of the harmonic measure is equivalent to the problem on the maximal module in the family Υ of quadrangles $G \subset U \setminus \{0, a_1, a_2\}$ having distinguished sides on $\mathbb{T}$ and such that curves $\gamma \subset G$ joining the distinguished sides are homotopic to a slit along the radius $L(\psi) = \{z = te^{i\psi},\ 0 < t < 1\}$, where $0 < \psi < 2\pi$, on $\overline{U} \setminus \{0, a_1, a_2\}$.

Certainly, the latter problem is a special case of the Jenkins Module problem , it can be considered as a problem on the maximal module in a corresponding family of doubly connected domains on the sphere $\overline{\mathbb{C}}$. This implies that a necessary condition of extremality can be obtained from the gradient theorem (1.3).

Summarizing these results we get

Lemma 9 (Preparation lemma). *A continuum E minimizing the harmonic measure in the family $\mathcal{E}(\theta)$, $\pi < \theta < 2\pi$, consists of at most three critical trajectories γ_1, γ_2, γ_3 of a quadratic differential $Q(z)\,dz^2$ having the form (up to rotation about the origin)*

$$Q(z)\,dz^2 = -\frac{e^{-i(\varphi_1+\varphi_2)/2}(z - e^{i\varphi_1})(z - e^{i\varphi_2})}{z(z - r_1)(1 - r_1 z)(z - r_2 e^{i\theta})(1 - r_2 e^{-i\theta})}\,dz^2. \tag{5.2}$$

Here the parameters φ_1 and φ_2 satisfy the conditions

$$0 \le \varphi_1 \le \theta - \pi, \qquad \pi \le \varphi_2 \le \theta$$

and at least one of the following

$$\varphi_1 \le (\theta - \pi)/2 \qquad \text{or} \qquad \varphi_2 \ge (\theta + \pi)/2.$$

Moreover,

$$\arg\left\{\frac{(r_1 - e^{i\varphi_1})(r_1 - e^{i\varphi_2})}{(r_1 - r_2 e^{i\theta})(1 - r_1 r_2 e^{-i\theta})}\right\} = (\varphi_1 + \varphi_2 - \pi)/2 \qquad (\mathrm{mod}\ 2\pi),$$

$$\arg\left\{\frac{(r_2 e^{i\theta} - e^{i\varphi_1})(r_2 e^{i\theta} - e^{i\varphi_2})}{(r_1 - r_2 e^{i\theta})(1 - r_1 r_2 e^{i\theta})}\right\} = (\varphi_1 + \varphi_2 + \pi)/2 \qquad (\mathrm{mod}\ 2\pi).$$

The trajectories γ_1, γ_2 join the simple poles $z = r_1$ and $z = r_2 e^{i\theta}$ with the zeros $z = e^{i\varphi_1}$ and $z = e^{i\varphi_2}$ respectively. The trajectory γ_3 is the arc $\mathbb{T}(\varphi_1, \varphi_2)$. In the cases $r_1 = 1$ and/or $r_2 = 1$ we have $\varphi_1 = 0$ and/or $\varphi_2 = \theta$ and the corresponding trajectories γ_1 and/or γ_2 are absent.

Figure 6: Problem FRW, Preparation lemma

There exists a unique domain G in the domain structure of the quadratic differential (5.2) considered on $\overline{U}$. This domain G is the extremal quadrangle of the corresponding problem on the maximal module of a quadrangle.

5.2. Problem FRW on the maximal index φ_0 such that the harmonic measure admits "nice" estimate (5) was posed in [11] where the inequality $\varphi_0 \geq \pi$ was proved. This question was also discussed by Marshall and Sundberg [33], and by Jenkins [29]. Below we shall give a lower bound for φ_0 based on a result of D. Hamilton [15] which is much better then π (nevertheless not sharp).

Let $0 < \rho \leq 1$, $\pi \leq \theta < 2\pi$, and $a = \rho e^{i\theta}$. Let $\Gamma(a)$ denote the class of closed Jordan curves homotopic to a slit along the ray $\{z : \arg z = \psi\}$, $\theta < \psi < 2\pi$, on $\mathbb{C} \setminus \{0, 1, a\}$. We shall consider the problem on the maximal module $\mathcal{M}$ in the family of doubly connected domains associated with $\Gamma(a)$ which separate the pair 0, ∞ from 1, a.

Let $\lambda(\tau)$ be the function mapping the curvilinear triangle $\{\tau : 0 < \operatorname{Re} \tau < 1, |\tau - 1/2| > 1/2\}$ onto H such that ∞ maps to 1. Then extend λ to H. Let

$$\mu(r) = \frac{\mathbf{K}(r)}{\mathbf{K}'(r)}.$$

The following lemma is just a reformulation of the result by Hamilton [15, Theorem 1] in terms of the moduli.

Lemma 10 (Hamilton). *There exists a value $\theta_1 \approx 4.602$ such that for $\pi < \theta \leq \theta_1$ the following inequality holds*

$$\mathcal{M}(re^{i\theta}) \leq \mathcal{M}(e^{i\theta}) \quad \textit{if} \quad 0 < r < 1.$$

For $\theta_1 < \theta < 2\pi$

$$\max_{0<r<1} \mathcal{M}(re^{i\theta}) > \mathcal{M}(e^{i\theta}).$$

The transition value of θ is defined by the equality

$$\theta_1 = 4 \arcsin r_1$$

where $r_1 \approx 0.913$ *solves the equation*

$$\lambda(\tau)\,\lambda''(\tau) - (\lambda'(\tau))^2 = 0, \qquad \tau = 1/2 + 2i\mu(r_1).$$

Theorem 9. *The maximal index of Problem FRW is no less then* $\theta_1 \approx 4.602$.

Proof. Let $\pi < \theta \leq \theta_1$. By Lemma on a triad and Preparation lemma we need only prove the inequality

$$\mathrm{mod}\,(G) \geq \mathrm{mod}\,(G(\theta)). \tag{5.3}$$

Here $G \in \Upsilon$ separates the point $z = 0$ from the points $a_1 = r_1,\ a_2 = r_2 e^{i\theta},\ 0 < r_1,\ r_2 \leq 1$. $G(\theta)$ denotes the quadrangle $U \setminus L(\pi + \theta/2)$ having the arcs $\mathbb{T}(\theta, \pi + \theta/2)$ and $\mathbb{T}(\pi + \theta/2, 2\pi)$ as its distinguished sides. Extend G across the distinguished sides to a doubly connected domain D symmetric with respect to $\mathbb{T}$. Then

$$\mathrm{mod}\,(G) = 2\,\mathrm{mod}\,(D). \tag{5.4}$$

Let $\Gamma(a_1, a_2)$ be the class of closed Jordan curves on $\mathbb{C} \setminus \{0, a_1, a_2\}$ homotopic to the slit along the ray $\{z :\ \arg z = \pi + \theta/2\}$. Let $M(a_1, a_2)$ be the maximal module in the family of doubly connected domains associated with $\Gamma(a_1, a_2)$. Clearly,

$$M(a_1, a_2) \geq \mathrm{mod}\,(D) \tag{5.5}$$

and equality here holds only in the case $|a_1| = |a_2| = 1$.

Using Lemma 10 we get the relation

$$M(a_1, a_2) \leq M(1, e^{i\theta}) = (1/2)\,\mathrm{mod}\,G(\theta),$$

that, together with (5.4) and (5.5), gives (5.3). $\square$

As the inequality (5) does not hold for large indices in general, then it is natural to consider subclasses of continua for which (5) is true. Below we give some results of this kind. The first one concerns continua with "symmetric" endpoints.

Theorem 10. *If a continuum* E *contains a Jordan curve* γ *ended at* $a_1 = r$ *and* $a_2 = re^{i\theta}$, $0 < \theta < 2\pi$, *such that* $\mathrm{ind}_0 \gamma = \theta$, *then*

$$\omega_E(0) \geq \theta/2\pi. \tag{5.6}$$

Equality in (5.6) is attained only in the case when E *is an arc of* $\mathbb{T}$.

The proof is similar to one of Theorem 9 but much simpler because it does not use the deep result of Hamilton.

The following theorem also may be useful to estimate the harmonic measure of continua with large indices.

Theorem 11. *Let* h *be the hyperbolic geodesic in* U *ended at the points* $z = e^{\pm i\alpha}$, $0 < \alpha < \pi/2$ *and let* $U_{1,\alpha}$ *be the connected component of* $U \setminus h$ *such that* $0 \in U_{1,\alpha}$. *Let a Jordan arc* $\gamma \subset U_{1,\alpha}$ *end at points* $a, b \in h$ *and let* $D(\gamma, h)$ *be the domain in* $\mathbb{C}$ *bounded by* γ *and by the arc* $h_{a,b} \subset h$ *joining the points* a *and* b. *Then*

$$\omega_\gamma(z) \geq \omega(z, \mathbb{T}(\alpha, 2\pi - \alpha), U) = 1 - \alpha/\pi \tag{5.7}$$

for all points $z \in D(\gamma, h)$.

Equality in (5.7) occurs only in the case $\gamma = \mathbb{T}(\alpha, 2\pi - \alpha)$.

Proof. Let us consider the function

$$\omega(z) = \omega_\gamma(z) - \omega_{\mathbb{T}(\alpha,2\pi-\alpha)}(z),$$

which is harmonic in $D(\gamma, h)$. Clearly,

$$\omega(z') \geq 0$$

for $z' \in \gamma$.

Let $z' \in h_{a,b}$. For suitable $\beta \in \mathbb{R}$ the Möbius mapping

$$z \mapsto e^{i\beta} \frac{z - z'}{1 - \overline{z}' z}$$

takes h into the diameter $[-1, 1]$ and γ into some curve $\gamma' \subset U_+$ having ends on the radii $[-1, 0]$ and $(0, 1]$.

By Theorem 1 in [11]

$$\omega_{\gamma'}(0) \geq \omega(0, \mathbb{T}(0, \pi), U),$$

that by the conformal invariance of the harmonic measure implies the inequality

$$\omega(z') \geq 0 \qquad \text{for} \qquad z' \in h_{a,b}.$$

Now the maximum principle for harmonic functions shows that $\omega(z) \geq 0$ for all $z \in D(\gamma, h)$. Besides, if $\omega(z_0) = 0$ at some point z_0 then $\omega(z) \equiv 0$ and therefore $\gamma = \mathbb{T}(\alpha, 2\pi - \alpha)$. The proof is complete. □

5.3. The last theorem of the paper solves the "local" version of Problem FJ for an arbitrary index $< 2\pi$. It seems an important step in studying Problems FJ and FRW.

Theorem 12. *Let a Jordan arc $\gamma \subset \overline{U}$ have endpoints $a_1 = r_1,\ a_2 = r_2 e^{i\theta},\ 0 < r_1,\ r_2 \leq 1$, and let $\mathrm{ind}_0 \gamma = \theta,\ 0 < \theta < 2\pi$. Then for all $r_1,\ r_2$ closed enough to 1 and such that $r_1 \cdot r_2 < 1$ the following inequality holds*

$$\omega_\gamma(0) > \theta/2\pi. \tag{5.8}$$

Proof. Let $r_k = 1 - \varepsilon_k$ where $k = 1, 2$ and $\varepsilon_k > 0$ is small enough. Let $E(\varepsilon_1, \varepsilon_2)$ be a continuum minimizing the harmonic measure among all curves E ended at the points $a_1 = r_1,\ a_2 = r_2 e^{i\theta}$ and such that $\mathrm{ind}_0 E = \theta$. Preparation lemma and usual arguments of proving the continuity dependence of quadratic differentials on parameters (see [32,36]) show that $E(\varepsilon_1, \varepsilon_2)$ consists of three critical trajectories $\gamma_k = \gamma_k(\varepsilon_1, \varepsilon_2),\ k = 1, 2, 3$, of a quadratic differential $Q(z)\,dz^2$ having the form (5.2) where $e^{i\varphi_1} = e^{i\delta_1},\ e^{i\varphi_2} = e^{i(\theta - \delta_2)}$ and $\delta_k > 0,\ \delta_k \to 0$ when $\varepsilon_k \to 0$ (see Figure 6).

Let α_1 denote the angle at $z = r_1$ formed by $\mathbb{R}_+$ and the trajectory γ_1 starting from this point. Analogously, α_2 is the angle formed by the trajectory γ_2 starting from $z = r_2 e^{i\theta}$ and by the ray $\{z :\ \arg z = \theta\}$.

I claim that $0 < \alpha_1 < \pi/2$ for all sufficiently small ε_1 and for all $r_2,\ 0 < r_2 \leq 1$. Of course, the similar assertion holds for the angle α_2 too.

If our assumption is false, then there are sequences $\varepsilon_{1,n} = 1 - r_{1,n} \to 0$ and $r_{2,n} = 1 - \varepsilon_{2,n} \to r_{2,\theta}$ as $n \to \infty$ such that

$$\pi/2 \leq \alpha_{1,n} \leq \pi \tag{5.9}$$

for corresponding values of $\alpha_{1,n}$. (The inequality $0 < \alpha_1 < \pi$ easily follows via the auxiliary mapping $z \mapsto \sqrt{z}$ and the connected polarization of a quadrangle.)

Changing the variable

$$z = \frac{(1 + r_{1,n}) + (1 - r_{1,n})\zeta}{(1 + r_{1,n}) - (1 - r_{1,n})\zeta} \tag{5.10}$$

and multiplying by a positive constant one can bring the differential $Q(z)\,dz^2$ to the form

$$Q_n(\zeta)\,d\zeta^2 = \frac{-i(\zeta - ip_n)(1 - i\zeta/q_n)}{(\zeta^2 - 1)(1 - \zeta^2/a_n^2)(1 - \zeta/b_n)91 + \zeta/\overline{b}_n)}\,d\zeta^2, \tag{5.11}$$

where $p_n > 0$, $q_n > 0$, $a_n > 0$, and b_n are the preimages of the points $e^{i\varphi_1}$, $e^{i\varphi_2}$, 0 and $r_{2,n}e^{i\theta}$ under the mapping (5.10).

Let

$$Q_0(\zeta)\,d\zeta^2 = \frac{-i(\zeta - ip_0)}{\zeta^2 - 1}\,d\zeta^2 \tag{5.12}$$

be the limit quadratic differential for (5.11) as $n \to \infty$. One can easily show that $0 < p_0 < \infty$ and that $Q_0(\zeta)\,d\zeta^2$ has three critical trajectories starting from the simple zero $z = ip_0$. The trajectory structure of the quadratic differentials (5.11) and (5.12) and the corresponding notation are shown in Figures 7.

Now we shall prove the uniqueness the value p_0 such that the quadratic differential (5.12) has the trajectory structure mentioned above. To do this we apply the method used for similar problems in [14,38]. Suppose that there are two such values p_0' and p_0'', $p_0' \neq p_0''$, and let $Q_0'(\zeta)\,d\zeta^2$ and $Q_0''(\zeta)\,d\zeta^2$ be the corresponding quadratic differentials. As follows from the relation

$$\mathrm{Im}\,[Q_0'(\zeta)/Q_0''(\zeta)] = 0,$$

or from the equivalent relation

$$(p_0'' - p_0')\,\mathrm{Re}\,\zeta = 0,$$

the points of tangency of the trajectories of the quadratic differential $Q_0'(\zeta)\,d\zeta^2$ with the trajectories of $Q_0''(\zeta)\,d\zeta^2$ lies on the imaginary axis.

On the other hand, an obvious argument using Pirl's lemma [34] and taking the trajectory structure into account shows that under our assumption there exists a point of tangency, say c, such that $\mathrm{Re}\,c \neq 0$. This contradiction proves the uniqueness of p_0 (which implies that $p_1 \to p_0$ as $\varepsilon_1 \to 0$).

Computing the angle α_0 at $\zeta = -1$ formed by $\mathbb{R}_+$ and by the critical trajectory l_1 (see Figure 7b) we find the equality

$$\alpha_0 = \arg\{i(1 - ip_0)\} \qquad (\mathrm{mod}\ 2\pi),$$

which shows that $0 < \alpha_0 < \pi/2$.

The last inequality, together with the limit relation $\alpha_{1,n} \to \alpha_0$ as $n \to \infty$, leads to a contradiction to the hypothesis (5.9). Hence the assertion on α_1 has been proved. Now the theorem follows from the gradient formula (1.3). $\square$

Remark 4. In fact we have proved a little more than (5.8): it is proved that the harmonic measure of the extremal continua decreases when endpoints of these continua run to the unit circle $\mathbb{T}$ and when they are sufficiently close to $\mathbb{T}$.

This remark holds true for an arbitrary index θ, $0 < \theta < 2\pi$. So the proof of Theorem 12 makes clear the reason of difficulties in computing the transition phenomenon φ_0: there is no

FIGURE 7: LOCAL VERSION OF PROBLEM FRW

singularity on $\mathbb{T}$ which separates one qualitative situation from another. For large values of θ there exist at least two local minima of the harmonic measure (one of them is at the point $e^{i\theta}$). Thus to solve the problem under consideration we must select the least of these minima, which is a problem of the global (and not local) analysis. Thus, in our opinion, computation of φ_0 is possible only with the aid of a computer.

5.4. The extremal problems concerning the harmonic measure are widely presented in literature. Besides the above mentioned papers, some new problems can be found in [1, 8, 41]. By the way, we note that a solution of Problem 8.1 of [8] (in a more general form) based on the $*$-function method of A. Baernstein and A. E. Fryntov is published in [43].

In conclusion we would like to pose the following

Open Problem. Let $a_1, \dots, a_n \in U_R$ and let γ_k be a continuum in U_R joining a_k to $\mathbb{T}_R$.

(1) Give a qualitative description of configurations $\cup_{k=1}^n \gamma_k$ that realize the minimum in

$$\Phi_R(a_1, \dots, a_n) = \min_{\gamma_1, \dots, \gamma_n} \omega(0, \cup_{k=1}^n \gamma_k, U_R).$$

(2) Let $a_k^* = e^{i2\pi(k-1)/n}$, $k = 1, \dots, n$. Find the asymptotic behavior of $\Phi_R(a_1^*, \dots, a_n^*)$ for $R \to \infty$ and for $R \to 1+0$.

(3) Find $\Phi_1(a_1, a_2)$ for $a_1,\ a_2 \in U \setminus \{0\}$ (or at least in the case $a_1 = -a_2$).

Initially the author conjectured that the extremal configuration of the problem on $\Phi_1(ra_1^*, \dots, ra_n^*)$, $0 < r < 1$, is the "regular" system of n equidistributed segments (it is well known that such a system realizes the maximum of the harmonic measure in the problem posed by Gonchar, see [8]). Our conjecture was based on the belief that the general principle "*symmetrical data imply symmetrical consequences*" can not be violated in such problems. Surprisingly enough this conjecture is wrong as follows from the next simple example communicated to us by A. E. Fryntov.

Example. Let $E(r,n) = \cup_{k=1}^{n}[ra_k^*, a_k^*]$ and let $F(r,n) = \mathbb{T}(\pi/n, (2n-1)\pi/n) \cup [-1,-r]$. Then for fixed r, $0 < r < 1$, and $n \to \infty$

$$\omega(0, E(r,n), U) = 1 - \frac{4}{\pi} r^{n/2} + o(r^{n/2}), \tag{5.13}$$

$$\omega(0, F(r,n), U) \le 1 - \frac{\sqrt{r}}{4(1+r)} \frac{\pi}{n^2} + o(n^{-2}), \tag{5.14}$$

and, therefore,

$$\Phi_1(ra_1^*, \dots, ra_n^*) < \omega(0, E(r,n), U)$$

for all sufficiently large n.

The relation (5.13) can be easily obtained from the identity $\omega(0, E(r,n), U) = \omega(0, [r^n, 1], U)$ via a conformal mapping of the domain $U \setminus [r^n, 1]$ onto H.

Let $D_0 = \mathbb{C} \setminus (-\infty, -r) \setminus \mathbb{T}(\pi/n, (2n-1)\pi/n)$. By Carleman's principle

$$\omega(0, F(r,n), U) < \omega(0, F(r,n), D_0),$$

that leads to (5.14) after some routine calculations concerning the conformal mapping of D_0 onto H.

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